"Le equazioni di Hamilton-Jacobi che si integrano per separazione di variabili," Rend. Circ. mat. Palermo (1) **33** (1912), 341-351.

# The Hamilton-Jacobi equations that can be integrated by separation of variables

By **Francesco Aurelio Dall'Acqua** (Mantua)

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Translated by D. H. Delphenich

## Preface

In 1891, **Paul Stäckel** ([1]) posed the problem: "Which **Hamilton-Jacobi** equations can be integrated by means of separating the variables?" That was a generalization of a question that **Liouville** had posed many years before. **Stäckel** found a noteworthy solution (namely, in the orthogonal case) and gave the conditions for the solution to be possible.

Years later ([2]), **Levi-Civita** deduced those conditions in a more convenient form by separating the terms that involved the potential from the other ones and solved the problem in another noteworthy case.

In 1908 ([3]), I myself exhausted the problem in three variables by fortuitous artifices, but laborious calculations.

Only a year later, **Burgatti** ([4]), guided by a brilliant intuition "more than by rigorous logic" (as he phrased it), gave $n + 1$ types of solutions to the problem in $n$ variables without, however, succeeding in proving that they would be the only possible ones "although I have no doubt of that (he stated)."

We shall revisit the problem in order to solve it with the rigor that would ensure the complete exhaustion of the argument *and recover all of **Burgatti**'s types, and only them.*

([1]) **P. Stäckel**, "Ueber die Integration der HAMILTON-JACOBI'schen Differentialgleichung mittels Separation der Variablen," Habilitationsschrift, Halle a. S., 1891.

([2]) **T. Levi-Civita**, "Sulla integrazione della equazione di HAMILTON-JACOBI per separazione di variabili," Math. Ann. **59** (1904), 383-397.

([3]) **F. A. Dall'Acqua**, , "Sulla integrazione della equazione di HAMILTON-JACOBI per separazione di variabili," Math. Ann. **67** (1908), 398-415.

([4]) **P. Burgatti**, "Determinazione dell'equazione di HAMILTON-JACOBI integrabili mediante la separazione delle variabili," Rend. R. Accad. Lincei (Roma), vol. XX, 1st semester 1911, pp. 108-111.

If (as is natural) we say *geodetic* to mean the cases in which no forces are acting (i.e., the cases in which the trajectories of motion are geodetic) then, with **Levi-Civita**, we will find that: *If a dynamical problem with constraints that are independent of time is the one being studied then it will only be the corresponding* ***geodetic problem*** *in which the potentials are annulled with no further analysis.*

As is known, the converse proposition is not true. There exist geodetic problems that cannot be attributed a potential, although they have the type that we have proposed to study (i.e., types in which the potential is actually zero). I call them *essentially geodetic*, and I shall prove that *the only essentially geodetic case is the* ***Levi-Civita*** *case.*

I shall also prove that: *In the other cases, the most generic expression for the potential depends upon a certain number of arbitrary functions, each of which has only one variable, and that number characterizes those cases.*

If that number is $n$, i.e., *if the potential can depend upon all of the variables, then one will have the* ***Stäckel*** *case.*

As for the methods: We shall take from **Burgatti** the idea of determining $n$ first integrals of the equation rather than the coefficients of the equation: Eliminating $n - 1$ constants from them will lead to the desired equation. I shall benefit from the divisibility criteria of entire (*intere*) functions that proved so fruitful for **Levi-Civita**, and finally I shall recall from my own research into the subject the idea of obtaining the arbitrary constants and the arbitrary functions of only one variable by annulling all, or all but one, of the independent variables: Those constants and functions will then be determined in a more obvious way, and an intimate link with the problem being treated will result from that.

## The equations of the problem.

**1.** – Let:

$$H\left(x_1, x_2, \dots, x_n; \frac{\partial W}{\partial x_1}, \frac{\partial W}{\partial x_2}, \dots, \frac{\partial W}{\partial x_n}\right) = h_0$$

($h_0$ = const.) be the **Hamilton-Jacobi** equation. To say that it is integrable by separation of variables is to say that any $\partial W / \partial x_r$ depends upon only $x_r$ . That being the case, if $p_{(r)}$ denotes a function of only $x_r$ then one writes:

$$\frac{\partial W}{\partial x_r} = p_{(r)} .$$

Recall the conditions that $H$ must satisfy. Differentiate it with respect to $x_r$ and observe that $H$ depends upon $x_r$ both directly and by way of $p_{(r)}$ , so one will have:

$$\frac{\partial H}{\partial x_r} + \frac{\partial H}{\partial p_{(r)}} \frac{dp_{(r)}}{dx_r} = 0 .$$

If one sets ([5]):

$$\frac{\partial H}{\partial x_r} : \frac{\partial H}{\partial p_{(r)}} = \rho_r ,$$

for brevity, then that can be written:

(1)
$$\frac{dp_{(r)}}{dx_r} = -\rho_r .$$

Differentiate this once more with respect to $x_r$ ($s \neq r$) and observe that since $r$ depends upon $x$ both directly and by way of $p_{(r)}$ :

$$\frac{\partial \rho_r}{\partial x_s} + \frac{\partial \rho_r}{\partial p_{(r)}} \frac{dp_{(r)}}{dx_s} = 0 ,$$

so from the preceding:

(2)
$$\frac{\partial \rho_r}{\partial x_s} - \frac{\partial \rho_r}{\partial p_{(r)}} \rho_s = 0 .$$

That must be satisfied for any value of $x$. In addition, since (1) defines the derivatives, the $p$ do not allow any arbitrariness beyond their initial values, and they must remain arbitrary precisely when $W$ is a complete integral of $H = h_0$, as one supposes. Since the $p$ must be independent, (2) must be satisfied identically for any values of the $x$ and $p$.

**2.** – From now on, suppose that the equation $H = h_0$ corresponds to a dynamical problem whose constraints do not depend upon time. Denote the *vis viva* of the system by:

$$T = \tfrac{1}{2} \sum_{r,s} a_{rs}\, x'_r\, x'_s ,$$

as usual, and let $a^{(rs)}$ denote the coefficients of the reciprocal form to 2 $T$, and set:

(3)
$$K = \tfrac{1}{2} \sum_{r,s} a^{(rs)}\, p_{(r)}\, p_{(s)} .$$

As is known, if one denotes the force potential by $U$ then one will have:

(4)
$$H = K - U$$

and

(5)
$$\frac{\partial H}{\partial p_{(r)}} = \frac{\partial K}{\partial p_{(r)}} = x'_r ,$$

which will give:

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([5]) Suppose, as is legitimate, that one has $\partial H / \partial p_{(r)} \neq 0$ in the domain considered.

(6)
$$\rho_r = \frac{\partial H}{\partial x_r} : x_r' .$$

**3.** – It is obvious that the functions $x_r'^2 \frac{\partial \rho_r}{\partial x_s}$, $x_r'^2 \frac{\partial \rho_r}{\partial p_{(s)}}$, $x_s' \rho_s$ are entire functions in the $p$. When equation (2) is multiplied by $x_r' x_s'$, it will then become entire:

(7)
$$x_s'^2 \left[ x_r'^2 \frac{\partial \rho_r}{\partial x_s} \right] - \left[ x_r'^2 \frac{\partial \rho_r}{\partial p_{(s)}} \right] (x_s' \rho_s) = 0 .$$

However, that must be verified for any value of the $p$ : Meanwhile, one can annul the parts of varying degrees separately. One soon sees that the terms that do not contain the $U$ are all of a certain degree (viz., four), while the ones that contain it have lower degree. One can then annul the part that does not contain the $U$, while neglecting the ones that do. That is equivalent to setting the potential equal to zero.

With **Levi-Civita**, we conclude that:

*If a dynamical problem with the characteristic function $H = K – U$ is integrable by separation of variables then the same property will be true of the equation $K = h_0$ that defines the geodesics.*

We then say (as I already mentioned in the preface) that the problem is *geodetic* when $K = h_0$ ($U = 0$), and *essentially geodetic* when it is not possible to associate it with any potential function with $K = h_0$ .

## The essentially-geodetic case and the cases that admit a potential.

**4.** – The left-hand side of (7) is an entire function of the $p$, and therefore of the $x'$, as well, but since the first term in it is obviously divisible by $x_s'$, as well, that will also be true of the second one, and that will present two possible cases for any value of $s$ according to whether one or the other of the two factors that constitute those terms is divisible by $x_s'$.

One can also separate the indices into two groups:

*Group 1: The ones for which the function $x_s' \rho_s$ is divisible by $x_s'$ (or zero).*

*Group 2: The ones for which that function is non-zero and not divisible by $x_s'$.*

(One of the two groups might possible be missing.)

For the first group, if $L_s$ denotes an entire function of the $p$ then:

(I) $$\rho_s = L_s \qquad (s \text{ from group 1}),$$

or, from (6), (4), if one isolates the parts of different degrees in the *p* :

($a_1$) $$\frac{\partial U}{\partial x_s} = 0$$

$(s$ from group 1),

($b_1$) $$\frac{\partial K}{\partial x_s} = x'_s\, L_s\,.$$

As is obvious (since the left-hand side is homogeneous of degree two in the *p*), $L_s$ will be a homogeneous linear function in the *p*.

As is shown by ($a_1$), *if group 2 is missing then the problem will be essentially geodetic*, and ($b_1$) is characteristic of the **Levi-Civita** case ([6]).

**5.** – I propose to prove that:

*The* ***Levi-Civita*** *case is the only essentially-geodetic one.*

More precisely, I will prove that:

*Any* $a^{(qq)}$ *(in which q is an index from group 2) will satisfy the condition equations for a potential and can then be assumed to be such a thing.*

If *s* belongs to group 2 (so $x'_s\,\rho_s$ will not be divisible by $x'_s$) then $x'^2_r \dfrac{\partial \rho_r}{\partial p_{(s)}}$ will be divisible by $x'_s$. That is, if $M_{rs}$ denotes a suitable function that is entire in the *p* then one will have:

(II) $$x'^2_r \frac{\partial \rho_r}{\partial p_{(s)}} = x'_s\, M_{rs} \qquad (s \text{ is from group 2, } r \neq s).$$

When one recalls (4), (5), (6) and separates the parts of differing degree in the *p*, that will give:

($a_2$) $$\frac{\partial U}{\partial x_r}\, a^{(rs)} = 0\,,$$

($s$ in group 2, $r \neq s$)

([6]) Cf., **Levi-Civita**, *loc. cit.* ([2]), **§ 5**, page 388.

$$(b_2) \qquad \frac{\partial^2 K}{\partial x_r\, \partial p_{(s)}} x'_r - \frac{\partial K}{\partial x_r} a^{(rs)} = x'_s M_{rs} \;.$$

($b_2$) shows immediately that the $M$ are homogeneous linear in the $p$. If one then substitutes the expressions that were given by ($b_2$) in (7) then one can split (7) into ([7]):

$$(a_3) \qquad \frac{\partial^2 U}{\partial x_r\, \partial x_s} x'_r - \frac{\partial U}{\partial x_r} \frac{\partial^2 U}{\partial x_s\, \partial p_{(r)}} - \frac{\partial U}{\partial x_s} M_{rs} = 0 \,,$$

($s$ in group 2, $r \neq s$)

$$(b_3) \qquad \frac{\partial^2 U}{\partial x_r\, \partial x_s} x'_r - \frac{\partial U}{\partial x_r} \frac{\partial^2 U}{\partial x_s\, \partial p_{(r)}} - \frac{\partial U}{\partial x_s} M_{rs} = 0 \,.$$

**6.** – When one differentiates the last one and ($b_2$) and eliminates the higher-order derivatives [i.e., differentiates ($b_2$) with respect to $x_s$ and ($b_3$) with respect to $p_{(s)}$ and subtracts], that will give:

$$(8) \qquad \frac{\partial K}{\partial x_s} \frac{\partial M_{rs}}{\partial p_{(s)}} + 2 \frac{\partial^2 K}{\partial x_r\, \partial p_{(s)}} \frac{\partial^2 K}{\partial x_s\, \partial p_{(r)}} - 2 \frac{\partial^2 K}{\partial x_r\, \partial x_s} a^{(rs)} = x'_s \frac{\partial M_{rs}}{\partial x_s} \qquad (s \text{ in group 2, } r \neq s),$$

which can be associated with the preceding ones.

**7.** – It will also be convenient to determine the functions $M$.

If one differentiates ($b_2$) with respect to $p_{(s)}$ then one will have:

$$\frac{\partial a^{(ss)}}{\partial x_r} x'_r = a^{(ss)} M_{rs} + x'_s \frac{\partial M_{rs}}{\partial p_{(s)}} \qquad (s \text{ in group 2, } r \neq s),$$

and differentiating once more (with respect to $p_{(s)}$, as always) will give:

$$(9) \qquad \frac{\partial a^{(ss)}}{\partial x_r} a^{(rs)} = 2\, a^{(ss)} \frac{\partial M_{rs}}{\partial p_{(s)}} \qquad (s \text{ in group 2, } r \neq s),$$

and one will have the desired expression for the $M$:

$$(10) \qquad M_{rs} = \frac{1}{2\, a^{(ss)2}} \frac{\partial a^{(ss)}}{\partial x_r} (2\, a^{(ss)} x'_r - a^{(rs)} x'_r) \qquad (s \text{ in group 2, } r \neq s).$$

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([7]) Separate the terms of varying degree and annul them separately, as usual.

## Simplifying the calculations.

**8.** – One can simplify the calculations that are still missing by observing that the right-hand side of (8) is divisible by $x'_s$. That will then be true of the functions on the left-hand side. However, when one replaces $\frac{\partial^2 K}{\partial x_r\, \partial x_s}$, $\frac{\partial^2 K}{\partial x_r\, \partial p_{(s)}}$, and $M$, and their derivatives with the expressions that are given by ($b_3$), ($b_2$), (10), and (9), respectively, it will reduce (neglecting the terms that are already divisible by $x'_s$ each time) to:

$$\frac{3}{2}\frac{\partial K}{\partial x_s}\frac{1}{a^{(ss)}}\,a^{(rs)}\frac{\partial a^{(ss)}}{\partial x_r} \qquad (s \text{ in group 2, } r \neq s),$$

which must either be divisible by $x'_s$ or equal to zero.

However, none of its factors is divisible by $x'_r$ ([8]). The term will then be zero identically, and therefore its first three factors cannot be, so one will have:

$$a^{(rs)}\frac{\partial a^{(ss)}}{\partial x_r} = 0 \qquad (s \text{ in group 2, } r \neq s).$$

$M_{rs}$ (if one examines its expression) is then divisible by $x'_r$, and therefore from ($b_2$), $\frac{\partial K}{\partial x_r}\,a^{(rs)}$ is either divisible or zero: It is divisible if $r$ belongs to group 1 and zero ([9]) if $r$ belongs to group 2.

**9.** – I say that $\frac{\partial a^{(ss)}}{\partial x_r} = 0$ in the first case ($r$ is in group 1, $s$ is in group 2). If $a^{(rs)} \neq 0$ then that will be obvious from the previous equation: One can then make $a^{(rs)} = 0$, and it will suffice to differentiate ($b_1$) twice with respect to $p_{(s)}$ and recall that $\frac{\partial x'_r}{\partial p_{(s)}} = a^{(rs)} = 0$, and that $L_r$ is linear in the $p$.

If $q$ denotes an index from group 2 then one can write:

$$\frac{\partial a^{(qq)}}{\partial x_r} = 0 \qquad (r \text{ from group 1, } q \text{ from group 2}). \tag{11}$$

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([8]) The only factor that contains the $p$, and therefore the $x'$, is $\partial K / \partial x_s$, and that is neither divisible by $x'_s$ nor zero when $s$ is an index in group 2.

([9]) Cf., the preceding footnote.

In the second case ($r$ belongs to group 2, $\frac{\partial K}{\partial x_r} a^{(rs)} = 0$), one has:

$$a^{(rs)} = 0 \qquad (r, s \text{ from group 2, } r \neq s). \tag{12}$$

First (10) and then ($b_2$) will then give:

$$(b_2') \qquad \begin{cases} M_{rs} = \dfrac{\partial \log a^{(ss)}}{\partial x_r} x_r', \\[2ex] \dfrac{\partial^2 K}{\partial x_s \, \partial p_{(r)}} = \dfrac{\partial \log a^{(rr)}}{\partial x_s} x_r', \end{cases} \qquad (r, s \text{ from group 2, } r \neq s)$$

from which ($a_3$), ($b_3$) will take the form (when one divides by $x_r'$):

$$(a_3') \qquad \frac{\partial^2 U}{\partial x_r \, \partial x_s} - \frac{\partial U}{\partial x_r} \frac{\partial \log a^{(rr)}}{\partial x_s} - \frac{\partial U}{\partial x_s} \frac{\partial \log a^{(ss)}}{\partial x_r} = 0 ,$$

$$(r, s \text{ from group 2, } r \neq s)$$

$$(b_3') \qquad \frac{\partial^2 K}{\partial x_r \, \partial x_s} - \frac{\partial K}{\partial x_r} \frac{\partial \log a^{(rr)}}{\partial x_s} - \frac{\partial K}{\partial x_s} \frac{\partial \log a^{(ss)}}{\partial x_r} = 0 .$$

When the last one is differentiated twice with respect to $p_{(q)}$, that will give:

$$\frac{\partial^2 a^{(qq)}}{\partial x_r \, \partial x_s} - \frac{\partial a^{(qq)}}{\partial x_r} \frac{\partial \log a^{(rr)}}{\partial x_s} - \frac{\partial a^{(qq)}}{\partial x_s} \frac{\partial \log a^{(ss)}}{\partial x_r} = 0 \tag{13}$$

($q$, $r$, $s$ are from group 2, $r \neq s$).

**10.** – We are then in a position to show that:

*The function $U = a^{(qq)}$ ($q$ is from group 2) satisfies all the conditions* ($a$) *that the potential must satisfy, so that can be assumed.*

Indeed, we have: From (11), ($a_1$) is satisfied, and when $r$ belongs to group 1, ($a_2$) will be. From (13), ($a_3'$) will be satisfied, and therefore ($a_3$), while the remaining one ($a_2$) (when $r$ belongs to group 2) will reduce to an identity by virtue of (12).

Therefore, *if the second group of indices exists (even if it consists of only one index) then the problem will admit a potential and will not belong to the essentially-geodetic type then: The only such type is then the **Levi-Civita** type*, as was stated before.

## The solution in the essentially-geodetic case.

**11**. – Mark the functions that either depend upon (at most) one variable $x_r$ or will become such things when one equates all of the other variables to zero with an index $|_{(r)}$ that is included in parenthesis (as we already did for the $p$). Analogously, denote the constants or the functions that will become constant when one annuls all of the variables that they depend upon by the index $|_0$ (as we already did with $h_0$).

**12.** – In particular, if one denotes the initial values of the $p_{(r)}$ by $c_r$ (they are, so to speak, *arbitrary constants*) then one will have:

$$p_{(r)0} = c_r .$$

For the $p$ in the first group, one will have $\left(\frac{dp_{(r)}}{dx_r} = \frac{\partial K}{\partial x_r} : x_r' = -L_r\right)$:

$$\frac{dp_{(r)}}{dx_r} + L_r = 0 . \tag{14}$$

It is easy to see that $L_r$ is independent of the $p$ in the second group. Indeed, (11) shows that [cf., (10)]:

$$M_{rs} = 0 \qquad (r \text{ from group 1}, s \text{ from group 2}).$$

($b_2$) will then reduce to its left-hand side, and when one expresses the $\partial K / \partial x_r$ in terms of $L_r$, to:

$$\frac{\partial L_r}{\partial p_{(s)}} = 0 \qquad (r \text{ from group 1}, s \text{ from group 2}),$$

which proves my assertion.

If $b\ p_{(r)}$ denotes the terms in $L_r$ that contain the $p_{(r)}$ and $\lambda_r$ denotes the remaining terms then (14) will become:

$$\frac{dp_{(r)}}{dx_r} + b\ p_{(r)} + \lambda_r = 0 ,$$

and when one sets all of the variables equal to zero, while singling out the $r^{\text{th}}$ one:

$$\frac{dp_{(r)}}{dx_r} + b_{(r)}\ p_{(r)} + \lambda_{(r)} = 0 ,$$

in which the $\lambda_{(r)}$, which is already homogeneous linear in the $p$ of group 1, except for $p_{(r)}$, will then be homogeneous linear in their initial values (viz., the first group of constants).

Multiply that by $\partial_{(r)} = e^{\int b_{(r)}\,dx_r} \neq 0$, and integrate between 0 and $x_r$ :

$$\partial_{(r)}\, p_{(r)} - \partial_0\, c_r + \int_0^{x_r} \lambda_{(r)}\, dx_r = 0\ .$$

If $\alpha$ denotes the first group of constants and $f_{(r)}\ (\alpha)$ denotes a linear form in the constants of that group (which is a form that depends upon only the variable $x_r$) then that will be an equation of the type:

(*A*) $$p_{(r)} = f_{(r)}\ (\alpha)\ .$$

*If the second group is missing then* (*A*) *will give the necessary conditions for* ***all*** *of the p. They are also sufficient* (as we will see later) *and solve the essentially-geodetic problem.*

## The solution in the general case.

**13.** – In order to get the *p* from the other group, we shall try to obtain a general expression for the *p* from which we can easily deduce the **Hamilton-Jacobi** equation under our hypotheses.

Indeed, that equation is written:

$$\sum_{s,t} a^{(st)} p_{(s)}\, p_{(t)} = 2\ U + 2\ h_0\ .$$

If one annuls all of the *x* except for the $r^{\text{th}}$ one then $p_{(r)}$ will be unaltered, while the remaining *p* will change into their arbitrary initial values, and all other functions will change into functions of only $x_r$ . Highlight the terms in the summation that contain $p_{(r)}$ [while at most changing the parameters conveniently ([10])] and set:

$$f_{(r)} = -\sum_{s \neq r} a_{(r)}^{(rs)}\, c_s\ ,$$

$$\Phi_{(r)} = \sum_{s,t \neq r} a_{(r)}^{(ss)}\, c_s\, c_t\,,$$

for brevity, and the aforementioned equation will assume the form:

(15) $$p_{(r)}^2 - 2 p_{(r)}\ f_{(r)} + \Phi_{(r)} - 2U_{(r)} - 2h_0 = 0\ .$$

**14.** – We shall attempt to specify when *r* belongs to the second group by means of the functions $f_{(r)}$ and $\Phi_{(r)}$ .

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([10]) If one changes the parameter $x_r$ then one can replace $a^{(rr)}$ with $a^{(rr)} : a_{(r)}^{(rr)}$. When one annuls the *x* that are different from $x_r$, that will become $a_{(r)}^{(rr)} : a_{(r)}^{(rr)} = 1$.

For the $f$ (recalling that $a^{(rs)} = 0$ when $r$ and $s$ are distinct indices of the second group), one has:

$$f_{(r)} = f_{(r)}(\alpha),$$

in which the $f_{(r)}(\alpha)$ have meanings that are analogous to the ones that they had in the preceding number.

As for the $\Phi$, if $\sum'$ and $\sum''$ denote summations that extend over only indices in the first group and only indices in the second group, respectively, then one can decompose them into sums of three terms:

$$\Phi_{(r)} = \sum_{s,t}{}' a_{(r)}^{(st)} c_s\, c_t + \sum_{s,t\neq r}{}'' a_{(r)}^{(st)} c_s\, c_t + 2\sum_t{}' \sum_{s\neq r}{}'' a_{(r)}^{(st)} c_s\, c_t . \tag{16}$$

The first summation is a quadratic form in the constants of the group $\alpha$ [call it $\Phi'_{(r)}(\alpha)$] and as one sees, it does not need to be specified any further. The second one will reduce to $\sum''_{s\neq r} a_{(r)}^{(ss)} c_s^2$, from the observation that was made before about the $a^{(rs)}$. The third one must be transformed, as we shall soon see.

**15.** – If one recalls that $\partial K / \partial p_{(s)} = x'_s$ [form. (5), **§ 2**] then $(b'_2)$ can be written:

$$\frac{\partial \log x'_s}{\partial x_r} = \frac{\partial \log a^{(ss)}}{\partial x_r} \qquad (r, s \text{ from group 2; } r \neq s)$$

or more simply:

$$\frac{\partial (x'_s ; a^{(ss)})}{\partial x_r} = 0 \qquad (r, s \text{ from group 2; } r \neq s).$$

Differentiate that with respect to $p_{(t)}$ and invert the derivatives $\left(\dfrac{\partial x'_s}{\partial p_{(t)}} = a^{(st)}\right)$:

$$\frac{\partial (a^{(st)} : a^{(ss)})}{\partial x_r} = 0 \qquad (r, s \text{ from group 2; } r \neq s).$$

If one annuls all of the $x$ except for the $r^{\text{th}}$ one and integrates from 0 to $x_r$ then that will give:

$$a_{(r)}^{(st)} : a_{(r)}^{(ss)} = a_0^{(st)} : a_0^{(ss)},$$

or also when one observes that $a_{(s)}^{(ss)} = 1$ will make $a_0^{(ss)} = 1$:

$$a_{(r)}^{(st)} = a_{(r)}^{(ss)}\, a_0^{(st)} \qquad (r, s \text{ from group 2; } r \neq s).$$

One can replace $a_{(r)}^{(st)}$ with the expression that was just found in the third summation in (16).

One can then write:

$$\Phi_{(r)} = \Phi'_{(r)}(\alpha) + \sum_{s\neq r}'' a_{(r)}^{(ss)} [c_s^2 + 2 \sum_t' a_0^{(st)} c_s\, c_t] \ .$$

*The quantities enclosed in square brackets are constants. There are just as many constants in the second group as there are independent functions* (as is easy to verify): *They can then be usefully substituted for the latter*. If one denotes the group by $\beta$, and $\varphi'_{(r)}(\beta)$ denotes a linear form in them with coefficients that depend upon only $x_r$ then one will have:

$$\Phi_{(r)} = \Phi'_{(r)}(\alpha) + \varphi'_{(r)}(\beta) .$$

**16.** – It remains for us to determine $h_0$ . When all of the variables have been annulled, the **Hamilton-Jacobi** equation will give only ([11]):

$$\sum_{s,t} a_0^{(st)} c_s\, c_t = 2\, h_0 \ .$$

Thus, $2h_0$ can also be written:

$$(C) \qquad \left\{ \begin{aligned} 2h_0 &= \sum_{s,t}' a_0^{(st)} c_s\, c_t + \sum_{s,t}'' a_0^{(st)} c_s\, c_t + 2 \sum_{s,t}' \sum_{s,t}'' a_0^{(st)} c_s\, c_t \\ &= \Phi'_0(\alpha) + \sum_{s,t}'' [c_s^2 + 2 \sum_t' a_0^{(st)} c_s\, c_t] = \Phi'_0(\alpha) + \varphi'_0(\beta). \end{aligned} \right.$$

If (15) is substituted for $f$, $\Phi'$, and $h_0$, and the expressions that are found are solved for $p_{(r)}$ then that will give:

$$(B) \qquad p_{(r)} = f_{(r)}(\alpha) \pm \sqrt{F_{(r)}(\alpha) + \varphi_{(r)}(\beta) + u_{(r)}}$$

for the $p$ in the second group, in which the terms that are quadratic in the $\alpha$ and linear in the $\beta$ are collected into the terms $F$ and $\varphi$, respectively, and $u_{(r)}$ is the term that is independent of the constants $\alpha$ and $\beta$.

**The problem is solved.**

**17.** – One can, if one prefers, give ($C$) a more convenient form. Indeed, it is easy to see that a homogeneous linear substitution of the constants of the group $\alpha$ can give the orthogonal form $\sum \alpha^2$ to $\Phi'_0$, while leaving the type of ($A$), ($B$) invariant.

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([11]) The right-hand side will be $2\, U_0 + 2\, h_0$ , but if one includes $U_0$ in $h_0$ then it can take the form that is written.

*One can then write:*

(*A*) $p_{(r)} = f_{(r)}\,(\alpha)$ (*r* is from group 1),

(*B*) $p_{(r)} = f_{(r)}(\alpha) \pm \sqrt{F_{(r)}(\alpha) + \varphi_{(r)}(\beta) + u_{(r)}}$ (*r* is from group 2),

(*C*) $2\,h_0 = \sum \alpha^2 + \sum \beta$.

*That will solve the problem under the conditions that the discriminants of the two systems that are comprised of the* $f_{(r)}$ *(r is an index from group 1) and the* $\varphi_{(r)}$ *are both zero.*

Indeed, under those hypotheses, (*A*) will give the constants of the group $\alpha$ as expressions that are linear and homogeneous in the *p* and therefore quadratic (and homogeneous) in the $\alpha^2$. Therefore, when one isolates the radical in the right-hand side and squares it, (*B*) will give linear equations that are soluble for the $\beta$ that will prove to be, on the one hand, homogeneous quadratic in the *p* and of degree zero in the *p*, on the other, and the same thing will be true of their sum and therefore the expression for $2h_0$ .

Therefore, *the elimination of the constants of the groups* $\alpha$ *and* $\beta$ *from* (*A*), (*B*), (*C*) *will give a* ***Hamilton-Jacobi*** *equation that is integrable by separation of variables*, and since (*A*), (*B*), (*C*) represent necessary conditions, *our problem is solved completely.*

**18.** – We add only that: *The absence of the group* $\alpha$ *characterizes the* ***Stäckel*** *type. The absence of the group* $\beta$ *characterizes the* ***Levi-Civita*** *type. Our equations reduce to the form of the ones in* ***Burgatti****, with which these coincide substantially*, so one solves (*C*) with respect to a constant of group $\alpha$ in the **Levi-Civita** case and with respect to group $\beta$ in any other case, while replacing (*A*) with the expression thus found in the first case and replacing (*B*) in the others.

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**F. A. DALL'ACQUA**

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