From G. Darboux, Leçons sur la théorie générale des surfaces, Gauthier-Villars, Paris, 1896, Partie IV, Note XI, pp. 505-516.

## NOTE XI

## ON THE AUXILIARY EQUATION

1.     - In an article that was inserted on March 1883 in volume XCVI of the Comptes rendus $\left({ }^{1}\right)$, I introduced a notion that seems useful to me, namely, that of the auxiliary equation of an ordinary differential equation or of a partial differential equation that contains an arbitrary number of independent variables. Since the auxiliary equation intervenes in the study of two problems of geometry that defined the main subject of the last part of this book, I would like to say a few words about it, without entering into a detailed study and expanding upon its various applications, moreover.

To fix ideas, consider an arbitrary differential equation, whether ordinary or partial, that is defined for a function $z$ of one or more independent variables. If one replaces $z$ with $z+\varepsilon z^{\prime}$, develops that in powers of $\varepsilon$, and equates the coefficient of $\varepsilon$ to zero then one will have a homogeneous linear equation with respect to $z^{\prime}$ that I shall call the auxiliary equation of the proposed equation. The auxiliary equation defines solutions that are infinitely close to a given solution. Consequently, it has a significance that does not depend upon the choice of independent variables in any way and will persist after an arbitrary change of variables.

The notion of auxiliary equation can be generalized with no difficulty and can be extended to any system of ordinary or partial differential equations. Each system of unknown functions or independent variables of that type, no matter what the number of equations in it, admits an auxiliary system that defines what one can call all of the solutions that are infinitely close to a given solution. In order to obtain the auxiliary system, one replaces each unknown function $u_{i}$ with $u_{i}+\varepsilon u_{i}^{\prime}$, and equates the derivative of each equation with respect to $\varepsilon$ to zero while setting $\varepsilon=0$ after the derivation $\left({ }^{2}\right)$.
2. - When one knows how to completely integrate a system of ordinary or partial differential equations, one will obviously know how to integrate the auxiliary system. In order to do that, it will suffice to replace each of the finite equations that constitute the integral with its first variation, which is obtained by varying all of the arbitrary constants or functions that enter into that equation

[^0]and all of the unknown functions $u_{i}$, and one then replaces the variations $\delta u_{i}$ with $u_{i}^{\prime}$, conforming to the notation that was used before. For example, if one is dealing with a second-order differential equation whose general integral is defined by the formula:
$$
u=f\left(x, c, c_{1}\right)
$$
then its auxiliary equation will have the integral:
$$
u^{\prime}=\frac{\partial f}{\partial c} c^{\prime}+\frac{\partial f}{\partial c_{1}} c_{1}^{\prime}
$$
in which $c^{\prime}$ and $c_{1}^{\prime}$ denote two new constants that are independent of $c$ and $c_{1}$.
On the contrary, if one is dealing with a partial differential equation that admits an integral that is defined by equations of the following form:
\[

$$
\begin{aligned}
& z=f\left[\alpha, \beta, \varphi(\alpha), \varphi^{\prime}(\alpha), \ldots, \psi(\beta), \psi^{\prime}(\beta), \ldots\right], \\
& x=f_{1}\left[\alpha, \beta, \varphi(\alpha), \varphi^{\prime}(\alpha), \ldots, \psi(\beta), \psi^{\prime}(\beta), \ldots\right], \\
& y=f_{2}\left[\alpha, \beta, \varphi(\alpha), \varphi^{\prime}(\alpha), \ldots, \psi(\beta), \psi^{\prime}(\beta), \ldots\right],
\end{aligned}
$$
\]

in which $\varphi(\alpha), \psi(\beta)$ denote two arbitrary functions, then one will have to combine those three equations with the following ones:

$$
\begin{aligned}
z^{\prime} & =\frac{\partial f}{\partial \alpha} \alpha^{\prime}+\frac{\partial f}{\partial \beta} \beta^{\prime}+\sum \frac{\partial f}{\partial \varphi^{(i)}(\alpha)} \varphi_{0}^{(i)}(\alpha)+\sum \frac{\partial f}{\partial \psi^{(k)}(\beta)} \psi_{0}^{(k)}(\beta), \\
0 & =\frac{\partial f_{1}}{\partial \alpha} \alpha^{\prime}+\frac{\partial f_{1}}{\partial \beta} \beta^{\prime}+\sum \frac{\partial f_{1}}{\partial \varphi^{(i)}(\alpha)} \varphi_{0}^{(i)}(\alpha)+\sum \frac{\partial f_{1}}{\partial \psi^{(k)}(\beta)} \psi_{0}^{(k)}(\beta) \\
0 & =\frac{\partial f_{2}}{\partial \alpha} \alpha^{\prime}+\frac{\partial f_{2}}{\partial \beta} \beta^{\prime}+\sum \frac{\partial f_{2}}{\partial \varphi^{(i)}(\alpha)} \varphi_{0}^{(i)}(\alpha)+\sum \frac{\partial f_{2}}{\partial \psi^{(k)}(\beta)} \psi_{0}^{(k)}(\beta),
\end{aligned}
$$

in which $\varphi_{0}(\alpha), \psi_{0}(\beta)$ denote two new arbitrary functions, and eliminate the variations $\alpha^{\prime}, \beta^{\prime}$ of $\alpha$ and $\beta$, resp., from them $\left({ }^{1}\right)$. We shall represent the complete derivatives with respect to $\alpha$ and $\beta$ by $\frac{\partial f}{\partial \alpha}, \frac{\partial f}{\partial \beta}, \ldots$

[^1]3. - Since the auxiliary system is linear, it is relatively easy to study it, and it can provide some precise conclusions relative to the proposed system. For example, suppose that one is given just one partial differential equation, and one demands that the equation must admit a general integral in which some arbitrary functions appear, along with their derivatives up to orders that are defined for each of the functions, but with no integration sign. The same thing must be true for the auxiliary equation in $z^{\prime}$ when one replaces $z$ with an arbitrary solution of the proposed equation.

As we have seen for the case of two independent variables, that condition translates analytically into certain relations between the invariants of the auxiliary equation. Upon writing out those relations, we will obtain some new partial differential equations in $z$ that must be verified at the same time as the proposed equation. The solution to the question that was posed can thus be reduced to simple eliminations.
4. - Leaving aside the numerous applications that one can make of those remarks, I shall study the following two problems of geometry, more especially.

Consider a surface $(\Sigma)$ and look for all of the infinitely-close surfaces that can form a Lamé family with $(\Sigma)$, i.e., a family of a triply-orthogonal system. Let $\rho, \rho_{1}, \rho_{2}$ denote the parameters of the three families that comprise an orthogonal system, and let the linear element of space is given by the formula:

$$
d s^{2}=H^{2} d \rho^{2}+H_{1}^{2} d \rho_{1}^{2}+H_{2}^{2} d \rho_{2}^{2},
$$

in which $H, H_{1}, H_{2}$ satisfy some relations in second-order partial derivatives that we proved already in no. $149\left({ }^{1}\right)$. Suppose that the surface $(\Sigma)$ belongs to the family whose parameter is $\rho_{2}$. Since the surfaces of the parameters $\rho$ and $\rho_{1}$ intersect along lines of curvature, one can say that the variables $\rho, \rho_{1}$, and the functions $H$ and $H_{1}$ can be regarded as known at each of its points, and in order to solve the problem that was posed, it will suffice to determine, at all points of $(\Sigma)$, the function $\mathrm{H}_{2}$ that will give the distance from each point of $(\Sigma)$ to the desired infinitely-close surface when it is multiplied by the constant $d \rho_{2}$, which is a surface that we shall call ( $\Sigma^{\prime}$ ). Now, the function $H_{2}$ satisfies (nos. 149, 1039) the equation:

$$
\begin{equation*}
\frac{\partial^{2} H_{2}}{\partial \rho \partial \rho_{1}}=\frac{1}{H} \frac{\partial H}{\partial \rho_{1}} \frac{\partial H_{2}}{\partial \rho}+\frac{1}{H_{1}} \frac{\partial H_{1}}{\partial \rho} \frac{\partial H_{2}}{\partial \rho_{1}}, \tag{1}
\end{equation*}
$$

which is both necessary and sufficient, which we shall assume here, for brevity, in such a way that the problem will reduced to the complete integration of that partial differential equation in $H_{2}$.

That equation is one of the ones to which one can reduce the following problem:
Find all of the surfaces that admit the same spherical representation as the surface ( $\Sigma$ ).
${ }^{(1)}$ See also nos. 1039, 1047, and 1054.

That is because if one assumes that it has solutions $x, y, z$ then it will be nothing but the pointlike equation relative to the conjugate system that is composed of lines of curvature of $(\Sigma)$ and does not differ from, for example, equation (6) of no. 948 (see also no. 950).

We shall then establish a relation between two problems that seem, on first glance, to be completely different: On the one hand, the determination of all surfaces that admit the same spherical representation as $(\Sigma)$, and on the other, the determination of the surfaces $\left(\Sigma^{\prime}\right)$ that are infinitely-close to $(\Sigma)$ in any Lamé family. The explanation for that fact will seem to be immediate when one recalls the theorem of Ribaucour that was proved no. 972. From that proposition, the osculating circles to the orthogonal trajectories of the surfaces of a Lamé family at the points where those trajectories meet one of the $(\Sigma)$ will form a cyclic system. As a result, the given surface $(\Sigma)$ and the desired surface ( $\Sigma^{\prime}$ ) can be considered to be two infinitely-close trajectories of the circles that belong to a cyclic system, and the following way of generating $\left(\Sigma^{\prime}\right)$ will result from that:

Construct the most general cyclic system that is composed of circles normal to ( $\Sigma$ ), and the desired surface $\left(\Sigma^{\prime}\right)$ will be the ones that we learned how to construct in nos. 951, et seq., and are normal to all of the circles in the system.

Since one knows that the search for all of the previous cyclic systems reduces to the determination of all surfaces that admit the same spherical representation as ( $\Sigma$ ), the desired explanation is thus provided in a complete manner, and the relation that is established between the two problems conforms a result that was established already by following a different path (no. 981), but which becomes obvious here: Once the problem of spherical representation has been solved for a surface $(\Sigma)$, the same thing will be true for all of the surfaces that are inverse to $(\Sigma)$.
5. - Now consider a different problem: the search for surfaces that can be mapped ${ }^{( }{ }^{\dagger}$ ) to a given surface ( $\Sigma^{\prime}$ ). If we begin by looking for the surfaces that can be mapped to $(\Sigma)$ and the ones that are infinitely-close to $(\Sigma)$ then we know that the solution to the problem reduces, by definition, to the integration of an equation in equal invariants:

$$
\begin{equation*}
\frac{\partial^{2} \theta}{\partial \alpha \partial \beta}=k \theta \tag{2}
\end{equation*}
$$

The surfaces for which one knows how to solve that problem can be divided into different classes. For each of them, one knows the expressions for the rectangular coordinates $x, y, z$ of a point on the surface as functions of the parameter $\alpha$ and $\beta$ of the asymptotic lines. The expressions contain at least four arbitrary functions of $\alpha$ and $\beta$. For the surfaces of class $p$, the general integral of the equation in $\theta$ consists of two arbitrary functions of $\alpha$ and $\beta$, with their derivatives up to order $p-1$, and the expressions for $x, y, z$ contain $2 p+4$ arbitrary functions (or $2 p+2$, if one chooses the parameters $\alpha$ and $\beta$ suitably). Now, it is clear that if one considers all of the surfaces that can
${ }^{\dagger}{ }^{+}$Translator: The sense of the word "map" here is that is that of "isometry," in particular.
be mapped to a given surface that are determined by equations that contain only arbitrary functions, along with their derivatives up to a well-defined order, then the problem of the infinitely-small deformation can be solved by some formulas that always have the same form and the same nature for each of them. All of those surfaces must then belong to one of the classes that were just defined, and one can obtain them by establishing some finite or differential relations between the arbitrary functions of $\alpha$ and $\beta$ that appear in the general expressions for the coordinates $x, y, z$ of a point of one of those surfaces when they are expressed by means of $\alpha$ and $\beta$. That is the path that we shall follow and formulate analytically.
6. - Recall the formulas of no. 883:
(3)

$$
\left\{\begin{array}{l}
x=\int\left(\theta_{2} \frac{\partial \theta_{3}}{\partial \alpha}-\theta_{3} \frac{\partial \theta_{2}}{\partial \alpha}\right) d \alpha-\left(\theta_{2} \frac{\partial \theta_{3}}{\partial \beta}-\theta_{3} \frac{\partial \theta_{2}}{\partial \beta}\right) d \beta \\
y=\int\left(\theta_{3} \frac{\partial \theta_{1}}{\partial \alpha}-\theta_{1} \frac{\partial \theta_{3}}{\partial \alpha}\right) d \alpha-\left(\theta_{3} \frac{\partial \theta_{1}}{\partial \beta}-\theta_{1} \frac{\partial \theta_{3}}{\partial \beta}\right) d \beta \\
z=\int\left(\theta_{1} \frac{\partial \theta_{2}}{\partial \alpha}-\theta_{2} \frac{\partial \theta_{1}}{\partial \alpha}\right) d \alpha-\left(\theta_{1} \frac{\partial \theta_{2}}{\partial \beta}-\theta_{2} \frac{\partial \theta_{1}}{\partial \beta}\right) d \beta
\end{array}\right.
$$

in which $\theta_{1}, \theta_{2}, \theta_{3}$ are solutions of an equation of the form (2).
The quantities $x_{1}, y_{1}, z_{1}$ are defined by the relations:

$$
\left\{\begin{array}{l}
x_{1}=\int\left(\theta_{1} \frac{\partial \omega}{\partial \alpha}-\omega \frac{\partial \theta_{1}}{\partial \alpha}\right) d \alpha-\left(\theta_{1} \frac{\partial \omega}{\partial \beta}-\omega \frac{\partial \theta_{1}}{\partial \beta}\right) d \beta \\
y_{1}=\int\left(\theta_{2} \frac{\partial \omega}{\partial \alpha}-\omega \frac{\partial \theta_{2}}{\partial \alpha}\right) d \alpha-\left(\theta_{2} \frac{\partial \omega}{\partial \beta}-\omega \frac{\partial \theta_{2}}{\partial \beta}\right) d \beta  \tag{4}\\
z_{1}=\int\left(\theta_{3} \frac{\partial \omega}{\partial \alpha}-\omega \frac{\partial \theta_{3}}{\partial \alpha}\right) d \alpha-\left(\theta_{3} \frac{\partial \omega}{\partial \beta}-\omega \frac{\partial \theta_{3}}{\partial \beta}\right) d \beta
\end{array}\right.
$$

in which $\omega$ is the most general integral of equation (2) that is satisfied by $\theta_{1}, \theta_{2}, \theta_{3}$, gives the most general solution to the total differential equation:

$$
\begin{equation*}
d x d x_{1}+d y d y_{1}+d z d z_{1}=0 \tag{5}
\end{equation*}
$$

in such a way that if $\varepsilon$ denotes an infinitely-small constant then:

$$
x+\varepsilon x_{1}, \quad y+\varepsilon y_{1}, \quad z+\varepsilon z_{1}
$$

will be the coordinates of a point on a surface $\left(\Sigma^{\prime}\right)$ that is infinitely-close to $(\Sigma)$ and can be mapped to it.
7. - Having said that, if one has established some relations between the arbitrary functions that are contained in formulas (3) such that all of the surfaces $(\Sigma)$ can be mapped to each other, and if one varies not only the arbitrary functions, which still persist, but also the parameters $\alpha$ and $\beta$, then the expressions:

$$
\begin{aligned}
& x+\delta x+\frac{\partial x}{\partial \alpha} \delta \alpha+\frac{\partial x}{\partial \beta} \delta \beta \\
& y+\delta y+\frac{\partial y}{\partial \alpha} \delta \alpha+\frac{\partial y}{\partial \beta} \delta \beta \\
& z+\delta z+\frac{\partial z}{\partial \alpha} \delta \alpha+\frac{\partial z}{\partial \beta} \delta \beta
\end{aligned}
$$

in which $\delta x, \delta y, \delta z$ denote the variations that produce the change of form in the arbitrary functions, will also be the coordinates of a point on a surface that can be mapped to $(\Sigma)$ and is infinitely-close to it. In order for the surface to coincide with ( $\Sigma^{\prime}$ ), it is necessary and sufficient that one can arrange the $\delta \alpha, \delta \beta$ in such a manner as to satisfy the equations $\left({ }^{1}\right)$ :

$$
\left\{\begin{array}{l}
x+\delta x+\frac{\partial x}{\partial \alpha} \delta \alpha+\frac{\partial x}{\partial \beta} \delta \beta=x_{1} \\
y+\delta y+\frac{\partial y}{\partial \alpha} \delta \alpha+\frac{\partial y}{\partial \beta} \delta \beta=y_{1}  \tag{6}\\
z+\delta z+\frac{\partial z}{\partial \alpha} \delta \alpha+\frac{\partial z}{\partial \beta} \delta \beta=z_{1}
\end{array}\right.
$$

and conversely, whenever it is possible to satisfy those equations, all of the surfaces $(\Sigma)$ can be mapped to each other.
8. - If one notes that $\theta_{1}, \theta_{2}, \theta_{3}$ are the direction parameters of the normal to $(\Sigma)$ then one can replace the preceding three equations by the single equation:

$$
\begin{equation*}
\Theta=\theta_{1}\left(\delta x-x_{1}\right)+\theta_{2}\left(\delta y-y_{1}\right)+\theta_{3}\left(\delta z-z_{1}\right)=0 \tag{7}
\end{equation*}
$$

that is obtained by adding the three equations after multiplying them by $\theta_{1}, \theta_{2}, \theta_{3}$, respectively.

[^2]The last equation contains quadrature signs, but one can make them disappear by differentiation. Indeed, one has:

$$
\left\{\begin{align*}
\frac{\partial^{2} \Theta}{\partial \alpha \partial \beta}-k \Theta & =\mathbf{S} \frac{\partial \theta_{1}}{\partial \alpha}\left(\delta \frac{\partial x}{\partial \beta}-\frac{\partial x_{1}}{\partial \beta}\right)+\mathbf{S} \frac{\partial \theta_{1}}{\partial \beta}\left(\delta \frac{\partial x}{\partial \alpha}-\frac{\partial x_{1}}{\partial \alpha}\right)+\mathbf{S} \theta_{1}\left(\delta \frac{\partial^{2} x}{\partial \alpha \partial \beta}-\frac{\partial^{2} x_{1}}{\partial \alpha \partial \beta}\right) \\
& =2\left|\begin{array}{lll}
\delta \theta_{1} & \delta \theta_{2} & \delta \theta_{3} \\
\frac{\partial \theta_{1}}{\partial \alpha} & \frac{\partial \theta_{2}}{\partial \alpha} & \frac{\partial \theta_{3}}{\partial \alpha} \\
\frac{\partial \theta_{1}}{\partial \beta} & \frac{\partial \theta_{2}}{\partial \beta} & \frac{\partial \theta_{3}}{\partial \beta}
\end{array}\right|-2 \frac{\partial \omega}{\partial \alpha} \mathbf{S} \theta_{1} \frac{\partial \theta_{1}}{\partial \beta}+2 \frac{\partial \omega}{\partial \beta} \mathbf{S} \theta_{1} \frac{\partial \theta_{1}}{\partial \alpha} \tag{8}
\end{align*}\right.
$$

identically, and consequently one must first have that the functions $\omega, \theta_{1}, \theta_{2}, \theta_{3}, \delta \theta_{1}, \delta \theta_{2}, \delta \theta_{3}$ verify the relation:

$$
\left|\begin{array}{lll}
\delta \theta_{1} & \delta \theta_{2} & \delta \theta_{3}  \tag{9}\\
\frac{\partial \theta_{1}}{\partial \alpha} & \frac{\partial \theta_{2}}{\partial \alpha} & \frac{\partial \theta_{3}}{\partial \alpha} \\
\frac{\partial \theta_{1}}{\partial \beta} & \frac{\partial \theta_{2}}{\partial \beta} & \frac{\partial \theta_{3}}{\partial \beta}
\end{array}\right|-\frac{\partial \omega}{\partial \alpha} \mathbf{S} \theta_{1} \frac{\partial \theta_{1}}{\partial \beta}+\frac{\partial \omega}{\partial \beta} \mathbf{S} \theta_{1} \frac{\partial \theta_{1}}{\partial \alpha}=0
$$

identically, which is devoid of any integration sign. When that equation is verified, the main difficulty in the problem will have disappeared. Nonetheless, it still remains for one to verify the original equation (9), which is not by any means a consequence of the preceding one, but whose left-hand side $\Theta$ must the satisfy the equation:

$$
\frac{\partial^{2} \Theta}{\partial \alpha \partial \beta}=k \Theta
$$

by virtue of the identity (8).
9. - Let us next apply that general method to the surfaces of the first class, for which one has:

$$
\begin{equation*}
\theta_{1}=A_{1}+B_{1}, \quad \theta_{2}=A_{2}+B_{2}, \quad \theta_{3}=A_{3}+B_{3}, \tag{10}
\end{equation*}
$$

in which $A_{1}, A_{2}, A_{3}$ are functions of $\alpha$, and $B_{1}, B_{2}, B_{3}$ are functions of $\beta$. The equation of equal invariant that $\theta_{1}, \theta_{2}, \theta_{3}$ must satisfy is this one:

$$
\begin{equation*}
\frac{\partial^{2} \theta}{\partial \alpha \partial \beta}=0 \tag{11}
\end{equation*}
$$

and one has:

$$
\left\{\begin{array}{l}
x=A_{3} B_{2}-A_{2} B_{3}+\int\left(A_{2} d A_{3}-A_{3} d A_{3}\right)-\int\left(B_{2} d B_{3}-B_{3} d B_{3}\right)  \tag{12}\\
y=A_{1} B_{3}-A_{3} B_{1}+\int\left(A_{3} d A_{1}-A_{1} d A_{3}\right)-\int\left(B_{3} d B_{1}-B_{1} d B_{3}\right) \\
z=A_{2} B_{1}-A_{1} B_{2}+\int\left(A_{1} d A_{2}-A_{2} d A_{1}\right)-\int\left(B_{1} d B_{2}-B_{2} d B_{1}\right)
\end{array}\right.
$$

Let us see if we can establish some relations between the functions $A$ and the function $B$ such that all of the corresponding surfaces can be mapped to each other. Since an arbitrary function of $\alpha$ and an arbitrary function of $\beta$ must remain, it is clear that can obtain only one relation between the functions $A$ and only one between the functions $B$.

One has:

$$
\begin{equation*}
\omega=A+B \tag{13}
\end{equation*}
$$

here, and the fundamental equation will take the form:

$$
\left|\begin{array}{ccc}
\delta A_{1}+\delta B_{1} & \delta A_{2}+\delta B_{2} & \delta A_{3}+\delta B_{3}  \tag{14}\\
A_{1}^{\prime} & A_{2}^{\prime} & A_{3}^{\prime} \\
B_{1}^{\prime} & B_{2}^{\prime} & B_{3}^{\prime}
\end{array}\right|-A^{\prime} \mathbf{S} \theta_{1} \frac{\partial \theta_{1}}{\partial \beta}+B^{\prime} \mathbf{S} \theta_{1} \frac{\partial \theta_{1}}{\partial \alpha}=0
$$

Since, by its very nature, it will decompose into relations that are linear in the functions of $\alpha$ and the functions of $\beta$, and since on the other hand, $A^{\prime}$ and $B^{\prime}$ are annulled at the same time as the $\delta A_{i}$ and the $\delta B_{i}$, one can assume that $A^{\prime}$ depends linearly $\delta A_{1}, \delta A_{2}, \delta A_{3}$, and that $B^{\prime}$ depends linearly $\delta B_{1}, \delta B_{2}, \delta B_{3}$, and as a result, the preceding equation will decompose into the following two:

$$
\begin{align*}
& \left|\begin{array}{ccc}
\delta A_{1} & \delta A_{2} & \delta A_{3} \\
A_{1}^{\prime} & A_{2}^{\prime} & A_{3}^{\prime} \\
B_{1}^{\prime} & B_{2}^{\prime} & B_{3}^{\prime}
\end{array}\right|-A^{\prime} \mathbf{S}\left(A_{i}+B_{i}\right) B_{i}^{\prime}=0,  \tag{15}\\
& \left|\begin{array}{rrr}
\delta B_{1} & \delta B_{2} & \delta B_{3} \\
A_{1}^{\prime} & A_{2}^{\prime} & A_{3}^{\prime} \\
B_{1}^{\prime} & B_{2}^{\prime} & B_{3}^{\prime}
\end{array}\right|+B^{\prime} \mathbf{S}\left(A_{i}+B_{i}\right) A_{i}^{\prime}=0 . \tag{16}
\end{align*}
$$

If one gives an arbitrary, but fixed, value to $\alpha$ in the first one then it will take the form:

$$
\left(B_{1}+m_{1}\right) B_{1}^{\prime}+\left(B_{2}+m_{2}\right) B_{2}^{\prime}+\left(B_{3}+m_{3}\right) B_{3}^{\prime}=0,
$$

in which $m_{1}, m_{2}, m_{3}$ denote constants. Upon integrating that, one will then have:

$$
\left(B_{1}+m_{1}\right)^{2}+\left(B_{2}+m_{2}\right)^{2}+\left(B_{3}+m_{3}\right)^{2}=\text { const. }
$$

However, since it is permissible to combine the constants $m_{i}$ with the functions $A_{i}$ in the expressions for $\theta_{i}$, one can suppose that those constants are zero and reduce the preceding equation to the form:

$$
\begin{equation*}
B_{1}^{2}+B_{2}^{2}+B_{3}^{2}=2 h, \tag{17}
\end{equation*}
$$

in which $h$ denotes a constant. Equation (15) will then take the form:

$$
\left|\begin{array}{ccc}
\delta A_{1} & \delta A_{2} & \delta A_{3} \\
A_{1}^{\prime} & A_{2}^{\prime} & A_{3}^{\prime} \\
B_{1}^{\prime} & B_{2}^{\prime} & B_{3}^{\prime}
\end{array}\right|-A^{\prime} \mathrm{S} A_{i} B_{i}^{\prime}=0,
$$

and since no other relation can exist between the functions $B_{i}$, one must annul the coefficient of each derivative $B_{i}^{\prime}$, which will give:

$$
\left\{\begin{array}{l}
A_{3}^{\prime} \delta A_{2}-A_{2}^{\prime} \delta A_{3}-A^{\prime} A_{1}=0  \tag{18}\\
A_{1}^{\prime} \delta A_{3}-A_{3}^{\prime} \delta A_{1}-A^{\prime} A_{2}=0 \\
A_{2}^{\prime} \delta A_{1}-A_{1}^{\prime} \delta A_{2}-A^{\prime} A_{3}=0
\end{array}\right.
$$

Upon multiplying those three equations by $A_{1}^{\prime}, A_{2}^{\prime}, A_{3}^{\prime}$, respectively, one can deduce that:

$$
A_{1} A_{1}^{\prime}+A_{2} A_{2}^{\prime}+A_{3} A_{3}^{\prime}=0,
$$

and as a result:

$$
\begin{equation*}
A_{1}^{2}+A_{2}^{2}+A_{3}^{2}=2 h_{1}, \tag{19}
\end{equation*}
$$

in which $h_{1}$ denotes a new constant.
Since equation (9) is verified by virtue of the relations (17) and (19), it must now amount to equation (7). One has:

$$
\begin{aligned}
& x_{1}=A B_{1}-B A_{1}-\int\left(A d A_{1}-A_{1} d A\right)+\int\left(B d B_{1}-B_{1} d B\right), \\
& y_{1}=A B_{2}-B A_{2}-\int\left(A d A_{2}-A_{2} d A\right)+\int\left(B d B_{2}-B_{2} d B\right), \\
& z_{1}=A B_{3}-B A_{3}-\int\left(A d A_{3}-A_{3} d A\right)+\int\left(B d B_{3}-B_{3} d B\right),
\end{aligned}
$$

here.

When one takes into account the relations (18) and the analogous relations relative to the functions $B_{i}$, an easy calculation will give:

$$
\delta x-x_{1}=\left(A_{2}+B_{2}\right)\left(\delta A_{3}-\delta B_{3}\right)-\left(A_{3}+B_{3}\right)\left(\delta A_{2}-\delta B_{2}\right)+(A+B)\left(A_{1}-B_{1}\right) .
$$

One deduces from this that:

$$
\Theta=\mathbf{S} \theta_{1}\left(\delta x-x_{1}\right)=(A+B)\left(2 h_{1}-2 h\right) .
$$

It will then suffice to take $h=h_{1}$, and one will then recover the propositions that were stated in nos. $\mathbf{7 6 9}$ and 770, to the extent that is essential.
10. - For the surfaces of the second class, one has:

$$
\left\{\begin{array}{l}
\theta_{1}=A_{1}^{\prime}+B_{1}^{\prime}-2 \frac{A_{1}-B_{1}}{\alpha-\beta},  \tag{20}\\
\theta_{2}=A_{2}^{\prime}+B_{2}^{\prime}-2 \frac{A_{2}-B_{2}}{\alpha-\beta}, \\
\theta_{3}=A_{3}^{\prime}+B_{3}^{\prime}-2 \frac{A_{3}-B_{3}}{\alpha-\beta}
\end{array}\right.
$$

The formulas that give the coordinates are even more complicated.
For example, the value of $x$ is:
$x=\int\left(A_{2}^{\prime} A_{3}^{\prime \prime}-A_{3}^{\prime} A_{2}^{\prime \prime}\right) d \alpha-\int\left(B_{2}^{\prime} B_{3}^{\prime \prime}-B_{3}^{\prime} B_{2}^{\prime \prime}\right) d \beta+A_{3}^{\prime} B_{2}^{\prime}-A_{2}^{\prime} B_{3}^{\prime}+2 \frac{\left(A_{3}-B_{3}\right)\left(A_{2}^{\prime}-B_{2}^{\prime}\right)-\left(A_{2}-B_{2}\right)\left(A_{3}^{\prime}-B_{3}^{\prime}\right)}{\alpha-\beta}$,
and one will get the corresponding values of $y$ and $z$ by performing circular permutations of the indices $1,2,3$. Similarly, the value of $x_{1}$ is:

$$
x_{1}=\int\left(A_{1}^{\prime} A^{\prime \prime}-A^{\prime} A_{1}^{\prime \prime}\right) d \alpha-\int\left(B_{1}^{\prime} B^{\prime \prime}-B^{\prime} B_{1}^{\prime \prime}\right) d \beta+A^{\prime} B_{1}^{\prime}-A_{1}^{\prime} B^{\prime}+2 \frac{(A-B)\left(A_{1}^{\prime}-B_{1}^{\prime}\right)-\left(A_{1}-B_{1}\right)\left(A^{\prime}-B^{\prime}\right)}{\alpha-\beta} .
$$

Equation (9), which is to be solved, also takes a form that is much less simple.
One can nonetheless succeed in finding a solution to it by adopting the hypothesis that:

$$
\begin{equation*}
\delta A_{i}=A A_{i}^{\prime}-A^{\prime} A_{i}, \quad \delta B_{i}=B B_{i}^{\prime}-B^{\prime} B_{i}, \tag{21}
\end{equation*}
$$

which would lead to the following values for the arbitrary functions:

$$
\left\{\begin{array}{lll}
A_{1}=i \frac{A^{2}-1}{2 A^{\prime}}, & A_{2}=i \frac{A^{2}+1}{2 A^{\prime}}, & A_{3}=\frac{i A}{A^{\prime}},  \tag{22}\\
B_{1}=i \frac{B^{2}-1}{2 B^{\prime}}, & B_{2}=\frac{B^{2}+1}{2 B^{\prime}}, & B_{3}=\frac{i B}{B^{\prime}} .
\end{array}\right.
$$

$A$ and $B$ denote arbitrary functions (but different, of course) of the ones that appear in the preceding expressions for $\delta A_{i}, \delta B_{i}$.

The solution that corresponds to those values of the functions $A_{i}, B_{k}$ is not distinct from the one that we gave in nos. 1078-1080 that is due to Weingarten. It coincides on the surfaces that can be mapped to the paraboloid with a generator that is tangent to the circle at infinity.
11. - More generally, one can demand to find what the values would be for the functions $\theta_{i}$ that correspond to the new solutions that were considered by Weingarten, Baroni, and Goursat, which are solutions that we made known in Book VIII, Chap. XIII. One will easily see that when one recalls the formulas of no. 916, if $p$ is the general solution to equation (57) of no. 1074:

$$
\begin{equation*}
\frac{\partial^{2} p}{\partial \alpha \partial \beta}=\frac{\psi^{\prime \prime}(p)}{(1+\alpha \beta)^{2}} \tag{23}
\end{equation*}
$$

then one can take:

$$
\left\{\begin{array}{l}
\theta_{1}=\frac{(1+\alpha \beta)^{2}}{2 \sqrt{i}}\left(\frac{\partial p}{\partial \alpha} \frac{\partial C}{\partial \beta}-\frac{\partial p}{\partial \beta} \frac{\partial C}{\partial \alpha}\right), \\
\theta_{2}=\frac{(1+\alpha \beta)^{2}}{2 \sqrt{i}}\left(\frac{\partial p}{\partial \alpha} \frac{\partial C^{\prime}}{\partial \beta}-\frac{\partial p}{\partial \beta} \frac{\partial C^{\prime}}{\partial \alpha}\right),  \tag{24}\\
\theta_{3}=\frac{(1+\alpha \beta)^{2}}{2 \sqrt{i}}\left(\frac{\partial p}{\partial \alpha} \frac{\partial C^{\prime \prime}}{\partial \beta}-\frac{\partial p}{\partial \beta} \frac{\partial C^{\prime \prime}}{\partial \alpha}\right),
\end{array}\right.
$$

in which $C, C^{\prime}, C^{\prime \prime}$ are the direction cosines that are given by formulas (52) (no. 1074).


[^0]:    $\left(^{1}\right)$ G. DARBOUX, "Sur les équations aux dérivées partielles," C. R. Acad. Sci. Paris 96 (19 March 1883), pp. 755.
    $\left({ }^{2}\right)$ One can generalize that notion of an auxiliary system by varying not only the unknown functions in the equations, but also certain arbitrary constants or functions. However, one can reduce that more general method to the one that we employ in the text by introducing some new unknowns.

[^1]:    $\left({ }^{1}\right)$ One can likewise use all of the incomplete solutions, provided that they contain arbitrary constants or functions that one can vary.

[^2]:    $\left({ }^{1}\right)$ We suppose that the constant $\varepsilon$ has been combined with the arbitrary functions that appear in $\omega$ as a multiplier.

