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The five-dimensional universe and wave mechanics

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Abstract. – The goal of this article is to present the remarkably simple form that mechanics takes in both its older aspect and its newer wave-like aspect when one adopts the idea of a five-dimensional universe that was put forth by Kaluza and Kramers. The most attractive consequence of that idea is to make the notion of force disappear from mechanics entirely and to replace it with some geometric concepts, even in the case of the motion of a point-like electric charge in an electromagnetic field. Furthermore, thanks to the theory of a five-dimensional universe, it is possible give a very satisfactory form to the laws of propagation in the new wave mechanics. That fact has been pointed out already in an interesting paper by O. Klein, but the equation of propagation that he proposed appears to need modification in the sense that is indicated in the present paper.

I. – INTRODUCTION.

1. The notion of force and general relativity. – The essential consequence of the principle of equivalence is to make the metaphysical notion of force disappear from the theory of gravitation. In Einstein's conception of things, the motion of a material point in a gravitational field is defined by the simple condition that the space-time world-line that represents that motion should be a geodesic. If the gravitational field is zero then space-time will be Euclidian, and the geodesics will be straight lines, as the principle of inertia would demand. If the gravitational field is not zero then the space-time will not be Euclidian and the geodesics will no be longer straight, in such a way that the spatial trajectories of material points will be curved.

The success of this beautiful interpretation of the gravitational field makes it very tempting to expel the notion of force from physics entirely by replacing it with some notions that are drawn from metric geometry, but there is a complication that one must certainly take into account. In the present state of our knowledge, it seems that all of the forces that have revealed themselves in experiments come down to only two types: The force of gravitation and the electromagnetic force. As we said, the former type of force is reducible entirely to the notion of space-time curvature, but the same thing is not true for the latter. Indeed, the trajectory of a point-like charge in a region where an electromagnetic field prevails, but the gravitational field is imperceptible, is not rectilinear, although the space-time is Euclidian; the world-line of the charge is not a geodesic then. Moreover, material particles with the same mass that carry different charges will have different trajectories, in such a way that no class of space-time curves that are defined by an *intrinsic* property can be identified with the possible world-lines of point-like charges, which are lines that depend upon the nature of the moving body by the intermediary of the ratio e / m_0 of the charge to the proper mass.

In order to perfect Einstein's work and reduce the electromagnetic force to geometric quantities, Kaluza and Kramers ([1]) developed a theory that is bold, but beautiful: viz., the theory of five-dimensional relativity. O. Klein ([2]) showed that this five-dimensional relativity permits one to write the equations of the new wave mechanics in a remarkably symmetric fashion. Since all of these concepts are not very well known to physicists, and since I prefer to write some of the equations in Klein's paper differently, moreover, I believe that it would be useful to summarize the dynamical side of that question here.

II. – VIEWPOINT OF NON-WAVE MECHANICS.

2. Motion of a point in a gravitational field. – Imagine the motion of a material point of proper mass m_0 in a gravitational field for which one has the metric relation:

$$ds^2 = g_{ik}\, dx^i\, dx^k, \tag{1}$$

with the usual summation convention on the indices.

Upon changing the sign that is usually chosen, we will call the curvilinear integral:

$$A\,(M) = \int_0^M m_0\, c\, ds\,, \tag{2}$$

which is taken along a world-line from a starting point O to a point M, the *Hamiltonian action of a point.* Hamilton's principle then shows that the world-lines are space-time geodesics.

One obtains the equations of motion very easily by writing:

$$\delta \int_0^M m_0\, c\, ds \;=\; \delta \int_0^M m_0\, c \frac{dx^i}{ds} dx^k \;=\; \delta \int_0^M m_0\, c\, g_{ik}\, u^i\, u^k ds = 0 \tag{3}$$

and remarking that *since the desired extremals are geodesics, A (M)* will not vary (up to first order) when one varies the line of integration. The classical Euler-Lagrange equations that correspond to (3) are then:

$$\frac{d}{ds}(m_0\, c\, g_{ik}\, u^k) = \frac{1}{2} m_0 c \frac{\partial g_{\mu\nu}}{\partial x^i} u^\mu\, u^\nu\,. \tag{4}$$

These are the equations of motion, and it is easy to verify that in the first approximation they lead to Newton's theory of gravitation. That point is sufficiently well-known today that it seems pointless to elaborate.

([1]) Kaluza, Sitzber. d. Berl. Akad. (1921), pp. 966. – Kramers, Proc. Amsterdam **23** no. 7 (1922).
([2]) O. Klein, Zeit. Phys. **36** (1926), pp. 893.

3. Motion of point-like charge in an electromagnetic field. – In order to simplify, suppose that the gravitational field is negligible and take rectangular axes. From the relativistic viewpoint, the electromagnetic field is defined by a space-time vector $\boldsymbol{\varphi}$ whose components are given as functions of the scalar potential ψ and the vector potential $\boldsymbol{a}$ by the relations:

$$\varphi_4 = \varphi^4 = \frac{\psi}{c}, \quad \varphi_1 = -\varphi^1 = -\frac{a_x}{c}, \qquad \varphi_2 = -\varphi^2 = -\frac{a_y}{c}, \qquad \varphi_3 = -\varphi^3 = -\frac{a_z}{c}. \tag{5}$$

We know that the electric and magnetic fields are defined by the formulas:

$$\boldsymbol{h} = -\operatorname{grad}\psi - \frac{\partial a}{\partial t}, \qquad \boldsymbol{H} = \operatorname{rot}\boldsymbol{a}. \tag{6}$$

According to Lorentz, the force that acts upon the charge e is:

$$\boldsymbol{f} = e\,\{\boldsymbol{h} + \frac{1}{c}[\boldsymbol{v}\times\boldsymbol{H}]\}. \tag{7}$$

The Hamiltonian action of the moving body can be written:

$$A(M) = \int_0^M (m_0\,c + e\,\varphi_s)\,ds \tag{8}$$

here, in which φ_s is the component of the $\boldsymbol{\varphi}$ that is tangential to the world-line. Since φ_s generally depends upon the x_i , the world-line will no longer be a geodesic and will depend upon the ratio e / m_0 .

Since the extremals of the Hamiltonian variational problem are no longer geodesics, we will write:

$$\delta\int_{t_0}^{t_M} L\,ds = 0, \tag{9}$$

$$L = m_0 c^2\sqrt{1-\beta^2} + e\,(c\,\varphi_4 + v_x\,\varphi_x + v_y\,\varphi_y + v_z\,\varphi_z),$$

in which v_x , v_y , v_z are the components of the velocity in the usual sense. The Lagrange equations are then:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial v^i}\right) = \frac{\partial L}{\partial x^i} \qquad (i = 1, 2, 3), \tag{10}$$

and one easily verifies that they have the vectorial form:

$$\frac{d}{dt}\left(\frac{m_0\mathbf{v}}{\sqrt{1-\beta^2}}\right) = e\,\boldsymbol{f}. \tag{11}$$

These are the equations of the dynamics of the electron when the notion of force still prevails.

4. Five-dimensional relativity. – Imagine, like Kaluza and Kramers, that in order to represent the set of world-events, one must employ a five-dimensional multiplicity; i.e., one must add a fifth dimension to space-time that corresponds to a fifth variable x^0. The variations of that fifth variable are completely lost to our senses, in such a way that two world-points that correspond to the same four space-time variables, but with different values of the variable x^0 will be indiscernible to us. We will then be confined to our four-dimensional space-time multiplicity, and we will perceive only the projections of the five-dimensional world-points onto that space-time.

Under those conditions, the world line element will be given by the formula:

$$d\sigma^2 = \gamma_{00}\,(dx^0)^2 + 2\sum_{i=1}^{4}\gamma_{0i}\,dx^0 dx^i + \sum_{i=1}^{4}\sum_{k=1}^{4}\gamma_{ik}\,dx^i dx^k\ , \tag{12}$$

in which we suppose that all of the γ are independent of the coordinate x^0.

Upon passing from one system of reference axes to another one that is animated with an arbitrary motion with the respect to the first one, we can perform all sorts of changes of variables in space-time, but those changes of variables will have no effect on the variable x^0. It will then seem logical to say that all of the changes of variables that are *humanly* possible will have the form:

$$x'_i = f_i\,(x_1, x_2, x_3, x_4) \qquad (i = 1, 2, 3, 4). \tag{13}$$

The quantity:

$$\gamma_{00}\,(dx_0)^2 + \sum_{i=1}^{4}\gamma_{0i}\,dx^0 dx^i$$

is an invariant *for us* then, and we will see, in turn, that it is the same as:

$$ds^2 = \left(\gamma_{ik} - \frac{\gamma_{0i}\,\gamma_{0k}}{\gamma_{00}}\right) dx^i\,dx^k. \tag{14}$$

In formula (14), and in what follows, one must always perform that summation only for values 1, 2, 3, 4 that correspond to space-time.

We then set:

$$d\theta = \sqrt{\gamma_{00}}\,dx^0 + \frac{\gamma_{0i}}{\sqrt{\gamma_{00}}}\,dx^i\ . \tag{15}$$

That is an invariant for us, and we can obviously write:

$$d\sigma^2 = ds^2 + d\theta^2. \tag{16}$$

In order to arrive at the interpretation of electromagnetic phenomena, it is necessary to introduce the potentials. The γ_{0i} transform like space-time vectors, so one will be led to set:

$$\gamma_{0i} = \alpha\, \gamma_{00}\, \varphi_i\,, \tag{17}$$

in which the φ_i are defined by (5), and α is a homogeneity constant.

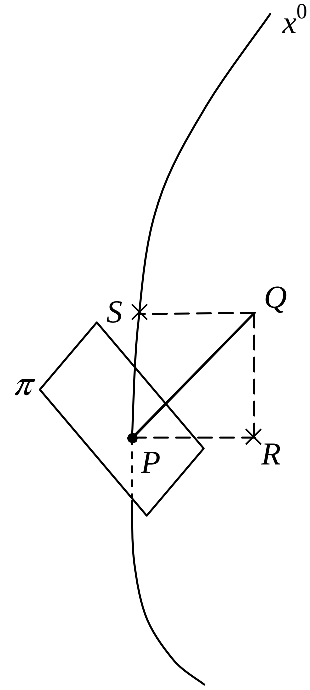

Figure 1. – $\overline{PQ} = d\sigma$, $\overline{PS} = d\theta$, $\overline{PR} = ds$.

The equation $d\theta = 0$ expresses the idea that the infinitesimal displacement considered is normal to the direction x^0 ; i.e., to the intersection of the surfaces x^1 = const., x^2 = const., x^3 = const., x^4 = const., which is an intersection that is not generally normal to our space-time x^0 = const. From the definition (17) of γ_{0i} , the equation $d\theta = 0$ will be integrable only if the electric and magnetic fields are zero; in general, there will therefore exist no four-dimensional multiplicities that are normal to the direction x^0 at any point. However, it is natural to suppose that the element of any world-line satisfies the condition $d\theta = 0$ (i.e., it is everywhere normal to the fifth dimension) to an extent that is determined entirely by the gravitational phenomena, and that it is, in turn, given by the expression $g_{ik}\, dx^i\, dx^k$ in Einstein's theory. From (16) and (14), that will lead us to set:

$$\gamma_{ik} = g_{ik} + \frac{\gamma_{0i}\,\gamma_{0k}}{\gamma_{00}} = g_{ik} + \gamma_{00}\, \alpha^2\, \varphi_i\, \varphi_k\,. \tag{18}$$

It is interesting to represent things geometrically by means of the figure above (Fig. 1). A coordinate line x^0 is traced in that figure. A small portion of the four-dimensional multiplicity x^0 = const. that passes through a point P of that line is represented by a plane element π that is oblique to the direction x^0. $\overline{PQ}$ is an element of the world-line of length $d\sigma$, $\overline{PS}$ is its orthogonal projection onto the direction x^0 (viz., the *covariant* component of $d\sigma$ along x^0), and $\overline{PR}$ is its projection normal to the direction x^0. Formula (16), in turn, shows that one will have:

$$\overline{PS} = d\theta, \qquad \overline{PR} = ds.$$

We will see later [formula (26)] that the tangent $d\theta / ds$ of the angle $\widehat{QPR}$ is proportional to the ratio e / m_0 of the material point whose world-line element is $\overline{PR}$. It will then result that the world-line of any moving body will make the same angle with the direction x^0 at each point, which will be a right angle when the electric charge is zero.

It is then obvious that the x^0 coordinate lines possess an absolute character, in a certain sense, so our space-time can be considered to be an arbitrary four-dimensional slice of the five-dimensional universe, and the intersections of the slice with the x^0 coordinate lines have only one absolute sense. That can be interpreted by assuming, with Klein, that the x^0 lines are closed and their total length is less than anything that one can measure; the universe will be, in some way, filamentary in the direction that is normal to the fifth dimension.

One can deduce Einstein's law of gravitation from the conception of the five-dimensional universe with the aid of a variational principle, and at the same time, Maxwell's equations ([1]). I shall not insist upon this aspect of the question here and will confine myself to noting that the following consequence will result from that argument: The product $\gamma_{00}\,\alpha^2$ is coupled with the usual gravitational constant G by the relation:

$$\gamma_{00}\,\alpha^2 = \frac{16\pi G}{c^2}. \tag{19}$$

5. The world-line of any material point is a geodesic. – When one assumes the existence of a fifth dimension to the universe, one can state the following principle: *The world-line of any material point is a geodesic in the five-dimensional universe.* We shall verify that by showing that this principle indeed leads to Einsteinian dynamics under the condition that one must make a hypothesis that determines the geometric significance of the electric charge.

We define a world-velocity with five components:

$$u^i = \frac{dx^i}{d\sigma} \qquad (i = 0, 1, 2, 3, 4), \tag{20}$$

and write:

$$\delta\int_O^M \tfrac{1}{2}\,d\sigma = \delta\int_O^M \tfrac{1}{2}[\gamma_{00}(u^0)^2 + 2\gamma_{0i}\,u^0\,u^i + \gamma_{ik}\,u^i\,u^k]\,d\sigma = 0, \tag{21}$$

([1]) O. KLEIN, *loc. cit.*, pp. 897-899.

in order to define the geodesics, in which O and M are two fixed points in the universe. Since the extremals are geodesics, we have to vary an integral of the form $\int L\,d\sigma$ when it is taken between fixed limits (see paragraph 2), and we can then write the Lagrange equations:

$$\frac{d}{d\sigma}\left(\frac{\partial L}{\partial u^i}\right) = \frac{\partial L}{\partial x^i} \qquad (i = 0, 1, 2, 3, 4). \tag{22}$$

Upon taking (15) into account, one will first get:

$$\gamma_{00}\, u^0 + \gamma_{0i}\, u^i = \sqrt{\gamma_{00}}\,\frac{d\theta}{d\sigma} = \text{const.} = p_0\,. \tag{23}$$

One will then find that:

$$\frac{d}{d\sigma}(g_{ik}\, u^k + \alpha\, p_0\, \varphi_i) = \frac{1}{2}u^\mu u^\nu \frac{\partial g_{\mu\nu}}{\partial x^i} + \alpha\, p_0\, u^\mu \frac{\partial \varphi_\mu}{\partial x^i} \tag{24}$$

for the variables with indices 1, 2, 3, 4.

If the electromagnetic field is zero then one only has to take s to be the independent variable instead of σ in order to find equation (4) of Einstein's theory. If the gravitational field is zero then the g_{ik} will constant, and one will have:

$$\frac{d}{d\sigma} g_{ik}\, u^k = \alpha\, p_0\, \varphi_i \left(\frac{\partial \varphi_\mu}{\partial x^i} - \frac{\partial \varphi_i}{\partial x^\mu}\right) = \alpha\sqrt{\gamma_{00}}\,\frac{d\theta}{d\sigma}\left(\frac{\partial \varphi_\mu}{\partial x^i} - \frac{\partial \varphi_i}{\partial x^\mu}\right)u^\mu\,. \tag{25}$$

Suppose that the inclination of the world-line of a moving body of mass m_0 and charge e to the direction x^0 is defined by the fundamental relation:

$$\frac{d\theta}{ds} = \frac{e}{\alpha\sqrt{\gamma_{00}}\, m_0 c}\,. \tag{26}$$

Equation (25) can then be written:

$$-\frac{d}{ds}\left(\frac{dx^i}{ds}\right) = \frac{e}{m_0 c}\cdot\frac{dx^\mu}{ds}\cdot\left(\frac{\partial \varphi_\mu}{\partial x^i} - \frac{\partial \varphi_i}{\partial x^\mu}\right) \qquad (i = 1, 2, 3), \tag{27}$$

and we will easily recover equations (11) in rectangular coordinates with the aid of the definitions (5).

Therefore, with the geometric significance that we have attributed to the potentials and the ratio e / m_0, the five-dimensional world-lines of material points will always be geodesics. *The notion of force has been banished completely from mechanics.*

Since the constants α and γ_{00} cannot depend upon the properties of the moving body in question, equation (26) will lead us to pose:

$$m_0\, c = I\,\frac{ds}{d\sigma}, \qquad \frac{e}{\alpha\sqrt{\gamma_{00}}} = I\,\frac{d\theta}{d\sigma}, \tag{28}$$

in which I is an invariant that is such that (1):

$$I^2 = m_0^2\, c^2 + \frac{e^2}{\alpha^2 \gamma_{00}}, \tag{29}$$

by reason of (16).

In order to recover the Hamiltonian action of formula (2) in the case of zero charge, we set:

$$A\,(M) = \int_O^M I\,d\sigma = \int_O^M I\left(\frac{d\theta}{d\sigma}d\theta + \frac{ds}{d\sigma}ds\right) = \int_O^M I\left(\frac{e}{\alpha\sqrt{\gamma_{00}}}d\theta + m_0 c\,ds\right)$$

$$= \frac{e}{\alpha}x_0 + f\,(x_1, x_2, x_3, x_4)\,. \tag{30}$$

III. – VIEWPOINT OF WAVE MECHANICS.

6. The wave mechanics and four-dimensional relativity. – I have presented the principles of wave mechanics in a previous article (2). The essential idea of that new doctrine is to consider matter as a phenomenon of an undulatory nature that is represented by a function that obeys certain equations of propagation. When conditions that permit us to study the solutions of the equation of propagation by means of the procedures of geometrical optics are realized, any material entity can be compared to a group of monochromatic waves with frequencies that are found in a very small interval $\nu - \delta\nu$, $\nu + \delta\nu$. The superposition of waves in the group gives rise to a singular point of phase agreement that displaces along one of the *rays* of the central wave of frequency ν, and that singular point, which is the analytical translation of the material point, moves according to the old dynamical laws. When the approximations of geometrical optics are no longer valid, one must rigorously study the solutions to the equations of propagation; that is when the new mechanics generalizes the old one, and the fecundity of that generalization, which has already been attested to by Schrödinger's beautiful results, has yet to be exhausted, moreover.

In the case of motion in the absence of any field, we have found the equation of propagation (in rectangular Galilean coordinates) to be:

$$\Delta u - \frac{1}{c_1}\frac{\partial^2 u}{\partial t^2} = \frac{4\pi^2 m_0^2\, c^2}{h^2}u\,, \tag{31}$$

(1) The quantities (28) are the components of the vector of length I that is carried tangentially to the world-line in the x^0 direction and normal to that direction. That vector is obviously the five-dimensional generalization of the world-impulse.

(2) Journal de Physique **7** (November 1926), pp. 321.

in which m_0 is a characteristic constant of the moving body (viz., its proper mass). Upon employing the notations of (four-dimensional) relativity, one can write (31) in the form:

$$g^{ik}\frac{\partial^2 u}{\partial x^i\,\partial x^k} = -\frac{4\pi^2}{h^2}m_0^2\,c^2\,u\,, \tag{32}$$

when one sets:

$$ds^2 = c^2\,dt^2 - dx^2 - dy^2 - dz^2.$$

Suppose that one has a gravitational field, but not an electromagnetic field. We can preserve the form (32), because the $\dfrac{\partial^2 u}{\partial x^i\,\partial x^k}$ are not the components of a tensor. Upon introducing the covariant derivative of the gradient of u, one will be led to replace equation (32) by the invariant equation:

$$g^{ik}\left[\frac{\partial^2 u}{\partial x^i\,\partial x^k} - \left\{\begin{matrix} i\,k \\ r \end{matrix}\right\}\frac{\partial u}{\partial x^r}\right] = -\frac{4\pi^2}{h^2}m_0^2\,c^2\,u\,, \tag{33}$$

with the well-known notation:

$$\left\{\begin{matrix} i\,k \\ r \end{matrix}\right\} = \frac{1}{2}g^{r\mu}\left(\frac{\partial g_{\mu i}}{\partial x^k} + \frac{\partial g_{\mu k}}{\partial x^i} - \frac{\partial g_{ik}}{\partial x^\mu}\right). \tag{34}$$

Equation (33) will be the equation of propagation of material waves in the gravitational field. If we verify that up to the degree of approximation of geometrical optics then we will recover Einstein's dynamics of the gravitational field. Indeed, the central wave of the group that corresponds to a material entity must then be written:

$$u = C\,e^{\frac{2\pi i}{h}?}. \tag{35}$$

We substitute (35) in (33), and since we are using the geometrical optics approximation, the only terms in the left-hand side that we shall keep will have degree two in the first derivatives of u; that will give:

$$g^{ik}\frac{\partial\varphi}{\partial x^i}\frac{\partial\varphi}{\partial x^k} = m_0^2\,c^2\,, \tag{36}$$

so we can infer that:

$$\varphi = m_0\,c\int_O^M u_i\,dx^i = \int_O^M m_0\,c\,ds\,. \tag{37}$$

φ is then identical to the Hamiltonian action that is defined by (2), and the rays of the central wave (35) are geodesics of space-time. Since the trajectory of the material point is one of those rays, we will find Einstein's dynamics of the material point in a gravitational field.

7. Wave mechanics and the five-dimensional universe. – Equation (33) is valid when one abstracts from electromagnetic phenomena. In order to take it into account, by introducing the ideas of Kaluza and Kramers, it will suffice to generalize equation (33) by assuming that in the five-dimensional universe, the material periodic phenomenon is a solution to the equation:

$$\gamma^{ik}\left[\frac{\partial^2 u}{\partial x^i\,\partial x^k}-\begin{Bmatrix} i\,k \\ r \end{Bmatrix}\frac{\partial u}{\partial x^r}\right]=-\frac{4\pi^2}{h^2}I^2\,u\,, \tag{38}$$

in which I is the invariant that is defined by (29), which is an invariant that is introduced naturally in place of $m_0\,c$ in the case of electrified points ([1]).

If the procedures of geometrical optics are valid then one can show, as in the preceding paragraph, that the central wave of the group that defines a material point has the expression:

$$C\,e^{\frac{2\pi i}{h}A}=f\,(x,y,z,t)\;e^{\frac{2\pi i}{h}\frac{e\,x_0}{\alpha}}\,,$$

in which A is the action in formula (30). It will then result that the five-dimensional world-line of any material point is a geodesic, and as a result, we will recover the dynamics of Einstein's gravitation and the dynamics of the electron in the geometrical optics approximation.

O. Klein wrote equation (38) with no right-hand side, and concluded that the world-lines are null-length geodesics. There seems to be no doubt that the right-hand side of (38) is necessary and that the world-lines are geodesics, but not *null-length* geodesics.

Let us see what equation (38) will become in the absence of a gravitational field when we no longer suppose that geometrical optics is applicable. The g_{ik} will then take their Galilean values in rectangular coordinates, and the relations (17) and (18) will then give the γ_{ik} ; one easily infers the γ^{ik} from that. Since, from (19), the product $\gamma_{00}\,\alpha^2$ is on the order of 10^{-27} cgs, one can suppose that the terms of that order are negligible with respect to unity. We shall no longer suppose that u has the form:

$$C\,e^{\frac{2\pi i}{h}A}\,,$$

but we shall further consider u to be a product of a function of x, y, z, t with $\sin\frac{2\pi}{h}\frac{e\,x_0}{\alpha}$. Under those conditions, a simple calculation will permit us to write (38) in the form:

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}-\Delta u+\left[\frac{1}{\gamma_{00}}+\frac{\alpha^2}{c^2}(\psi^2-a^2)\right]\frac{\partial^2 u}{\partial x_0^2}-2\sum_{x,y,z}\alpha\frac{a_x}{c}\frac{\partial^2 u}{\partial x^0\,\partial x}-\frac{2\alpha\psi}{c^2}\frac{\partial^2 u}{\partial x^0\,\partial t}=-\frac{4\pi^2 I^2}{h^2}u\,, \tag{39}$$

to the chosen approximation, or upon taking the assumed form for u into account:

([1]) Naturally, the indices vary from 0 to 4 in equation (38).

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} - \Delta u - \frac{4\pi i}{h}\sum_{x,y,z}\frac{e\,a_x}{c}\frac{\partial u}{\partial x} - \frac{4\pi i}{h}\frac{e\psi}{c^2}\frac{\partial u}{\partial t} + \frac{4\pi^2}{h^2}\left[m_0^2\,c^2 - \frac{e^2}{c^2}(\psi^2 - a^2)\right]u = 0. \qquad (40)$$

That is equation (59) in my previously-cited paper in *Journal de Physique*. After quoting the equation, I added: "One must point out that equation (59) contains imaginary terms, and that might raise some objections from a physical viewpoint." One sees that the anomaly has disappeared here, since (40) is only a degenerate form of (39).

8. Conclusion. – Equation (38), which can be written in the very remarkable form:

$$\gamma^{ik}\left[\frac{\partial^2 u}{\partial x^i\,\partial x^k} - \begin{Bmatrix} i\,k \\ r \end{Bmatrix}\frac{\partial u}{\partial x^r}\right] + \frac{4\pi^2}{h^2}\left[m_0^2 + \frac{e^2}{16\pi G}\right]u = 0, \qquad (41)$$

by virtue of (29) and (19), seems to be the general equation of wave mechanics for a material point. In order to go deeper into the problem of matter and its atomic structure, it will undoubtedly be necessary to systematically assume the viewpoint of the five-dimensional universe, which seems more fruitful that that of Weyl. If one were to arrive at an interpretation of the manner by which the constants e, m_0 , c, h, and G enter into equation (41) then one would be very close to having understood one of the more troubling secrets of nature.

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