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ON THE CIRCULATION THEOREM OF V. BJERKNES AND THE THEORY OF INTEGRAL INVARIANTS

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Introduction and summary

One knows the importance that the problems in the calculus of variations have taken on in mathematical physics since **Hamilton** showed that the motion of a holonomic and conservative material system is governed by a principle that is analogous to **Fermat**'s principle for light.

That fundamental principle and the consequences that result from it have not ceased to dominate research in mechanics for over more than a century. If one would like to extend those properties and methods to other physical problems then one would necessarily be led to demand to know whether one can or cannot deduce their equations from a variational principle.

The question comes down to the following mathematical problem that is called *the inverse* problem in the calculus of variations:

If one is given a system of differential equations in ordinary or partial derivatives then what are the conditions under which that system can be deduced from a problem in the calculus of variations?

However, in that form, the problem is too vague and demands to be made more precise. To fix ideas, imagine a system of *n* second-order differential equations in *n* unknown functions q^i of the independent variable *t*:

(a)
$$F_i(t,q^j,\dot{q}^j,\ddot{q}^j) = 0$$

$$\left(q^{j} = q^{j}(t), \dot{q}^{j} = \frac{dq^{j}}{dt}, \ddot{q}^{j} = \frac{d^{2}q^{j}}{dt^{2}}\right) \qquad (i, j = 1, 2, ..., n)$$

One can pose the following question:

A. – Does there exist a "Lagrangian function" $L = L(t,q^{j},\dot{q}^{j})$ whose variational derivatives $\delta L/\delta q^{i}$ are **identical** $F_{i}(^{1})$:

$$\frac{\delta L}{\delta q^{i}} = \frac{\partial L}{\partial q^{i}} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}^{i}} = F_{i}?$$

One can prove that the answer to that question is in the affirmative on the necessary and sufficient condition that the system of variations of F_i must be self-adjoint (pp. 24), which is a condition that is not generally satisfied. For example, take a holonomic material system with n degrees of freedom $q^1, ..., q^n$ that is characterized by a kinetic energy $T = T(t, q^j, \dot{q}^j)$ and is subject to forces $K_i = K_i(t, q^j)$. The equations of motion will then have the form:

(b)
$$F_i = \frac{\partial T}{\partial q^i} - \frac{d}{dt} \frac{\partial T}{\partial \dot{q}^i} + K_i = 0$$

and the self-adjointness condition here amounts to the existence of a function $V = V(t, q^{j})$ (potential energy) such that:

$$K_i = - \frac{\partial V}{\partial q^i} \; .$$

In other words, the system considered must be conservative. The function L is then given by:

$$L=T-V.$$

One can further pose the question:

B. – Does there exist a "Lagrangian function" $L = L(t, q^j, \dot{q}^j)$ such that the equations $\delta L / \delta q^i = 0$ are equivalent to the proposed equations?

That question obviously admits an affirmative response in many cases that are broader in scope than the original one, but it is much more difficult, and it has been solved in only the particular cases where n = 1 and n = 2. We are then confined to the problems that relate to question A.

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The non-existence of a function L that answers question A in the general case of a holonomic material system has immediate repercussions in the mechanics of inviscid fluids. The set of trajectories of the fluid particles can indeed be considered to be a triply-infinite family of trajectories of a holonomic material system with three degrees of freedom q^1 , q^2 , q^3 that is characterized by:

$$T = \frac{1}{2} [(\dot{x}^{1})^{2} + (\dot{x}^{2})^{2} + (\dot{x}^{3})^{2}], \qquad K_{i} = -\frac{\partial \varphi}{\partial x^{i}} - v \frac{\partial p}{\partial x^{i}} \qquad (i = 1, 2, 3).$$

 $\varphi = \varphi(x^i, t)$, $v = v(x^i, t)$, $p = p(x^i, t)$ represent the field of force per unit mass (one then supposes that this force is derived from a potential), the specific volume, and the pressure, respectively. The K_i are or are not derived from a potential $V = V(x^i, t)$ according to whether there does or does not exist a relationship between the specific volume and pressure, respectively; in other words, *according to whether the fluid motion is barotropic or baroclinic, resp.*

In the former case, the trajectories constitute a field of extremals of the integrals:

$$\int L dt, \qquad L = T - \varphi - U, \qquad U = \int v dp,$$

but that is no longer true in the latter case. From the mathematical standpoint, that is the essential difference between barotropic fluids and baroclinic fluids, or to use an expression of **V. Bjerknes**, between *classical hydrodynamics* and *physical hydrodynamics*.

In barotropic fluids, the theorem of the conservation of circulation and **Helmholtz**'s theorem (which says that vortex lines displace like fluid lines) are consequences of the existence of a variational principle. The first one results from the properties of **H. Poincaré**'s relative linear integral invariant. The second one comes from some properties of the **Hargreaves-Cartan** relative linear integral invariant.

Those theorems will cease to be true for baroclinic fluids. They have nonetheless been generalized by **V. Bjerknes** into a theorem that gives the variation of circulation per unit time and a theorem by **W. Thomson** that gives a necessary and sufficient condition for a vortex line to displace like a fluid line.

The goal of the present work consists of showing how those two propositions can be attached to the theory of integral invariants despite the non-existence of the function L that is a solution to the problem A. To that end, we shall utilize a method that was pointed out by **H. Bateman** in 1931.

* *

Bateman's method is based upon the fact that any system of equations (a) can be prolonged *uniquely* by the adjunction of *n* differential equations:

$$G_i(t;q^j,s^j;\dot{q}^j,\dot{s}^j;\cdots)=0$$

4

that include *n* new unknowns $s^{i}(t)$ in such a way that there will exist a function $\mathcal{L} = \mathcal{L}(t;q^{j},s^{j};\dot{q}^{j},\dot{s}^{j};\ddot{q}^{j},\ddot{s}^{j})$ (²):

$$\frac{\delta \mathcal{L}}{\delta q^i} \equiv 0, \qquad \frac{\delta \mathcal{L}}{\delta s^i} \equiv F_i \,.$$

As one sees, that method amounts to considering the problem that is governed by equations (a) in the *n*-dimensional space of $q^1, ..., q^n$ to be the projection onto that space onto a problem in the 2*n*-dimensional space $q^1, ..., q^n, s^1, ..., s^n$, which is a problem for which question A will admit an affirmative answer.

In the application of that method to baroclinic fluids, there is good reason to consider the threedimensional fluid motion to be the projection of a fictitious fluid motion in six-dimensional space. All of the properties that result from the existence of the function \mathcal{L} will be true for that fictitious motion. In particular, that motion possesses a relative linear integral invariant, and we will show that in three-dimensional space the properties of that invariant will translate into the circulation theorem of **V. Bjerknes** and **W. Thomson**'s theorem on vorticity (*).

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This treatise is divided into two chapters.

The first one reviews the properties of the calculus of variations and integral invariants that are involved with the theorem of the conservation of circulation and **Helmholtz**'s theorem (as well as a particular case of **Lagrange**'s theorem). The style of presentation is inspired by the **E. Cartan**'s *Leçons sur les invariants intégraux*, as well as the work of **Th. Lepage** on multiple integrals (³).

The second chapter contains a presentation of **Bateman**'s method in which we have drawn attention to the invariant properties and to the link between the geometric properties of the space of (q) and those of the prolonged space (q, s) (⁴). We will then have the common thread that links the general theorems of the calculus of variations to physical hydrodynamics.

* *

The results that are presented here have been extracted from a doctoral thesis (**) that was prepared under the direction of professor **F. van Dungen**. It is a pleasant task for me to acknowledge all of the advice and encouragement that he never ceased to provide me with.

^(*) **H. Ertel** (³⁰) has tried to attach the theorem of **V. Bjerknes** to canonical equations. He did not utilize complementary equations but appealed to "pre-canonical equations." By contrast, the method of **Bateman** will give us the true canonical equations.

^(**) Defended at l'Université libre du Bruxelles in June 1948. The editing of it has been revised completely.

I would also like to thank some other people who have contributed to this work by their teaching or advice, and in particular, J. Géhéniau, Th. Lepage, P. Libois, and J. van Mieghem.

	Page
Introduction and summary	1
CHAPTER II. – Review of some theorems from the calculus of variations and the theory of integral invariants	7
§ 1. A first problem in the calculus of variations and the exterior differential calculus	7
§ 2. Integral invariants	10
§ 3. Variational problems of the type $L(t, q^i, \dot{q}^i)$	13
§ 4. Extremal fields	15
CHAPTER II. – Bateman's method and its application to hydrodynamics	21
§ 1. Preliminaries	21
§ 2. The inverse problem in the calculus of variations	23
§ 3. Bateman's method	25
§ 4. The theorems of V. Bjerknes and W. Thomson	29
APPENDIX. – On the variational method in the hydrodynamics of viscous fluids	39
NOTES	43

CHAPTER I

Review of some theorems from the calculus of variations and the theory of integral invariants.

1. – A first problem in the calculus of variations and the exterior differential calculus.

1. – Consider a linear differential form, or *Pfaff form*, in the *N*-dimensional space \mathcal{E}_N of variables x^1, \ldots, x^N (⁵):

(1.1)
$$\omega = a_1 dx^1 + \dots + a_N dx^N = a_V dx^V \qquad (n = 1, 2, \dots, N)$$

in which the coefficients $a_1, ..., a_N$ are functions of $x^1, ..., x^N$. (⁶).

The curvilinear integral:

(1.2) [sic]
$$I = \int_{C} \omega$$

is defined for any arc of the curve C: $x^r = x^r(t)$ $(t_1 \le t \le t_2)$ in \mathcal{E}_N that joins the point $P_1(t_1)$ to the point $P_2(t_2)$.

The simplest problem in the *calculus of variations* consists of looking for extremals of the integral (1.1).

In order to do that, one considers a family of curves C that depend upon a parameter u and join the extremities P_1 and P_2 :

(1.2)
$$\begin{aligned} x' &= x'(t,u) & (t_1 \le t \le t_2), \\ x^r(t_1,u) & \text{and} & x^r(t_2,u) & \text{are independent of } u \end{aligned}$$

The integral I will become a function of u whose derivative with respect to that variable will have the expression:

$$\left(\frac{\partial I}{\partial u}\right)_0 = \int_C \left(\frac{\partial a_s}{\partial x^r} - \frac{\partial a_r}{\partial x^s}\right) \frac{\partial x^r}{\partial u} dx^s$$

when $u = u_0$ (⁷). The integral on the right-hand side must be taken along the curve C_0 that corresponds to the value $u = u_0$. By definition, that curve will be an *extremal* if that derivative is zero for any family $x^r = x^r(t,u)$ that satisfies the boundary conditions (1.2) and gives C_0 for $u = u_0$. In order for that situation to occur, it is necessary and sufficient that the following relations should exist between the coordinates x^r of a point on the curve and (homogeneous) components dx^r of a tangent vector:

(1.3)
$$\left(\frac{\partial a_s}{\partial x^r} - \frac{\partial a_r}{\partial x^s}\right) dx^s = 0$$

That system of Pfaff equations is attached to the form ω intrinsically. That system plays an essential role in the study of the form, and its integration amounts to solving the variational problem that was posed (⁸).

2. – Consider an arbitrary surface in \mathcal{E}_N that is referred to curvilinear coordinates u^1 , u^2 :

$$x^r = x^r(u^1, u^2) ,$$

and let d_1 and d_2 denote two differentiation symbols that refer to u^1 and u^2 , respectively. In regard to the permutability of the operations d_1 and d_2 [symbolically, $d_1 d_2 f - d_2 d_1 f = \frac{\partial^2 f}{\partial u^1 \partial u^2} du^1 du^2$ for

any function $f(x^r)$ (⁶)], one has:

$$d_1 \ \omega \ (d_2) - d_2 \ \omega \ (d_1) = \frac{1}{2} \left(\frac{\partial a_s}{\partial x^r} - \frac{\partial a_r}{\partial x^s} \right) \cdot \begin{vmatrix} d_1 x^r & d_1 x^s \\ d_2 x^r & d_2 x^s \end{vmatrix}$$

.

or one can set:

$$[d \ \omega] = \frac{1}{2} \left(\frac{\partial a_s}{\partial x^r} - \frac{\partial a_r}{\partial x^s} \right) [dx^r \ dx^s] ,$$

to abbreviate.

The latter notation is symbolic. It will immediately lead to a very important method of differential and integral calculus that was introduced by **E. Cartan** and **H. Poincaré** (1899) and bears the name of the *exterior differential calculus* (⁹). The left-hand side is the symbolic product of the operator *d* and the form ω . It bears the name of the *exterior differential* of the form ω . In the right-hand side, the symbolic product $[dx^r dx^s]$ is an *exterior* product (one also says *alternating*), which verifies the **Grassmann** rule of multiplication:

$$[dx^r dx^s] + [dx^s dx^r] = 0,$$
$$[dx^r dx^r] = 0.$$

and in particular $(^{10})$:

That fact will be verified immediately when one reverts to the notation in the form of determinants.

In practice, when no ambiguity is possible, one suppresses the brackets in the symbolic notation and writes simply:

(2.1)
$$d\omega = \frac{1}{2} \left(\frac{\partial a_s}{\partial x^r} - \frac{\partial a_r}{\partial x^s} \right) dx^r dx^s,$$

(2.2) (¹⁰)
$$dx^r dx^s + dx^s dx^r = 0, \quad dx^r dx^r = 0.$$

The right-hand side of (2.1) is a particular case of a quadratic exterior differential form. The most-general quadratic exterior differential form of \mathcal{E}_N is written:

$$\Omega = A_{rs} \, dx^r \, dx^s \, .$$

The A_{rs} in that expression are *functions of the independent variables* $x^1, ..., x^N$ that are taken to have their meaning in mathematical analysis, while the $dx^1, ..., dx^N$ are *indeterminates* in the sense of modern algebra that are subject to the rules of calculation (2.2)

The same remark applies to the a_r and the dx^r in the Pfaff form ω or linear exterior differential form. The operator d of exterior differentiation makes the linear form ω correspond to a quadratic form $d\omega$ whose coefficients are, by definition:

$$A_{rs} = \frac{1}{2} \left(\frac{\partial a_s}{\partial x^r} - \frac{\partial a_r}{\partial x^s} \right)$$

One can define exterior differential forms of degree $\mu = 1, 2, ..., N$ more generally, but it will suffice to consider the cases $\mu = 1, 2$ here.

The exterior differential forms of degree μ are expressions that one encounters under a μ -uple integral sign (¹¹), and the operation of exterior differentiation permits one to write the *Stokes formula* in its simplest form.

Let a portion of a surface *S* be bounded by a closed contour *C*. That formula is written:

$$\int_{C} a_{r} dx^{r} = \int_{S} \frac{1}{2} \left(\frac{\partial a_{s}}{\partial x^{r}} - \frac{\partial a_{r}}{\partial x^{s}} \right) dx^{r} dx^{s} .$$

With the notations that were just introduced, the expressions under the signs can be written ω and $d\omega$, respectively. On the other hand, the contour *C* bounds the surface *S*, so it is determined completely by the surface. It is often represented by the symbol $\partial S = C$ (*). The Stokes formula will then take the remarkable form:

$$\int_{\partial S}\omega=\int_{S}d\omega$$

3. – The left-hand sides of equations (1.3) are nothing but the coefficients of the dx^r in the form $d\omega$ that was defined in (2.1). That system bears the name of the *characteristic system* of the form $d\omega$, and the rank (which is always even) 2ρ of the antisymmetric determinant:

^(*) The symbol ∂ is read "del."

$$\left|\frac{\partial a_s}{\partial x^r} - \frac{\partial a_r}{\partial x^s}\right|$$

is the *rank* (or *class*) of $d\omega$.

The system (1.3) will possess $N - 2\rho$ linearly-independent solutions for the unknowns dx^r at a point $P(x^r)$. Those solutions will define an $N - 2\rho$ -dimensional planar element at each point.

One proves $(^{12})$ that the characteristic system (1.3) is *completely integrable*. That means that one and only one $N - 2\rho$ -dimensional manifold passes through each point *P* that is tangent to the corresponding planar element at that point. Those manifolds are called the *characteristic manifolds* of the form $d\omega$.

It is obvious that any curve that satisfies the equations (1.3) is situated entirely in a characteristic manifold. Conversely, if a curve is situated entirely in a characteristic manifold then it will satisfy those equations. We can then conclude that:

A necessary and sufficient condition for a curve C to be an extremal of the integral (1.1) is that it must belong to a characteristic manifold of the exterior differential d ω of the Pfaff form ω (¹³).

One can prove that there always exist either 2ρ functions $X^{\alpha}(x^{r})$, $Y_{\alpha}(x^{r})$ ($a = 1, 2, ..., \rho$) or $2\rho + 1$ functions $X^{\alpha}(x^{r})$, $Y_{\alpha}(x^{r})$, $Z(x^{r})$ that are *mutually independent* and such that one will have:

$$\omega = Y_a dX^a$$
 or $\omega = Y_a dX^a + dZ$ $(a = 1, 2, ..., \rho)$

In both cases, one will have:

$$d\omega = dY_a dX^a$$
,

and the characteristic system will be written:

$$dX^a = 0 , \qquad dY_a = 0 .$$

The characteristic manifolds are the $N - 2\rho$ -dimensional manifolds that are defined by $X^a = \text{const.}, Y_a = \text{const.} ({}^{14}\text{)}.$

4. – Recall the Pfaff form:

$$\omega = a_r dx^r$$

and consider a system of differential equations:

(4.1)
$$\frac{dx^{1}}{X^{1}} = \frac{dx^{2}}{X^{2}} = \dots = \frac{dx^{N}}{X^{N}},$$

in which the $X^1, X^2, ..., X^N$ are functions of the $x^1, x^2, ..., x^N$. One knows that one and only one integral curve or *trajectory* of that system passes through each point of \mathcal{E}_N .

The set of trajectories that pass through the points of a curve C that joins the points P_1 and P_2 (Fig. 1) constitutes a surface Σ . Displace the points of the curve C in an arbitrary, continuous manner along their P'

trajectories up to a position C' with extremities P'_1 and P'_2 .

In general, one will have:

$$I_1 = \int_C \omega \neq I'_1 = \int_{C'} \omega$$

When the \neq sign can be replaced with the = sign, no matter what the initial curve *C* and final position, one says that *I*₁ is a *linear integral invariant* of the equations (4.1). When one has:

$$J_1 = \int_C \omega = J'_1 = \int_{C'} \omega,$$





no matter what the initial closed curve C and final closed

curve C' that is obtained by an arbitrary, continuous displacement of C along the tube T then one says that J_1 is a relative linear integral invariant (4.1). By contrast, an invariant of type I_1 is called *absolute*.



5. – Now consider a portion of the surface S that is bounded by the closed curve C (Fig. 1). Deform that surface into a position S' such that each point is displaced along the corresponding trajectory of the system (4.1). The surface S' is bounded by a closed curve that is situated along the tube T of the trajectories that pass through the points of C.

If one is given a double integral:

$$I_2 = \int A_{rs} \, dx^r \, dx^s$$
$$[A_{rs} = A_{rs}(x^1, \dots, x^N), r, s = 1, 2, \dots, N]$$

then one will have, in general:

$$I_{2} = \int_{S} A_{rs} dx^{r} dx^{s} \neq I_{2}' = \int_{S'} A_{rs} dx^{r} dx^{s}$$

When the \neq sign can be replaced with the = sign for any portion of the initial surface S and the portion of the final surface S', one says that I_2 is a *double integral invariant* of equations (4.1).

That invariant is said to be *absolute*. When the = sign can be employed only when the surfaces S and S' are closed, the invariant is said to be *relative*.

Take the particular case:

(5.1)
$$A_{rs} = \frac{1}{2} \left(\frac{\partial a_s}{\partial x^r} - \frac{\partial a_r}{\partial x^s} \right)$$

The **Stokes** formula will give us:

$$J_1 = \int_C \omega = \int_{\partial S} \omega = \int_S d\omega = I_2 .$$

For the A_{rs} that are given by (5.1), the equalities $I_2 = I'_2$ and $J_1 = J'_1$ are then equivalent.

Any relative linear integral invariant corresponds to an absolute double integral invariant.

6. – When one is given the Pfaff form $\omega a \text{ priori}$, one can look for all differential systems (4.1) that admit the relative linear and absolute double invariants:

(6.1)
$$J_1 = \int_C \omega = \int_{\partial S} \omega, \qquad I_2 = \int_S d\omega.$$

That problem can be called the *"integral invariant problem"* relative to the Pfaff form ω .

One proves that the systems (4.1) that answer the question are subject to the necessary and sufficient condition $(^{15})$:

(6.2)
$$\left(\frac{\partial a_s}{\partial x^r} - \frac{\partial a_r}{\partial x^s}\right) X^s = 0.$$

That amounts to saying that equations (1.3) must be consequences of (4.1). Therefore, in regard to the results of the first section, one can say:

A necessary and sufficient condition for a system of differential equations (4.1) to admit the integral invariants J_1 and I_2 (6.1) is that the trajectories of that system must be situated entirely in the characteristic manifolds of the exterior differential form $d\omega$ of the Pfaff form ω .

Upon comparing that result to the one in no. 3, we can conclude that:

The problem in the calculus of variations that was posed in no. 1 relative to the form ω and the integral invariant problem that relates to that same form are entirely equivalent.

Those problems amount to the integration of the system (1.3) (¹⁶).

§ 3. – Variational problems of the type $L(t, q^i, \dot{q}^i)$.

7. – In the variational problem of the type that is described by Hamilton's principle, one is given a function $L(t, q^i, \dot{q}^i)$ (i = 1, 2, ..., n) that is defined in the 2n + 1-dimensional space \mathcal{E}_{2n+1} of the t, q^i, \dot{q}^i and one looks for the curves $q^i = q^i(t)$ (which are called *extremals*) in the n + 1-dimensional space of \mathcal{E}_{n+1} of t, q^i that annul the first variation of the integral (¹⁷):

(7.1)
$$\int L\left(t,q^{i},\frac{dq^{i}}{dt}\right)dt.$$

E. Cartan ameliorated the result of **A.** C. Dixon (¹⁸) by proving a remarkable result that said that the problem that one poses in \mathcal{E}_{n+1} can be reduced to a problem of the type that was considered in no. 1 in the space \mathcal{E}_{2n+1} (2n + 1 = N) for the Pfaff form:

(7.2)
$$\omega = L dt + \frac{\partial L}{\partial \dot{q}^i} (dq^i - \dot{q}^i dt) \; .$$

Meanwhile, one can suppose that a certain determinant is non-zero:

(7.3)
$$\left| \frac{\partial^2 L}{\partial \dot{q}^i \, \partial \dot{q}^k} \right| \neq 0 ,$$

i.e., that the problem L is a regular problem in the calculus of variations $(^{19})$.

Let us specify the proposition that gives the key to some properties of the integral invariants of mechanics.

Any curve $q^i = q^i(t)$ in \mathcal{E}_{n+1} (viz., space-time) possesses a curve in \mathcal{E}_{2n+1} (viz., the space of contact elements) that is the image of the equations $q^i = q^i(t)$, $\dot{q}^i = dq^i/dt$ that verifies the *n* Pfaff equations:

$$dq^i - \dot{q}^i dt = 0.$$

Conversely, a line $q^i = q^i(t)$, $\dot{q}^i = dq^i / dt$ in the space \mathcal{E}_{2n+1} is the image of a curve in \mathcal{E}_{n+1} on the condition that it must satisfy equations (7.4).

The Dixon-Cartan theorem says that:

The extremals of the problem (7.1) that was posed in \mathcal{E}_{n+1} have images in \mathcal{E}_{2n+1} that are extremals of the variational problem that relates to the form ω (7.2), and conversely.

The proof of that involves writing the characteristic system of the exterior differential form $d\omega$. One confirms that $d\omega$ has rank 2n and that by means of (7.3), that system is composed of equations (7.4), completed by the Euler-Lagrange equations (²⁰):

(7.5)
$$\frac{\partial L}{\partial q^i} dt - d \frac{\partial L}{\partial \dot{q}^i} = 0 .$$

The calculations can be performed by utilizing the *n canonical variables*, or *momentoids* that are defined by:

$$p_i = \frac{\partial L}{\partial \dot{q}^i} = p_i(t, q^j, \dot{q}^j)$$

in place of the \dot{q}^i , so by virtue of (7.3):

(7.6)
$$\dot{q}^i = \dot{q}^i (t, q^j, p_j)$$

The Pfaff form (7.2) will then become:

(7.7)
$$\omega = p_i dq^i - H dt,$$

in which *H* is the *Hamiltonian function*:

$$H = H(t, q^j, p_j) = p_i \dot{q}^i - L.$$

One has:

$$d\omega = dp_i dq^i - \left(\frac{\partial H}{\partial q^i} dq^i + \frac{\partial H}{\partial p_i} dp_i\right) dt$$
.

Therefore, the characteristic system is obtained by annulling the coefficients of the dq^i , dp_i , dt:

$$dp_i + rac{\partial H}{\partial q^i} dt = 0 , \qquad - dq^i + rac{\partial H}{\partial p_i} dt = 0 , \qquad - rac{\partial H}{\partial q^i} dq^i - rac{\partial H}{\partial p_i} dp_i = 0 .$$

One then recovers **Hamilton**'s 2n canonical equations [which are equivalent to the 2n equations (7.4) and (7.5), as one knows], plus one last equation that is nothing but the *energy* equation. It is a consequence of the first 2n equations, and it can be written:

$$dH - \frac{\partial H}{\partial t} dt = 0$$

8. – By virtue of what we saw in no. **2**, the extremals of (7.1), or what amounts to the same thing, the integral of the canonical equations can also be characterized by the property that they admit the relative linear integral invariant:

$$\int_{\partial S} \omega = \int_{\partial S} L dt + \frac{\partial L}{\partial \dot{q}^{i}} (dq^{i} - \dot{q}^{i} dt) = \int_{\partial S} p_{i} dq^{i} - H dt.$$

In the study of conservative holonomic material systems, Hamilton's principle is nothing but the variational principle (7.1) for which L = T - V and V are the kinetic and potential energies, respectively. Time t plays a privileged role in the statement of that principle.

The results that we have recalled permit us to replace *Hamilton's principle* with two equivalent principles that enjoy the remarkable property that the time *t* no longer plays a special role: *The variational principle that relates to the form* ω (7.2) and the "*integral invariant principle*" that relates to that same form. The former is what **E. Cartan** called *the principle of the conservation of the momentum and energy* (²¹) because the coefficients p_i , – *H* of the form (7.6) are nothing but the components of the momentum and energy. When the form ω is considered to be an invariant, those n + 1 quantities are components of a covariant vector with respect to the group of general coordinates transformations of spacetime \mathcal{E}_{n+1} :

$$q^{n} = q^{n}(q^{i},t), \qquad t' = t'(q^{i},t) .$$

§ 4. – *Fields of extremals*.

9. – There are ∞^{2n} extremals of the problem (7.1). One and only one of them passes through each point of \mathcal{E}_{2n+1} , and ∞^n of them pass through each point of \mathcal{E}_{n+1} .

One says *field of extremals* to mean a family of extremals such that one and only one of them passes through each point of \mathcal{E}_{n+1} . When such a field is given, it will be possible to calculate the coefficients that are coordinates of the extremal that corresponds to each point in \mathcal{E}_{n+1} :

$$\frac{dq^i}{dt} = Q^i(q^j,t) \,.$$

Conversely, being given the functions Q^i will permit one to recover the field by integrating the preceding system.

When one passes to canonical variables, the field must be characterized by the functions:

$$P_i = P_i(q^j, t) = \left\{\frac{\partial L}{\partial \dot{q}^i}\right\},$$

in which the curly brackets signify that one has replaced \dot{q}^i with $Q^i(q^j,t)$.

If one chooses *n* functions $Q^i(q^j,t)$ or $P_i(q^j,t)$ arbitrarily then they will not define a field of extremals.

In order for *n* functions $P_i(q^j, t)$ to define a field, it is necessary and sufficient that the integrals $q^i = q^i(t)$ of the system:

(9.1)
$$\frac{dq^{i}}{dt} = \left[\frac{\partial H}{\partial p_{i}}\right]$$

- in which the brackets signify that one has replaced the p_i with $P_i(q^j,t)$ - when completed by the system:

$$p_i = P_i \left[q^j(t), t \right] \,,$$

should be solutions to the canonical equations:

$$\frac{dq^i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q^i}.$$

That leads to the condition:

That leads to the condition:
(9.2)
$$\frac{\partial P_i}{\partial q^i} \left[\frac{\partial H}{\partial p_i} \right] + \frac{\partial P_i}{\partial t} + \left[\frac{\partial H}{\partial q^i} \right] = 0.$$

A field of extremals is therefore characterized by n momentoid functions $P_i = P_i(q^j, t)$ that verify the n partial differential equations (9.2). The trajectories of the field are the integrals of the system (9.1).

10. – If we are given a field of extremals C then we can consider it to be something that is defined by the trajectories of a fluid motion in spacetime \mathcal{E}_{n+1} .

It is clear that this fluid motion admits the relative linear integral invariant:

(10.1)
$$\int_{\partial S} [\omega] = \int_{\partial S} P_i \, dq^i - [H] \, dt$$

and the absolute double invariant:

$$\int_{S} d[\omega] = \int_{S} dP_{i} dq^{i} - d[H] dt$$

Similarly, the trajectories annul the first variation of integral:

$$\int [\omega]$$
.

When one sets dt = 0 in (10.1), one restricts one's considerations to only those curves $(\partial S)_0$ that are situated in the hyperplane t = const., and one will find nothing other than the *theorem of the conservation of circulation*:

$$\frac{d}{dt}\int_{(\partial S)_0} [\omega]_0 = \frac{d}{dt}\int_{(\partial S)_0} P_i dq^i = 0,$$

which corresponds to H. Poincaré's relative linear integral invariant.

However, whereas one has only one system of differential equations (namely, the canonical system) that enjoy those properties of invariance and being an extremum *vis* \dot{a} *vis* the Pfaff form ω that is defined by (7.2) or (7.7) in \mathcal{E}_{2n+1} , the same thing is not true for the form $[\omega]$ that is defined in \mathcal{E}_{n+1} . Indeed, the systems in question are the ones whose trajectories are found in the characteristic manifolds of the form $d[\omega]$.

If one takes (9.2) into account then one will have:

$$d\left[\omega\right] = \left(\frac{\partial P_j}{\partial q^i} - \frac{\partial P_i}{\partial q^j}\right) dq^i \, \varpi^j, \qquad \varpi^j = dq^j - \left[\frac{\partial H}{\partial p_j}\right] dt.$$

The characteristic system is obtained by annihilating the coefficients of dq^i and dt and will reduce to (²²):

(10.2)
$$\left(\frac{\partial P_j}{\partial q^i} - \frac{\partial P_i}{\partial q^j}\right) \boldsymbol{\sigma}^j = 0.$$

It obviously admits the solution $\pi^{j} = 0$ that defines the trajectories of the field in question C.

When the rank of the system (10.2) is equal to *n* (which can happen only when *n* is even), the solution $\varpi^{j} = 0$ will be the only solution. By contrast, there exist other solutions whenever the tank 2*p* of (10.2) is less than *n* (which will necessarily be the case when *n* is odd).

Example. – (E. Cartan, *loc. cit.*, 1922). Recall the example of hydrodynamics and suppose that the fluid is barotropic (see the introduction pp. 2). For a well-defined motion, the fluid trajectories constitute a field of extremals for the problem that is defined by:

$$L = T - V - U.$$

Let $u_i(q^j, t)$ (i, j = 1, 2, 3) denote the components of the velocity that plays the role of the $P_i(^{23})$. Upon setting dt = 0 in $[\omega]$, it will become:

$$[\omega]_0 = u_i \, dx^i \, .$$

When the integral of $[\omega]_0$ is extended over a closed curve in the space \mathcal{E}_1 of the (x^i, t) whose points are all taken at the same instant (dt = 0), it will not change in value when the points of that curve displace with the fluid particles. That property constitutes the *theorem of the conservation* of circulation that imagines only curves that are taken at the same instant. Upon utilizing the form:

$$[\omega] = u_i \, dx^i + u_0 \, dt \, ,$$

in which:

$$-u_0 = [H] = \frac{1}{2}[(u_1)^2 + (u_2)^2 + (u_3)^2] + \varphi + U,$$

that property will generalize to arbitrary curves in spacetime that displace arbitrarily along a tube of fluid trajectories.

The characteristic system of $d [\omega]$ is written:

(10.3)
$$\left(\frac{\partial u_j}{\partial x^i} - \frac{\partial u_i}{\partial x^j}\right) (dx^i - u^j dt) = 0.$$

In the general case, the coefficients of that system, i.e., the components of the rotation of the velocity, are not all zero, and the rank will be equal to 2 (viz., rotational motion). The characteristic manifolds constitute a doubly-infinite family of surfaces S such that one and only one of them passes through each point in spacetime \mathcal{E}_4 . All of the systems of differential equations whose trajectories are contained entirely in those surfaces admit relative linear integral invariants and absolute double ones:

$$\int_{\partial S} [\omega] \quad \text{and} \quad \int_{S} d[\omega] ,$$

so the trajectories of those systems will be extremals of:

$$\int [\omega]$$
,

in addition.

The same thing will be true for the system:

$$dx^i - u^i dt = 0$$

that defines the fluid trajectories. Another remarkable system that enjoys those properties defines the vortex lines:

$$\frac{dx^1}{\frac{\partial u_2}{\partial x^3} - \frac{\partial u_3}{\partial x^2}} = \frac{dx^2}{\frac{\partial u_3}{\partial x^1} - \frac{\partial u_1}{\partial x^3}} = \frac{dx^3}{\frac{\partial u_1}{\partial x^2} - \frac{\partial u_2}{\partial x^1}} = \frac{dt}{0}.$$

One observes that this system is obtained by setting dt = 0 in (10.3).

Any characteristic surface S can be just as well generated by the displacement of a trajectory that intersects a fixed vortex line as by the displacement of a vortex line that intersects a given trajectory.

In the latter mode of generation, *the vortex line moves with the fluid particles and remains a vortex line*. That property constitutes **Helmholtz's theorem**.

11. – Those properties easily generalize to an arbitrary field of extremals C.

We have already indicated the generalization of the theorem of the conservation of circulation in the preceding section.

Let $2p (\leq n)$ be the rank of the system (10.2). We know that it is completely integrable, and that property will persist when we prolong the system by the completely-integrable equation dt = 0. We will then obtain the completely-integrable system:

(11.1)
$$\left(\frac{\partial P_j}{\partial q^i} - \frac{\partial P_i}{\partial q^j}\right) dq^j = 0, \qquad dt = 0.$$

The infinitesimal vectors (dq^i, dt) that have their origin at a point *P* in spacetime \mathcal{E}_{n+1} and satisfy equations (11.1) generalize the tangent vectors to a vortex line in the preceding example. One can call them *vorticity vectors*. There are ∞^{n-2p} of them, and they are distributed throughout an n - 2p-dimensional planar element that passes through the point *P*. Only and only one n - 2p-dimensional manifold passes through each point *P* (that one can call the *vorticity manifold*) that is situated in the hyperplane t = const. that passes through *P* and is tangent to the n - 2p-dimensional planar element that points.

The characteristic manifolds of the system (10.2) are n - 2p + 1-dimensional. One can just as well generate them by displacing a trajectory of the field C that intersects a fixed vorticity manifold as by displacing a vorticity manifold along a given trajectory.

In the latter mode of generation, *the vorticity manifold moves with the fluid particles and remains a vorticity manifold*. That property generalizes **Helmholtz**'s theorem.

12. – The preceding considerations supposed that the rank 2*p* of the system (10.2) is constant. Meanwhile, that rank can be lowered at certain singular points of \mathcal{E}_{n+1} .

Let *P* be one of those points. It can happen that *the rank* 2p' < 2p *is constant along the extremal* of the field *C* that passes through *P* (²⁴).

One can then speak of "singular trajectories" that are loci of singular points. The vorticity vectors that are attached to a point *P* of one such trajectory will be situated in an n - 2p'(>n - 2p)-dimensional planar element, but it will no longer be possible to define an n - 2p'-dimensional "vorticity manifold." When applied to this degenerate case, **Helmholtz**'s theorem will correspond to the following property: *When one displaces the origin and extremity of an (infinitely-small) vorticity vector like a fluid particle, that vector will constantly remain a vorticity vector* (²⁵).

The most-advanced case of degeneracy is the one in which 2p' = 0. It will occur at a point *P* where the components of the rotation of the velocity:

$$\frac{\partial P_j}{\partial q^i} - \frac{\partial P_i}{\partial q^j}$$

are annulled. At such a point, any infinitely-small vector will be a vorticity vector, and that situation will be necessarily reproduced at any point of the trajectory that passes through *P* (viz., *Lagrange's theorem*).

When 2p' = 0 at every point of an *n*-dimensional hypersurface in \mathcal{E}_{n+1} , which is a hypersurface that intersects the trajectories of the field C, one will necessarily have 2p' = 0 in all of spacetime \mathcal{E}_{n+1} . The field of extremals C will then be called a *geodesic field*. The Pfaff form $[\omega]$ reduces to the differential dS of a function $S = S(q^i, t)$ for which:

$$P_i = \frac{\partial S}{\partial q^i}, \qquad [H] = -\frac{\partial S}{\partial t}.$$

The function S is the solution to the Hamilton-Jacobi partial differential equation $(^{26})$:

$$\frac{\partial S}{\partial t} + H\left(t, q^i, \frac{\partial S}{\partial q^i}\right) = 0 \; .$$

One has $d[\omega] = 0$, and the integral:

 $\int [\omega]$

will be an *independent integral*, i.e., its value will not depend upon the path that is followed in order to join two well-defined points of \mathcal{E}_{n+1} (²⁷).

CHAPTER II

Bateman's method and its applications to hydrodynamics.

§ 1. – Preliminaries.

13. – Here, we must consider the variational problem:

(13.1)
$$\delta \int L dt = 0 ,$$

in which L depends upon not only the derivatives \dot{q}^i of q^i , but also on the second derivatives \ddot{q}^i :

(13.2)
$$L = L(t, q^i, \dot{q}^i, \ddot{q}^i).$$

Recall that in this case, the differential equations of the extremals (which are generally of fourth order) are [see **Th. De Donder** (³¹)]:

(13.3)
$$\frac{\delta L}{\delta q^i} \equiv \frac{\partial L}{\partial q^i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i} + \frac{d^2}{dt^2} \frac{\partial L}{\partial \ddot{q}^i} = 0.$$

14. – As a result, the system of equations:

(14,1)
$$F_a(t,q^i,\dot{q}^i,\ddot{q}^i) = 0 \qquad (a,i,j=1,2,...,n)$$

that we must consider will govern a certain physical or geometric problem in the space E_n of the $q^1, ..., q^n$. One can perform transformations T on the q^i that belong to a certain group Γ and have the equations (²⁸):

(14.2)
$$q'' = q''(q^1, \dots, q^n), \qquad \frac{\partial(q'')}{\partial(q^i)} \neq 0.$$

We assume that the equations (14.1) consist of annulling the components of a certain tensor **F**. That means that equations (14.1) will become:

$$F'_{a}(t,q'^{i},\dot{q}'^{i},\ddot{q}'^{i}) = 0$$

after a transformation (14.2), in which F'_a are deduced from F_a by formulas of the type:

(14.3)
$$F'_a = \tau^{\cdot b}_a F_b, \qquad |\tau^{\cdot b}_a| \neq 0,$$

In other words, any transformation *T* corresponds to a matrix $\tau = || \tau_a^{\cdot b} ||$ that provides the transformation law (14.3) for the components of **F** and whose components $\tau_a^{\cdot b}$ are functions of q^1 , ..., q^n .

In order for that law to be coherent, the correspondence $T \rightarrow \tau$ cannot be arbitrary. Indeed, suppose that after the transformation T (14.2) takes the q^i to q'^i , the transformation T' takes q'^i to q''^i :

$$q''^i = q''^i(q'^i)$$

One can pass directly from the q^i to the q'' by the transformation $T'' = T' \cdot T$:

$$q''^{i} = q''^{i}[(q'^{j}(q^{k})]]$$
.

Let $\| \tau_a^{i,b} \|$, $\| \tau_a^{',b} \|$, $\| \tau_a^{'',b} \|$ denote the matrices that are associated with the transformations *T*, *T'*, *T''*. In order for the law (14.3) to be coherent, it is necessary and sufficient that one should obtain the same components F_a'' by passing from the q^i to the q'' by the intermediary of the q'', so one must then have:

$$\tau_a^{\prime\prime\,b} = \tau_a^{\prime\,c} \cdot \tau_c^{\,b}$$
 or $\tau_a^{\prime\prime} = \tau^{\prime} \cdot \tau$.

That expresses the idea that the set γ of the matrices τ that are associated with the transformations T constitute a group and that the correspondence $T \rightarrow \tau$ is a homomorphism of Γ onto γ .

The group γ , when endowed with the homomorphism $h: T \xrightarrow{h} \tau$, defines the *variance* (one also says *nature*) of the tensor **F**.

That variance corresponds to another one that is called the *opposite* one, and is such that if **G** is a tensor with that new variance and components G^a then the product:

$$F_a G^a = F_1 G^1 + \dots + F_n G^n$$

will be an invariant. The law of transformation of the components G^a into the transformation (14.2) is:

$$G'^a = \overline{\tau}^a_{\cdot b} G^b,$$

in which $\|\bar{\tau}_{b}^{a}\|$ denotes the inverse matrix of $\|\tau_{a}^{b}\|$ (²⁹).

The most-important variances (which are also opposite to each other) are those of the *covariant* vectors (with components A_i) and the *contravariant vectors* (with components B^i) that one obtains by associating the transformations (14.2) with the linear transformation:

$$A'_i = rac{\partial q^j}{\partial q'^i} A_j$$
 and $B'^i = rac{\partial q'^i}{\partial q^j} B^j$.

For example, if the function L defined in (13.2) is an invariant then the variational derivatives $\delta L / \delta q^i$ that were defined in (13.3) will be the components of a covariant vector with respect to the group Γ of transformations (14.2) [cf., **Th. De Donder** (³¹)].

The variational problem (13.1) that is defined in the space E_n of $q^1, ..., q^n$ on which the group Γ in (14.2) operates will lead to equations (13.3) that consist of annulling the components of a covariant vector.

15. – When one considers the space E (we shall drop the index n from now on) of $q^1, ..., q^n$ on which the group Γ of coordinate transformations (14.2) operates, and on which one defines a tensorial variance like (14.3), it will be possible to define the 2*n*-dimensional space \mathcal{F} of $q^1, ..., q^n, f_1, ..., f_n$ on which the group Γ^* operates:

$$q'^{i} = q'^{i}(q^{1},...,q^{n}),$$
$$f'_{a} = \tau_{a}^{\cdot b}f_{b} = f'_{a}(q^{1},...,q^{n},f_{1},...,f_{n}).$$

One can similarly consider the space \mathcal{F} of $q^1, \ldots, q^n, s^1, \ldots, s^n$ on which group $\overline{\Gamma}^*$ operates:

$$q'^{i} = q^{N}(q^{1},...,q^{n}),$$
$$s'^{a} = \overline{\tau}^{a}_{\cdot b} s^{b} = s'^{a}(q^{1},...,q^{n},s^{1},...,s^{n}).$$

When the variance (14.3) is that of covariant (contravariant, resp.) vectors, the space \mathcal{F} will be the space whose "points" are covariant (contravariant, resp.) vectors that are tangent to the various points of *E*, and the space \mathcal{G} is the space whose "points" are the contravariant (covariant, resp.) vectors that are tangent to the various points of *E*.

§ 2. – *The inverse problem in the calculus of variations.*

16. – If one is given the system (14.1), which consists of annulling the components of the tensor **F** with respect to a group Γ , then one can pose the following two questions, which both deserve the name of *the inverse problem in the calculus of variations:*

A. – Does there exist a function $L = L(t, q^i, \dot{q}^i)$ whose variational derivatives are *identical* to (14.1)?

B. – Does there exist a function $L = L(t, q^i, \dot{q}^i)$ such that the equations that are obtained by annulling its variational derivatives are **equivalent** to (14.1)?

In order for the first one to have any invariant sense, the tensor \mathbf{F} must obviously be a covariant vector. By contrast, the second one will possess an invariant sense regardless of the variance of \mathbf{F} .

The answer to the first question involves the notions of the *system of variations* and the *adjoint* system $(^{31})$.

One intends the phrase system of variations of the F_a to mean the *n* expressions:

$$\Phi_a(t,q^i,\dot{q}^i,\ddot{q}^i;\eta^i,\dot{\eta}^i,\ddot{\eta}^i) = \frac{\partial F_a}{\partial q^i}\eta^i + \frac{\partial F_a}{\partial \dot{q}^i}\dot{\eta}^i + \frac{\partial F_a}{\partial \ddot{q}^i}\ddot{\eta}^i$$

that are linear and homogeneous in the η^i , $\dot{\eta}^i$, $\ddot{\eta}^i$, which denote the new variables.

By the phrase *adjoint system* to the Φ_a , one intends that to mean a system of *n* expressions:

$$\Gamma_i(t,q^i,\dot{q}^i,\ddot{q}^i,\ddot{q}^i,\ddot{q}^i,q^{i};\eta^i,\dot{\eta}^i,\dot{\eta}^i)$$

that is linear and homogeneous in the variables, such that there exists an expression:

$$I(t,q^{i},\dot{q}^{i},\ddot{q}^{i},\ddot{q}^{i},\ddot{q}^{i},\overset{(iv)}{q}^{i};\eta^{i},\dot{\eta}^{i},\zeta^{a},\dot{\zeta}^{a})$$

that is linear in both the η^i , $\dot{\eta}^i$ and the ζ^a , $\dot{\zeta}^a$, and gives rise to the identity:

(16.1)
$$\zeta^a \Phi_a - \eta^i \Gamma_i \equiv \frac{dI}{dt},$$

in which the operator d / dt is defined by:

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \dot{q}^{i} \frac{\partial}{\partial q^{i}} + \dots + \frac{q^{i}}{q^{i}} \frac{\partial}{\partial \ddot{q}^{i}} + \dot{\eta}^{i} \frac{\partial}{\partial \eta^{i}} + \ddot{\eta}^{i} \frac{\partial}{\partial \dot{\eta}^{i}} + \dot{\zeta}^{i} \frac{\partial}{\partial \zeta^{i}} + \ddot{\zeta}^{i} \frac{\partial}{\partial \dot{\zeta}^{i}}.$$

Upon taking the variational derivatives of the two sides of (16.1) with respect to η^i , one will see that if the Γ_i exist then they will be given by:

(16.2)
$$\Gamma_{i} = \frac{\delta}{\delta\eta^{i}} \zeta^{a} \Phi_{a} = \frac{\partial F_{a}}{\partial q^{i}} \zeta^{a} - \frac{d}{dt} \left(\frac{\partial F_{a}}{\partial \dot{q}^{i}} \zeta^{a} \right) + \frac{d^{2}}{dt^{2}} \left(\frac{\partial F_{a}}{\partial \dot{q}^{i}} \zeta^{a} \right).$$

One then verifies that those expressions indeed answer the question and that one must take:

$$I = \frac{\partial F_a}{\partial \dot{q}^i} \zeta^a \eta^i - \frac{d}{dt} \left(\frac{\partial F_a}{\partial \ddot{q}^i} \zeta^a \eta_i \right) + 2 \frac{\partial F_a}{\partial \ddot{q}^i} \dot{\eta}^i \zeta^a.$$

That being the case, one shows that:

The question A gets an affirmative response under the necessary and sufficient condition that the system of variations of the F_a must be self-adjoint (³²),

i.e., that one must have:

$$\Phi_a(t,q^i,\dot{q}^i,\ddot{q}^i;\eta^i,\dot{\eta}^i,\ddot{\eta}^i) = \Gamma_i(t,q^i,\dot{q}^i,\ddot{q}^i,\ddot{q}^i,q^i,\eta^i,\dot{\eta}^i,\dot{\eta}^i,\dot{\eta}^i).$$

One notes that the Γ_i do not depend upon the \ddot{q}^i , $\overset{(iv)}{q}^i$ explicitly in this case.

As for the question B, it is much more difficult and up till now it has been solved only in the case of n = 1 by **G. Darboux** (³³) and in the case of n = 2 by **J. Douglas** (³⁴). We shall not go into that here, but we point it out simply to show that expression "inverse problem in the calculus of variations" demands to be made more precise.

§ 3. – Bateman's method.

17. – Recall the system (14.1). Bateman's method starts from the following property $(^{35})$: Consider the variational problem:

(17.1)
$$\delta \int \mathcal{L} dt = 0 , \quad \mathcal{L} = s^a F_a$$

in the space of $q^1, \ldots, q^n, s^1, \ldots, s^n$. The extremals are given by the equations:

$$\begin{cases} \frac{\delta \mathcal{L}}{\delta q^{i}} \equiv G_{i}(t, q^{i}, \dot{q}^{i}, \ddot{q}^{i}, \ddot{q}^{i}, \ddot{q}^{i}, \dot{q}^{i}, \dot{q}^{i}, \dot{s}^{a}, \dot{s}^{a}, \ddot{s}^{a}) = 0, \\ \frac{\delta \mathcal{L}}{\delta s^{a}} \equiv F_{a}(t, q^{i}, \dot{q}^{i}, \ddot{q}^{i}) = 0. \end{cases}$$

The last *n* of them are identical to the proposed equations (14.1), while the first *n* involve the *n* auxiliary unknowns $s^{a}(t)$.

Any system of differential equations (14.1) can then be prolonged into a system that is derived from the calculus of variations in the sense of problem A.

One easily confirms that the G_i are linear and homogeneous in the s^a , \dot{s}^a , \ddot{s}^a , and will reduce to the Γ_i of the system that is adjoint to the system of variations of the F_a after one has replaced the ζ^a with the s^a . In regard to (16.2), that property is equivalent to the easily-verified identity:

$$\frac{\delta}{\delta q^i}(s^a F_a) \equiv \frac{\delta}{\delta \eta^i}s^a \Phi_a$$

18. – If we perform a coordinate transformation (14.2) from the group Γ then the system (14.1) will become:

(18.1)
$$F'_{a}(t,q'^{i},\dot{q}'^{i},\ddot{q}'^{i}) = 0,$$

with

$$F_a' = \tau_a^{\cdot b} F_b.$$

Let us apply Bateman's method to the system (18.1). We then construct the system of 2n equations in 2n unknowns:

$$\frac{\delta \mathcal{L}}{\delta q'^{i}} \equiv G_{i}(t, q'^{i}, \dots; s'^{a}, \dots) = 0,$$
$$\frac{\delta \mathcal{L}}{\delta s'^{a}} \equiv F_{a}'(t, q'^{i}, \dots) = 0$$

and define the extremals of the integral:

$$\int \mathcal{L} \, dt \,, \qquad \qquad \mathcal{L} = \, s'^a \, F'_a$$

That problem can be considered to be the transform of the problem (17.1) under the coordinate transformation $(q^i, s^a) \rightarrow (q'^i, s'^a)$:

(18.2)
$$\begin{cases} q'^i = q'^i(q^j), \\ s'^a = \overline{\tau}^a_{\cdot b} s^b, \end{cases}$$

which prolongs (14.2) and ensures the invariance of the Lagrangian function:

$$\mathcal{L} = \mathcal{L}'.$$

Under that transformation, the G_i , F_a transform into the G'_i , F'_a , resp., like the components of a covariant vector:

$$G'_{i} = \frac{\partial q^{j}}{\partial q^{\prime i}} G_{j} + \frac{\partial s^{b}}{\partial q^{\prime i}} F_{b},$$
$$F'_{a} = 0 + \frac{\partial s^{b}}{\partial s^{\prime a}} F_{b} \quad (=\tau_{a}^{\cdot b} F_{b})$$

We will then be led to consider the variational problem (17.1) to be something that is defined in the space \mathcal{G} of the $q^1, ..., q^n, s^1, ..., s^n$ that we encountered in no. **15** and on which the group $\overline{\Gamma}^*$ of transformations (18.2) operates. In addition, there is good reason to consider the operation that associates the point (q^i, s^a) of Γ with the point (q^i) in *E*. In that way, one will see that the projection $(q^i, s^a) \rightarrow (q^i)$ of $\mathcal{G} \rightarrow E$ is an intrinsic operation, i.e., it is independent of the coordinates.

One likewise sees that the link between the problem (14.1) that was defined in E and Bateman's variational problem (17.1) that was defined in G is independent of the coordinates, i.e., that link is intrinsic. In summary:

In Bateman's method:

1. One starts from a system of differential equations (14.1) that are defined in the space *E* on which the group Γ operates. The equations consist of annulling the components of a tensor **F** whose variance is characterized by a group γ of matrices that is homomorphic to Γ .

2. One associates it intrinsically with the variational problem (17.1) that is defined in the space Γ on which the group Γ^* operates. The extremals of that problem are the curves in Γ that project into *E* along the solutions to the proposed system (14.1).

19. – The geometric properties of the space \mathcal{G} depend upon both the geometric properties of the *E* and the nature of the tensor **F**. Here are some examples. The one that is denoted by (*b*) will be particularly useful for us in the context of the theorems of **V**. Bjerknes and **W**. Thomson.

a) Suppose that the group Γ is the general group of transformations of the type:

(19.1)
$$q'^i = q'^i (q^j),$$

and that the tensor \mathbf{F} is a contravariant vector.

We represent its components by F^i , in which the subscript *a* has been replaced with a superscript *i*. More generally, under the present hypothesis, we replace a subscript or superscript *a*, *b* with a superscript or subscript *i*, *j*, respectively. That is how s^a will become s_i (a = i) and that the $\tau_a^{,b}$, $\overline{\tau}_{,b}^a$ will become:

$$\tau^{i}_{\cdot j} = \frac{\partial q'^{i}}{\partial q^{j}}, \qquad \overline{\tau}^{\cdot j}_{i} = \frac{\partial q^{j}}{\partial q'^{i}} \qquad (a = i, b = j).$$

The transformations (18.2) of the group $\overline{\Gamma}^*$ will become:

$$q'^{i} = q'^{i}(q^{j}), \qquad s'_{i} = \frac{\partial q^{j}}{\partial q'^{i}}s_{j}$$

and leave the Pfaff form invariant:

$$\omega = s_i dq^i$$

a') If we confine ourselves to the group Γ of linear transformations:

$$q'^i = \tau^i_{\cdot i} q^j$$

then the group $\overline{\Gamma}^*$ will be written:

$$q^{\prime i} = \tau^i_{\,\,\,j} \, q^j, \quad s_i = \overline{\tau}^{\,\,\,j}_i \, s_j.$$

That group will leave the following metric invariant:

$$(dS)^2 = dq^i ds_i$$

in which the *n*-planes $dq^i = 0$ and $ds_i = 0$ are isotropic *n*-planes.

b) Now suppose that the group Γ is the group of linear transformation (19.2), but that **F** is a covariant vector. We represent its components by F_i and in a general manner, any index a, b, \ldots of the general theory will be replaced by an index i, j, \ldots that is located in the same place under the present hypothesis.

The transformations of the group $\overline{\Gamma}^*$ are written:

$$q'^{i} = \tau^{i}_{.\,i} q^{j}, \quad s'^{i} = \tau^{i}_{.\,i} s^{j}.$$

Suppose, in addition, that the metric:

$$(ds)^2 = g_{ij} dq^i dq^j$$
 ($g_{ij} = g_{ji} = \text{constants}$)

is defined in E [viz., the space of (q)]. We associate it intrinsically with the following metric:

$$(dS)^2 = 2g_{ij}\,dq^i\,ds^j\,,$$

which is defined in G [viz., the space of (q, s)]. The vectors that verify one of the groups of equations $dq^i = 0$ or $ds^i = 0$ are once more isotropic vectors. More generally, a pair of orthogonal vectors a^i and b^i (one will then have $g_{ij}a^ib^j = 0$) will correspond intrinsically in \mathcal{G} to two isotropic vectors whose components are (a^i, b^j) and (b^i, a^j) .

If a^i , b^i are the contravariant components of a vector in G then the covariant components will be:

$$a_i = g_{ii} b^j, \qquad b_i = g_{ii} a^j.$$

If we consider an *n*-plane in Γ that verifies the equations (which are invariant under $\overline{\Gamma}^*$):

$$dq^i = ds^i$$

then there will exist one and only one point in that *n*-plane that projects onto a given point in E, and the lengths S and s of the two image vectors will have a ratio of 2 to 1 under that correspondence.

We shall call the *n*-planes whose equations are $q^i - s^i = \text{const.}$ diagonals. Any figure that is situated in a diagonal *n*-plane will be itself diagonal.

§ 4. – The theorems of V. Bjerknes and W. Thomson.

20. – Consider an inviscid fluid in motion in three-dimensional Euclidian space *E* that is referred to (rectangular or oblique) Cartesian coordinates x^1 , x^2 , x^3 . The fundamental metric form will be:

$$(ds)^2 = g_{ii} dx^i dx^j$$
 $(i, j = 1, 2, 3; g_{ij} = g_{ji} = \text{constant}).$

On the other hand, if a *particular motion* is in question then it will be characterized by a velocity field whose contravariant and covariant components are:

$$u^{i} = u^{i}(x^{1}, x^{2}, x^{3}, t)$$
 and $u_{i} = u_{i}(x^{1}, x^{2}, x^{3}, t) = g_{ij}u^{j}$,

and by fields of pressure p and specific volume v:

$$p = p(x^{1}, x^{2}, x^{3}, t), \quad v = v(x^{1}, x^{2}, x^{3}, t)$$

that are supposed to be known. In addition, the motion is supposed to take place in a specific force field that is derived from a potential φ :

$$\varphi = \varphi(x^1, x^2, x^3, t) \, .$$

The trajectories of the particular fluid motion are the integrals of the differential system:

(20.1)
$$\frac{dx^{1}}{u^{1}} = \frac{dx^{2}}{u^{2}} = \frac{dx^{3}}{u^{3}} = \frac{dt}{1}$$

and will verify the dynamical equations:

(20.2)
$$g_{ij}\frac{d^2x^j}{dt^2} - K_i = 0$$
, in which $K_i = -\frac{\partial\varphi}{\partial x^i} - v\frac{\partial p}{\partial x^i}$,

in addition.

The trajectories of the fluid motion considered constitute a "field of solutions" of equations (20.2) (³⁷).

When the fluid is barotropic, there will exist a relation f(p, v) = 0, and as a result, the integral:

$$\int_{(x_0^1, x_0^2, x_0^3)}^{(x_1^1, x_0^2, x_0^3)} K_i \, dx^i$$

will be independent of the integration path that joins the fixed point (x_0^1, x_0^2, x_0^3) to the variable one (x^1, x^2, x^3) . It is a function $V(x^i, t)$ for which:

$$K_i = - \frac{\partial V}{\partial x^i}.$$

In that case, the system of variations of the left-hand side of equations (20.2) will be self-adjoint. That system admits a solution to the inverse problem A (no. 16) for which:

$$L = T - V, \qquad T = \frac{1}{2} g_{ii} \dot{x}^i \dot{x}^j.$$

It will then result immediately that by virtue of what we saw in Chapter I, § 4, a barotropic fluid will verify the theorem of the conservation of circulation and Helmholtz's theorem (conservation of the vorticity lines).

When the fluid is baroclinic, there will no longer be the relation f(p, v) = 0, the function V, nor the function L, and those theorems will break down. Meanwhile, it is possible for us to examine their generalization that Bateman's method provides.

21. – We begin by specifying that we shall assume that the Cartesian or linear transformations (group Γ) act upon the x^i :

$$x^{\prime i} = \tau^i_{\cdot i} x^j.$$

On the other hand, we see that the left-hand sides of (20.2) are the components of a covariant vector. We will then find ourselves within the scope of the conditions in example *b*) in no. **19**.

The Bateman function \mathcal{L} is:

$$\mathcal{L} = y^i (g_{ij} \ddot{x}^j - K_i) = g_{ij} \ddot{x}^j y^i - K_i y^i$$

here, in which the y^i play the role of the x^i in the general theory. The space Γ is the sixdimensional space of the x^i , y^i in which the group Γ^* operates:

$$x^{\prime i} = \tau^i_{\cdot i} x^j. \quad y^{\prime i} = \tau^i_{\cdot i} y^j.$$

That space is endowed with the metric:

(21.1)
$$(dS)^2 = 2g_{ii} dx^i dx^j.$$

Observe that one has:

$$\mathcal{L} = \frac{d}{dt} (g_{ij} \dot{x}^i y^j) - (g_{ij} \dot{x}^i \dot{y}^j + K_i y^i)$$

and that one will not modify the extremals when one replaces that function with:

$$-(g_{ij}\dot{x}^i\dot{y}^j+K_iy^i)$$

since the integral:

$$\int_{P_0}^{P} \frac{d}{dt} (g_{ij} \dot{x}^i y^j) dt = (g_{ij} \dot{x}^i y^j) \Big|_{P_0}^{P}$$

does not depend upon the path that is followed during the integration. In order to avoid a cumbersome – sign, we finally set:

(21.2)
$$\mathcal{L} = g_{ii} \dot{x}^i \dot{y}^j + K_i y^i.$$

We then find that in the space \mathcal{G} , we are down to a problem of the type that was envisioned in Chap. I, § 3. The extremals are solutions to the second-order differential system:

(21.3)
$$\begin{cases} \frac{\delta \mathcal{L}}{\delta x^{i}} \equiv g_{ij} \ddot{y}^{j} - \frac{\partial K_{j}}{\partial x^{i}} y^{j} = 0, \\ \frac{\delta \mathcal{L}}{\delta y^{i}} \equiv g_{ij} \ddot{x}^{j} - K_{i} = 0. \end{cases}$$

If we introduce canonical variables that are nothing but the covariant components of the velocity [when one takes the metric (21.1) into account]:

$$p_i = rac{\delta \mathcal{L}}{\delta \dot{x}^i} = g_{ij} \dot{y}^j, \qquad q_i = rac{\delta \mathcal{L}}{\delta \dot{y}^i} = g_{ij} \dot{x}^j,$$

and the Hamiltonian function:

(21.4)
$$\mathcal{H} = p_i \dot{x}^i + q_i \dot{y}^i - \mathcal{L} = g^{ij} p_i q_j - K_i y^i,$$

then equations (21.3) will be equivalent to the canonical equations:

(21.5)
$$\frac{dx^{i}}{\frac{\partial \mathcal{H}}{\partial p_{i}}} = \frac{dy^{i}}{\frac{\partial \mathcal{H}}{\partial q_{i}}} = -\frac{dp_{i}}{\frac{\partial \mathcal{H}}{\partial x^{i}}} = -\frac{dq_{i}}{\frac{\partial \mathcal{H}}{\partial y^{i}}} = dt.$$

The fluid motion considered in the space *E* corresponds to a field of solutions (³⁷) of the equations (20.2). We shall associate it with a field of extremals of the problem (21.2) in the space G:

 $\delta \int \mathcal{L} \, dt = 0 \; .$

We let:

$$U^{i} = U^{i}(x^{j}, y^{j}, t), \quad V^{i} = V^{i}(x^{j}, y^{j}, t)$$

denote the contravariant components of the velocity of a moving point on a trajectory of the field (those six quantities correspond to the *n* quantities Q^i in no. 9). The covariant components of that velocity will be denoted by:

$$U_i = U_i(x^j, y^j, t) = g_{ij}V^j, \quad V_i = V_i(x^j, y^j, t) = g_{ij}U^j$$

(which are quantities that are analogous to the P_i of no. 9).

A *first condition* to impose upon that field of extremals will be to project it onto the space E along the trajectories of the fluid motion considered, i.e., along the field of solutions of (20.2). That amounts to saying that:

$$U^{i}(x^{j}, y^{j}, t) = u^{i}(x^{j}, t) ,$$

in which the u^i are the functions in (20.1). It will then result from that hypothesis that the U^i do not depend upon the y^i explicitly. In terms of covariant components, that condition is expressed by:

(21.6)
$$V_i(x^j, y^j, t) = u^i(x^j, t)$$

As a second condition, we impose the following relation at the instant t_0 :

$$V^{i}(x^{j}, y^{j}, t_{0}) = u^{i}(y^{j}, t_{0}),$$

or, in covariant components:

(21.7)
$$U^{i}(x^{j}, y^{j}, t_{0}) = u^{i}(y^{j}, t_{0}).$$

That condition can be interpreted in the following manner: Call the pairs of points (x^i, y^i) and (y^i, x^i) of \mathcal{G} symmetric. Similarly, call two vectors that have symmetric points for their origins and the same components symmetric: (a^i, b^i) and (b^i, a^i) . If the first condition is fulfilled then the second one will signify that the velocities at symmetric points are symmetric at the instant t_0 .

Those two conditions will determine a field of extremals in space G completely. Indeed, if one is given an arbitrary vector field whose components are:

$$a^{i} = a^{i}(x^{j}, y^{j}), \qquad b^{i} = b^{i}(x^{j}, y^{j})$$

then there will exist one and only one trajectory $x^i = x^i(t)$ of equations (21.3) that pass through an arbitrary point x_0^i , y_0^i at the instant t_0 and have the vector (a^i, b^i) for its velocity (\dot{x}^i, \dot{y}^i) at that point. In addition, the set of those trajectories constitutes a field of extremals in the space of x^i , y^i , t for values of t that are close to t_0 .

One will then see that in order to satisfy the two conditions, there is good reason to adopt the initial conditions:

$$a^{i}(x^{j}, y^{j}) = u^{i}(x^{j}, t_{0}), \qquad b^{i}(x^{j}, y^{j}) = u^{i}(y^{j}, t_{0}).$$

The functions $U^{i}(x^{j}, y^{j}, t)$, $V^{i}(x^{j}, y^{j}, t)$ are determined completely with that.

The trajectories of the field, thus-defined, enjoy the following important property: The ones that pass through a point of the 3-plane P_3 whose equations are $x^i = y^i$ at the instant t_0 are tangent to that 3-plane. The corresponding velocity vectors are diagonal vectors. Meanwhile, that property will no longer be verified at the final instant.

22. – That being the case, consider a closed curve γ in \mathcal{G} (viz., a locus of instantaneous points) that projects along a close curve *c* (viz., a locus of likewise-instantaneous points) in *E*. We the following circulations in \mathcal{G} and *E*:

$$\Gamma = \int_{\gamma} U_i \, \delta x^i + V_i \, \delta y^i, \qquad C = \int_c u_i \, \delta x^i,$$

in which the symbols δ denote differentiations along γ and c. If the curve γ is situated in the 3-plane $P_3(x^i = y^i)$ then one will have:

(22.1)
$$(\delta x^i)_{\gamma} = (\delta y^i)_{\gamma} = (\delta x^i)_c,$$

(22.2)
$$\Gamma = 2C$$

at the instant t_0 .

When the curves γ and *c* are displaced in conformity with the associated fields in the spaces Γ and *E*, the relations (22.1) will remain true at the instant $t_0 + dt$ due to the property of the trajectories that they are tangent to P_3 . By contrast, the relation (22.2) will cease to be true at that instant because one will no longer have $U_i = V_i = u_i$. Meanwhile, by virtue of (22.1) and (21.6), one will have:

$$\int_{\gamma} V_i \,\delta \, y^i = \int_c u_i \,\delta x^i \qquad (\text{at the instant } t_0 + dt).$$

We will then have:

$$\frac{d}{dt}\int_{c}u_{i}\,\delta x^{i}=\frac{d}{dt}\int_{\gamma}V_{i}\,\delta y^{i}$$

at the instant t_0 , or by virtue of the theorem of the conservation of circulation in G:

$$\frac{d}{dt}\int_{c}u_{i}\,\delta x^{i}=-\frac{d}{dt}\int_{\gamma}U_{i}\,\delta x^{i}\,.$$

Performing the calculations on the right-hand side of that will now produce the:

Theorem of V. Bjerknes. – *The variation of the circulation (in E) per unit time along a closed curve c is equal to the number of isobaric-isosteric solenoids that are encircled by that curve.*

By virtue of (21.4), (21.5), (22.1):

$$\frac{d}{dt}\int_{\gamma}V_{i}\,\delta y^{i} = \int_{\gamma}\frac{dV_{i}}{dt}\,\delta y^{i} - \delta V_{i}\frac{dy^{i}}{dt} = \int_{\gamma}-\frac{\partial\mathcal{H}}{\partial y^{i}}\,\delta y^{i} - \frac{\partial\mathcal{H}}{\partial V_{i}}\,\delta V_{i} = \int_{\gamma}K_{i}\,\delta x^{i} - \delta\left(\frac{1}{2}g^{ij}U_{i}U_{j}\right)$$

$$= \int_{\gamma} K_i \, \delta x^i = -\int_c v \, \delta p = \int_c p \, \delta v \, .$$

Similarly:

$$\frac{d}{dt} \int_{\gamma} U_i \,\delta x^i = \int_{\gamma} \frac{dU_i}{dt} \,\delta x^i - \delta U_i \frac{dx^i}{dt} = \int_{\gamma} -\frac{\partial \mathcal{H}}{\partial x^i} \,\delta x^i - \frac{\partial \mathcal{H}}{\partial U_i} \,\delta U_i = \int_{\gamma} -\delta \mathcal{H} + \left(\frac{\partial \mathcal{H}}{\partial y^i} \,\delta y^i + \frac{\partial \mathcal{H}}{\partial V_i} \,\delta V_i\right)$$
$$= \int_c v \,\delta p = -\int_c p \,\delta v \,.$$

Each point P of c corresponds to a value of v and a value of p, so to a point P' of Clapeyron's (p, v) diagram in thermodynamics. The locus of point P' is a curve c' that bounds a region S in the (p, v) plane. The area of that region is what the meteorologists denote by the symbol $N_c(p, v)$ and call the number of isobaric-isosteric solenoids that are encircled by the curve c in the space E.

The theorem will finally result from this that one has:

$$N_c(p, v) = \int_{S} dp \, dv = \int_{c'} p \, dv = \int_{c} p \, \delta v$$

23. – We recover the theorem of W. Thomson analogously, which can be stated as follows:

Theorem of W. Thomson. – A vorticity vector will remain a vorticity vector when one displaces its origin and its extremity like fluid particles on the necessary and sufficient condition that the rotation of the acceleration must be proportional to the rotation of the velocity.

We shall prove that for a vorticity vector that displaces from the instant t_0 to the instant $t_0 + dt$. We know that **Helmholtz**'s theorem (Chap. I, § 4) is applicable to the field of extremals in the space \mathcal{G} .

On the other hand, the vorticity vectors with their origin at (x^i, y^i) at the instant *t* are given by the following equations, which are analogous to (11.1):

(23.1)
$$\begin{cases} \left(\frac{\partial U_i}{\partial x^j} - \frac{\partial U_j}{\partial x^i}\right) \delta x^j + \left(\frac{\partial U_i}{\partial y^j} - \frac{\partial V_j}{\partial x^i}\right) \delta y^j = 0, \\ \left(\frac{\partial V_i}{\partial x^j} - \frac{\partial U_j}{\partial y^i}\right) \delta x^j + 0 \cdot \delta y^i = 0, \end{cases}$$

which reduces to:

(23.2)
$$\begin{cases} 0 \cdot \delta x^{j} + \left(\frac{\partial u_{i}}{\partial x^{j}} - \frac{\partial u_{j}}{\partial x^{i}}\right) \delta y^{j} = 0, \\ \left(\frac{\partial u_{i}}{\partial x^{j}} - \frac{\partial u_{j}}{\partial x^{i}}\right) \delta x^{j} + 0 \cdot \delta y^{i} = 0, \end{cases}$$

for $t = t_0$ and $x^i = y^i$.

The rank of the system (23.1) is generally equal to six, but under the conditions of (23.2), it will reduce to four or zero according to whether the rotation of the fluid motion in E is non-zero or zero, resp. In addition, we have seen that this number if constant along the corresponding trajectories in \mathcal{G} (no. 12, pp. 19).

Let π be a point in the 3-plane P_3 that projects onto a point P in E where we suppose that:

$$\left(\frac{\partial u_i}{\partial x^j} - \frac{\partial u_j}{\partial x^i}\right) \neq 0$$

at the instant t_0 , to fix ideas (38). The rank of (23.2) in π is equal to four, and the ∞^2 vorticity vectors distribute themselves in a plane that cuts P_3 along a line. They correspond to ∞^1 diagonal vorticity vectors that are given up to a factor:

(23.3)
$$\frac{\delta x^{1}}{\frac{\partial u_{2}}{\partial x^{3}} - \frac{\partial u_{3}}{\partial x^{2}}} = \frac{\delta x^{2}}{\frac{\partial u_{3}}{\partial x^{1}} - \frac{\partial u_{1}}{\partial x^{3}}} = \frac{\delta x^{3}}{\frac{\partial u_{1}}{\partial x^{2}} - \frac{\partial u_{2}}{\partial x^{1}}} = \frac{\delta y^{1}}{\frac{\partial u_{2}}{\partial x^{3}} - \frac{\partial u_{3}}{\partial x^{2}}} = \frac{\delta y^{2}}{\frac{\partial u_{3}}{\partial x^{1}} - \frac{\partial u_{1}}{\partial x^{3}}} = \frac{\delta z^{3}}{\frac{\partial u_{1}}{\partial x^{2}} - \frac{\partial u_{2}}{\partial x^{1}}}$$

Each of them projects onto E along a vorticity vector with components δx^i at the point P.

After an time interval dt, the point π will go to the point $\overline{\pi}$ (which is again situated in P_3), whereas the point P will go to \overline{P} , which is the projection of $\overline{\pi}$. The vectors (23.3) will remain vorticity vectors (i.e., Helmholtz's theorem) and diagonal [cf., (21.6) and (21.7)]. Their components will become:

$$\overline{\delta x^i} = \delta x^i + d \, \delta x^i, \quad \overline{\delta y^i} = \delta y^i + d \, \delta y^i,$$

and are solutions to the equations:

(23.4)
$$\left(\frac{\overline{\partial U_i}}{\partial x^j} - \frac{\overline{\partial U_j}}{\partial x^i}\right) \overline{\delta x^j} + \left(\frac{\overline{\partial U_i}}{\partial y^j} - \frac{\overline{\partial V_j}}{\partial x^i}\right) \overline{\delta y^j} = 0,$$

(23.5)
$$\left(\frac{\overline{\partial V_i}}{\partial x^j} - \frac{\overline{\partial U_j}}{\partial y^i}\right)\overline{\delta x^j} + 0 \cdot \overline{\delta y^i} = 0,$$

(23.6)
$$\overline{\delta x^i} = \overline{\delta y^i}.$$

In all of those relations, the overbars indicate that the values must be taken at the point $\overline{\pi}$ and at the instant $t = t_0 + dt$. The projections of those vectors onto *E* will cease to be vorticity vectors in *E*, in general.

The vorticity vectors δx^i at \overline{P} and $t = t_0 + dt$ are indeed, given by the equations:

(23.7)
$$\left(\frac{\overline{\partial u_i}}{\partial x^j} - \frac{\overline{\partial u_j}}{\partial x^i}\right) \delta x^j = 0.$$

We see that in order for the (infinitely-small) vorticity vector δx^i to remain a vorticity vector when its origin and its extremity are displaced with the fluid in *E*, it is necessary and sufficient that equations (23.5) should be equivalent to (23.7) or rather, by virtue of (21.6), to:

(23.8)
$$\left(\frac{\overline{\partial V_i}}{\partial x^j} - \frac{\overline{\partial V_j}}{\partial x^i}\right) \overline{\delta x^j} = 0.$$

Now, upon adding (23.4) and (23.5) and utilizing (23.6), one will get the identity:

(23.9)
$$\left[\left(\frac{\overline{\partial V_i}}{\partial x^j} - \frac{\overline{\partial V_j}}{\partial x^i} \right) + \left(\frac{\overline{\partial U_i}}{\partial x^j} - \frac{\overline{\partial U_j}}{\partial x^i} \right) + \left(\frac{\overline{\partial U_i}}{\partial y^j} - \frac{\overline{\partial U_j}}{\partial y^i} \right) \right] \overline{\partial x^j} = 0.$$

As a result, the condition (23.8) can be replaced by the following two relations, which must be satisfied simultaneously:

(23.10)
$$\begin{cases} \alpha \equiv \left(\frac{\overline{\partial V_i}}{\partial x^j} - \frac{\overline{\partial V_j}}{\partial x^i}\right) \overline{\delta x^j} = 0, \\ \beta \equiv \left[\left(\frac{\overline{\partial U_i}}{\partial x^j} - \frac{\overline{\partial U_j}}{\partial x^i}\right) + \left(\frac{\overline{\partial U_i}}{\partial y^j} - \frac{\overline{\partial U_j}}{\partial y^i}\right)\right] \overline{\delta x^j} = 0. \end{cases}$$

One ultimately finds that:

$$\frac{\overline{\partial V_i}}{\partial x^j} = \frac{\partial V_i}{\partial x^j} + d \frac{\partial V_i}{\partial x^j} = \frac{\partial V_i}{\partial x^j} + \frac{\partial}{\partial x^j} dV_i - \frac{\partial V_i}{\partial x^k} \frac{\partial U^k}{\partial x^j} dt = \frac{\partial V_i}{\partial x^j} + \frac{\partial K_i}{\partial x^j} dt - \frac{\partial V_i}{\partial x^k} \frac{\partial U^k}{\partial x^j} dt ,$$

$$\frac{\overline{\partial U_i}}{\partial x^j} = \frac{\partial U_i}{\partial x^j} + \frac{\partial^2 K_h}{\partial x^i \partial x^j} y^h dt = \frac{\partial^2 K_h}{\partial x^i \partial x^j} y^h dt ,$$

$$\frac{\partial U_i}{\partial y^j} = \frac{\partial U_i}{\partial y^j} + \frac{\partial K_h}{\partial x^i} dt - \frac{\partial U_i}{\partial y^k} \frac{\partial V^k}{\partial y^j} dt.$$

Then, upon neglecting the second-order elements with respect to the differentiation symbol d, the relations (23.10) will give:

$$\alpha = \left(\frac{\partial V_i}{\partial x^j} - \frac{\partial V_j}{\partial x^i}\right) d\,\delta x^j + \left(\frac{\partial K_i}{\partial x^j} - \frac{\partial K_j}{\partial x^i}\right) dt\,\delta x^j - \left(\frac{\partial V_i}{\partial x^k} \frac{\partial U^k}{\partial x^j} - \frac{\partial V_j}{\partial x^k} \frac{\partial U^k}{\partial x^i}\right) dt\,\delta x^j = 0,$$

$$\beta = \left(\frac{\partial U_i}{\partial y^j} - \frac{\partial U_j}{\partial y^i}\right) d\,\delta x^j + \left(\frac{\partial K_i}{\partial x^j} - \frac{\partial K_j}{\partial x^i}\right) dt\,\delta x^j - \left(\frac{\partial U_i}{\partial y^k} \frac{\partial V^k}{\partial y^j} - \frac{\partial U_j}{\partial y^k} \frac{\partial V^k}{\partial y^i}\right) dt\,\delta x^j = 0.$$

Upon subtracting them and taking (21.6) and (21.7) into account, that will give:

$$\alpha - \beta = 2 \cdot \left(\frac{\partial K_i}{\partial x^j} - \frac{\partial K_j}{\partial x^i} \right) \delta x^j dt = 0.$$

On the other hand, since the δx^{j} verify the equations:

$$\left(\frac{\partial u_i}{\partial x^j} - \frac{\partial u_j}{\partial x^i}\right) \delta x^j = 0,$$

that can be true only if there exists a number k such that (³⁹):

$$\left(\frac{\partial u_i}{\partial x^j} - \frac{\partial u_j}{\partial x^i}\right) = k \cdot \left(\frac{\partial K_i}{\partial x^j} - \frac{\partial K_j}{\partial x^i}\right)$$

We have thus established that the condition in **Thomson**'s theorem is indeed necessary.

Conversely, suppose that the number k exists. That means that in the first approximation, we will have:

 $\alpha - \beta = 0$.

$$\alpha + \beta = 0$$

identically, from (23.9), it will result that equations (23.10) are verified in the first approximation. That is precisely what had to be established since we have remarked that (23.10) are necessary and sufficient conditions.

Q. E. D.

APPENDIX

On the variational method in the hydrodynamics of viscous fluids.

The considerations that were developed in the first three sections of Chapter II extend easily to systems of partial differential equations (⁴⁰).

For example, consider a viscous fluid whose motion is governed (in rectangular axes) by the **Navier-Stokes** equations (⁴¹):

(1)
$$\left(\frac{\partial u^{i}}{\partial x^{j}}u^{j} + \frac{\partial u^{i}}{\partial t}\right) = \rho K_{i} - \frac{\partial p}{\partial x^{i}} + (\lambda + \mu)\frac{\partial}{\partial x^{i}}\frac{\partial u^{j}}{\partial x^{j}} + \mu \frac{\partial^{2} u^{i}}{\partial x^{j} \partial x^{j}} = 0.$$

The notations are identical to the ones in Chapter II, § 4, along with $\rho = 1 / v$ (specific mass), and λ , μ are the two coefficients of viscosity (which are constants).

When ρ , p, λ , μ are known, one will be dealing with a system of three equations in three unknowns $u^i = u^i(x^1, x^2, x^3, t)$. One immediately confirms that the equations of variation are not self-adjoint, and consequently, those equations cannot be deduced from a variational principle in the sense of problem A (Chap. II, § 2) (⁴²).

When the motion is *permanent* $\left(\frac{\partial u^i}{\partial t} = 0\right)$ and *slow*, one assumes that the rectangular terms

 $\frac{\partial u^i}{\partial x^i}u^j$ are negligible in equations (1). They will then reduce to three linear equations:

(2)
$$F_i \equiv \mu \frac{\partial^2 u^i}{\partial x^j \partial x^j} + (\lambda + \mu) \frac{\partial}{\partial x^i} \frac{\partial u^j}{\partial x^j} + \rho K_i - \frac{\partial p}{\partial x^i} = 0$$

in three unknowns $u^i = u^i(x^1, x^2, x^3)$. The equations of variation will be self-adjoint, and the Lagrangian function *L* that is a solution to the problem A can be obtained in the following manner upon starting from **Bateman**'s method:

When **Bateman**'s method is applied to equations (2), it will lead one to define the function:

$$\mathcal{L} = v^i F_i ,$$

in which the $v^i = v^i(x^j)$ are three unknown auxiliary functions.

Assume, to begin with, that the forces of pressure are in equilibrium with the external forces:

$$\rho K_i = \frac{\partial p}{\partial x^i}.$$

In other words, we suppose that equations (2) are homogeneous. The equations of variations are no different from the equations themselves. It is clear that under those conditions, the function:

$$L = \frac{1}{2}u^i F_i$$

that corresponds to the variational problem:

(3)
$$\delta \iiint L \, dx^1 \, dx^2 \, dx^3 = 0$$

will give rise to:

$$\frac{\delta L}{\delta u^i} \equiv F_i \,,$$

but it will contain second derivatives of the u^i .

Now, one will not alter the problem (3) by adding the following divergence (an expression whose variational derivatives are identically zero) to the function L:

$$-\frac{1}{2}\frac{\partial}{\partial x^{j}}\left[\mu u^{i}\left(\frac{\partial u^{i}}{\partial x^{j}}+\frac{\partial u^{j}}{\partial x^{i}}\right)+\lambda u^{i}\frac{\partial u^{j}}{\partial x^{j}}\right],$$

and upon likewise changing the sign of the Lagrangian function, that will give:

$$L = \frac{1}{2} \left[\mu \frac{\partial u^{i}}{\partial x^{j}} \left(\frac{\partial u^{i}}{\partial x^{j}} + \frac{\partial u^{j}}{\partial x^{i}} \right) + \lambda \frac{\partial u^{i}}{\partial x^{i}} \frac{\partial u^{j}}{\partial x^{j}} \right] = \frac{1}{2} \psi ,$$

in which the function ψ is nothing but Lord Rayleigh's *dissipation function*, which is written in the form:

$$\psi = \lambda \left(\sum_{i} \frac{\partial u^{i}}{\partial x^{i}} \right)^{2} + \sum_{i \neq j} \left(\frac{\partial u^{i}}{\partial x^{j}} + \frac{\partial u^{j}}{\partial x^{i}} \right)^{2} + 2\mu \sum_{i} \left(\frac{\partial u^{i}}{\partial x^{i}} \right)^{2}.$$

When the forces of pressure are not in equilibrium with the external forces, i.e., when equations (2) are not homogeneous, there will be good reason to take:

$$L = \frac{1}{2}\psi - u^{i}\left(\rho K_{i} - \frac{\partial p}{\partial x_{i}}\right).$$

We thus find the extension of a theorem that is due to **J. Kravtchenko** that related to the incompressible and homogeneous case (45) to the case of a compressible fluid. In addition, we see

that the author's hypothesis of making the external forces derivable from a potential is superfluous (⁴⁴). We then assert the:

Theorem:

When one knows the distribution of mass and pressure under the slow and permanent motion of a viscous fluid, the velocity field will be defined by the variational principle:

(4)
$$\delta \iiint_{V} \left[\frac{1}{2} \psi - u^{i} \left(\rho K_{i} - \frac{\partial p}{\partial x_{i}} \right) \right] dx^{1} dx^{2} dx^{3} = 0$$

in which the variations δu^i of the velocity are annulled on the boundary of the volume V.

The hypothesis that makes the forces K_i derivable from a potential Φ :

(5)
$$K_i = \frac{\partial \Phi}{\partial x^i}$$

enters into a theorem of **Helmholtz** (⁴³) that relates to incompressible and homogeneous fluids (so $\frac{\partial u_i}{\partial x^i} = 0, \rho = \text{const.}$).

Helmholtz's theorem:

Let an incompressible and homogeneous viscous fluid that is subject to the action of external forces that are derived from a potential Φ be in slow, permanent motion inside of a volume V that is bounded by a surface S. For a given distribution of velocity on the bounding surface S, the motion of the fluid inside of V will realize the minimum of the dissipation, i.e., the integral:

(6)
$$\iiint_V \psi \, dx^1 \, dx^2 \, dx^3$$

over all possible incompressible, homogeneous motions.

Remark. – The given distribution of velocity on the boundary *S* must be compatible with the incompressibility. One must then have:

$$\iint_{S} u^{1} dx^{2} dx^{3} + u^{2} dx^{3} dx^{1} + u^{3} dx^{1} dx^{2} = 0.$$

Helmholtz's theorem is composed of two parts:

(a) The first variation of (6) is zero when one gives variations δu^i to the u^i that are zero on S and verify the incompressibility condition:

(7)
$$\frac{\partial}{\partial x^i} \delta u^i = 0$$

(b) The total variation of (6) is positive-definite.

(a) This part results from (4) because when one takes (5) and (7) into account, one will find that:

$$\delta \iiint_V u^i \left(\rho K_i - \frac{\partial p}{\partial x_i} \right) dx^1 dx^2 dx^3 = 0.$$

Indeed, after an integration by parts, the left-hand side can be written:

$$\iiint_{V} \frac{\partial}{\partial x_{i}} \left[\delta u^{i} (\rho \Phi - p) \right] dx^{1} dx^{2} dx^{3} - \iiint_{V} (\rho \Phi - p) \frac{\partial}{\partial x_{i}} \delta u^{i} dx^{1} dx^{2} dx^{3}.$$

The first integral transforms into a surface integral that extends over *S*, and it will be zero by virtue of the condition that $\delta u^i = 0$ on *S*. The second integral will be zero by virtue of (7).

(b) By virtue of the quadratic character of the function ψ and the vanishing of the first variation, the total variation of (6) will be written:

$$\iiint_V \overline{\psi} \, dx^1 \, dx^2 \, dx^3,$$

in $\overline{\psi}$ is the function ψ when one has replaced u^i with δu^i . The property will then result from this that ψ is a positive-definite function. [See **H. Villat** (⁴¹), pp. 76.]

Remark. – **Helmholtz**'s theorem is not a true variational principle because it considers *constrained variations* of the u^i . It does not permit one to obtain equations of motion in the same way that the principle (4) does. If one changes the pressure field and the force field without modifying the given u^i on the surface *S* then one will obtain a different motion that will once more verify **Helmholtz**'s theorem.

One can compare **Helmholtz**'s theorem to the following problem: Find a minimum of the function $z = x^2 + y^2$ when one gives variations to the *x* and *y* that are coupled by the condition that dx = 0. Regardless of the constant *a*, the point x = a, y = 0 will realize such a minimum.

NOTES

1) If the differential system has order 3 or 4 then one must find an $L = L(t, q^i, \dot{q}^i, \ddot{q}^i)$ such that:

$$\frac{\delta L}{\delta q^{i}} \equiv \frac{\partial L}{\partial q^{i}} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}^{i}} + \frac{d^{2}}{dt^{2}} \frac{\partial L}{\partial \ddot{q}^{i}} \equiv F_{i}.$$

2) The complementary equations $G_i = 0$ must generally include fourth derivatives of the *q*. That is why the function *L* is a second-order differential; see Chap. II.

3) **Th. Lepage**, "Sur les champs géodésiques du calcul des variations," Bull. Acad. R. Belg., Cl. Sc. (5) **22** (1936). "Sur les champs géodésiques des intégrales multiples," *ibid.*, **27** (1941). "Champs stationaires, champs géodésiques et formes intégrables," *ibid.* **28** (1942).

A presentation of the methods of **Th. Lepage**, as applied to simple integrals, is due to **H. Boerner**, "Variationsrechnung aus dem Stokesschen Satz," Math. Zeit. **46** (1940).

4) See also on that subject: **P. Dedecker**, "Sur une méthode de Bateman dans le problème inverse du calcul des variations," Bull. Acad. R. Belg., Cl. Sc. (5) **35** (1949), 774-792.

5) We shall adopt the classical convention that we sum over a repeated index.

6) All functions that are considered in this work will be supposed to be continuouslydifferentiable a sufficient number of times. They are defined in a certain simply-connected region of space that defines the limits of one's considerations.

7)
$$\frac{\partial I}{\partial u} = \frac{\partial}{\partial u} \int_{P_1}^{P_2} \omega = \int_{P_1}^{P_2} \frac{\partial a_r}{\partial x^s} \frac{\partial x^s}{\partial u} dx^r + \int_{P_1}^{P_2} a_r d \frac{\partial x^r}{\partial u}$$

The last integral becomes:

$$-\int_{P_1}^{P_2} \frac{\partial a_r}{\partial x^s} \frac{\partial x^s}{\partial u} dx^r + \int_{P_1}^{P_2} d\left(a_r \frac{\partial x^r}{\partial u}\right) \, .$$

The formula in the text will result from the fact that the last integral in that expression is zero from the boundary conditions.

8) See E. Goursat, Leçons sur le problème de Pfaff, Hermann, Paris, 1922, Chap. I.

It is useful to observe that the extremal property persists when one takes varied curves that join points P'_1 , P'_2 that are close to P_1 , P_2 , resp., and are defined by starting from them and giving them displacements $(dx^i)_1$, $(dx^i)_2$, resp., that verify:

$$a_r(dx^r)_1 = a_r(dx^r)_2 = 0$$
.

9) E. Cartan, Leçons sur les Invariants Intégraux, Hermann, Paris, 1922. Les systèmes différentiels extérieures et leurs applications géométriques, Hermann, Paris, 1945 (A. S. I. no. 994). Th. De Donder, Théorie des Invariants Intégraux, Gauthier-Villars, Paris, 1927. E. Goursat, loc. cit., note 8. For a modern interpretation, see S. S. Chern, "Some new viewpoints in differential geometry in the large," Bull. Amer. Math. Soc. 52 (1946), 1-30.

10) One does *not* sum over the repeated index r in this formula.

11) For the sake of clarity, one should briefly recall the definition of a μ -uple integral and take the case of $\mu = 2$, to fix ideas.

Let $x^r = x^r(u, v)$ be the parametric equations of a portion of the surface S when the point (u, v) traverses a domain R in the u, v-plane.

When calculating the integral:

$$I_2 = \iint_S \Omega = \iint_S A_{rs} \, dx^r \, dx^s$$

there is good reason to:

1. Replace the x as functions of u, v in the A and to replace the symbolic products $dx^r dx^s$ with the expressions:

$$\frac{\partial(x^r, x^s)}{\partial(u, v)} du dv$$

One will then come down to a double integral:

$$I'_{2} = \iint_{R} A \, du \, dv \,, \qquad A = A_{rs} \frac{\partial (x^{r}, x^{s})}{\partial (u, v)}$$

that extends over the region R in the u, v-plane. In the latter expression, the product du dv will be an ordinary product that is equal to the area of a rectangle whose sides are parallel to the axes and have lengths of du, dv, resp.

2. Choose an "orientation" on the surface *S*, which can be done by choosing a sense of traversal along the line *C* that is the boundary of *S*.



If that sense corresponds on the image curve C' in the u, *v*-plane with the one that leads directly from the positive *u*axis to the positive *v*-axis (Fig. 3) then one sets, by definition:

$$I_2 = I'_2$$

and in the contrary case (Fig. 4), one sets:

$$I_2 = -I'_2$$
,

It is due to the antisymmetry of the (r, s) in the expressions:

$$\frac{\partial(x^r, x^s)}{\partial(u, v)} \, du \, dv$$

that one is led to the rule (2.2)

For more details, see e.g., **E Goursat**, *Cours d'analyse*, vol. I, Gauthier-Villars, Paris and **W. D. H. Hodge**, *The theory and applications of harmonic integrals*, Cambridge University Press, 1941, Chap. II.

12) E. Cartan, E. Goursat, loc. cit., note 9.

13) *Example:* we can associate the hydrodynamical problem that was considered in the introduction (pp. 3), with the Pfaff form that is defined in the three-dimensional space of x^1 , x^2 , x^3 for each constant value of t:

$$\omega = K_1 dx^1 + K_2 dx^2 + K_3 dx^3 = K_i dx^i,$$

with

$$K_i = -\frac{\partial \varphi}{\partial x^i} - v \frac{\partial p}{\partial x^i} \qquad (i = 1, 2, 3).$$

The extremals of $\int \omega$ are then given by:

(a)
$$\left(\frac{\partial p}{\partial x^{i}}\frac{\partial v}{\partial x^{j}}-\frac{\partial p}{\partial x^{j}}\frac{\partial v}{\partial x^{i}}\right)dx^{j}=0,$$

or, if the coefficients of those equations are not all zero (i.e., if the v-field is baroclinic) then by:

$$dv \equiv \frac{\partial v}{\partial x^i} dx^i = 0$$
, $dp \equiv \frac{\partial p}{\partial x^i} dx^i = 0$.

The extremals, or what amounts to the same thing, the characteristics, are nothing but the isobaricisosteric lines.

When the field is barotropic, equations (a) will be satisfied identically, and any curve will be an extremal. That is not surprising, and one notes that the barotropic condition expresses the idea that the integral $\int_{R}^{P_2} \omega$ is independent of the path that joins the points P_1 and P_2 .

14) In the example that was just considered in the preceding note, one had:

$$\rho = 1, \quad \omega = dZ + Y \, dX, \qquad Z = -\varphi, \qquad Y = v, \quad X = p$$

in a baroclinic field, and $\rho = 0$, $\omega = dZ$, $Z = -(\varphi + U)$, $U = \int v dp$ in a barotropic field.

15) The proof can be achieved as follows:

a) Necessary condition. If the closed curve C can be reduced to a point by continuous deformation along the tube T, and therefore without leaving it (case of the contour γ , Fig. 2), or if the surface S is situated entirely on the tube (case of the surface a bounded by γ), one must have $J_1 = I_2 = 0$. Take the case of an infinitely-small parallelogram that is traced on the tube and whose opposite sides coincide with the elements of the trajectories. The value of $J_1 = I_2$ will then reduce to the bilinear covariant:

$$d_1 \ \omega \ (d_2) - d_2 \ \omega \ (d_1) = \frac{1}{2} \left(\frac{\partial a_s}{\partial x^r} - \frac{\partial a_r}{\partial x^s} \right) \begin{vmatrix} d_1 x^r & d_1 x^s \\ d_2 x^r & d_2 x^s \end{vmatrix},$$

in which, for example, d_1 indicates a variation along a trajectory and d_2 indicates a transverse variation. That expression must be zero for any d_1x^r , so the d_2x^r must be proportional to the X^r . Hence, one has the condition (6.2).

b) Sufficient condition. The relation (6.2) implies that $I_2 = 0$ for any portion of the surface that is situated on the tube T; for example, the surface Σ that is generated by displacing C until it occupies the position C'. By virtue of Stokes's formula, that double integral will be the simple integral of ω that is extended over the boundary contour of Σ . That boundary is composed of the combination of the two closed curves C and C', when the latter is traverses in the opposite sense to C. Therefore:

$$\int_{C-C'} \omega = \int_C \omega + \int_{-C'} \omega = \int_C \omega - \int_{C'} \omega = 0.$$
Q. E. D.

16) In the examples of notes 13 and 14, the integrals:

$$\int_{C} \omega = \int_{C} K_{i} dx^{i} \qquad (C \text{ is a closed curve}),$$
$$\int_{S} d\omega = \iint_{S} - dv dp$$

were invariant for just the differential system dv = dp = 0, whose trajectories were the isobaricisosteric lines. Those integrals represent the number N(p, v) of isobaric-isosteric lines that are encircled by the contour C or traversed by the surface S. Those integrals will be identically zero in a barotropic field. 17) More rigorously, consider a family of curves C that depend upon a parameter u and join two points $P_1(q_1^i, t_1)$, $P_2(q_2^i, t_2)$ of \mathcal{E}_{n+1} , and are such that:

1. The curves in the family have equations of the form:

$$q^i = q^i(t,u)$$
 for $t_1 \le t \le t_2$, $u_0 - \varepsilon \le u \le u_0 + \varepsilon$,

in which ε is an arbitrary positive number.

2. The expressions:

$$q_1^i = q^i(t_1, u), \qquad q_2^i = q^i(t_2, u)$$

are independent of u.

3. The functions are both continuously-differentiable in the domain *R* that is defined by the inequalities in 1. (One can make an even-weaker hypothesis in regard to differentiability.)

When one calculates along *C*, the integral (7.1) will become a function I(u). The curve C_0 that corresponds to $u = u_0$ is called an extremal of (7.1) if one has:

$$\left(\frac{\partial I}{\partial u}\right)_0 = 0$$

for any family of curves C that verify the conditions 1., 2., 3., and reduce to C_0 for $u = u_0$.

As one knows, the extremals are given by the Euler-Lagrange equations:

$$\frac{\delta L}{\delta q^i} \equiv \frac{\partial L}{\partial q^i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i} = 0 \; .$$

 A. C. Dixon, "On the relation between Pfaff's problem and the calculus of variations," Proc. London Math. Soc. 7 (1909). – E. Cartan, *Leçons sur les Invariants Intégraux*, Hermann, Paris, 1922.

19) Upon recalling note 8, one will see that in \mathcal{E}_{2n+1} , one can take the variations at the limits such that $dq^i = 0$, $dp_i =$ arbitrary.

The theorem of **Dixon-Cartan** extends to multiple integrals in the calculus of variations. **P. Dedecker**, "Sur les intégrales multiples du calcul des variations," Comptes-Rendus du Congrés national des Sciences de Belgique, sec. Math., Brussels, May-June 1950; Liége, Desoer (1951).

- 20) See H. Boerner, *loc. cit.*, note 3.
- 21) E. Cartan, loc. cit., note 18, Selecta, pp. 104, Gauthier-Villars, Paris, (1939).

22) The coefficient of dt:

$$\left(\frac{\partial P_j}{\partial q^i} - \frac{\partial P_i}{\partial q^j}\right) \left[\frac{\partial H}{\partial p_i}\right] dq^j$$

is zero by virtue of (10.2).

23) Due to the particular form of the function *T*, the contravariant components u^i of the velocity (which are analogous to the Q^i) are equal to the covariant components u_i (which are analogous to the P_i).

24) In order to prove that, make a change of variables:

$$u^i = u^i (q^j, t) \qquad (t'=t),$$

in which the u^i are *n* independent first integrals of the system (9.1). $d[\omega]$ will become a form in du^i , dt with coefficients that are functions of the u^i , *t*. That form will give rise to an absolute double integral invariant for the equations of extremals of the field. Now, it will result that the coefficients of that form cannot depend upon *t* (and that the coefficient of dt of must be zero). As another consequence, the rank of $d[\omega]$ will depend exclusively upon the u^i , which was to be proved.

In terms of the variables u^i , t, one will have $d[\omega] = [i, j] du^i du^j$, in which the [i, j] denote the *Lagrange brackets*, which verify the "Lagrange equations": $\partial / \partial t [i, j] = 0$. See **C. Carathedory**, *Variationsrechnung und partielle Differentialgleichungen erster Ordnung*, Teubner, Leipzig and Berlin, 1935, no. 45, pp. 46.

25) Indeed, in terms of the variables u^i , t of the preceding note, the components du^i , dt (= 0) of that vector will be constants in the course of that motion, and the property will result that the Lagrange brackets will be independent of t.

26) One will observe that when one sets $P_i = \partial S / \partial q^i$, equation (9.2) will be verified automatically if the function S is a solution to the Hamilton-Jacobi equation.

27) C. Caratheodory, *loc. cit.*, note 24, no. 242, pp. 208.

28) A group Γ is a set of elements a, b, c, ... that is endowed with a law of composition that associates any ordered pair of elements a, b with a third element c that one calls the *product* of a and b, and which one writes $a \cdot b = c$. That law of composition must possess the following properties:

1. $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ (associativity).

2. There exists an element *e* (which is called *unity*) such that for any element *a*, one will have $a \cdot e = e \cdot a = a$.

3. Any element *a* corresponds to an element a^{-1} (which is called the *inverse* of *a*) such that $a \cdot a^{-1} = e$. One does not need to have $a \cdot b = b \cdot a$.

The set of transformations $q'^i = q'^i(q^j)$ that are continuously-differentiable a sufficient number of times, have non-zero Jacobian, and transform a certain region of the space of q^i into itself constitutes a group in which the product $T' \cdot T$ of the transformations $T : q'^i = q'^i(q^j)$ and $T' : q''^i = q''^i(q'^j)$ is the transformation $T'' : q''^i = q''^i(q'^k(q^j))$.

If one is given two groups Γ , Γ' then a correspondence that associates any element a in Γ with an element a' in Γ' in such a way that the element of Γ' that is associated with the product $(a \cdot b)$ of two elements of Γ will be the product $(a' \cdot b')$ of the elements a', b' that are associated with aand b, resp., is called a *homomorphism* of Γ into Γ' . If distinct elements correspond to distinct elements, in addition, then one will be dealing with an *isomorphism*.

Any group Γ is associated with a group Γ' that is called its *opposite* and is defined over the same elements but is such that the product $c = a \cdot b$ in Γ' will be equal to the product $b \cdot a$ in Γ , by definition.

The set of square matrices $\tau = \|\tau_i^j\|$ with non-zero determinant corresponds to two opposite groups. In one of them, the product $\rho = \sigma \cdot \tau$ is defined by the matrix $\|\rho_i^j\| = \|\sigma_i^k \cdot \tau_k^j\|$, and in the other one, by the matrix $\|\rho_i'^j\| = \|\sigma_k^i \cdot \tau_i^k\|$. In order to distinguish the two, we will let $\|\tau_i^{j}\|$ denote a matrix when it is considered to belong to the first group, and let $\|\tau_{j}^i\|$ denote the same matrix when it is considered to belong to the second one.

For more details, see a treatise on modern algebra. For example, Van der Waerden, Moderne Algebra, Springer, 1937.

29) The variance of G is defined by the group $\overline{\gamma}$ that is opposite to γ and endowed with the homomorphism \overline{h} of Γ into $\overline{\gamma}$ that is obtained by composing the homomorphism h of Γ into γ with the canonical homomorphism of γ into $\overline{\gamma}: \Gamma \to \gamma \to \overline{\gamma}$.

30) **H. Ertel**, "Hydrodynamische Gleichungen in prae-kanonischer Form und Variationsprincipen der atmosphärischen Dynamik," Meteor. Zeit. (1939).

31) Frobenius, "Ueber adjungierte lineare Differentialausdrücke," J. f. Math. 84 (1878). – Th. De Donder, *Théorie invariantive du calcul des variations*, Gauthier-Villars, Paris, 1935.

32) For the proof of this, see: **A. Hirsch**, Math. Ann. **49** (1897). – **J. Kürschak**, *ibid.* **60** (1965). – **L. Koenigsberger**, *Die Principien der Mechanik*, Teubner, Leipzig, 1961. – **D. R. Davis**, "The inverse problem of the calculus of variations in higher space," Trans. Amer. Math. Soc. **30**

(1928). – L. La Paz, *ibid.*, v. 32. – P. Dedecker, "Sur un problème inverse du calcul des variations," Bull. Acad. Roy. Belg., Cl. Sc. (5) 36 (1950), 63-76.

33) For n = 1, the answer is always affirmative: **G. Darboux**, *Théorie des Surfaces*, Gauthier-Villars, Paris, t. 3, pp. 53.

34) For n = 2, the answer is generally affirmative, but with exceptions: **J. Douglas**, Proc. Nat. Acad. Sc. U. S. A. **26** (1940); Trans. Amer. Math. Soc. **50** (1941).

35) **H. Bateman**, "On dissipative dynamical systems and related variational principles," Phys. Rev. (2) **38** (1931). Similarly, see the second paragraph in the introduction to that author's treatise: *Partial differential equations of mathematical physics* and his article "Hamilton's work in dynamics and its influence on modern thought," Scripta Math. (1944).

F. H. van den Dungen, "Les équations canoniques di résonateur linéaire," Bull. Acad. Roy. Belg. (1945). – J. Géhéniau, "La quantification des systèmes non canoniques," *ibid.* (1945). – **P. Dedecker**, *loc. cit.*, note 4.

36) *Example.* – In the case of the equation in one unknown q = q(t):

$$F = a\ddot{q} + b\dot{q} + cq = 0 \qquad (a, b, c \text{ constant}),$$

one will have:

$$\mathcal{L} \equiv s(a\ddot{q} + b\dot{q} + cq), \qquad G \equiv a\ddot{s} - b\dot{s} + cs.$$

One indeed recognizes that G is the adjoint polynomial to F:

$$s \cdot F(q) - q \cdot G(s) \equiv \frac{d}{dt} a(\dot{q} s - \dot{s} q) + b q s.$$

37) The term "field of solutions," which is a notion that generalizes the "field of extremals," obviously means a triply-infinite family of solutions to equations (20.2) such that one and only one solution to the family will pass through each point in the space (x^i, t) .

38) The argument can be extended to the case in which the rotation is zero.

39) One utilizes the hypothesis here that n (viz., the dimension of the space E) = 3.

It would hardly be necessary to point out that if one studies the condition of the conservation of vorticity in a certain domain that encircles the point *P* at the instant t_0 then that number *k* will become a function of the x^i , *t*.

- 40) **P. Dedecker**, *loc. cit.*, note 4.
- 41) **H. Villat**, *Leçons sur les fluides visqueux*, Gauthier-Villars, Paris, 1943.

42) That fact was proved by a method that is beautiful, but lengthy, by **C. B. Millikan**, "On the study motion of viscous incompressible fluids, with particular reference to a variation principle," Phil. Mag. (1929). – See also, **H. Villat**, *loc. cit.*, note 41, Chap. III.

43) Helmholtz, Wissenschaftliche Abhandlungen, Bd. I, J. A. Barth, Leipzig, 1882, pp. 224.

44) J. Kravtchenko, "Sur un principe variationnel de l'hydrodynamique des fluides visqueux," C. R. Acad. Sci. Paris 213 (1941), pp. 977. – H. Villat, *loc. cit.*, note 41.

Those authors suppose that the K_i are derived from a potential, which is a superfluous hypothesis, as one can see.

45) We intend the terms *incompressible* and *homogeneous* to mean that ρ is constant in the mass of the fluid. By virtue of the continuity equation and the existence of the potential Φ , one will then have:

$$\frac{\partial u^i}{\partial x^i} = 0 , \qquad \rho K_i = \frac{\partial}{\partial x^i} (\rho \Phi) .$$