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On the solution of the equation $dx^2 + dy^2 + dz^2 = ds^2$ and some analogous equations

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I.

In a short article “Sur la résolution de l’équation $dx^2 + dy^2 = ds^2$ et de quelques équations analogues” that was inserted in tome XVIII (2nd series, pp. 236) of this Journal in 1873, I successively considered some differential equations that were all of the following type:

$$f(dx_1, dx_2, \ldots, dx_n) = 0 , \tag{1}$$

in which f denotes an arbitrary homogeneous function with constant coefficients of the differentials $dx_1, dx_2, \ldots, dx_n$, and I showed how one can integrate those equations by supposing that $x_1, x_2, \ldots, x_n$ are unknown functions of the same independent variable. I propose to return to the results that I indicated in order to complete them and deduce some new consequences.

First of all, it is necessary to make the problem that was posed more precise. Obviously, one can either give $x_3, x_4, \ldots, x_n$ as functions of x_2 or give $x_2, x_3, \ldots, x_n$ as functions of a parameter t. Equation (1) will then give dx_1 as a function of x_2 and dx_2 in the fist case and as a function of t and dt in the second, and as a result x_1 will be determined by a quadrature. Such solutions will be excluded in the rest of this article, and we propose to express the most general expressions in $x_1, \ldots, x_n$ that satisfy equation (1) as functions of an arbitrary parameter *when those expression do not contain any quadrature signs.* That is how Euler integrated the first equation:

$$dx^2 + dy^2 = ds^2$$

and J.-A. Serret integrated the analogous equation:

$$dx^2 + dy^2 + dy^2 = ds^2 .$$

[*See* tome XIII (1st series, pp. 353) of this Journal.]

In order to solve equation (1) in that way, we set:

(2) $$dx_1 = a_1\, dx_n\,, \qquad dx_2 = a_2\, dx_n\,, \qquad \ldots, \qquad dx_{n-1} = a_{n-1}\, dx_n\,.$$

$a_1, a_2, \ldots, a_{n-1}$, must verify the equation:

(3) $$f(a_1, a_2, \ldots, a_{n-1}, 1) = 0\,.$$

Now introduce the quantities b_i that are defined by the relations:

(4) $$\begin{cases} a_1\, x_n - x_1 = b_1\,, \\ a_2\, x_n - x_2 = b_2\,, \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ a_{n-1}\, x_n - x_{n-1} = b_{n-1}\,. \end{cases}$$

If one differentiates those relations while taking into account equations (2) then they will give:

(5) $$\frac{db_1}{da_1} = \frac{db_2}{da_2} = \ldots = \frac{db_{n-1}}{da_{n-1}} = x_n\,.$$

The proposed question is then found to come down to the following one: *Determine the most general functions a_i, b_i of a certain parameter that satisfies equations* (3) *and* (5). Once those functions are known, one will have:

$$x_n = \frac{db_i}{da_i}\,,$$

and one can then determine $x_1, x_2, \ldots, x_{n-1}$ from equations (4). It is easy to recognize that formulas (4) and (5) lead to equations (2), and when they are combined with the relation (3), they will give equation (1), which amounts to a solution.

Take $n - 1$ functions $a_1, a_2, \ldots, a_{n-1}$ of a certain parameter t that is subject to verifying equation (3). It remains for one to express the most general values of $b_1, b_2, \ldots, b_{n-1}$ that satisfy the relations:

(6) $$\frac{db_1}{da_1} = \frac{db_2}{da_2} = \ldots = \frac{db_{n-1}}{da_{n-1}}$$

without any integral sign. Now for a long time, geometry has taught us how to solve that problem, at least in the case where $n - 1$ is equal to 2 or 3.

Set:

(7)
$$U = \begin{vmatrix} b_1 & b_2 & \cdots & b_{n-1} \\ \dfrac{da_1}{dt} & \dfrac{da_2}{dt} & \cdots & \dfrac{da_{n-1}}{dt} \\ \dfrac{d^2a_1}{dt^2} & \dfrac{d^2a_2}{dt^2} & \cdots & \dfrac{d^2a_{n-1}}{dt^2} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{d^{n-2}a_1}{dt^{n-2}} & \dfrac{d^{n-2}a_2}{dt^{n-2}} & \cdots & \dfrac{d^{n-2}a_{n-1}}{dt^{n-2}} \end{vmatrix} .$$

If one develops that determinant in the elements of the first row then one will have:

(8)
$$U = \lambda_1\, b_1 + \lambda_2\, b_2 + \ldots + \lambda_{n-1}\, b_{n-1} ,$$

in which the functions λ verify the relations:

(9)
$$\begin{cases} \lambda_1\, da_1 & + \cdots + & \lambda_{n-1}\, da_{n-1} = 0, \\ \lambda_1\, d^2a_1 & + \cdots + & \lambda_{n-1}\, d^2a_{n-1} = 0, \\ \ldots\ldots\ldots & \ldots\ldots & \ldots\ldots\ldots\ldots\ldots\ldots \\ \lambda_1\, d^{n-2}a_1 & + \cdots + & \lambda_{n-1}\, d^{n-2}a_{n-1} = 0 \end{cases}$$

from the elementary properties of determinants. One encounters that set of formulas in either the theory of contact or in the theory of linear differential equations. As one knows, one can deduce a series of equations of the following type by repeated differentiations:

(10)
$$d^i\lambda_1\, d^k a_1 + d^i\lambda_2\, d^k a_2 + \cdots + d^i\lambda_{n-1}\, d^k a_{n-1} = 0 ,$$

in which one has:

$$i + k < n - 1 , \qquad k \geq 1 , \qquad i \geq 0 .$$

When i is equal to 0, one must replace $d^0\lambda_h$ with λ_h .

Let us write out all of those relations for which k is equal to 1. We will get the system:

(11)
$$\begin{cases} \lambda_1\, da_1 & + \cdots + & \lambda_{n-1}\, da_{n-1} = 0, \\ d\lambda_1\, da_1 & + \cdots + & d\lambda_{n-1}\, da_{n-1} = 0, \\ \ldots\ldots\ldots & \ldots\ldots & \ldots\ldots\ldots\ldots\ldots\ldots \\ d^{n-3}\lambda_1\, da_1 & + \cdots + & d^{n-3}\lambda_{n-1}\, da_{n-1} = 0 . \end{cases}$$

If we now replace the differentials $da_1, \ldots$ with the differentials $db_1, \ldots$ then we will be led to the following system:

$$\text{(12)}\qquad \begin{cases} \lambda_1\, db_1 + \cdots + \quad \lambda_{n-1}\, db_{n-1} = 0, \\ d\lambda_1\, d_1 + \cdots + \quad d\lambda_{n-1}\, db_{n-1} = 0, \\ \dots\dots\dots\dots\dots\dots\dots\dots, \\ d^{n-3}\lambda_1\, db_1 + \cdots + d^{n-3}\lambda_{n-1}\, db_{n-1} = 0, \end{cases}$$

which will lead us to the desired result.

Indeed, if one differentiates equation (8) while successively taking into account the various formulas (12) then one will get the following system:

$$\text{(13)}\qquad \begin{cases} U = b_1\,\lambda_1 & + \cdots + \quad b_{n-1}\,\lambda_{n-1}, \\ \dfrac{dU}{dt} = b_1\,\dfrac{d\lambda_1}{dt} & + \cdots + \quad b_{n-1}\,\dfrac{d\lambda_{n-1}}{dt}, \\ \dfrac{d^2U}{dt^2} = b_1\,\dfrac{d^2\lambda_1}{dt^2} & + \cdots + \quad b_{n-1}\,\dfrac{d^2\lambda_{n-1}}{dt^2}, \\ \quad\cdots & \quad\cdots \qquad \cdots \\ \dfrac{d^{n-2}U}{dt^{n-2}} = b_1\,\dfrac{d^{n-2}\lambda_1}{dt^{n-2}} & + \cdots + \quad b_{n-1}\,\dfrac{d^{n-2}\lambda_{n-1}}{dt^{n-2}}, \end{cases}$$

and it will obviously suffice to solve those $n - 1$ equations with respect to $b_1, b_2, \ldots, b_{n-1}$ in order to obtain the values of the unknowns when expressed in terms of the completely-arbitrary function U and its first $n - 2$ derivatives.

First, suppose that the determinant:

$$\text{(14)}\qquad \Delta = \begin{vmatrix} \lambda_1 & \cdots & \lambda_{n-1} \\ \dfrac{d\lambda_1}{dt} & \ddots & \dfrac{d\lambda_{n-1}}{dt} \\ \vdots & \vdots & \vdots \\ \dfrac{d^{n-2}\lambda_1}{dt^{n-2}} & \cdots & \dfrac{d^{n-2}\lambda_{n-1}}{dt^{n-2}} \end{vmatrix}$$

is non-zero. Formulas (13) will then provide well-defined values for $b_1, b_2, \ldots, b_{n-1}$. However, none of the minors of Δ with respect to the elements of the last row can be zero, moreover, so equations (11) and (12) will also unambiguously determine the mutual ratios of the $da_1, da_2, \ldots, da_{n-1}$ or the $db_1, db_2, \ldots, db_{n-1}$. Formulas (12) can be considered to be a simple consequence of equations (13). For example, it suffices to differentiate the first equation in (13). One will recover the first of formulas (12) upon taking into account the second one, and so on. A comparison of the relations (11) and (12) will then immediately show that the differentials of the quantities $b_1, \ldots, b_{n-1}$ that are determined by the formulas (13) are proportional to those of $a_1, \ldots, a_{n-1}$. The problem that we posed is then solved completely.

In the case where the quantities $a_1, a_2, \ldots, a_{n-1}$, which must satisfy only equation (3), have been chosen in such a manner that the determinant Δ is zero, one can argue in the following way:

Let:

(15) $$\Delta' = \begin{vmatrix} a_1 & a_2 & \cdots & a_{n-1} \\ \dfrac{da_1}{dt} & \dfrac{da_2}{dt} & \cdots & \dfrac{da_{n-1}}{dt} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{d^{n-2}a_1}{dt^{n-2}} & \cdots & \cdots & \dfrac{d^{n-2}a_{n-1}}{dt^{n-2}} \end{vmatrix}.$$

If one sets:

(16) $$A_{i,k} = \frac{d^i\lambda_1}{dt^i}\frac{d^k a_1}{dt^k} + \cdots + \frac{d^i\lambda_{n-1}}{dt^{n-1}}\frac{d^k a_{n-1}}{dt^k}$$

then one will have:

$$\Delta\Delta' = \begin{vmatrix} A_{00} & A_{01} & A_{0,n-2} \\ A_{10} & \cdots & A_{1,n-2} \\ \vdots & \vdots & \vdots \\ A_{n-2,0} & \cdots & A_{n-2,n-2} \end{vmatrix}.$$

By virtue of formula (10), one will have:

$$A_{i,k} = 0,$$

whenever i and k satisfy the inequalities:

$$i + k < n - 1, \quad k \geq 1, \quad i \geq 0.$$

One will then have:

(17) $$\Delta\Delta' = \begin{vmatrix} A_{00} & 0 & \cdots & \cdots & 0 \\ A_{10} & 0 & \cdots & 0 & A_{1,n-2} \\ A_{20} & 0 & \cdots & A_{2,n-3} & A_{2,n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_{n-2,0} & A_{n-2,1} & \cdots & \cdots & A_{n-2,n-2} \end{vmatrix}$$

and consequently:

(18) $$\Delta\Delta' = \pm A_{00}\, A_{1,\,n-2}\, A_{1,\,n-3} \ldots A_{n-2,\,1}.$$

If one now applies the obvious formula:

$$\frac{dA_{ik}}{dt} = A_{i+1,\,k} + A_{i,\,k+1}$$

to the elements $A_{0,\,n-2}$, $A_{1,\,n-3}$, …, $A_{1,\,n-3}$, which are zero, then one will have:

$$A_{1,\,n-2} + A_{0,\,n-1} = 0 \,,$$
$$A_{2,\,n-3} + A_{1,\,n-2} = 0 \,,$$
$$\dots\dots\dots\dots\dots\dots\dots,$$
$$A_{n-2,\,1} + A_{n-3,\,2} = 0 \,,$$

i.e.:

$$A_{n-2,\,1} = -A_{n-3,\,2} = \ldots = \pm A_{1,\,n-2} = \mp A_{0,\,n-1} \,.$$

The product $\Delta\Delta'$ will then take the very simple form:

$$\Delta\Delta' = \pm A_{00}\,(A_{0,\,n-1})^{n-2}. \tag{19}$$

In order for Δ to be zero, it is necessary that one must have either:

$$A_{00} = \lambda_1\, a_1 + \ldots + \lambda_{n-1}\, a_{n-1} = 0$$

or

$$A_{0,\,n-1} = \lambda_1 \frac{d^{n-1}a_1}{dt^{n-1}} + \cdots + \lambda_{n-1} \frac{d^{n-1}a_{n-1}}{dt^{n-1}} = 0 \,.$$

If one refers to the definition of the quantities λ by formulas (7) and (8) then that will give the first condition:

$$\begin{vmatrix} a & \cdots & a_{n-1} \\ \dfrac{da_1}{dt} & \cdots & \dfrac{da_{n-1}}{dt} \\ \vdots & \cdots & \vdots \\ \dfrac{d^{n-2}a_1}{dt^{n-2}} & \cdots & \dfrac{d^{n-2}a_{n-1}}{dt^{n-2}} \end{vmatrix} = 0 \,.$$

As one knows, one deduces from that the existence of one or more linear homogeneous relations with constant coefficients between the quantities a_i .

Similarly, the second condition then gives:

$$\begin{vmatrix} \dfrac{da_1}{dt} & \cdots & \dfrac{da_{n-1}}{dt} \\ \vdots & \cdots & \vdots \\ \dfrac{d^{n-1}a_1}{dt^{n-1}} & \cdots & \dfrac{d^{n-1}a_{n-1}}{dt^{n-1}} \end{vmatrix} = 0 \,,$$

and it also expresses the idea that there exist one or more linear relations with constant coefficients between the a_i . However, those relations are no longer necessarily homogeneous.

In summary, one sees that the exceptional case in which the determinant of equations (13) is zero can present itself only if there exist one or more relations of the form:

$$k_1 a_1 + k_2 a_2 + \ldots + k_{n-1} a_{n-1} + k_n = 0$$

between the quantities a_i , in which $k_1, \ldots, k_n$ denote constants. Upon replacing the a_i with their values that are deduced from formulas (2), one will have:

$$k_1 dx_1 + \ldots + k_n dx_n = 0 ,$$

or, upon integrating:

$$k_1 x_1 + \ldots + k_n x_n = k .$$

One can then eliminate a certain number of the quantities x_i from the proposed equations (1) by means of those relations. One will then be led to solve an equation of the same form that contains fewer variables and for which the exceptional case does not present itself.

One can replace the preceding argument, which offers the advantage of exhibiting an interesting relation between the determinants Δ, Δ', with the following one, which is much simpler. If one has:

$$\Delta = 0$$

then, as one knows, one will get one or more linear homogeneous relations between the quantities λ_i . Let:

$$h_1 \lambda_1 + h_2 \lambda_2 + \ldots + h_{n-1} \lambda_{n-1} = 0$$

be any one of them. Upon replacing the λ with their values, one will have:

$$\begin{vmatrix} h_1 & h_2 & \cdots & h_{n-1} \\ da_1 & da_2 & \cdots & da_{n-1} \\ d^2 a_1 & \cdots & \ddots & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ d^{n-2} a_1 & \cdots & \cdots & d^{n-2} a_{n-1} \end{vmatrix} = 0 .$$

That relation keeps the same form, while only the values of the constants h_i change, when one performs an arbitrary linear substitution on the a_i . We choose that substitution in such a way that all of the constants except for h_1 reduce to zero. The equation will become:

$$\begin{vmatrix} da'_2 & \cdots & da'_{n-1} \\ \vdots & \ddots & \vdots \\ d^{n-2} a'_2 & \cdots & d^{n-2} a'_{n-1} \end{vmatrix} = 0 .$$

In that form, one immediately recognizes that there must exist at least one linear relation between the quantities a_i', and as a result, between the quantities a_i .

II.

We shall first apply the preceding general method to the Euler equation:

$$dx^2 + dy^2 = ds^2. \tag{20}$$

One solves that equation *algebraically* by setting:

$$dx = ds \cos\theta, \qquad dy = ds \sin\theta. \tag{21}$$

One then sets:

$$\begin{cases} x - s\cos\theta = a, \\ y - s\sin\theta = b, \end{cases} \tag{22}$$

and upon differentiating, one will have:

$$s = \frac{db}{\cos\theta\, d\theta} = \frac{-da}{\sin\theta\, d\theta}. \tag{23}$$

In order to determine a and b, one takes:

$$a\cos\theta + b\sin\theta = U, \tag{24}$$

and when that equation is differentiated and one takes the preceding one into account, that will give:

$$-a\sin\theta + b\cos\theta = \frac{dU}{d\theta}. \tag{25}$$

The formulas thus-obtained determine a, b, s, x, y as functions of θ. One then deduces the system:

$$\begin{cases} x\sin\theta - y\cos\theta + \dfrac{dU}{d\theta} = 0, \\[2mm] x\cos\theta + y\sin\theta + \dfrac{d^2U}{d\theta^2} = 0, \\[2mm] s = U + \dfrac{d^2U}{d\theta^2}, \end{cases} \tag{26}$$

which has been known and employed for some time now.

Since one has:

$$(27) \qquad \begin{cases} \tan\theta = \dfrac{dy}{dx}, \\ U = s - x\cos\theta - y\sin\theta, \end{cases}$$

one sees that for any algebraic curve whose arc-length is algebraic, U is necessarily an algebraic function of the trigonometric lines of θ. The Euler formulas then exhibit all of the algebraic planar lines whose arc-length is an algebraic function of the coordinates of the contact point.

Every algebraic curve whose arc-length is algebraic obviously has involutes (*développantes*) that are all algebraic. One will then get all of the algebraic curves that are algebraically rectifiable by considering all of the developments (*développées*) of the algebraic curves. However, one can take a very different viewpoint in the search for those curves, and propose, for example, to determine all of the planar curves of given class or degree that are algebraically rectifiable. We shall be content to point out that interesting question to the geometers, whose solution will undoubtedly be made possible by the beautiful propositions that are due to Halphen on the developments of algebraic plane curves.

Now consider the equation:

$$(28) \qquad dx^2 + dy^2 + dz^2 = ds^2 ,$$

which was the subject of the studies of J.A.- Serret, as we recalled already.

We set:

$$(29) \qquad \frac{dx}{ds} = x_0 , \qquad \frac{dy}{ds} = y_0 , \qquad \frac{dz}{ds} = z_0 ,$$

and take x_0 , y_0 , z_0 to be three functions of the same parameter t that are subject to the single condition that they must verify the equation:

$$(30) \qquad x_0^2 + y_0^2 + z_0^2 = 1 .$$

Conforming to the general method, we then set:

$$(31) \qquad \begin{cases} x = x_0\, s + X_0 , \\ y = y_0\, s + Y_0 , \\ z = z_0\, s + Z_0 . \end{cases}$$

Upon differentiation, we will have:

$$(32) \qquad -s = \frac{dX_0}{dx_0} = \frac{dY_0}{dy_0} = \frac{dZ_0}{dz_0} .$$

In order to determine the functions X_0, Y_0, Z_0 without quadrature, it will suffice to interpret the preceding relations geometrically. The two curves (Γ_0), which is described by the point (X_0, Y_0, Z_0), and (γ_0), which is described by the point (x_0, y_0, z_0), must have parallel tangent planes at each instant, and as a result, parallel osculating planes. Therefore, the curve (Γ_0) will be the edge of regression of a developable surface whose tangent planes are all parallel to the osculating planes of (γ_0). From that, if one considers the equation:

$$X\left(\frac{dy_0}{dt}\frac{d^2z_0}{dt^2}-\frac{dz_0}{dt}\frac{d^2y_0}{dt^2}\right)+Y\left(\frac{dz_0}{dt}\frac{d^2x_0}{dt^2}-\frac{dx_0}{dt}\frac{d^2z_0}{dt^2}\right)+Z\left(\frac{dx_0}{dt}\frac{d^2y_0}{dt^2}-\frac{dy_0}{dt}\frac{d^2x_0}{dt^2}\right)-U=\Phi=0\,,$$

in which U is an arbitrary function of t, then the values of X, Y, Z that are deduced from the three equations:

$$\text{(33)} \qquad \Phi = 0\,, \qquad \frac{d\Phi}{dt} = 0\,, \qquad \frac{d^2\Phi}{dt^2} = 0$$

will be precisely those of X_0, Y_0, Z_0 . The geometric method agrees completely with our general theory here.

Once the values of X_0, Y_0, Z_0 have been determined, the formulas will exhibit s and x, y, z as functions of the variable parameter t.

For any algebraic curve whose arc-length is algebraic, x_0, y_0, z_0 will obviously be algebraic functions of one conveniently-chosen parameter. The same will be true for X_0, Y_0, Z_0, by virtue of formulas (31), and consequently U as well. Therefore, in order to obtain all of the skew algebraic curves whose arc-length is algebraic, one must take U, x_0, y_0, z_0 to be algebraic functions of the same parameter that are subject to the single condition that they must verify equation (30).

We can recover the solution that we just gave along an entirely geometric route and establish with no calculation the relations that exist between the curves (γ_0), (Γ_0), and the desired curve (C), which is the locus of the point (x, y, z). If we are given the spherical curve (γ_0) then, as was indicated above, we construct the curve (Γ_0) whose tangents are parallel to those of (γ_0). The osculating planes to the corresponding points of the two curves will be parallel. As a result, the contingency angles and the torsion that relate to two infinitely-close corresponding arcs will be equal. Now, one knows that in order to obtain an arbitrary development of a skew curve, one must draw a normal to each point of that curve that an angle that is equal to:

$$\int \frac{ds}{\tau}$$

with the principal normal, in which ds denotes the differential of arc-length and τ is the radius of torsion.

It will then follow that if one draws two parallel normals at two corresponding points of (γ_0) and (Γ_0), one of which touches a development of (γ_0), while the other envelopes a development of

(Γ_0). In particular, consider those of the developments of (γ_0) that reduce to a point, namely, the center of the sphere on which (γ_0) is described. We will then be led to the following proposition:

Given the spherical curve (γ_0) *and the arbitrary curve* (Γ_0), *whose tangents are parallel to those of* (γ_0), *through each point of* (Γ_0), *draw that parallel to the radius of the sphere that contains* (γ_0) *and passes through the corresponding point of the latter curve. That parallel will envelope a development of* (Γ_0).

That development is precisely the curve (C) that the preceding analytical method showed us how to determine. Conversely, if we are given a curve (C) whose arc-length is expressed with no quadrature then we construct one of its involutes (Γ_0) and draw parallels to the tangents of (C) through the center of a sphere of radius 1 that determine the curve (γ_0) on the sphere. It is clear that (C) is deduced by means of (γ_0) and (Γ_0) by the construction that we just indicated, and as a result, that construction will indeed give all of the curves whose arc-length is expressed without quadrature.

It remains for us to indicate how we will determine the definitive values of x, y, z, s as functions of t. Equations (33) have the form:

$$(34)\qquad \begin{cases} X_0(y_0'\,z_0'' - z_0'\,y_0'') & + \cdots \quad = U, \\ X_0(y_0'\,z_0''' - z_0'\,y_0''') & + \cdots \quad = U', \\ X_0(y_0'\,z_0^{\mathrm{IV}} - z_0'\,y_0^{\mathrm{IV}} + y_0''\,z_0''' - z_0''\,y''') & + \cdots \quad = U''. \end{cases}$$

When solved for X_0, Y_0, Z_0, they will always give values for those quantities that we always write in the following manner:

$$(35)\qquad \begin{cases} X_0 = \lambda\,x_0' + \mu\,x_0'' + \nu\,x_0''', \\ Y_0 = \lambda\,y_0' + \mu\,y_0'' + \nu\,y_0''', \\ Z_0 = \lambda\,z_0' + \mu\,z_0'' + \nu\,z_0''', \end{cases}$$

in which λ, μ, ν are chosen conveniently. In order to determine those three parameters, we substitute the values of X_0, Y_0, Z_0 in the equations that they must verify and obtain the following result:

$$(36)\qquad \begin{cases} \nu\,(123) & = U, \\ -\mu\,(123) & = U', \\ \lambda\,(123) - \mu\,(124) - \nu\,(134) & = U'', \end{cases}$$

in which we denotes the determinant:

$$\begin{vmatrix} x^{(i)} & x^{(k)} & x^{(l)} \\ y^{(i)} & y^{(k)} & y^{(l)} \\ z^{(i)} & z^{(k)} & z^{(l)} \end{vmatrix},$$

which is defined by the derivatives of order i, k, l of x, y, z, by (ikl), to abbreviate.

Set:

(37) $$(123) = \Delta , \qquad (134) = D .$$

Upon differentiating the first of those equations, we will have:

$$\begin{aligned} (124) &= \Delta', \\ (134) + (125) &= \Delta'', \end{aligned}$$

and formulas (36) will then give us:

(38) $$\begin{cases} \mu = \dfrac{-U}{\Delta}, \\ \nu = \dfrac{U}{\Delta}, \\ \lambda = \dfrac{U''}{\Delta} - \dfrac{U'\Delta'}{\Delta^2} + \dfrac{DU}{\Delta^2} = \dfrac{d}{dt}\left(\dfrac{U'}{\Delta}\right) + \dfrac{DU}{\Delta^2}. \end{cases}$$

Upon substituting those values in λ, μ, ν in equations (35), we will have:

(39) $$\begin{cases} X_0 = \left[\dfrac{d}{dt}\left(\dfrac{U'}{\Delta}\right) + \dfrac{DU}{\Delta^2}\right] x_0' - \dfrac{U'\,x_0''}{\Delta} + \dfrac{U\,x_0'''}{\Delta}, \\ Y_0 = \left[\dfrac{d}{dt}\left(\dfrac{U'}{\Delta}\right) + \dfrac{DU}{\Delta^2}\right] y_0' - \dfrac{U'\,y_0''}{\Delta} + \dfrac{U\,y_0'''}{\Delta}, \\ Z_0 = \left[\dfrac{d}{dt}\left(\dfrac{U'}{\Delta}\right) + \dfrac{DU}{\Delta^2}\right] z_0' - \dfrac{U'\,z_0''}{\Delta} + \dfrac{U\,z_0'''}{\Delta}. \end{cases}$$

It remains for us to find the value of s. We differentiate the last of equations (34) while replacing dX_0, dY_0, dZ_0 with their values $-s\,dx_0$, $-s\,dy_0$, $-s\,dz_0$. We find that:

$$s\,\Delta = -U''' + 2\lambda\,(124) - 2\nu\,(234) - \mu\,(125) - \nu\,(135) .$$

The equation:

$$(134) = D,$$

which serves to define D, will give us:

$$(135) + (234) = D'$$

upon differentiation, or:

$$(135) = D' - E$$

upon setting:

$$(234) = E .$$

The expression for s then takes the form:

(40) $$s = -\frac{U'''}{\Delta} + \frac{2\Delta'}{\Delta^2}U'' + U'\left(\frac{\Delta''}{\Delta^2} - \frac{2\Delta'}{\Delta^3} - \frac{D}{\Delta^2}\right) + U\left(\frac{2D\Delta'}{\Delta^3} - \frac{E}{\Delta^2} - \frac{D'}{\Delta^2}\right),$$

or, more simply:

(40.cont.) $$s = -\frac{d^2}{dt^2}\left(\frac{U'}{\Delta}\right) - \frac{d}{dt}\left(\frac{U\,D}{\Delta^2}\right) - \frac{DU}{\Delta^2}.$$

It will now suffice to substitute the values of s, X_0, Y_0, Z_0 in formulas (31) in order to obtain the definitive expressions for x, y, z.

III.

The general method that we just applied to the two remarkable examples can be modified in an advantageous manner in certain special cases. As an example, we choose the equation:

(41) $$dx^2 + dy^2 + dz^2 = dx_1^2 + dy_1^2 + dz_1^2 ,$$

on the subject of which, we must modify and complete the results that were given in our previous work.

One can solve it by taking:

(42) $$\begin{cases} dx = a\,dx_1 + a'\,dy_1 + a''\,dz_1 , \\ dy = b\,dx_1 + b'\,dy_1 + b''\,dz_1 , \\ dz = c\,dx_1 + c'\,dy_1 + c''\,dz_1 , \end{cases}$$

in which a, b, c, … are functions of one parameter t that are subject to verifying the equations:

(43) $$\begin{cases} a^2 + b^2 + c^2 = 1, \quad aa' + bb' + cc' = 0, \\ \dots\dots\dots\dots \quad \dots\dots\dots\dots\dots \end{cases}$$

In what follows, we shall suppose that the determinant of the nine quantities a, b, c, … is equal to 1. If it were otherwise, it would suffice to change of the signs of x_1, y_1, z_1 .

We introduce the arbitrary α, β, γ that are defined by the relations:

(44) $$\begin{cases} x = a\,x_1 + a'\,y_1 + a''\,z_1 + \alpha , \\ y = b\,x_1 + b'\,y_1 + b''\,z_1 + \beta , \\ z = c\,x_1 + c'\,y_1 + c''\,z_1 + \gamma , \end{cases}$$

and differentiate those relations. Upon taking the preceding ones (42) into account, we will have:

(45)
$$\begin{cases} 0 = x_1\, da + y_1\, da' + z_1\, da'' + d\alpha\,, \\ 0 = x_1\, db + y_1\, db' + z_1\, db'' + d\beta\,, \\ 0 = x_1\, dc + y_1\, dc' + z_1\, dc'' + d\gamma\,. \end{cases}$$

From some well-known propositions, one can express the differentials *da*, *db*, … as functions of three arbitrary parameters *p*, *q*, *r* using the relations:

(46)
$$\begin{cases} da \;= (br - cq)\,dt, & db = (cp - ar)\,dt, & dc = (aq - bp)\,dt, \\ da' \;= (b'r - c'q)\,dt, & \ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots \\ da'' = (b''r - c''q)\,dt, & \ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots \end{cases}$$

If one substitutes those values in equations (45) then one will have, upon taking into account formulas (44):

(47)
$$\begin{cases} q\,(z-\gamma) - r\,(y-\beta) = \dfrac{d\alpha}{dt}, \\ r\,(x-\alpha) - p\,(z-\gamma) = \dfrac{d\beta}{dt}, \\ p\,(y-\beta) - q\,(x-\alpha) = \dfrac{d\gamma}{dt}. \end{cases}$$

In order for the proposed equation (41) to be verified, it will then suffice for x, y, z ; x_1, y_1, z_1 to satisfy equations (44) and (47). However, it is important to remark that α, β, γ cannot be chosen arbitrarily. Indeed, if one adds the preceding equations, after multiplying them by *p*, *q*, *r*, respectively, then one will get the condition:

(48)
$$p\,\frac{d\alpha}{dt} + q\,\frac{d\beta}{dt} + r\,\frac{d\gamma}{dt} = 0$$

that α, β, γ must satisfy.

In order to get the values of α, β, γ that verify the preceding equation with no quadrature, it will suffice to set:

(49)
$$p\,\alpha + q\,\beta + r\,\gamma = U\,.$$

Upon differentiating that and taking equations (48) into account, one will have:

(50)
$$\alpha\,\frac{dp}{dt} + \beta\,\frac{dq}{dt} + \gamma\,\frac{dr}{dt} = \frac{dU}{dt}\,,$$

and the values of α, β, γ must verify the last two equations, in which *U* is an entirely arbitrary function.

In summary, one takes a, b, c ; a', b', c' ; a'', b'', c'' to be arbitrary functions of t that are subject to the single condition that that they must verify the well-known conditions (43) between the nine cosines. One chooses α, β, γ to be three new functions that satisfy the two equations (49) and (50). The values of x, y, z will be given by the system (47), and those of x_1, y_1, z_1 by the system (44). Since the three equations (47) reduce to two and cannot determine x, y, z completely, it is necessary to add an arbitrary relation to them:

$$f(x, y, z) = 0$$

that will permit one to determine x, y, z. Hence, the curve that is locus of the point (x, y, z) can be traced on an arbitrary surface.

A very simple geometric interpretation will shed much light on the preceding solution.

If one regards x, y, z and x_1, y_1, z_1 in the equation to be solved as the rectangular coordinates of two points then the problem that was posed can be stated in the following manner: *Determine (without quadrature) two curves in space that correspond point-by-point in such a manner that the corresponding arc-lengths of the two curves are equal.*

Let us now examine the solution. Consider x, y, z in formulas (44) to be the coordinates of a point in space when referred to moving axes Ox, Oy, Oz, and consider x_1, y_1, z_1 to be the coordinates of the same points when referred to fixed axes $O_1\,x_1$, $O_1\,y_1$, $O_1\,z_1$. Those formulas define a displacement in which the variable t plays the role of time, and at each instant, the quantities p, q, r denote the components of the infinitely-small rotation of the moving system when taken with respect to the moving axes. Having said that, equations (42), which served as the starting point, express the idea that there exists a curve (C) of the moving figure that rolls along a curve (C_1) in the fixed system, and that consequently, the contact point of the two curves at the instant considered will necessarily have a zero velocity. In order for that to be true, it is obviously necessary that all of the successive infinitely-small motions of the moving system should not be helicoidal motions but should reduce to simple rotations. That is the condition that is expressed by equation (48), which one can obtained immediately by writing out that the velocity of the origin of the moving axes is perpendicular to the direction of the axis of rotation.

We have indicated the means for solving that equation without quadrature, and we thus know the motions in which each infinitely-small displacement is equivalent to a rotation. As one knows, one will get all of those motions by rolling a ruled surface (K) on another ruled surface (K_1) that is applicable to the first one. The equation of the surface (K) will result from the elimination of t from the two equations to which the system (47) reduces. One will get the surface K_1 by eliminating t, x, y, z from equations (44) and (47), or what amounts to the same thing, by eliminating t from the equations:

$$\text{(51)}\qquad \left\{\begin{aligned} q(c\,x_1 + c'\,y_1 + c''\,z_1) - r(b\,x_1 + b'\,y_1 + b''\,z_1) &= \frac{d\alpha}{dt}, \\ r(a\,x_1 + a'\,y_1 + a''\,z_1) - p(c\,x_1 + c'\,y_1 + c''\,z_1) &= \frac{d\beta}{dt}, \\ p(b\,x_1 + b'\,y_1 + b''\,z_1) - q(a\,x_1 + a'\,y_1 + a''\,z_1) &= \frac{d\gamma}{dt}, \end{aligned}\right.$$

which likewise reduce to two.

One deduces a first consequence from the preceding remarks that deserves to be pointed out: *One can obtain (without quadrature) the most general equations of two ruled surfaces that are applicable to each other.*

We now cut the ruled surface (K) with an arbitrary surface:

$$f(x, y, z) = 0 .$$

We will get a certain curve (C) that rolls on the corresponding curve (C_1) of the surface (K_1). *The two curves* (C) *and* (C_1) *are precisely the ones that our analytical solutions showed us how to determine.*

In conclusion, we recall a solution to the same question that is completely different from the one that we gave.

If one sets:

$$\text{(52)} \qquad \left\{\begin{aligned} x - x_1 &= X , \\ y - y_1 &= Y , \\ z - z_1 &= Z , \end{aligned}\right.$$

$$\text{(53)} \qquad \left\{\begin{aligned} x + x_1 &= X_1 , \\ y + y_1 &= Y_1 , \\ z + z_1 &= Z_1 \end{aligned}\right.$$

in equation (41) then it will take the form:

$$\text{(54)} \qquad dX\, dX_1 + dY\, dY_1 + dZ\, dZ_1 = 0 .$$

Take X_1, Y_1, Z_1 to be arbitrary functions of one parameter t, and set:

$$\text{(55)} \qquad X\, dX_1 + Y\, dY_1 + Z\, dZ_1 = U\, dt .$$

If one differentiates that equation, while taking the preceding one into account then one will have:

$$\text{(56)} \qquad X\, d^2X_1 + Y\, d^2Y_1 + Z\, d^2Z_1 = dU\, dt ,$$

and conversely, the two equations (55) and (56) will imply the relation (54).

It will then suffice to take X, Y, Z to be functions that satisfy the two equations (55) and (56). That will permit one, for example, to determine one of them arbitrarily or to give an arbitrary relation between X, Y, Z :

$$\varphi(X, Y, Z) = 0$$

a priori.

To point out one application, suppose that one would like to find two curves whose arc-lengths are equal, and the corresponding points are always found at the same distance l from each other. One sets:

$$X^2 + Y^2 + Z^2 = l^2, \tag{57}$$

and one will have to solve the three equations (55), (56), and (57).

Take the auxiliary unknown to be the determinant:

$$\begin{vmatrix} \dfrac{dX_1}{dt} & \dfrac{dY_1}{dt} & \dfrac{dZ_1}{dt} \\ \dfrac{d^2X_1}{dt^2} & \dfrac{d^2Y_1}{dt^2} & \dfrac{d^2Z_1}{dt^2} \\ X & Y & Z \end{vmatrix} = \Delta . \tag{58}$$

If one squares that and sets:

$$ds_1^2 = dX_1^2 + dY_1^2 + dZ_1^2, \qquad \left(\frac{d^2X_1}{dt^2}\right)^2 + \left(\frac{d^2Y_1}{dt^2}\right)^2 + \left(\frac{d^2Z_1}{dt^2}\right)^2 = H^2,$$

to abbreviate, then one will find that:

$$\Delta^2 = \begin{vmatrix} \left(\dfrac{ds_1}{dt}\right)^2 & \dfrac{ds_1}{dt}\dfrac{d^2s_1}{dt^2} & U \\ \dfrac{ds_1}{dt}\dfrac{d^2s_1}{dt^2} & H^2 & \dfrac{dU}{dt} \\ U & \dfrac{dU}{dt} & l^2 \end{vmatrix} .$$

Δ will then be known, and in order to determine X, Y, Z, it will suffice to append the first-degree equation (58) to the equations (55) and (56).

One can further combine those two equations, which give the solution to the problem that was posed, with an arbitrary relation between X, Y, Z ; X_1, Y_1, Z_1 . Suppose, for example, that one demands to determine the two curves (C), (C_1) in such a manner that two arbitrary corresponding points of the two curves are always at the same distance from the origin. One will have:

$$x^2 + y^2 + z^2 = x_1^2 + y_1^2 + z_1^2$$

or

$$X X_1 + Y Y_1 + Z Z_1 = 0 .$$

That relation, when combined with equations (55) and (56), will give X, Y, Z with no difficulty.

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