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Study of integral invariants

(Part Two)

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PREFACE AND SUPPLEMENTS

This article is a continuation of my "Étude sur les invariants intégraux," which appeared in volume **15** of these "Rendiconti" (session on 17 March 1901).

First of all, I shall complete the bibliographic information that I gave in that study. To that effect, I shall cite the following articles:

"Sur les Invariants intégraux des groupes continus de transformations," by **K. Zorawski** (Bull. de l'Académie des Sciences de Cracovie, 1895).

"Ueber die Erzeugung der Invarianten durch Integration," by **A. Hurwitz** (Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, 1897).

"Die Theorie der Integral-Invarianten ist ein Corollar der Theorie der Differential-Invarianten," by **S. Lie** (Berichte der Sächsischen Gesell. zu Leipzig, 1897).

"Die Theorie der Integral-Invarianten und ihre Verwertung für die Theorie der Differential-Gleichungen," by **S. Lie** (*ibid.*, 1897).

"Invariante Curvenintegrale bei infinitesimal Transformationen in drei Veränderlichen *x*, *y*, *z*, und deren Verwertung," by **C. Heineck** (Dissertation, Leipzig, 8°, 1899).

An example will serve to show how those works differ from those of **H. Poincaré**. Suppose the equations are given:

(T) $$\frac{\delta x_i}{X_i} = \frac{\delta z_i}{Z_i} = \delta t \qquad \begin{pmatrix} i = 1, \dots, n \\ k = 1, \dots, m \end{pmatrix},$$

in which the X_i and the Z_i are functions of t, x, and z.

S. Lie and his disciples wrote the system, or *infinitesimal transformation* (T), in the following form:

$$Tf = \sum_i X_i \frac{\partial f}{\partial x_i} + \sum_k Z_k \frac{\partial f}{\partial z_k}.$$

They did not write the term $\partial f / \partial t$ because they considered t to be a parameter. On the other hand, they supposed that the z are arbitrary functions of the x, which leads one to suppose that in:

$$I_n = \int M \, dx_1 \cdots dx_n,$$

for example, M includes not only the x and z, but also certain *partial derivatives* of the z_k with respect to the x_i. For more simplicity, suppose that there is only one function z of $x_1, \dots, x_n$, and that M contains only the first derivatives p_i of z with respect to x_i .

We will then have (no. **18**) (*):

$$\frac{\delta I_n}{\delta t} = \frac{\delta M}{\delta t} \frac{\partial (x_1, \dots, x_n)}{\partial (\lambda_1, \dots, \lambda_n)} + M \frac{\delta}{\delta t} \frac{\partial (x_1, \dots, x_n)}{\partial (\lambda_1, \dots, \lambda_n)},$$

$$\frac{\delta M}{\delta t} \equiv TM = \sum_i \frac{\partial M}{\partial x_i} X_i + \frac{\partial M}{\partial z} Z + \sum_i \frac{\partial M}{\partial p_i} \frac{\delta p_i}{\delta t},$$

$$\frac{\delta}{\delta t} \frac{\partial x_i}{\partial \lambda_i} = \frac{\partial X_i}{\partial \lambda_i} = \sum_i \left(\frac{\partial X_1}{\partial x_i} + \frac{\partial X_1}{\partial z} \frac{\partial z}{\partial x_i} \right) \frac{\partial x_i}{\partial \lambda_1}.$$

In the results obtained (no. **18**), one must then replace $\frac{\partial X_1}{\partial x_i}$ with $\frac{\partial X_1}{\partial x_i} + \frac{\partial X_1}{\partial z} \frac{\partial z}{\partial x_i}$. Finally, we must once more calculate $\frac{\delta p_i}{\delta t}$. In order to do that, we identify the two sides of:

$$\frac{\delta}{\delta t} \left(dz - \sum p \, dx \right) = \rho \left(dz - \sum p \, dx \right),$$

or

(*) Here, I suppose that M does not include t explicitly in order for the formulas to be identical to the ones that were given by the authors that I just cited.

$$dZ - \sum_i \frac{\delta p_i}{\delta t} dx_i - \sum_i p_i \, dX_i = \rho \left(dz - \sum_i p_i \, dx_i \right) .$$

We then obtain:

$$\rho = \frac{\partial Z}{\partial z} - \sum_i p_i \frac{\partial X_i}{\partial z} ,$$

$$\frac{\delta p_i}{\delta t} = \rho \, p_i + \frac{\partial Z}{\partial x_i} - \sum_k p_k \frac{\partial X_k}{\partial x_i} \equiv P_i .$$

(*First extension*)

If M includes the second derivatives $\frac{\partial^2 z}{\partial x_i \, \partial x_k}$, or p_{ik}, of z with respect to x_i and x_k then one must calculate $\frac{\delta p_{ik}}{\delta t}$ (*second extension*). To that end, one identifies the two sides of each of the relations:

$$\frac{\delta}{\delta t}\left(dp_i - \sum_k p_{ik} \, dx_k \right) = \rho_0^i \left(dz - \sum_k p_k \, dx_k \right) + \sum_l \rho_l^i \left(dp_l - \sum_k p_{kl} \, dx_k \right) ,$$

in which the variation d is determined by the infinitesimal transformation (*prolonged* once):

$$\frac{\delta x_i}{X_i} = \frac{\delta z}{Z} = \frac{\delta p_i}{P_i} = \delta t .$$

From nos. **35** and **36** (*Étude*) (*), I have deduced **Poisson**'s celebrated theorem from a certain Jacobian. **Bühl** arrived at the same result in a note that was presented to the Paris Academy on 11 February 1901 and in his thesis (**). **P. Appell** (C. R. Acad. Sci. Paris, 5 August 1901) has deduced the Jacobian system that served as the starting point for **Bühl**'s research from **Poisson**'s theorem.

In number **49**, I indicated the necessary and sufficient conditions to the vortex lines to be conserved in the form: $\delta \, Dx = D \, \delta x$. **Appell** (***) and **Z. Zorawski** (†) carried out analogous studies.

The notation in my *Étude* (first memoir) can be simplified by making use of a certain symbolic calculus that was studied and utilized by **Lipschitz** (††) and **Cartan** ["Sur certaines expression différentielles et sur le problème de Pfaff," Annales de 'École Norm. Sup. (1899)]. For example, I will show how one must employ that calculation. Recall no. **33**. By virtue of the proposed equations, we will have:

(*) The manuscript of this article was submitted to **H. Poincaré** on 3 February 1901.
(**) "Sur les équations différentielles simultanées et la forme aux dérivées adjointe" (14 June 1901).
(***) "Sur les équations de l'Hydrodynamique et la théorie des tourbillons," J. math. pures et appl. (1896).
(†) "Erhaltung der Wirbelbewegung," Bull. Cracovie (1900).
(††) "Bemerkungen über die Differentiale von symbolic Ausdrücken, Berlin. Sitzungsber. (1890).

$$\frac{\delta}{\delta t}\sum_i dx_i\,dy_i\,dz_i = \sum_i (dX_i\,dy_i\,dz_i + dx_i\,dX_i\,dz_i + dx_i\,dy_i\,dZ_i)$$
$$= \sum_i\sum_k \left(\frac{\partial X_i}{\partial x_k}dx_k\,dy_i\,dz_i + \frac{\partial X_i}{\partial y_k}dy_k\,dy_i\,dz_i + \frac{\partial X_i}{\partial z_k}dz_k\,dy_i\,dz_i + \cdots\right).$$

If one recalls that:

$$dx_i\,dy_k\,dz_l = \frac{\partial(x_i, y_k, z_l)}{\partial(\lambda_1, \lambda_2, \lambda_3)}\,d\lambda_1\,d\lambda_2\,d\lambda_3$$

then one will understand that all of the terms that include $dx_i\,dx_i$, $dy_i\,dy_i$, $dz_i\,dz_i$ are *zero* and that any permutation that is performed on two of the differentials dx_i, dy_i, and dz_i in a term will change the sign of that term but will not alter the absolute value of that term (*). The following rule should be mentioned again, which permits one to transform a p-uple integral into a $(p + 1)$-uple one. For example, let $\sum_i\sum_j M_{ij}\,dx_i\,dx_j$ be a double integral element ($i, j = 1, \ldots, n$; $M_{ij} = - M_{ji}$). One can deduce the following triple integral element from that rule:

$$d\sum_i\sum_j M_{ij}\,dx_i\,dx_j = \sum_i\sum_j \frac{\partial M_{ij}}{\partial x_k}dx_k\,dx_i\,dx_j = \alpha\sum_{i,j,k}\left(\frac{\partial M_{ij}}{\partial x_k} + \frac{\partial M_{jk}}{\partial x_i} + \frac{\partial M_{ki}}{\partial x_j}\right)dx_k\,dx_i\,dx_j\,,$$

in which α is a numerical constant that plays no role in that theory.

I shall give summaries of the various chapters of this article at the beginnings of those chapters.

CHAPTER XII

Application to the conservation of given form of a system of differential equations.

Summary. – The three given forms that I shall consider are those of *canonical* equations, *characteristic* equations, which present themselves in the theory of first-order partial differential equations, and finally the equations that **S. Lie** called *infinitesimal contact transformations*. I shall study *all* changes of variables (i.e., transformations) that preserve each of the forms. The method employed is general. In order for it to be applicable, it suffices to known one or more relations (d δ) that *characterize* the proposed form.

51. Lemma. – If Φ is a function of the x_i then, by virtue of the equations:

$$\frac{\delta x_i}{X_i} = \delta t \qquad (i = 1, \ldots, n),$$

(*) One must add the following statement to (29): "and the functions X_i, Y_i, Z_i depend upon only x_i, y_i, z_i, and t."

one will have (*):

$$\frac{\delta\Phi}{\delta t} \equiv X\,\Phi = \sum_i \frac{\partial\Phi}{\partial x_i} X_i \,.$$

Take n new (distinct) variables: $z_i = z_i(x_1, \ldots, x_n)$. One will then have the new system (no. **19**):

$$\frac{\delta z_i}{\left(\sum_k \frac{\partial z_i}{\partial x_k} X_k\right)_1} = \delta t \,.$$

Let Φ_1 be what Φ becomes when expressed in terms of z. By virtue of the new system:

$$\frac{\delta\Phi_1}{\delta t} \equiv Z\,\Phi_1 = \sum_k \sum_i \frac{\partial\Phi_1}{\partial z_k}\left(\frac{\partial z_k}{\partial x_i} X_i\right).$$

One will then have:

$$X\,\Phi = Z\,\Phi_1 \,,$$

or more simply:

$$\delta\Phi = \delta\Phi_1 \,.$$

52. Theorem. – In order for the system of equations:

$$\frac{\delta x_i}{X_i} = \frac{\delta y_i}{Y_i} = \delta t \qquad (i = 1, \ldots, n)$$

to admit the relative integral invariant:

$$J = \int \sum_i y_i \, dx_i \,,$$

it is necessary and sufficient that this system should be *canonical*, i.e., it should have the form:

$$\frac{\delta x_i}{\frac{\partial H}{\partial y_i}} = \frac{\delta y_i}{-\frac{\partial H}{\partial x_i}} = \delta t \,, \tag{51′}$$

in which H is an arbitrary function of x, y, and t; it is called the *characteristic function*.

We say that:

$$\frac{\delta}{\delta t}\sum y\,dx = d\,W \tag{52}$$

is a *relation* ($d\ \delta$) *that characterizes* the canonical equations.

(*) For more simplicity, we suppose that the functions considered do not include t explicitly.

Upon performing the calculations, one will find that:

$$\sum (Y\,dx - X\,dx) + d\sum y\,X = d\,W\,,$$

so:

$$W = -\;H + \sum y\frac{\partial H}{\partial y}\,.$$

If we are given the canonical system (51′) then we propose to replace the x_i, y_k with $2n$ new variables:

$$\xi_i = \xi_i\,(x_1, \ldots, x_n, y_1, \ldots, y_n)\,,$$
$$\eta_i = \eta_i\,(x_1, \ldots, x_n, y_1, \ldots, y_n)$$

that preserves the canonical form of equations (51′).

In order for the ξ and the η to possess that property, it is necessary and sufficient that those $2n$ functions are distinct and that $\int\sum \eta\,d\xi$ is a relative invariant of the proposed system (51′). The latter condition is expressed analytically thanks to no. **31**. If one supposes that H is *arbitrary* in the equations of the conditions, thus-found, then they will become:

$$(53)\qquad \begin{cases} \displaystyle\sum_i \frac{\partial(\xi_i,\eta_i)}{\partial(x_p,x_q)} = 0 & (p, q = 1, \ldots, n), \\ \displaystyle\sum_i \frac{\partial(\xi_i,\eta_i)}{\partial(y_p,y_q)} = 0, & \\ \displaystyle\sum_i \frac{\partial(\xi_i,\eta_i)}{\partial(x_p,y_q)} = 0 & (p \neq q), \\ \displaystyle\sum_i \frac{\partial(\xi_i,\eta_i)}{\partial(x_p,y_q)} = \text{the same numerical constant } k. & \end{cases}$$

Those conditions signify that:

$$\int\sum \eta\,d\xi = \alpha\iint\sum dx\,dy = \alpha_1\int\sum y\,dx + dS\,,$$

in which α and α_1 are numerical constants.

If $k \neq 0$ then upon setting $\alpha_1 = 1$, one will have:

$$\sum \eta\,d\xi = \alpha_1\sum y\,dx + dS\,.$$

The transformation, thus-defined, is called (*) *a contact transformation in* x, p (here: in x, y).

(*) *Leçons sur l'intégration des équations aux dérivées partielles*, by **E. Goursat**, Chap. XI.

One will deduce from the preceding identity (no. **19**) that:

$$\frac{\partial(\xi_1,\ldots,\xi_n,\eta_1,\ldots,\eta_n)}{\partial(x_1,\ldots,x_n,y_1,\ldots,y_n)} \neq 0 .$$

Hence, one has the:

Theorem (*)**:**

*The **only** changes of variables that preserve the form of a canonical equation are the contact transformations in* x, p.

Corollary:

The absolute invariant $I_2 = \int \sum dx\, dy$ *is the only absolute invariant that characterizes the canonical equations.*

The preceding will permit one to easily show that the changes of variables that were indicated by **H. Poincaré** (**) leave the canonical form invariant.

Theorem:

In order for the system (51) *to be reducible to the canonical form, it is necessary and sufficient that it must admit a relative invariant:*

$$I = \int \sum M\, dx + N\, dy$$

whose element here has class $2n$ (***), *i.e., it can be identified with* $\sum \eta\, d\xi$, *in which the* ξ *and* η *are* $2n$ *distinct functions of the* x *and* y.

Corollary:

The determinant of the ξ *and the* η *with respect to the* x *and* y *will be a multiplier for equations* (51).

(*) **Th. de Donder**, "Sur les invariants intégraux," C. R. Acad. Sci. Paris, 9 September 1901.
(**) *Méthodes Nouvelles*, t. I, pp. 15.
(***) Cited article by **E. Cartan** and a note by **Koenigs** (C. R. Acad. Sci. Paris, December 1895).

53. –

Theorem:

If one supposes that $\delta z - \sum y\,\delta x = 0$ (*) *then the relation* $(d\ \delta)$:

$$\frac{\delta}{\delta t}\left(dz - \sum y\,dx\right) = \theta\left[dH + \omega\left(dz - \sum y\,dx\right)\right] \tag{54}$$

will ***characterize*** *the equations:*

$$\frac{\delta x_i}{\theta \dfrac{\partial H}{\partial y_i}} = \frac{\delta y_i}{-\theta\left(\dfrac{\partial H}{\partial x_i} + y_i \dfrac{\partial H}{\partial z}\right)} = \frac{\delta z}{\theta \sum y \dfrac{\partial H}{\partial z}} = \delta t\,. \tag{55}$$

Indeed, consider the $2n + 1$ equations:

$$\frac{\delta x_i}{X_i} = \frac{\delta y_i}{Y_i} = \frac{\delta z}{Z} = \delta t\,.$$

Suppose that $Z = \sum y\,X$, and identify $dZ - \sum(Y\,dx + y\,dX)$, with the right-hand side of (54). We get (55) and:

$$\omega = -\frac{\partial H}{\partial z}\,.$$

Equations (55) will become the characteristic equations when one sets $\theta = 1$ in them. Those equations can be represented by the infinitesimal transformation:

$$\theta\,[H, f]\,. \tag{55′}$$

Theorem:

In order for equations (55) *to* ***preserve*** *their form after one replaces the* x, y, z *in them with* $2n + 1$ *other distinct variables* ξ, η, ζ, *it is necessary and sufficient that one has:*

$$\frac{\delta \zeta}{\delta t} = \sum \eta\,\frac{\delta \xi}{\delta t}\,,$$

(*) And not $dz - \sum y\,dx = 0$, because the d are *arbitrary*.

$$\frac{\delta}{\delta t}\left(d\zeta - \sum \eta\, d\xi\right) = \theta' \left[dK - \frac{\partial K}{\partial \zeta}\left(d\zeta - \sum \eta\, d\xi\right)\right]$$

by virtue of equations (55).

Example. Suppose that one has the identity (*):

$$d\zeta - \sum \eta\, d\xi = \rho\left(dz - \sum y\, dx\right) \qquad (\rho \neq 0).$$

The ξ, η, ζ will then define a (finite) *contact transformation*, in the language of **Lie**.

One knows (*) that:

$$\frac{\partial(\xi_1, \dots, \xi_n, \eta_1, \dots, \eta_n)}{\partial(x_1, \dots, x_n, y_1, \dots, y_n)} = \pm \rho^{n+1} \neq 0 .$$

On the other hand, by virtue of (55), one will have:

(57) $$\frac{\delta}{\delta t}\left(d\zeta - \sum \eta\, d\xi\right) = \rho\theta \left[dH + \left(\frac{1}{\rho^2 \theta}\frac{\delta \rho}{\delta t} - \frac{\partial H}{\rho\, \partial z}\right)\left(d\zeta - \sum \eta\, d\xi\right)\right] .$$

Finally, one deduces from the identity (56) upon replacing d with δ that:

$$\delta\zeta - \sum \eta\, \delta\xi = 0 .$$

By virtue of the foregoing, and above all (57), the new equations can be written:

(55″) $$\frac{\delta \xi_i}{\rho_1\, \theta_1 \dfrac{\partial H_1}{\partial \eta_i}} = \frac{\delta \eta_i}{-\rho_1\, \theta_1 \left(\dfrac{\partial H_1}{\partial \xi_i} + \eta_i \dfrac{\partial H_1}{\partial \zeta}\right)} = \frac{\delta \zeta}{\rho_1\, \theta_1 \sum \eta \dfrac{\partial H}{\partial \eta}} = \delta t ,$$

in which the index 1 indicates that one has replaced the x, y, z in the functions that carry that index with their values as functions of the ξ, η, ζ.

By virtue of the lemma and equations (55) and (55″), for $\theta = 1$, one will have:

$$\frac{\partial H_1}{\partial \zeta} = \frac{1}{\rho}\frac{\partial H}{\partial z} - \frac{1}{\rho^2}[H, \rho] ,$$

$$[H, f] = \rho\, [H_1, f_1] ,$$

(*) *Leçons* by **Goursat**, Chap. XI.

$$[H, \xi_i] = \rho \frac{\partial H_1}{\partial \eta_i}, \quad \text{etc.}$$

Hence, one will get the well-known relations that exist between the ξ_i, η_k, ζ, ρ upon replacing H with each of the latter quantities.

The classical example that we just studied *does not define the most general transformation that preserves the form of equations* (55).

Indeed, $2n + 1$ distinct functions ξ, η, ζ that verify the identity:

$$\delta\zeta - \sum \eta\, \delta\xi = \rho\left(dz - \sum y\, dx\right) + R\, dH,$$

in which R is an arbitrary function of x, y, z, will also possess that property.

54. –

Theorem:

The relation:

$$\frac{\delta}{\delta t}\left(dz - \sum y\, dx\right) = \omega\left(dz - \sum y\, dx\right),$$

*in which ω is an arbitrary function of z, x, and y, **characterizes** a system of $2n + 1$ equations have the form:*

$$\frac{\delta x_i}{\dfrac{\partial H}{\partial y_i}} = \frac{\delta y_i}{-\dfrac{\partial H}{\partial x_i} - y_i \dfrac{\partial H}{\partial z}} = \frac{\delta z}{\sum y \dfrac{\partial H}{\partial z} - H} = \delta t, \tag{58}$$

in which H is an arbitrary function of z, x, y.

That theorem is due to **S. Lie**, who stated it as follows (*):

*The most-general **infinitesimal contact transformation** has the form* (58) *or:*

$$[H, f] - H \frac{\partial f}{\partial z}.$$

*H is called the **characteristic function**.*

(*) *Theorie der Transformationsgruppen*, by **S. Lie**, with the collaboration of **F. Engel** (Teubner, Leipzig, 1890), t. II, pp. 251. (In what follows, I shall cite that book as Tgr.)

Upon performing the calculations, one will find that $\omega = -\partial H / \partial z$, so if H does not include z explicitly, one will have:

$$\frac{\delta}{\delta t}\left(dz - \sum y\,dx\right) = 0 \qquad \text{(no. } \mathbf{10}\text{)}.$$

Upon reasoning as before, one will find the necessary and sufficient conditions for the $2n + 1$ new distinct variables ξ, η, ζ to preserve the form of equations (58).

Example:

$$d\zeta - \sum \eta\, d\xi = \rho\left(dz - \sum y\,dx\right).$$

Exercises. – If the new variables satisfy the preceding identity then the new characteristic function will be ρH (*).

If one is given two infinitesimal contact transformations in x, y, z whose characteristic functions are H_1 and H_2 then show that one can deduce a third infinitesimal contact transformation from them whose characteristic function is:

$$[H_1, H_2] - \left(H_1 \frac{\partial H_2}{\partial z} - H_2 \frac{\partial H_1}{\partial z}\right).$$

*The **Jacobi-Mayer** identity will permit one to rapidly solve this last exercise* (**).

54. – **S. Lie** has treated two problems that have a strong analogy with the question that we just treated.

In one of those problems, he looked for the changes of variables that transform a *given* system of equations into another *given* system of equations; it can then be *identical* to the first one (***). Only *given functions* are involved in this problem.

The other problem (†) to which I alluded is very general. I shall return to it later on, but in order to give some idea of it now, I shall state the theorem: *The integral invariant $I_n = \int dx_1 \cdots dx_n$ **characterizes** the n equations:*

$$\frac{\delta x_i}{X_i} = \delta t,$$

in which:

(*) Tgr., Bd. II, pp. 276.
(**) Tgr., Bd. II, pp. 275.
(***) Tgr., Bd. I, pp. 327. *Theorie der Aehnlichkeit r-gliedriger Gruppen*.
(†) **S. Lie**, "Ueber Differentialinvarianten," Math. Ann. (1884).

$$\sum \frac{\partial X_i}{\partial x_i} = 0 \qquad \text{(no. **18**)}.$$

CHAPTER XIII

Application to the calculus of variations

Summary. – The theory of integral invariants can be considered to be the *inverse* of the calculus of variations since it provides all of the formulas with no integration by parts. It neatly points to the generalization in which one considers an arbitrary number of parameters λ. This chapter includes a generalization of the **Kelvin-Helmholtz** relative integral invariant, as well as the extension of the notion of a relative invariant to the case in which there are several independent variables.

That extension will become very useful later one when we set $W = 0$.

56. – Set:

$$\frac{\delta q_i}{q_i'} = \delta t \qquad (i = 1, \ldots, n),$$

and look for the system of differential equations that admits the relative invariant:

$$J = \int \sum_i N_i \, dq_i .$$

It is necessary and sufficient that:

$$\frac{\delta}{\delta t} \sum_i N_i \, dq_i = dW ,$$

in which W is an arbitrary function of t, q_i, and q_i'. Therefore:

$$J = \int \sum_i \frac{\partial W}{\partial q_i'} dq_i$$

is a relative invariant of:

$$\frac{\delta \dfrac{\partial W}{\partial q_i'}}{\dfrac{\partial W}{\partial q_i}} = \frac{\delta q_i}{q_i'} = \delta t .$$

Those equations have the form of the **Lagrange** equations.

That system will become canonical when one takes the q_i and $\partial W / \partial q_i' \equiv p_i$ (which are supposed to be distinct) for new variables. In mechanics, that change of variables is called the **Poisson-Hamilton** transformation (*).

If we adopt the notations of no. **52** then we will have:

$$H = -W + \sum_i p_i\, q_i',$$

in which H is a function of t, q_i, and p_i . Set:

$$j = \sum_i \frac{\partial W}{\partial q_i'}\, dq_i,$$

so

$$\frac{\delta j}{\delta t} = d\,W,$$

or

$$j_{t_1} - j_{t_0} = d \int_{t_0}^{t_1} W\,\delta t,$$

in which the integral is taken along one of the trajectories that are defined by equations (59) (**Hamilton**'s principle).

By virtue of no. **29**, we can set:

$$J = I + E,$$

in which:

$$E = \int dV.$$

One then deduces that:

(60) $$\sum p\,dq - dV = \text{constant},$$

$$\frac{\delta J}{\delta t} = \int dW = \int d\,\frac{\delta V}{\delta t},$$

$$\frac{\delta V}{\delta t} = W = \sum_i p_i \frac{\partial H}{\partial p_i} - H,$$

(V) $$V = V_0 + \int_{t_0}^{t} W\,\delta t.$$

(*) *Traité de Mécanique*, by **P. Appell**, 1896; t. II, no. 478. See also no. **57** of this article.

Let V_1 represent what V will become when it is expressed as a function of the q_i and n distinct integration constants $a_1, \ldots, a_n$ of equations (59) (*).

One has:

$$\frac{\delta V_1}{\delta t} = \frac{\partial V_1}{\partial t} + \sum_i \frac{\partial V_1}{\partial q_i}\frac{\partial H}{\partial p_i}.$$

Now:

$$\delta V_1 = \delta V,$$

so:

$$\frac{\partial V_1}{\partial t} + H + \sum_i \frac{\partial H}{\partial p_i}\left(\frac{\partial V_1}{\partial q_i} - p_i\right) = 0.$$

The relation (60) becomes:

$$\sum_i \left(\frac{\partial V_1}{\partial q_i} - p_i\right) dq_i + \frac{\partial V_1}{\partial a_i} da_i = \text{constant},$$

and by means of (V), one will find that:

$$\frac{\partial V_1}{\partial q_i} - p_i = \text{constant}. \tag{61}$$

The **Jacobi** equation:

$$\frac{\partial V_1}{\partial t} + H\left(t, q_k, \frac{\partial V_1}{\partial q_i}\right) = 0$$

will correspond precisely to the case in which one supposes that the n constants of (61) are identically *zero*. Thus:

$$\frac{\partial V_1}{\partial q_i} = p_i,$$

$$\frac{\partial V_1}{\partial a_i} = \text{constants } b_i,$$

$$V_0 = \sum_i a_i b_i,$$

in which a_i and b_i are the values of the p_i and q_i for the initial instant t_0.

57. – The relative invariant:

$$\int \sum \frac{\partial W}{\partial q'} dq$$

(*) Which is always possible (*Cours d'Analyse*, by **Jordan**, t. III, pp. 331) in equations (59) is the generalization of the **Helmholtz-Kelvin** relative invariant.

Equations (59) can be written (*):

$$(59') \qquad \frac{\delta q'_i}{q_i} = \frac{\delta q'_k}{q''_k} = \delta t \qquad (i, k = 1, \ldots, n),$$

$$\sum_i \left(\frac{\partial^2 W}{\partial q'_k \, \partial q_i} q'_i + \frac{\partial^2 W}{\partial q'_k \, \partial q'_i} q''_i \right) - \frac{\partial W}{\partial q_k} = 0 .$$

Thus:

$$q''_i = \frac{\Delta}{\left| \dfrac{\partial^2 W}{\partial q'_k \, \partial q'_i} \right|} .$$

However, one has:

$$I_{2n} = \int dp_1 \cdots dp_n \, dq_1 \cdots dq_n = \int \frac{\partial \left(\dfrac{\partial W}{\partial q'_1} \cdots \dfrac{\partial W}{\partial q'_n} \right)}{\partial (q'_1 \cdots q'_n)} dp'_1 \cdots dp'_n \, dq'_1 \cdots dq'_n .$$

Consequently, $\left| \dfrac{\partial^2 W}{\partial q'_k \, \partial q'_i} \right|$ is a *multiplier* is the **Lagrange** equations when one employs the variables q and q'. Represent that multiplier by M. The **Lagrange** equations become:

$$(59'') \qquad \frac{\delta q_i}{q'_i} = \frac{\delta q'_i}{\dfrac{\Delta_i}{M}} = \delta t .$$

In that form, one sees that q'_i, Δ_i / M define a solution for the first-order variations. Hence, $M q'_i$ and the Δ_i are (up to sign) the coefficients (M^i) of an integral invariant of order $(2n - 1)$ (no. **35**).

58. – The results of no. **56** are susceptible to several generalizations.

As before, one will find that:

$$J_1 = \int \sum_i \sum_l N_l^i \, dq_i^l \qquad \left\{ \begin{array}{l} i = 1, \ldots, n, \\ l = 0, \ldots, p-1, \\ q_i^0 \equiv q_i , \\ N_{-1}^i \equiv 0 \end{array} \right.$$

is a relative integration invariant of the system:

(*) I suppose that W does not include t explicitly.

$$(63)\qquad \left\{\begin{aligned} &\frac{\delta q_i}{q_i^1} = \frac{\delta q_i^1}{q_i^2} = \cdots = \frac{\delta q_i^{p-1}}{q_i^p} = \delta t, \\ &\frac{\partial W}{\partial q_i} - \frac{\delta}{\delta t}\frac{\partial W}{\partial q_i^1} + \cdots + (-1)^p \frac{\delta^p}{\delta t^p}\frac{\partial W}{\partial q_i^p} = 0, \end{aligned}\right.$$

when

$$N_0^i = \frac{\partial W}{\partial q_i^1} - \frac{\delta}{\delta t}\frac{\partial W}{\partial q_i^2} + \cdots + (-1)^{p-1}\frac{\delta^{p-1}}{\delta t^{p-1}}\frac{\partial W}{\partial q_i^p},$$

$$N_1^i = \qquad \frac{\partial W}{\partial q_i^2} + \cdots + (-1)^{p-2}\frac{\delta^{p-2}}{\delta t^{p-2}}\frac{\partial W}{\partial q_i^p},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,$$

$$N_{p-1}^i = \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \frac{\partial W}{\partial q_i^p}.$$

One can easily verify that by noting that:

$$\frac{\delta}{\delta t} N_l^i = \frac{\partial W}{\partial q_i^l} - N_{l-1}^i .$$

If one takes the N_l^i and the q_l^i to be variable then the system (63) will take the canonical form (**Jacobi**).

59. – One will then have:

$$(64)\qquad j_{t_1} - j_{t_0} = d\int_{t_0}^{t_1} W\,\delta t,$$

in which:

$$j = \sum_i \sum_l N_l^i \, dq_i^l .$$

Take the variation *d* of the two sides of (62), so (*):

$$dj_{t_1} - dj_{t_0} = j'_{t_1} - j'_{t_0} = d\int_{t_0}^{t_1} dW\,\delta t,$$

$$j' = dj = \sum \frac{\partial\, dW}{\partial q'_i} dq_i .$$

(*) I take $d^2 q_i = 0$ (**Jordan**'s *Cours*, t. III, pp. 503).

We now write the equations of variations of equations (59):

(65) $$\frac{\delta \dfrac{\partial dW}{\partial q_i'}}{\dfrac{\partial dW}{\partial q_i}} = \frac{\delta\, dq_i}{dq_i'} = \delta t\,.$$

This system admits the relative invariant:

$$J_1' = \int \sum \frac{\partial dW}{\partial q_i'} dq_i$$

hence, it can also be put into the canonical form.

60. – We say that:

$$j_\mu = \sum_i N_i^\mu\, dx_i \qquad \begin{cases} i = 1, \ldots, n, \\ \mu = 1, \ldots, \nu \end{cases}$$

are the ν elements of a first-order relative integral invariant of the system (no. **41**):

$$\delta x_i = \sum_\mu X_i^\mu\, \delta t_\mu$$

that is comprised of *n total* differential equations when:

(66) $$\sum_\mu \frac{\delta j_\mu}{\delta t_\mu} = dW\,.$$

Thus:

(66′) $$d\int^\nu W\, \delta t_1 \cdots \delta t_\nu = \int^\nu \sum_\mu \frac{\delta j_\mu}{\delta t_\mu}\, \delta t_1 \cdots \delta t_\nu\,.$$

The latter integral reduces immediately to an integral of order $\nu - 1$ that is extended over a closed manifold.

Let $\nu = 2$. Set:

(67) $$\delta x_i = x_i^1\, \delta t_1 + x_i^2\, \delta t_2\,.$$

As in no. **56**, one finds that equations (67), combined with the following equations:

(68) $$\frac{\partial W}{\partial x_i} - \frac{\delta}{\delta t_1}\frac{\partial W}{\partial x_i^1} - \frac{\delta}{\delta t_2}\frac{\partial W}{\partial x_i^2} = 0 ,$$

admit a relative invariant whose elements are:

(69) $$\begin{cases} j_1 = \sum \dfrac{\partial W}{\partial x_i^1}\partial x_i, \\ j_2 = \sum \dfrac{\partial W}{\partial x_i^2}\partial x_i. \end{cases}$$

Remark. – In the calculus of variations, one begins by giving the function W of t_μ, x_i, and $x_i^{\alpha_1+\cdots+\alpha_\nu}$ or $\dfrac{\delta^{\alpha_1+\cdots+\alpha_\nu} x_i}{\delta t_1^{\alpha_1}\cdots\delta t_\nu^{\alpha_\nu}}$. One then proposes to calculate the variation:

$$d\int W\ \delta t_1\cdots\delta t_\nu ,$$

in such a manner that no $dx_i^{\alpha_1+\cdots+\alpha_\nu}$ will appear under the integration sign. The preceding shows that this variation will be equal to a ν-tuple integral in which the left-hand sides of equations (69) enter in a generalized form and an integral of order $\nu - 1$ that one easily deduces from the right-hand side of (66′) and a generalization of formulas (69).

61. – We once more say that:

(70) $$\begin{cases} \delta x_i = \dfrac{\partial H_1}{\partial y_i}\delta t_1 + \dfrac{\partial H_2}{\partial y_i}\delta t_2, \\ \delta y_i = -\dfrac{\partial H_1}{\partial x_i}\delta t_1 - \dfrac{\partial H_2}{\partial x_i}\delta t_2 \end{cases}$$

are *canonical* equations in which H_1 and H_2 are the characteristic functions.

Theorem:

In order for the equations:

$$\delta x_i = X_i^1\,\delta t_1 + X_i^2\,\delta t_2 ,$$
$$\delta y_i = Y_i^1\,\delta t_1 + Y_i^2\,\delta t_2$$

to admit the relative invariant $j_1 = j_2 =$ *such that:*

$$\frac{\delta j_1}{\delta t_1} = d\,W_1\,,$$
$$\frac{\delta j_2}{\delta t_2} = d\,W_2\,,$$

it is necessary and sufficient that this system should be canonical.

Upon proceeding as in no. **56**, one can extend the **Jacobi** method of integration to equations (70). [*See* the paper by **Saltykow**, J. Math. pures et appl. (1899).]

CHAPTER XIV

Proof of a theorem by H. Poincaré

Resume. – In this chapter, I shall give a new proof of a fundamental theorem by **Poincaré**. The one that was given by the distinguished geometer does not seem as simple to me. The proof here is based upon several other theorems that have come about in recent times in some remarkable articles (*).

62. – Recall no. **42** and consider the system of linear differential equations (44), which we write:

(71) $$\frac{\delta \xi_i}{\sum_k X_{ik}\,\xi_k} = \delta t \qquad (i, k = 1, \ldots, n),$$

in which X_{ik} are n^2 *periodic* functions of period T. Let:

(72) $$\begin{cases} \xi_i = \psi_i^1, \\ \xi_i = \psi_i^2, \\ \quad\vdots \\ \xi_i = \psi_i^n \end{cases} \qquad (i = 1, \ldots, n)$$

be n linearly-independent solutions of equations (71). They will not change when one changes t into $t + T$, and the n solutions will become:

$$\xi_i = \psi_i^1(t+T)\,, \qquad \text{etc.}$$

(*) *Méthodes Nouvelles*, t.I, pp. 184-192. – **E. Lindelöf**, "Démonstration de quelques théomès sur les équations différentielles," J. math. pures appl. (1900). – **J. Hadamard**, "Sur les intégrales d'un système d'éq. diff. ord.," Bull. Soc. Math. France (1900).

They must then be linear combinations of the n solutions (72) in such a way that:

$$\begin{aligned}\psi_i^1(t+T) &= A_{11}\psi_i^1(t+T)+A_{12}\psi_i^2(t+T)+\cdots+A_{1n}\psi_i^n(t+T)\,,\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,\\ \psi_i^n(t+T) &= A_{n1}\psi_i^1(t+T)+A_{n2}\psi_i^2(t+T)+\cdots+A_{nn}\psi_i^n(t+T)\,,\end{aligned}$$

in which the A_{ik} are constants whose determinant is non-zero. Having said that, form the equation in S:

$$(S)\qquad \begin{vmatrix} A_{11}-S & \cdots & A_{1m} \\ \vdots & \ddots & \vdots \\ A_{n1} & \cdots & A_{nn}-S \end{vmatrix} = 0\,.$$

Let S_1 be one of the roots of that equation (S). Set:

$$S_1 = e^{\alpha_1 T}\,.$$

We have a particular solution of equations (71) that we can write (*):

$$\xi_i = e^{\alpha_1 t}\lambda_i^1(t) \qquad (i = 1, \ldots, n),$$

in which the λ_i^1 are periodic of period T. Such a solution is said to be of the *first type*. If α_1 is a root of order $p > 1$ then it will give solutions of the form $e^{\alpha_1 t}$, multiplied by an entire polynomial in t whose coefficients are period functions of t of period T. They are solutions of the *second type*. The roots α are called *characteristic exponents*.

63. Theorem of H. Poincaré:

If the X_i that enter into equations (1) *are uniform and periodic of period T, and if those equations admit a periodic solution of period T, in addition, as well as p uniform integrals* $F_1, \ldots, F_p$ *that do not include t explicitly then p of the characteristic exponents will be* ***zero****, unless all of the determinants that are contained in the matrix:*

(*) *Méthodes Nouvelles*, t. I, pps. 66 and 195.

$$(73) \qquad \begin{vmatrix} \dfrac{\partial F_1}{\partial x_1} & \cdots & \dfrac{\partial F_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial F_p}{\partial x_1} & \cdots & \dfrac{\partial F_p}{\partial x_n} \end{vmatrix}$$

are non-zero at all points of the periodic solution considered. – If the X_i do not include t explicitly then there will be at least $p + 1$ characteristic exponents that are ***zero****.*

Proof: Let $\psi_i^1, \ldots, \psi_i^n$ be n distinct solutions of (71); suppose that $p = 1$. One will have:

$$\sum_i F_{1i}\,\psi_i^1 \equiv c_1\,,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots,$$
$$\sum_i F_{1i}\,\psi_i^n \equiv c_n\,,$$

identically, in which F_{1i} represents what $\partial F_1 / \partial x_i$ will become when one replaces x_k with the periodic solution (viz., generator): $x_k = \varphi_k(t)$; $c_1, \ldots, c_n$ are well-defined constants.

At the (arbitrary) $t + T$, one will have:

$$\sum_i F_{1i}\,(A_{11}\,\psi_i^1 + \cdots + A_{1n}\,\psi_i^n) \equiv c_1\,,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$
$$\sum_i F_{1i}\,(A_{n1}\,\psi_i^1 + \cdots + A_{nn}\,\psi_i^n) \equiv c_n\,.$$

If one subtracts corresponding sides of the preceding two systems then one will see that equation (S) admits the solution $S = 1$. Hence, there will be a characteristic exponent that is zero that consequently corresponds to a solution to the periodic variations. The restriction in regard to (73) is obvious.

Let $p = 3$. If the theorem is supposed to have been proved for $p = 2$ then one knows that there are two characteristic exponents that are equal to zero that correspond to the two solutions:

$$\xi_i^1 = \Phi_i^1\,,$$

and

$$\xi_i^2 = t\,\Phi_i^1 + \Phi_i^2\,.$$

By virtue of no. **28**, one will have the integral invariant:

$$I_1 = \int \sum_{ijk} \frac{\partial(F_1, F_2, F_3)}{\partial(x_i, x_j, x_k)}\, dx_i\, dx_j\, dx_k \qquad (i, j, k = 1, \ldots, n).$$

Hence:

$$\sum_{ijk} F_{ijk}^{123} \begin{vmatrix} \Phi_i^1 & \Phi_j^1 & \Phi_k^1 \\ \Phi_i^2 & \Phi_j^2 & \Phi_k^2 \\ \psi_i^l & \psi_j^l & \psi_k^l \end{vmatrix} = \text{an integral} \quad (l = 3, \ldots, n).$$

Replace the x_k with the solution (i.e., generator) $\varphi_k(t)$. The left-hand side of the preceding expression will reduce to a well-defined constant c_l . Not all of the c_l can be zero at the same time.

If we increase t by T then we will get a new system that will give:

$$\sum_{ijk} F_{ijk}^{123} \begin{vmatrix} \Phi_i^1 & \Phi_j^1 & \Phi_k^1 \\ \Phi_i^2 & \Phi_j^2 & \Phi_k^2 \\ A_{l3}\psi_i^3 + \cdots + (A_{ll} - 1)\psi_i^l + \cdots + A_{ln}\psi_i^n & (\psi_j) & (\psi_k) \end{vmatrix} \equiv 0$$

when it is subtracted from the preceding one. The significance of (ψ_j) and (ψ_k) is easy to find. Those $n - 2$ expressions, which are linear and homogeneous in $A_{l3}, \ldots, A_{ll} - 1, \ldots, A_{ln}$, are compatible only if one has:

(75)
$$\begin{vmatrix} A_{33} - 1 & \cdots & A_{3n} \\ \vdots & \ddots & \vdots \\ A_{n3} & \cdots & A_{nn} - 1 \end{vmatrix} \equiv 0 .$$

Since Φ_i^1 is periodic, we will have:

$$A_{11} = 1, \qquad A_{12} = \ldots = A_{1n} = 0 .$$

The value of ψ_i^2 shows that:

$$A_{21} = 1, \qquad A_{22} = 1, \qquad A_{23} = \ldots = A_{2n} = 0 .$$

Therefore, equation (S) will become:

$$\begin{vmatrix} 1 - S & 0 & 0 & \cdots & 0 \\ T & 1 - S & 0 & \cdots & 0 \\ A_{31} & A_{32} & A_{33} - S & \cdots & A_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_{n1} & \cdots & \cdots & \cdots & A_{nn} - S \end{vmatrix} = 0 .$$

One sees immediately that $S = 1$ will annul all of the minors that relate to any two elements that are taken from the positive diagonal of the determinant (S). Therefore, $S = 1$ is a triple root of equation (S), so there will be three zero exponents.

Set:

$$\xi_i^1 = X_i ,$$
$$\xi_i^2 = t\,X_i + \Phi_i^2 ,$$
$$\xi_i^l = \psi_i^l \qquad (l = 3, \dots, n).$$

One has (no. **35**):

$$\sum_i F_{1i} X_i \equiv 0 ,$$
$$\sum_i F_{2i} X_i \equiv 0 ,$$

so:

$$(76) \qquad \sum_{i,j} \begin{vmatrix} F_{1i} & F_{1j} \\ F_{2i} & F_{2j} \end{vmatrix} \begin{vmatrix} X_i & X_j \\ \xi_i^1 & \xi_j^1 \end{vmatrix} \equiv 0 ,$$

as one can verify by performing the multiplication and replacing $\sum\limits_{i,j}$ with a double summation $\sum\limits_{i=1\cdots n} \sum\limits_{j=1\cdots n}$.

The integral invariant:

$$I_2 = \int \sum_{i,j} F_{ij}'^2 \, dx_i \, dx_j$$

will give:

$$\sum_{i,j} F_{ij}'^2 \begin{vmatrix} \xi_i^2 & \xi_j^2 \\ \xi_i^l & \xi_j^l \end{vmatrix} = \text{an integral.}$$

By reasoning as one did in the preceding case ($p = 3$) and making use of the identity (76), one will get the identity (75), after which, nothing will need to be changed in the proof.

The case in which p is arbitrary can be treated in the same way. One always begins by considering the integral invariant I_p that one can write:

$$\int dF_1 \cdots dF_p .$$

CHAPTER XV

Application to the Lagrange and Riemann's adjoint equation

Summary. – This chapter includes the synthesis of numerous studies (*) that have been made on the subject. It is an interesting application of the generalized calculus of variations. Some new simplifications are given in it. Note that this theory can be utilized in the case of an *arbitrary* system of ordinary differential equations when one knows a solution (no. **42**).

64. – Consider the *n linear* ordinary differential equations:

$$(E) \qquad \frac{\delta x_i}{\sum_k a_k^i x_k} = \delta t \qquad (k, i = 1, \ldots, n),$$

in which the a_k^i are functions of only *t*.

The coefficients of an integral invariant I_p of order *p* of the system will define a solution to $\frac{n!}{p!(n-p)!}$ *linear* ordinary differential equations *when one supposes that those coefficients are functions of only t*.

We call that system *the adjoint system $A_p E$*. Let $p = 1$. Formulas (8) will then give:

$$(A_1 E) \qquad \frac{\delta M_i}{-\sum_k a_i^k M_k} = \delta t ,$$

$$A_1 A_1 E \equiv E .$$

One has, in addition (no. **39**), that:

$$M_i = (-1)^i M \, \xi_{(n-1)}^i .$$

Let $p = n$. Nos. **18** and **39** will give:

$$M = \frac{1}{\xi_{(n)}} = \exp\left(-\int \sum a_k^k \, \delta t\right) = \frac{1}{\Delta} = \text{a multiplier.}$$

Δ represents the determinant that is formed from *n* distinct solutions of *E*. The solutions to *E* are, at the same time, solutions to the *variations* of *E*. Let $V_q E$ represent the system of *linear* ordinary differential equations that the solutions to the variations $\xi_{(q)}$ of order *q* of *E* satisfy; call that system the *associated system $V_q E$*.

Thanks to the formulas in nos. **8**, **18**, **23**, **25** (cont.), **34**, **35**, and **39**, one will get the following remarkable relations from some very simple calculations:

(*) **Schlesinger**, "Theorie der linearen Differentialgleichungen," Crelle's Journal, vols. 1 and (1901).

$$V_q\,E \equiv E\,,$$
$$A_p\,A_1\,E \equiv A_1\,A_p\,E \equiv V_p\,E\,,$$
$$A_p\,A_{n-1}\,E \equiv V_p\,V_{n-1}\,E,$$
$$A_1\,V_p\,E \equiv V_p\,A_1\,E \equiv A_p\,E\,.$$

One will also find the multipliers $A_p\,E$ or $V_p\,E$ just as easily since all of those multipliers are equal to Δ raised to various powers.

65. – Consider the n^{th}-order ordinary differential equation:

(77) $$a_0\,x_0 + a_1\,x_1 + a_2\,x_2 + \ldots + a_{n-1}\,x_{n-1} + a_n\,x_n = 0\,,$$

in which:

$$x_0 \equiv x \quad \text{and} \qquad x_p \equiv \frac{\delta^p x}{\delta t^p}\,.$$

The coefficient M_{n-1} in the integral invariant:

$$I_1 = \int M_0\,dx_0 + M_1\,dx_1 + \cdots + M_{n-1}\,dx_{n-1}$$

of equation (77) satisfies an n^{th}-order equation (*) that is the ***Lagrange*** *adjoint* of (77). The coefficient M_0 satisfies an n^{th}-order equation that was studied by **Jacobi** and later by **Darboux** and **Cels**.

The system $(A_1\,E)$ implies the very simple relation:

$$\frac{\delta M_0}{\delta t} - \frac{a_0}{a_n} M_{n-1} = 0\,,$$

which was utilized by **Cels**.

66. – Now suppose that there are two independent variables t_1 and t_2 . Consider a second-order partial differential equation that we write as follows:

(77) $$a_{00}\,x_{00} + a_{01}\,x_{01} + a_{10}\,x_{10} + a_{11}\,\mathrm{x}_{11} + a_{02}\,x_{03} + a_{20}\,x_{20} = 0\,,$$

in which:

(*) One can always find an equations of order at most n that is satisfied by one of the coefficients by differentiation and eliminating by means of determinants, even in the general case of an arbitrary system of n *linear* equations.

$$x_{00} \equiv x \qquad \text{and} \qquad \frac{\delta^q x_{ik}}{\delta t_2^q} = x_{i,k+q} ,$$

and in which the coefficients a are functions of t_1 and t_2 .

Set:

$$I_1 = M_{00}\, dx_{00} + M_{01}\, dx_{01} + M_{10}\, dx_{10} ,$$
$$I_2 = N_{00}\, dx_{00} + N_{01}\, dx_{01} + N_{10}\, dx_{10} ,$$

and look for the necessary and sufficient conditions for one to have:

$$\frac{\delta I_1}{\delta t_1} + \frac{\delta I_2}{\delta t_2} = 0 \qquad \text{(no. } \mathbf{60}\text{)}.$$

We find that:

$$(79) \qquad \begin{cases} \delta_1 M_{00} + \delta_2 N_{00} - N_{01} \dfrac{a_{00}}{a_{02}} = 0, \\ \delta_1 M_{10} + \delta_2 N_{01} + N_{00} - N_{01} \dfrac{a_{00}}{a_{02}} = 0, \\ \delta_1 M_{10} + \delta_2 N_{00} + M_{00} - N_{01} \dfrac{a_{00}}{a_{02}} = 0, \\ M_{01} + N_{01} - N_{01} \dfrac{a_{00}}{a_{02}} = 0, \\ M_{10} - N_{01} \dfrac{a_{00}}{a_{02}} = 0, \end{cases}$$

in which δ_1 and δ_2 are used in place of $\delta / \delta t_1$ and $\delta / \delta t_2$, resp.

The last equation gives:

$$(80) \qquad \begin{cases} M_{10} = y\, a_{20}, \\ N_{10} = y\, a_{02}, \end{cases}$$

in which y is a function of t_1 and t_2 that satisfies a second-order partial differential equation that the ***Riemann*** *adjoint* (*) of (78). That equation is obtained by eliminating the coefficients M and N of equations (79) and (80). We remark that:

$$M_{01} + N_{01} = y\, a_{11} ,$$

so those two coefficients enter into consideration only by way of their sum. An indeterminacy will then result that one can benefit from by taking, for example, $N_{10} \equiv 0$. As a result of that fact, the

(*) *Leçons* by **G. Darboux**, t. II.

coefficients M_{00} and N_{00} can have simpler values, and **Riemann**'s method of integration will have much to recommend it.

Paris, 21 February 1902

Th. de Donder.