"Sur le théorème d'indépendence de Hilbert," C. R. Acad. Sci. Paris **156** (1913), 609-611.

On Hilbert's independence theorem

Note [1] by **TH. DE DONDER**, presented by Appell

Translated by D. H. Delphenich

I. *First lemma.* – Consider the differential system:

$$\frac{dy_i}{y_i^{(1)}} = \frac{dy_i^{(1)}}{y_i^{(2)}} = \ldots = \frac{dy_i^{(\alpha)}}{\eta_i(t, y, y^{(1)}, \ldots, y^{(\alpha)})} = dt \qquad (i = 1, \ldots, n), \tag{1}$$

which is equivalent to n differential equations of order $(\alpha + 1)$. Upon utilizing a notation of Lie, that system can be written:

$$\eta f \equiv \frac{\partial f}{\partial t} + \sum_{i=1}^{n} \left(\frac{\partial f}{\partial y_i} y_i^{(1)} + \cdots + \frac{\partial f}{\partial y_i^{(\alpha)}} \eta_i \right). \tag{2}$$

Suppose that one knows n functions $\overline{y_1^{(1)}}, \ldots, \overline{y_n^{(1)}}$ of $y_1, \ldots, y_n$, and t that satisfy equations (1). Now consider the system:

$$\frac{dy_i}{\overline{\overline{y_i^{(1)}}}} = dt \qquad (i = 1, \ldots, n), \tag{3}$$

which one can also write:

$$\bar{\eta}\,\bar{f} \equiv \frac{\partial \bar{f}}{\partial t} + \sum_{i=1}^{n} \frac{\partial \bar{f}}{\partial y_i} \overline{y_i^{(1)}}. \tag{4}$$

Let J be a p-uple integral form such that one has:

$$\eta J = K, \tag{5}$$

where K is also a p-uple integral form. *One easily proves that one has:*

$$\bar{\eta}\,\bar{J} = \bar{K}. \tag{6}$$

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[1] Presented at the session in 17 February 1913.

The overbars indicate that one has replaced the $y_i^{(1)}, \ldots, y_i^{(\alpha)}$ ($i = 1, \ldots, n$) with the functions of $y_1, \ldots, y_n$, and t that were considered above and the ones that are derived from them; for example:

$$\overline{y_i^{(2)}} \equiv \frac{\partial \overline{y_i^{(1)}}}{\partial t} + \sum_{\lambda=1}^{n} \frac{\partial \overline{y_i^{(1)}}}{\partial y_\lambda} \overline{y_\lambda^{(1)}} .$$

Second Lemma. – Consider the differential system:

$$\frac{dy_i}{Y_i(t, y)} = dt \qquad (i = 1, \ldots, n) \tag{7}$$

and a 1-uple integral invariant:

$$\sum_{i=1}^{n} N_i \, \delta y_i , \tag{8}$$

in which $N_1, \ldots, N_n$ are functions of $y_1, \ldots, y_n$, and t. One has, by hypothesis, and by virtue of equations (7):

$$\frac{d}{dt} \sum_{i=1}^{n} N_i \, \delta y_i \equiv dW . \tag{9}$$

If t is the independent variable then one sets $\delta t \equiv 0$.

The theory of invariants teaches that ([1]):

$$\sum_{i=1}^{n} N_i \begin{vmatrix} \delta y_i & \delta t \\ Y_i & 1 \end{vmatrix} + W \, \delta t \tag{10}$$

is a 1-uple integral invariant of the system:

$$\frac{dy_i}{Y_i(t, y)} = \frac{dt}{1} = d\tau \quad (i = 1, \ldots, n) . \tag{11}$$

If τ is the independent variable then one sets $\delta\tau \equiv 0$ and $\delta t \neq 0$.

One can state the following lemma: *In order for the form* (10) *to be an exact differential*, it is *necessary and sufficient that the form* (8) *should be an exact differential*. In the first differential, one has $\delta t \neq 0$, while in the second, one has $\delta t \equiv 0$.

([1]) TH. DE DONDER, Bulletin de l'Académie royale de Belgique: Classe des sciences, February 1911 (see especially Chapters I and VI).

II. *Extension of Hilbert's independence theorem* ([1]). – To fix ideas, consider the 1-uple relative integral invariant ([2]):

$$J \equiv \sum_{i=1}^{n} \left(\frac{\partial F}{\partial y_i^{(1)}} - \frac{d}{dt} \frac{\partial F}{\partial y_i^{(2)}} \right) \delta y_i + \sum_{i=1}^{n} \frac{\partial F}{\partial y_i^{(2)}} \delta y_i^{(1)} \equiv \sum_{i=1}^{n} (M_i \, \delta y_i + Q_i \, \delta y_i^{(1)}) \tag{12}$$

of the differential equations that define the extremals of $\delta \int_{t_1}^{t_2} F \, dt = 0$, where F is an arbitrary function of t, y_i, $y_i^{(1)}$, $y_i^{(2)}$ $(i = 1, \ldots, n)$. If those equations are satisfied by the n functions $\overline{y_1^{(1)}}, \ldots, \overline{y_n^{(1)}}$ of $y_1, \ldots, y_n$, and t then the first lemma shows one how to deduce a 1-uple relative integral invariant $\bar{J}$ of system (3) from J. Thanks to the second lemma, one will deduce the following 1-uple relative integral invariant of the system (11) from $\bar{J}$:

$$\bar{J}' \equiv \sum_{i=1}^{n} \left(\bar{M}_i + \sum_{k=1}^{n} \bar{Q}_k \frac{\partial \overline{y_k^{(1)}}}{\partial y_i} \right) \delta y_i + \left[\bar{F} - \sum_{i=1}^{n} \left(\bar{M}_i + \sum_{k=1}^{n} \bar{Q}_k \frac{\partial \overline{y_k^{(1)}}}{\partial y_i} \right) \overline{y_k^{(1)}} \right] \delta t \ . \tag{13}$$

In addition, one finds that in order for $\bar{J}'$ to be an exact differential, it is necessary and *sufficient* that $\bar{J}$ should be an exact differential. Therefore, *if* $n = 1$ *then* $\bar{J}'$ *will always be an exact differential*.

([1]) O. Bolza, Rendiconti del Circolo matematico di Palermo, t. XXXI, 1st semester 1911.
([2]) Th. De Donder, Rendiconti del Circolo matematico di Palermo, 1st semester 1902 (see especially no. 58 in that paper).

"Sur le théorème d'indépendence de Hilbert," C. R. Acad. Sci. Paris **156** (1913), 868-870.

On Hilbert's independence theorem

Note by **TH. DE DONDER**, presented by P. Appell

Translated by D. H. Delphenich

I. – Thanks to the theory of integral invariants, we have extended Hilbert's independence theorem in these *Comptes rendus* (session on 17 February 1913) to the case in which the function F depends upon n functions $y_1, \ldots, y_n$ of the independent variables t, as well as their derivatives *up to an arbitrary order*. That theory also provides the extension of the results of Hilbert, Hahn, and Bolza that relate to the case in which F includes only the first derivatives.

II. – Now suppose that there are *several independent variables*, for example t_1 and t_2, and for more simplicity, suppose that F contains the partial derivatives only up to order *two*. Set:

$$\frac{\partial y_i}{\partial t_1} \equiv y_i^{(1)}, \quad \ldots, \quad \frac{\partial^2 y_i}{\partial t_1\,\partial t_2} \equiv y_i^{(12)}, \quad \frac{\partial^2 y_i}{\partial t_2^2} \equiv y_i^{(22)} \qquad (i = 1, \ldots, n)\,.$$

III. – Let:

$$\left\{\begin{aligned} j_1 &\equiv \sum_{i=1}^{m} N_{1i}\,\delta y_i\,, \\ j_2 &\equiv \sum_{i=1}^{m} N_{2i}\,\delta y_i \end{aligned}\right. \tag{1}$$

be a 1-uple relative integral invariant ([1]) of the total differential equation:

$$dy_i = Y_{1i}\,(t_1, t_2, y)\,dt_1 + Y_{2i}\,(t_1, t_2, y)\,dt_2 \qquad (i = 1, \ldots, n)\,,$$

which is immediately integrable.

One will have, by definition:

$$\frac{dj_1}{dt_1} + \frac{dj_2}{dt_2} = \delta F\,,$$

([1]) We have indicated that extension in a note that was presented by Appell to the Académie des Sciences de Paris (session on 9 September 1901).

in which F is a function of $y_1, \ldots, y_n, t_1, t_2$, and $\delta t_1 = \delta t_2 = 0$.

In order for ([1]):

$$\sum_{i=1}^{n} (N_{1i}\,\delta y_i\,\delta t_2 - N_{2i}\,\delta y_i\,\delta t_1) + \left[F - \sum_{i=1}^{n} (N_{1i}\,Y_{1i} + N_{2i}\,Y_{2i})\right]\delta t_1\,\delta t_2 \tag{2}$$

to be a 2-uple exact differential, it is necessary *and sufficient that* j_1 *and* j_2 *should be two exact differentials* (where $\delta t_1 = \delta t_2 \equiv 0$).

IV. – Consider the function F of no. II and identify:

$$\frac{d}{dt_1}\sum_{i=1}^{n} [P_{1i}\,\delta y_i + Q_{1i}\,\delta y_i^{(1)} + R_{1i}\,\delta y_i^{(2)}] + \frac{d}{dt_2}\sum_{i=1}^{n} [P_{2i}\,\delta y_i + Q_{2i}\,\delta y_i^{(1)} + R_{2i}\,\delta y_i^{(2)}] \equiv \delta F .$$

One finds n partial differential equations:

$$\frac{\partial F}{\partial y_i} - \frac{d}{dt_1}\frac{\partial F}{\partial y_i^{(1)}} - \frac{d}{dt_2}\frac{\partial F}{\partial y_i^{(2)}} + \frac{d^2}{dt_1^2}\frac{\partial F}{\partial y_i^{(11)}} + \frac{d^2}{dt_1\,dt_2}\frac{\partial F}{\partial y_i^{(12)}} + \frac{d^2}{dt_2^2}\frac{\partial F}{\partial y_i^{(22)}} = 0 , \tag{3}$$

and some equations that determine the $P_{1i}, \ldots, R_{2i}$ completely, with the exception of the R_{1i} and Q_{2i} , which are subject to only the conditions:

$$R_{1i} + Q_{2i} = \frac{\partial F}{\partial y_i^{(12)}} \qquad (i = 1, \ldots, n) .$$

V. – Suppose that one knows $2n$ functions $\overline{y_1^{(1)}}, \ldots, \overline{y_n^{(1)}}, \overline{y_1^{(2)}}, \ldots, \overline{y_n^{(2)}}$ of the $n + 2$ variables $y_1, \ldots, y_n, t_1, t_2$ that satisfy equations (3) identically. Set:

$$\overline{N_{1i}} \equiv \overline{P_{1i}} + \sum_{k=1}^{n}\left(\overline{Q_{1k}}\,\frac{\partial \overline{y_k^{(1)}}}{\partial y_i} + \overline{R_{1k}}\,\frac{\partial \overline{y_k^{(2)}}}{\partial y_i}\right) ,$$

and similarly for $\overline{N_{2i}}$.

One will have the following *independence theorem:* In order for:

([1]) The expression (2) is deduced from (1) by a process that is analogous to the one that was employed in our preceding note.

$$\text{(4)} \qquad \sum_{i=1}^{n} (\overline{N_{1i}}\,\delta y_i\,\delta t_2 - \overline{N_{2i}}\,\delta y_i\,\delta t_1) + \left[F - \sum_{i=1}^{n} (\overline{N_{1i}}\,\overline{y_i^{(1)}} + \overline{N_{2i}}\,\overline{y_i^{(2)}}) \right] \delta t_1\,\delta t_2$$

to be a 2-uple exact differential, it is necessary and sufficient that:

$$\sum_{i=1}^{n} \overline{N_{1i}}\,\delta y_i\,, \qquad \sum_{i=1}^{n} \overline{N_{2i}}\,\delta y_i$$

should be two exact differentials.

If $n = 1$ then the expression (4) will *always* be a 2-uple exact differential.
