"Sur une méthode de Bateman dans le problème inverse du calcul des variations," Bull. Acad. roy. Belgique (Cl. des Sc.) (5) **35** (1949), 774-792.

# On Bateman's method in the inverse problem in the calculus of variations.

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Translated by D. H. Delphenich

**1.** – In 1931, H. Bateman showed that any system of *n* partial differential equations of arbitrary order:

$$F_a\left(x^a; y^i, \frac{\partial y^i}{\partial x^{\alpha}}, \frac{\partial^2 y^i}{\partial x^{\alpha} \partial x^{\beta}}, \ldots\right) = 0 \qquad \qquad \begin{pmatrix} a, i = 1, 2, \ldots, n \\ \alpha, \beta = 1, 2, \ldots, \mu \end{pmatrix}$$

in the *n* unknowns  $y^i = y^i(x^{\alpha})$  can be included in a system of 2n equations in 2n unknowns that are derived from the calculus of variations (<sup>1</sup>).

The given equations define integral manifolds  $V_{\mu}$  of  $y^i = y^i(x^{\alpha})$  in the space  $E_{\mu+n}$  of  $x^{\alpha}$ ,  $y^i$ . The method amounts associating them with a variational problem in a  $(\mu + 2n)$ -dimensional space  $E^*_{\mu+2n}$ .

Those two spaces are coupled by a certain correspondence whose properties we shall study. We suppose that the given equations consist of annulling the components of a tensor  $\mathbf{F}$  with *n* components relative to a group  $\Gamma$  that operates in *E*. There will then exist a group  $\Gamma$  in  $E^*$  that is isomorphic to  $\Gamma$  but which depends upon the nature of the tensor  $\mathbf{F}$  as far as the group of transformations is concerned.

Another interpretation of the *n* auxiliary unknowns consists of regarding them as the components of a tensor **Z** with *n* components that is defined at point of a manifold  $V_{\mu}$ . Bateman's 2*n* equations then define manifolds of elements in  $E_{\mu+n}$ , if what we call an *element* is composed of a point and a tensor **Z** that is attached to that point. The variance of **Z** will be opposite to that of **F**, up to a multiplier.

In addition, we observe that Bateman's method is only a special case of the method of Lagrange multipliers.

<sup>(\*)</sup> Presented by F. H. van den Dungen.

<sup>(&</sup>lt;sup>1</sup>) **H. Bateman**, "On dissipative systems and related variational principles," Phys. Rev. (2) **38** (1931). See also the second sentence in the introduction to the book *Partial Differential Equations of Mathematical Physics* and the article "Hamilton's work in dynamics and its influence on modern thought," Scripta Math. (1944).

Likewise, see: **F. H. van den Dungen**, "Les équations canoniques du résonateur linéaire," Bull. Acad. Rot. Belgique (Cl. des Sc.) (1945) and **J. Géhéniau**, "La quantification des systems non canoniques," *ibid.* (1945).

**2.** *Review.* – Consider a  $\mu$ -uple integral in the calculus of variations:

(2.1) 
$$I = \int_{\mu} L(x^{\alpha}, y^{i}, y^{i}_{\alpha}, \dots, y^{i}_{\alpha_{1} \cdots \alpha_{r}}) dx^{1} \cdots dx^{\mu}$$

in which the Lagrangian function L depends upon the  $\mu$  independent variables  $x^{\alpha}$  ( $\alpha = 1, 2, ..., \mu$ ), the *n* dependent variables  $y^{i} = y^{i}(x^{\alpha})$  (i = 1, 2, ..., n), as well as their derivatives up to order *r*:

$$y^i_{\alpha} = \frac{\partial y^i}{\partial x^{\alpha}}, \qquad \dots, \qquad y^i_{\alpha_1 \cdots \alpha_r} = \frac{\partial^r y^i}{\partial x^{\alpha_1} \cdots \partial x^{\alpha_r}}.$$

The extremals are manifolds  $V_{\mu}$  whose equations are  $y^{i} = y^{i}(x^{1},...,x^{\mu})$  that are embedded in the space  $E_{\mu+n}$  of the  $x^{\alpha}$ ,  $y^{i}$ , and they are solutions to the variational derivative equations (<sup>1</sup>):

(2.2) 
$$\frac{\delta L}{\delta y^i} = 0$$

Perform a change of coordinates in  $E_{\mu+n}$  that takes the form:

(2.3) 
$$\begin{cases} \overline{x}^{\alpha} = \overline{x}^{\alpha}(x^{\beta}), & \frac{\partial(\overline{x}, \overline{y})}{\partial(x, y)} = \frac{\partial(\overline{x})}{\partial(x)} \cdot \frac{\partial(\overline{y})}{\partial(y)} \neq 0. \end{cases}$$

The integral *I* that one supposes to be *invariant* becomes:

$$I = \int_{\mu} \overline{L} \left( \overline{x}^{\alpha}, \overline{y}^{i}, \overline{y}^{i}_{\alpha}, \dots, \overline{y}^{i}_{\alpha_{1} \cdots \alpha_{r}} \right) d\overline{x}^{1} \cdots d\overline{x}^{\mu} \qquad \left( \overline{y}^{i}_{\alpha} = \frac{\partial y^{i}}{\partial \overline{x}^{\alpha}}, \dots \right),$$

in which the new Lagrangian function is given by:

(<sup>1</sup>) As is well known, the variational derivatives are given by:

$$\frac{\delta L}{\delta y^{i}} = \frac{\partial L}{\partial y^{i}} - \frac{d}{dx^{\alpha}} \frac{\partial L}{\partial y^{i}_{\alpha}} + \dots + (-1)^{r} \frac{d}{dx^{\alpha_{i}} \cdots dx^{\alpha_{r}}} \frac{\partial L}{\partial y^{i}_{\alpha_{i} \cdots \alpha_{r}}}$$
$$\frac{d}{dx^{\alpha}} \equiv \frac{\partial}{\partial x^{\alpha}} + y^{i}_{\alpha} \frac{\partial}{\partial y^{i}} + y^{i}_{\alpha\beta} \frac{\partial}{\partial y^{i}_{\beta}} + \dots$$

with:

(2.4) 
$$\overline{L}(\overline{x}^{\alpha}, \overline{y}^{i}, \overline{y}^{i}_{\alpha}, \ldots) = L(x^{\alpha}, y^{i}, y^{i}_{\alpha}, \ldots) \frac{\partial(x)}{\partial(\overline{x})},$$

while the equations of the extremals will become:

$$\frac{\delta \overline{L}}{\delta \overline{y}^{i}} = 0 ,$$

with

(2.5) 
$$\frac{\delta L}{\delta \overline{y}^{i}} = \frac{\partial y^{j}}{\partial \overline{y}^{i}} \frac{\delta L}{\delta y^{j}} \frac{\partial (x)}{\partial (\overline{x})}$$

**3.** Definitions. – The relations (2.4) and (2.5) indicate the variance of the function L and its variational derivatives with respect to the group of transformations (2.3). To use a terminology that was inspired by Craig (<sup>1</sup>), we will say that the function L is a scalar of weight one and that its variational derivatives are the components of a covariant vector of weight one.

More generally, we will appeal to the following definitions:

When a function  $F(x^{\alpha}, y^{i}, y^{i}_{\alpha}, ...)$  transforms into a function  $\overline{F}(\overline{x}^{\alpha}, \overline{y}^{i}, \overline{y}^{i}_{\alpha}, ...)$  such that:

(3.1) 
$$\overline{F} = F\left(\frac{\partial(x)}{\partial(\overline{x})}\right)^p,$$

we say that we are dealing with a *scalar* of *weight p*.

Let an entity be characterized in variables x, y by n components  $A_1, \ldots, A_n$  ( $B^1, \ldots, B^n$ , resp.) that are functions of  $x^{\alpha}$ ,  $y^i$ ,  $y^i_{\alpha}$ , ..., and in variables  $\overline{x}$ ,  $\overline{y}$ , it is characterized by n components  $\overline{A}_1, \ldots, \overline{A}_n$  ( $\overline{B}^1, \ldots, \overline{B}^n$ , resp.) that are functions of  $\overline{x}^{\alpha}$ ,  $\overline{y}^i$ ,  $\overline{y}^i_{\alpha}$ , ... We say that we are dealing with a *covariant (contravariant, resp.) vector of weight p* if we pass from the A to the  $\overline{A}$  by the formulas:

(3.2) 
$$\overline{A}_{i} = \left(\frac{\partial(x)}{\partial(\overline{x})}\right)^{p} \frac{\partial y^{j}}{\partial \overline{y}^{i}} A_{j}$$

(if we pass from the *B* to the  $\overline{B}$  by the formulas:

(3.3) 
$$\overline{B}^{i} = \left(\frac{\partial(x)}{\partial(\overline{x})}\right)^{p} \frac{\partial \overline{y}^{j}}{\partial y^{j}} B^{j},$$

respectively.)

<sup>(&</sup>lt;sup>1</sup>) **H. V. Craig**, Vector and Tensor Analysis, 1943.

More generally, consider a group  $\Gamma$  of transformations of the type (2.5). A *tensor* relative to  $\Gamma$ is characterized in each system of coordinates by a certain number r of components  $T_1, ..., T_r$  that transform linearly under a change of coordinates (<sup>1</sup>). Each transformation T of  $\Gamma$  will then correspond to a square matrix  $\|M_a^b\| = \|M\|$  (*a*, *b* = 1, 2, ..., *r*) whose elements depend upon the x, y (or the  $\overline{x}$ ,  $\overline{y}$ ) and are such that the new components T are deduced from the old ones by the formulas:

$$(3.4) \overline{T}_a = M_a^b T_b$$

The correspondence between the transformations T and the matrices || M || is not arbitrary. Let  $T_1$  and  $T_2$  be two transformations whose product is  $T_3 = T_2 \cdot T_1$ . It is necessary that the matrix  $||M_3||$  that is associated with  $T_3$  should be the product of the matrices  $||M_1||$  and  $||M_2||$  that are associated with  $T_1$  and  $T_2$ :  $||M_3|| = ||M_2|| \cdot ||M_1||$ . In other words, the set of matrices ||M|| must constitute a group  $\gamma$  that is isomorphic to  $\Gamma$ . That condition is likewise sufficient for the formulas (3.4) to define a tensor **T** in a coherent manner. The group of matrices  $\gamma$  defines the variance (one also says the *nature*) of the tensor **T**.

The tensor **T** is associated in a unique manner with a tensor **S** with *r* components  $S^{\alpha}$  such that the product  $T_{\alpha} \cdot S^{\alpha}$  will be a scalar of arbitrary weight p. The variance of S is defined by the group  $\Sigma_p$  of matrices  $\left\| \left[ \frac{\partial(x)}{\partial(\bar{x})} \right]^p \bar{M}_a^b \right\|$ , where  $\bar{M}_a^b$  denotes the normalized minor of  $M_a^b$  in  $\|M\|$ . One

will then have:

$$\overline{S}^{a} = \left[\frac{\partial(x)}{\partial(\overline{x})}\right]^{p} \overline{M}_{b}^{a} S^{b}.$$

We say that the variance of S is opposite to that of T, up to weight p. For example, a covariant vector of weight p and a contravariant vector of weight q have opposite variances, up to weight p +q.

*Remark.* – Except for no. 6, we will have to consider only the case in which r = n. Unless stated to the contrary, the indices a, b will then take the same values 1, 2, ..., n as the indices i, j, ... Nonetheless, we shall reserve the latter in order to indicate that we are dealing with covariant or contravariant vectors, while the letters a, b will be utilized when we are dealing with more-general tensors.

<sup>(1)</sup> It is implicit that the T can depend upon not only the  $x^{\alpha}$ ,  $y^{i}$ , but also the  $y^{i}_{\alpha}$ ,  $y^{i}_{\alpha\beta}$ , ...

**4.** *Inverse problem in the calculus of variations.* – Consider a system of *n* partial differential equations of order *s* whose unknowns are *n* functions  $y^i$  of the  $x^{\alpha}$ :

(4.1) 
$$F_{a}\left(x^{a}; y^{i}, \frac{\partial y^{i}}{\partial x^{\alpha}}, \frac{\partial^{2} y^{i}}{\partial x^{\alpha} \partial x^{\beta}}, \ldots\right) = 0 \qquad \left(\begin{array}{c}a, i = 1, 2, \ldots, n\\\alpha = 1, 2, \ldots, \mu\end{array}\right)$$

and whose left-hand sides are the components of a tensor **F** relative to the group  $\Gamma$ .

Two questions are posed that both deserve the name of the *inverse problem in the calculus of variations:* 

a) Does there exist a function L whose variational derivatives are *identical* to the left-hand sides of that (4.1)?

b) Does there exist a function L such that the corresponding equations (2.2) are *equivalent* to equations (4.4)?

In order for the first one to have an invariant sense, it is necessary that the tensor  $\mathbf{F}$  should be a covariant vector of weight one. By contrast, the second question will have an invariant sense regardless of the nature of the tensor  $\mathbf{F}$ . A complete answer to those two questions has not been given up to now, to our knowledge. As far as the first one is concerned, one knows a *necessary* condition for the existence of the Lagrangian function that is provided by the theory of *adjoint* systems.

Let:

(4.2) 
$$y^{i} = y^{i}(x^{\alpha}), \quad y^{i} = y^{i}(x^{\alpha}) + \delta y^{i} = y^{i}(x^{\alpha}) + \eta^{i}(x^{\alpha})$$

be the equations of an arbitrary manifold  $V_{\mu}$  in  $E_{\mu+n}$  that is or is not a solution to (4.1) and a neighboring manifold. The homogeneous linear expressions in the  $\eta^i$  and their derivatives (<sup>1</sup>):

$$\delta F_a \equiv \mathsf{F}_a\left(\eta\right) \equiv \frac{\partial F_a}{\partial y^i} \eta^i + \frac{\partial F_a}{\partial y^i_{\alpha}} \eta^i_{\alpha} + \dots + \frac{\partial F_a}{\partial y^i_{\alpha_1 \cdots \alpha_s}} \eta^i_{\alpha_1 \cdots \alpha_s},$$

whose coefficients will become functions of the  $x^{\alpha}$  upon taking (4.2) into account, constitute the system of variations of (4.2) along  $V_{\mu}$ .

One shows that if the problem (a) possesses a solution then the  $F_a$  will be self-adjoint polynomials for any  $V_{\mu}$  (<sup>2</sup>). In other words, there exist  $\mu$  alternating bilinear forms  $I^a$  in two groups

<sup>(1)</sup> We suppose that  $\delta x^{\alpha} \equiv 0$ .

<sup>(&</sup>lt;sup>2</sup>) **Th. De Donder**, *loc. cit.*, Chap. XV, pp. 204, Application.

of dependent variables  $\eta_1^i(x^{\alpha})$ ,  $\eta_2^i(x^{\alpha})$ , and their derivatives with coefficients that are functions of  $x^{\alpha}$ ,  $y^i$ ,  $y^i_{\alpha}$ , ... such that one will have (<sup>1</sup>):

(4.3) 
$$\eta_1^i \mathsf{F}_i(\eta_2) - \eta_2^i \mathsf{F}_i(\eta_1) \equiv \frac{d}{dx^{\alpha}} \mathbf{I}^{\alpha}$$

along all of  $V_{\mu}$ . Moreover, one can show that this condition is sufficient in a certain number of special cases (<sup>2</sup>).

**5.** *Bateman's method.* – Faced with the difficulty or impossibility of solving one of the two inverse problems, Bateman pointed out the following property:

Consider the variational problem that is associated with the integral:

(5.1) 
$$\int \mathbf{L} \, dx^1 \cdots dx^\mu = \int z^a F_a \, dx^1 \cdots dx^\mu$$

in the space  $E_{\mu+2n}^*$  of the  $x^{\alpha}$ ;  $y^i$ ,  $z^a$ . The search for extremals leads to the system of 2n partial differential equations:

$$\frac{\delta \mathbf{L}}{\delta y^{i}} \equiv G_{i}(x^{\alpha}; y^{i}, z^{a}; y^{i}_{\alpha}, z^{a}_{\alpha}; \cdots) = 0.$$

(5.2)

$$\frac{\delta \mathbf{L}}{\delta z^a} \equiv F_a(x^a; y^i, y^i_a, \cdots) = 0.$$

The last *n* of them are identical to (4.1), while the first  $\mu$  include the *n* auxiliary unknowns  $z^{a}(x^{\alpha})$ .

Any system of partial differential equations can then be prolonged into a system that is derived from the calculus of variations in the sense of problem (a).

$$[\delta y^i \delta F_i] \equiv \frac{d}{dx^{\alpha}} \mathbf{I}^{\alpha} ,$$

in which the  $I^{\alpha}$  are quadratic exterior differential forms.

(<sup>2</sup>) A. Hirsch,  $(n = 1, \mu = s = 2)$ , Math. Ann. 49 (1897); J. Kürschak,  $(n = 1, s = 2, \mu \text{ arbitrary})$  *ibid.* 60 (1905); D. R. David  $(n = 2, s = 2, \mu = 1)$  Trans. Amer. Math. Soc. 30 (1928); L. La Paz  $(n = 1, s = 2, \mu = \text{arbitrary})$  *ibid.*, v. 32; L. Königsberger gave equivalent conditions without using the notion of adjoint system and proved the converse in the case  $n = 1, s = 2, 4, \mu = 1; n = 2, s = 2, \mu = 1$ . *Die Principien der Mechanik*, Teubner, Leipzig, 1901;  $n = 1, s - 2, \mu = 2$ , Crelle's Journal 124 (1902). That author added that his method extended to the general case, but without performing the explicit calculations.

Likewise, see pp. 15, note  $(^1)$ .

As far as problem (b) is concerned, see G. Darboux (n = 1, s = 2,  $\mu = 1$ ), Théorie des Surfaces, t. 3, pp. 53; J. Douglas, (n arbitrary, s = 2,  $\mu = 1$ ), Proc. Nat. Acad. Sci. U.S.A. 26 (1940), Trans. Amer. Math. Soc. 50 (1941); Th. Lepage (n = 1, s = 2,  $\mu$  arbitrary), Bull. Acad. roy. Belg. 5 (1946).

<sup>(1)</sup> With the notations of **E. Cartan**'s exterior differential calculus, the relation (4.3) is written:

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One easily confirms that the  $G_i$  are homogeneous linear expressions in the  $z^a(x^{\alpha})$  and their derivatives with coefficients that are functions of the  $x^{\alpha}$ ;  $y^i$ ,  $y^i_{\alpha}$ , ...

There is reason to insist upon the fact that equations (5.2) are arranged in the order  $G_1 = 0, ..., G_n = 0, F_1 = 0, ..., F_n = 0$ , which is the order of the variational derivatives of the function **L** when the dependent variables are arranged in the order  $y^1, ..., y^n, ..., z^1, ..., z^n$ . That remark does not have merely a formal character. It is indispensable for developing the theory in a coherent and invariant manner, as will become apparent in what follows.

The mathematical significance of the  $G_i$  is given by the following proposition by Bateman:

### Theorem:

The expressions  $G_i$ , which are linear in the  $z^a$  and their derivatives, constitute the adjoint system to the system of variations of (4.1).

We express the idea that the system of variations of (5.2) that we represent by  $\Gamma_i(\eta, \zeta) \equiv \delta G_i$ ,  $F_a(\eta) \equiv \delta F_a$  is self-adjoint. There exist alternating bilinear forms  $\mathbf{I}^a$  in the two groups of variables  $\eta_1^i, \zeta_1^a; \eta_2^i, \zeta_2^a$  such that we will have:

$$\eta_1^i \cdot \Gamma_i(\eta_2, \zeta_2) + \zeta_1^a \,\mathsf{F}_a(\eta_2) - \eta_2^i \cdot \Gamma_i(\eta_1, \zeta_1) - \zeta_2^a \,\mathsf{F}_a(\eta_1) \equiv \frac{d}{dx^{\alpha}} \mathbf{I}^{\alpha}$$

identically. Set  $\eta_1^i = 0$ , and set  $\eta_2^i = \eta^i$ ,  $\zeta_1^a = \zeta^a$ . That will become:

(5.3) 
$$\zeta^{a} \mathbf{F}_{a}(\eta) - \eta^{i} \cdot G_{i}(\zeta) \equiv \frac{d}{dx^{\alpha}} \mathbf{I}^{*\alpha} \,.$$

In that relation, the  $G_i(z)$  are what the  $G_i$  and will become when one replaces the z in them with  $\zeta$ . The  $\mathbf{I}^{*a}$  are obtained by setting  $\eta_1^i = 0$  in  $\mathbf{I}^a$  and are necessarily independent of the  $\zeta_2(^1)$ . They are (generally not alternating) bilinear forms in the  $\eta$  and  $\zeta$ ; the identity (5.3) will then express the stated thesis.

*Example.* – In the case of the equation in one unknown y = y(z):

(5.4) 
$$a\ddot{y}+b\dot{y}+cy=0$$
 (a, b, c constants),

one has:

$$\mathbf{L} \equiv z(a\ddot{y}+b\dot{y}+cy); \qquad G \equiv a\ddot{z}-b\dot{z}+cz.$$

<sup>(1)</sup> Since  $\zeta_2^a$  does not appear in the left-hand side.

One indeed recognizes the adjoint polynomial to F in G:

$$z F(y) - y G(z) \equiv \frac{d}{dx} [a(\dot{y}z - y\dot{z}) + b y z] .$$

We further establish the following proposition:

#### Theorem:

If  $\mathbf{L}^{*}(x^{\alpha}; y^{i}, z^{a}; y^{i}_{\alpha}, z^{a}_{\alpha}; \cdots)$  is a function whose variational derivatives with respect to the  $z^{a}$  are identical to the  $F_{a}$  then the  $\frac{\delta \mathbf{L}^{*}}{\delta y^{i}} \equiv G^{*}_{i}$  will necessarily have the form:

$$G_i^* \equiv G_i + \frac{\delta L}{\delta y^i},$$

in which L depends upon only the  $x^{\alpha}$ ;  $y^{i}$ ,  $y^{i}_{\alpha}$ , ...

Indeed, upon reasoning as above after setting  $\delta G_i^* \equiv \Gamma_i^*$ , the formula (5.3) must be replaced with:

$$\zeta^{a} \mathsf{F}_{a}(\eta) - \eta^{i} \cdot \Gamma_{i}(0, \zeta) \equiv \frac{d}{dx^{\alpha}} \mathbf{I}^{\alpha},$$

in which  $\Gamma_i^*(0,\zeta) \equiv \Gamma_i(0,\zeta) \equiv G_i(\zeta)$ . The exact differential expression  $\partial G_i^* - \partial G_i$  is then independent of the  $\delta z^a$ ,  $\delta z_{\alpha}^a$ , ... The coefficients of the  $\delta y^i$ ,  $\delta y_{\alpha}^i$ , ... cannot depend upon the  $z^a$ ,  $z_{\alpha}^a$ , ... It will then follow that  $G_i^* \equiv G_i + g_i$ , in which the  $g_i$  are functions of only the  $x^{\alpha}$ ;  $y^i$ ,  $y_{\alpha}^i$ , ... They are the variational derivatives with respect to the  $y^i$  of the function  $L = \mathbf{L}^* - \mathbf{L}$ .

In the case where they contain the  $z^a$ ,  $z^a_{\alpha}$ , ... explicitly, one replaces those variables with arbitrarily-chosen functions  $z^a(x^{\alpha})$  and their derivatives (zero, for example) without modifying the  $\frac{\delta L}{\delta v^i}$ , so we have proved the stated thesis (<sup>1</sup>).

<sup>(&</sup>lt;sup>1</sup>) It can be useful to point out that when one replaces (5.1) with an *equivalent* system  $F_{\alpha}^* = 0$ , the corresponding complementary equations  $G_i^* = 0$  will no longer be equivalent to the  $G_i = 0$ , in general. Example:  $F^* = e^{(b/a)t} (a\ddot{y} + b\dot{y} + cy) = 0$  is equivalent to (5.4), but  $G^* = e^{(b/a)t} (a\ddot{z} + b\dot{z} + cz) = 0$  is no longer equivalent to G = 0. [Cf., **H. Bateman**, *loc. cit.*, Phys. Rev. (1931)]

That is due to the fact that **Bateman**'s method does not correspond to the study of just solutions to the given equations. It corresponds to the study of the tensor  $\mathbf{F}$  in all of the space E.

**6.** Link with the Lagrange multipliers. – Let us determine the extremals of an integral (2.1) from among the manifold  $V_{\mu}$  that are subjected to r equations of constraint (viz., a constrained extremum):

$$F_a(x^{\alpha}; y^i, y^i_{\alpha}, ...) = 0$$
 (*a* = 1, 2, ..., *r*)

The method of Lagrange multipliers consists of seeking the extremals of the integral:

$$\int (L+\lambda^a F_a)\,dx^1\cdots dx^\mu$$

in the  $(\mu + n + r)$ -dimensional space of  $x^{\alpha}$ ,  $y^{i}$ ,  $\lambda^{a}$ . If we set:

$$\lambda^a F_a = \mathbf{L}$$

then those extremals are given by the system of n + r equations in the n + r unknowns  $z^{a}(x^{\alpha})$ ,  $\lambda^{a}(x^{\alpha})$ :

$$\begin{cases} \frac{\delta L}{\delta y^{i}} + \frac{\delta \mathbf{L}}{\delta y^{i}} = 0, \\ \frac{\delta \mathbf{L}}{\delta \lambda^{a}} \equiv F_{a} = 0. \end{cases}$$

One verifies immediately that  $\delta \mathbf{L} / \delta y^i$  are expressions that homogeneous and linear in the  $\lambda^a$  and their partial derivatives. They constitute the adjoint system to the system of variations of the  $F_a$ .

When L is non-zero, one is obliged to suppose that r < n in order for there to be a variational problem effectively. However, when the function L is identically zero, one can consider the case of r = n, and one will then recover nothing but Bateman's method.

In what follows, we will always place ourselves in the Bateman case ( $L \equiv 0, r = n$ ), although some of the properties that we will point out will be extended to the general case.

7. *Invariant properties.* – Consider a change of coordinates **T** of the type (2.5) that belongs to the group  $\Gamma$  with respect to which the tensor **F** is defined. Equations (4.1) will become:

(7.1) 
$$\overline{F}_a(\overline{x}^{\alpha}; \overline{y}^i, \overline{y}^i_{\alpha}, \ldots) = 0,$$

with

(7.2) 
$$\overline{F}_a = M_a^b F_b.$$

Apply Bateman's method to the system (7.1). We then construct the system of 2n equations in 2n unknown:

(7.3) 
$$\begin{cases} \overline{G}_i(\overline{x}^{\alpha}; \overline{y}^i, \overline{z}^a; \overline{y}^i_{\alpha}, \overline{z}^i_{\alpha}, ...) = 0, \\ \overline{F}_a(x^{\alpha}; y^i, y^i_{\alpha}, ...) = 0 \end{cases}$$

that define the extremals of the integral:

$$\int \overline{\mathbf{L}} \, dx^1 \cdots dx^{\mu} \,, \qquad \overline{\mathbf{L}} = \overline{z}^a \, \overline{F}_a$$

We shall show that the transformation **T** of the  $x^{\alpha}$ ;  $y^{i}$  into the  $x^{\alpha}$ ;  $\overline{y}^{i}$  prolongs into a unique transformation **T**<sup>\*</sup> of the  $x^{\alpha}$ ;  $y^{i}$ ,  $z^{a}$  into the  $\overline{x}^{\alpha}$ ;  $\overline{y}^{i}$ ,  $\overline{z}^{a}$  such that:

1. The function **L** transforms into  $\overline{\mathbf{L}}$  like a scalar of weight one in the space  $E_{\mu+2n}^*$ .

2. The  $G_1, ..., G_n, F_1, ..., F_n$  transform into  $\overline{G}_1, ..., \overline{G}_n, \overline{F}_1, ..., \overline{F}_n$  like the 2n components of a covariant vector of weight one in  $E^*_{\mu+2n}$ .

3. When the transformation **T** generates the group  $\Gamma$ , the transformation **T**<sup>\*</sup> will generate an isomorphic group  $\Gamma^*$ .

*Proof.* – The first condition imposes a variance on the  $z^a$  that is opposite to that of the  $F_a$ , up to weight one (<sup>1</sup>). The transformation **T**<sup>\*</sup> (if it exists) will then necessarily be:

(7.4)  
$$\begin{cases} \overline{x}^{\alpha} = \overline{x}^{\alpha} (x^{\beta}), \\ \overline{y}^{i} = \overline{y}^{i} (x^{\beta}; y^{j}), \\ \overline{z}^{a} = \frac{\partial(x)}{\partial(\overline{x})} \overline{M}^{a}_{b} z^{b}, \end{cases}$$

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and will indeed verify 1., namely:

$$\overline{z}^a \, \overline{F}_a = z^a \, F_a \frac{\partial(x)}{\partial(\overline{x})} \, .$$

The condition 2 is necessarily satisfied by virtue of the variance of the variational derivatives that was recalled in no. **2**. One will then have:

<sup>(1)</sup> That conclusion will break down if equations (4.1) are not independent, which is a case that we eliminate.

(7.5)  
$$\begin{cases} \overline{G}_{i} = \left(\frac{\partial y^{j}}{\partial \overline{y}^{i}}G_{j} + \frac{\partial z^{b}}{\partial \overline{y}^{i}}F_{b}\right)\frac{\partial(x)}{\partial(\overline{x})},\\ \overline{F}_{a} = \left(0\cdot G + \frac{\partial z^{b}}{\partial \overline{z}^{a}}F_{b}\right)\frac{\partial(x)}{\partial(\overline{x})}. \end{cases}$$

The latter relation will reduce to (7.2), moreover  $(^1)$ .

Finally, the condition 3 is due to the fact that the group  $\overline{\gamma}_1$  of matrices  $\left\| \frac{\partial(x)}{\partial(\overline{x})} \overline{M}_b^a \right\|$  is

isomorphic to the group  $\Gamma$ , as well as what results from the definitions in no. **3**.

The group  $\Gamma^*$  is holohedrally isomorphic to  $\Gamma$ , but insofar as the group of transformations operates in  $E^*_{\mu+2n}$ , it will depend upon the variance of **F**. If one considers two systems of partial differential equations that are defined in the same space  $E_{\mu+n}$ , but for which the tensors **F** do not have the same variance, then the corresponding groups  $\Gamma^*$  will be isomorphic, but not similar.

We just showed that *Bateman's variational problem is associated with the system of partial differential equations* (4.1) *in an invariant manner.* 

The first relation (7.5) shows that the  $G_i$  are the components of a covariant vector of weight one when they are calculated *along an integral manifold of* (4.1), and that will be true regardless of the nature of **F**.

#### 8. Examples.

a) Recall the system that is governed by equation (5.4) and take the group  $\Gamma$  to be the group of homotheties  $\overline{y} = m \cdot y$ , along the *y*-axis, for which the independent variable *x* is invariant ( $\overline{x} = x$ ). In addition, suppose that the left-hand side *F* transforms into  $\overline{F} = m \cdot F$  (viz., a contravariant vector with one component).

Bateman's method leads us to study the system:

(8.1) 
$$\begin{cases} G \equiv a \ddot{z} - b \dot{z} + c z = 0, \\ F \equiv a \ddot{y} + b \dot{y} + c y = 0 \end{cases}$$

in relation to the group  $\Gamma^*$ :

$$\overline{y} = m y$$
,  $\overline{z} = \frac{1}{m} z$ .

<sup>(1)</sup> One sees that in order to verify that property, it is essential to put the left-hand sides of the system (5.2) into the order  $G_1, ..., G_n, F_1, ..., F_n$ . It will break down if one puts the *F* before the *G* since the dependent variables are always arranged in the order  $y^1, ..., y^n, z^1, ..., z^n$ .

It preserves the metric  $dS^2 = dz \cdot dy$  in the *yz*-plane, such that the *x* and *y* axes play the role of isotropic lines.

That leads us to perform the transformation:

$$y = Y + i Z$$
,  $z = Y - i Z$   $(i = \sqrt{-1})$ ,

which does not belong to  $\Gamma^*$ , but which will give the usual Euclidian form to  $dS^2$ :

(8.2) 
$$dS^2 = dz \cdot dy = dY^2 + dZ^2.$$

When equations (8.1) are considered to be equations that annul the components of a covariant vector in the *yz*-plane, they will become:

$$\begin{split} \mathbf{G} &\equiv \frac{\partial y}{\partial Y} G + \frac{\partial z}{\partial Y} F \equiv 2(a\ddot{Y} + i\mathbf{1}\cdot\dot{Z} + cY) = 0 , \\ \mathbf{F} &\equiv \frac{\partial y}{\partial Z} G + \frac{\partial z}{\partial Z} F \equiv 2(a\ddot{Z} - i\mathbf{1}\cdot\dot{Y} + cZ) = 0 . \end{split}$$

We can interpret them as defining the motion of a point of mass 2a in a Euclidian plane that is referred to rectangular axes. That point is subject to a position-dependent force that is derived from the potential  $V = c(Y^2 + Z^2)$  and has a motion-dependent forces with components  $-2bi1\dot{Z}$ ,  $2bi1\dot{Y}$ .

The latter is perpendicular to the velocity and therefore does no work. One can consider it to be something that produces a magnetic field of intensity *i* that is perpendicular to the *yz*-plane upon which the point that carries an electric charge 2b moves. Or rather, as a Coriolis force, such that the *Y*, *Z* axes are supposed to be animated with a rotation around the origin whose angular velocity:

$$\omega = \frac{bi}{2a}.$$

a') If one assumes that F transforms as a covariant vector with one component, i.e.,  $\overline{F} = \frac{1}{m}F$ , then equations (8.1) must be studied in relation to the group:

$$\overline{y} = m y, \qquad \overline{z} = m z,$$

which does not preserve the metric (8.2).

b) Return to equations (4.1) and suppose that:

1. The left-hand sides  $F_a$  are homogeneous linear expression in the  $y^i$  and their derivatives with coefficients that are functions of  $x^{\beta}$  (<sup>1</sup>).

2. The group  $\Gamma$  has the form:

$$\overline{x}^{\alpha} = \overline{x}^{\alpha}(x^{\beta}), \qquad \overline{y}^{i} = A^{i}_{i} y^{j},$$

in which the  $A_i^i$  can depend upon  $x^{\beta}$ .

3. The  $M_a^b$  that enter into (7.2) are independent of the  $y^i$ . (In other words, the property 1 is invariant under  $\Gamma$ .)

Due to 1, there is no reason to distinguish between the  $F_a$  and their system of variations. The  $G_i$  then constitute the adjoint system to the  $F_a$ . Formulas (7.5) will then reduce to:

$$\overline{G}_i = \frac{\partial(x)}{\partial(\overline{x})} A_i^j G_j, \qquad \overline{F}_a = M_a^b F_b.$$

One sees that the  $G_i$  possess the character of a covariant vector of weight one with respect to the group in question regardless of the nature of **F** (<sup>2</sup>).

c) Now suppose that the tensor **F** is a contravariant vector of weight p. We represent its components by  $F^i$ , so the lower index a has been replaced with the upper index i (see no. 3, remark). In a general fashion, for the sake of clarity in the formulas that follow, there is reason to replace every lower (upper, resp.) index  $a, b, \ldots$  with an upper (lower, resp.) index  $i, j, \ldots$ 

Therefore, the  $z^a$  will become  $z_i$  (i = a), and the  $M_a^b$  will become  $\left[\frac{\partial(x)}{\partial(\overline{x})}\right]^p \frac{\partial \overline{y}^i}{\partial y^j}$  (a = i, b = j).

The transformation (7.4) is written:

$$\overline{x}^{\alpha} = \overline{x}^{\alpha} (x^{\beta}),$$
  

$$\overline{y}^{i} = \overline{y}^{i} (x^{\beta}, y^{j}),$$
  

$$\overline{z}_{i} = \left[\frac{\partial(x)}{\partial(\overline{x})}\right]^{1-p} \frac{\partial y^{j}}{\partial \overline{y}^{i}} z_{j},$$

<sup>(&</sup>lt;sup>1</sup>) In this case, the sufficiency of the self-adjointness for the existence of a solution L to the inverse problem (a) (no. 4) is easily established with the aid of the foregoing. It suffices to take  $L = \frac{1}{2}[\mathbf{L}]$ , in which the brackets indicate that one has set z = (?) in the function L that was defined in (5.1).

 $<sup>\</sup>binom{2}{i}$  In order to satisfy that property, it is not necessary to calculate the  $G_i$  along a integral manifold in the general case.

so it will follow that the linear differential form:

$$\omega = z_i dy^i$$

is a scalar of weight 1 - p.

*c'*) Let us specialize the preceding hypothesis by taking p = 1. In addition, we will assume that we shall confine ourselves to linear transformations of the *y* with coefficients that might depend upon the *x*. The transformations  $T^*$  will then be written:

$$\begin{cases} \overline{x}^{\alpha} = \overline{x}^{\alpha} (x^{\beta}), \\ \overline{y}^{i} = A^{i}_{j} y^{j}, \\ \overline{z}_{i} = \overline{A}^{j}_{i} z_{j}. \end{cases}$$

They leave the quadratic form  $dz_i \cdot dy^i$  invariant. We can then define a metric in the manifolds  $x^{\alpha} = \text{const.}$  in the space  $E^*_{\mu+2n}$  (which are manifolds that we represent by  $W^*_{2n}$ ) that is invariant under the group  $\Gamma^*$  by setting:

(8.3) 
$$dS^2 = dz_1 dy^1 + \dots + dz_n dy^n.$$

In that metric, the manifolds  $y^i = \text{const.}(z_i = \text{const.})$  of  $W_{2n}^*$  are isotropic *n*-planes.

*c*") Along with the hypotheses (*c*), (*c*'), suppose that the manifolds  $W_n$  with equations  $x^{\alpha} =$  const. in the space  $E_{\mu+n}$  are Euclidian spaces that are endowed with the metric:

(8.4) 
$$ds^{2} = (dy^{1})^{2} + \dots + (dy^{n})^{2},$$

and that one confines oneself to orthogonal transformations of the y, i.e., ones for which  $A_i^i = \overline{A_i^j}$ .

In addition to the metric (8.3), the group  $\Gamma^*$  will preserve the *n*-plane in  $W_{2n}^*$  that has  $y^i = z_i$  for its equations (more generally, the *n*-planes  $y^i - z_i = \text{const.}$  or  $dy^i = dz_i$  are invariant). In that *n*-plane, knowing the  $y^i$  will suffice to fix a point, and one confirms that the distance between two neighboring points  $y^i$ ,  $y^i + dy^i$  is given by:

$$dS^2 = (dy^1)^2 + \dots + (dy^n)^2$$

**Remark.** – One might be tempted to regard the space  $E_{\mu+n}$  (the manifold  $W_n$ ) as the subspace of  $E_{\mu+2n}^*$  (the submanifolds of  $W_{2n}^*$ ) whose equations are  $z_i = 0$ . That convention would not be

appropriate in the present example. Indeed, a displacement  $dy^i$  of  $W_n$  possesses a length ds that is given by (8.4). The same displacement, when considered to be something that belongs to  $W_{2n}^*$ , will have components  $dz_i$  that are zero. It will possess a length dS that is zero, so it will be different from ds.

By contrast, we can consider  $W_n$  to be the subspace of  $W_{2n}^*$  whose equations are  $y^i = z_i$ . A displacement of  $W_n$  possesses components  $dy^i$ ,  $dz_i = dy^i$  in  $W_{2n}^*$ . The lengths ds and dS will be equal (<sup>1</sup>).

9. New interpretation. – Up to now, we have considered Bateman's equations (5.2) to be something that defines a manifold in the space  $E_{\mu+2n}^*$  of the  $x^{\alpha}$ ;  $y^i$ ,  $z^{\alpha}$  in which a group  $\Gamma^*$  of transformations of the type (7.4) operates. That interpretation makes the variables y, z play a symmetric role, even though that symmetry does not, in fact, exist. The transformation of the y is arbitrary, while that of the z is essentially linear. (Nonetheless, that asymmetry will not exist when one confines oneself to transformations that are linear in the y, as was the case in the examples (a), (a'), (b), (c'), (c'').

The following interpretation has the advantage of making that inconvenience disappear without the necessity of having to leave the space  $E_{\mu+n}$ .

Instead of regarding the  $x^{\alpha}$ ;  $y^{i}$ ,  $z^{a}$  as the coordinates of a point in an  $(\mu + 2n)$ -dimensional space, we shall regard the as the coordinates of a point in  $x^{\alpha}$ ,  $y^{i}$ , and the  $z^{a}$  as the components of an *n*-tensor **Z** that is attached to that point. From what we saw in no. 7, that interpretation will be invariant under the necessary and sufficient condition that *the variance of the tensor* **Z** *is opposite to that of* **F**, *up to weight one:* 

$$\overline{z}^{a} = \frac{\partial(x)}{\partial(\overline{x})} \overline{M}^{a}_{b} z^{b}.$$

For example, if  $\mathbf{F}$  is a covariant (contravariant, resp.) vector of weight one then  $\mathbf{Z}$  will be a contravariant (covariant, resp.) vector of weight zero.

Any solution of (5.2) defines a set of point of  $E_{\mu+n}$  that define a manifold  $V_{\mu}$  that is an integral of (4.1) with a tensor **Z** that is attached to each of its points. If we agree to call an entity that is composed of a point *P* in  $E_{\mu+n}$  and a tensor **Z** with its origin at *P* an *element* then we will say that a solution of the system (5.2) constitutes a  $\mu$ -dimensional *manifold of elements* whose *support* is an integral manifold  $V_{\mu}$  of (4.1).

The distribution of the tensor  $\mathbb{Z}$  on  $V_{\mu}$  is not arbitrary. Any property of that distribution reflects a property of the initial equations (4.1). For example, if the  $F_a$  are the variational derivatives of a certain function then the  $G_i = 0$  will be nothing but the equations of variation of the given system (no. 4). Let:

<sup>(&</sup>lt;sup>1</sup>) The latter remark will show its utility in an article that will be entitled "Invariants intégraux et champs baroclines," which will appear in the Mémoires de l'Institut Royal Météorologique de Belgique.

$$y^i = y^i(x^{\alpha}), \quad z^i = z^i(x^{\alpha})$$

be a solution of (5.2). The equations:

$$y^i = y^i(x^{\alpha}) + \varepsilon \cdot z^i(x^{\alpha})$$
,

in which  $\varepsilon$  is an infinitesimal, represent a solution of (4.1) that is close to  $y^i = y^i(x^{\alpha})$ . In other words, when the given equations are derived from the calculus of variations in the sense of the problem (a) (no. 4), the prolonged equations will define not only integral manifolds, but also the neighboring integral manifolds.

10. *Example.* – Let a material system with *n* degrees of freedom  $q^1, ..., q^n$  be subject to the non-holonomic constraints:

(10.1) 
$$a_{\alpha i} dq^i = 0$$
  $(a = 1, 2, ..., r < n)$ 

(this Pfaff system is not completely integrable) and be characterized by a kinetic energy:

$$T = T(t; q^1, ..., q^n; \dot{q}^1, ..., \dot{q}^n)$$

as well as being acted upon by a generalized force with components:

$$K_i = K_i(t; q^1, ..., q^n; \dot{q}^1, ..., \dot{q}^n).$$

The motion is determined by Routh's n + r equations (<sup>1</sup>):

(10.2) 
$$\begin{cases} F_i \equiv \frac{d}{dt} \frac{\partial T}{\partial \dot{q}^i} - \frac{\partial T}{\partial q^i} - K_i - \lambda^{\alpha} a_{\alpha i} = 0, \\ F_{\alpha} \equiv a_{\alpha i} \dot{q}^i = 0, \end{cases}$$

whose unknowns are the  $q^i = q^i(t)$ , as well as the *r* Lagrange multipliers or auxiliary quantities  $\lambda^{\alpha} = \lambda^{\alpha}(t)$ .

In order to apply Bateman's method, we introduce n + r quantities  $p^1, ..., p^n, \mu^1, ..., \mu^n$ , and form the function:

$$\mathbf{L} \equiv F_i p^i + F_a \mu^a.$$

<sup>(1)</sup> See, for example, Paul Appel, Traité de Mécanique rationelle, Gauthier-Villars, Paris.

The  $p^i$ ,  $\mu^a$  play the role of the z in the general theory. They are the components of a tensor in the space of q,  $\lambda$  whose nature one has good reason to determine.

We note that the  $a_{\alpha i}$  are not determined exactly: Indeed, we can replace the constraints (10.1) with the equivalent relations:

$$a'_{\alpha i} dq^i = 0$$
, in which  $a'_{\alpha i} = m^\beta_\alpha a_{\beta i}$ ,  $|m^\beta_\alpha| \neq 0$ .

It will then result that the transformation of the  $\lambda$  is:

$$\lambda^{\prime \alpha} = m^{\alpha}_{\beta} \lambda^{\beta}$$

in order for us to have the invariant relation:

(10.3) 
$$\lambda^{\alpha} a_{\alpha i} = \lambda^{\prime \alpha} a_{\alpha i}^{\prime}$$

If we perform the transformation:

$$\overline{q}^i = \overline{q}^i(t;q^1,\ldots,q^n) ,$$

we must perform the transformation of the *a*:

$$\overline{a}_{\alpha i} = \frac{\partial q^{j}}{\partial \overline{q}^{i}} a_{\alpha j}$$

in order to ensure the covariance of the expressions (10.3) that enter into the  $F_i$ .

In summary, we are led to utilize the group  $\Gamma$  in the space q,  $\lambda$  that is defined by:

$$ar{q}^i = ar{q}^i(t;q^1,\ldots,q^n)\,, \ ar{\lambda}^lpha = ar{m}^lpha_eta\,\lambda^eta\,,$$

in which the  $F_i$ ,  $F_a$  have the variances:

$$\overline{F}_i = \frac{\partial q^j}{\partial \overline{q}^i} F_j, \quad \overline{F}_\alpha = m_\alpha^\beta F_\beta,$$

resp. It will then result that the tensor  $(p, \mu)$  will have the variance:

(10.4) 
$$\overline{p}^{i} = \frac{\partial \overline{q}^{i}}{\partial q^{j}} p^{j}, \quad \overline{\mu}^{\alpha} = \overline{m}^{\alpha}_{\beta} \mu^{\beta}.$$

The  $\mu$ , like the  $\lambda$ , are auxiliary unknowns, and we are interested in only the p. As the first of equations (10.4) shows, those quantities are the components of a contravariant vector in the space  $E_n$  of the  $p^1, ..., p^n$ .

In the case of a non-holonomic material system, Bateman's method defines a manifold of elements in  $E_n$  that are composed of trajectories with a contravariant vector **p** defined at each of their points.

That curve and that vector are constrained to verify the equations:

$$\frac{\delta \mathbf{L}}{\delta q^i} = \frac{\delta \mathbf{L}}{\delta \lambda^{\alpha}} = \frac{\delta \mathbf{L}}{\delta p^i} = \frac{\delta \mathbf{L}}{\delta \mu^{\alpha}} = 0.$$

The second of them are written:

$$a_{\alpha i} p^{i} = 0$$

explicitly, which shows that the vector **p** is tangent to the constraint.