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The Helmholtz-Cartan theorem for a simple integral of higher order

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Abstract. – The equivalence between the *principles of classical mechanics* and the theorem that Élie Cartan called the *principle of conservation of impulse-energy* (E. Cartan’s relative integral invariant) originates in the possibility of replacing *Hamilton’s principle*, viz., $\delta I = 0$, $I = \int L(t, q^i, dq^i / dt)\,dt$ [a first-order variational problem P_1 in (t, q^i) -space] with a stronger and zero-order *variational principle* P_0, $\delta \int \omega = 0$, $\omega = L\,dt + \sum (\partial L/\partial \dot{q}^i)[dq^i - \dot{q}^i\,dt]dt$ $= \sum p_i\,dq^i - H\,dt$ in the phase space $(t, q^i, \dot{q}^i)$ [([1]), Chap. XVIII]. That possibility is extended here (and consequently all of the formalism of Hamilton-Jacobi-E. Cartan) to variational problems of order $s > 1$ whose Lagrangians L depend upon derivatives $d^k q^i / dt^k$ of order $k < s$ (Theorems 2.2 and 2.3). The result is subject to the local invertibility of a higher Legendre transformation that was used recently by Gelfand-Dikii ([6]) and S. Sternberg ([11]), that was already familiar to Th. de Donder (4) half a century ago, and the origin of which goes back to at least the manuscripts that were left by Carl G. J. Jacobi (1804-1851) and published posthumously by A. Clebsch ([8]).

1. – The method that Élie Cartan described in his *Leçons sur les invariants intégraux* (pages VII, 7, 13, 186) brings a fundamental element to the Hamiltonian formalism by identifying it with the integral invariants of Henri Poincaré, when duly and considerably enriched. It thus completes and clarifies one essential point in a set of theories or “doctrines” that are closely linked or equivalent to each other and constitute what some are calling the “Hamilton-Cartan formalism” and is probably more rightly called *the formalism of Euler-Lagrange-Hamilton-Jacobi-Cartan*, namely:

1. Classical analytical mechanics.
2. The second-order Euler-Lagrange equations.
3. The Legendre transformation and Hamilton’s first-order canonical differential equations.
4. The variational problem P_1 of order one that we will call the *weak Hamilton principle.*
5. The variational problem P_0 of order zero, or *strong Hamilton principle.*
6. The theorems of the relative integral invariant of order one and the absolute integral invariant of order two (viz., the principle of the conservation of energy-momentum).
7. The Hamilton-Jacobi partial differential equations.

8. The contact geometry of phase space, the symplectic geometry of extremal space.
9. The principles of Huyghens and Fermat, geometric optics, and the set to which they must be attached.
10. The mechanics of fluids, not just perfect ones, but also baroclinic or viscous ones ([4]). Elasticity, thermodynamics, field theory.
11. The foundations of quantum mechanics, etc.

Technically, the equivalence between the weak Hamilton principle P_1 and the strong one P_0 is fundamental, but it is hardly easy to recover its historical origin because few of the authors cited their sources. That equivalence is very clear in the work ([1]) of Élie Cartan, Chapter XVIII, no. 184, pp. 186 ([12]), but he had his antecedents. For example, L. Königsberger ([9]) associated the problem P_1 in the space (t, q^i) with the variational problem Q_1 of order one in $(t, q^i, \dot{q}^i)$ that is defined by the Lagrangian $\mathcal{L} = \mathcal{L}(t, q^i, \dot{q}^i, dq^i / dt, d\dot{q}^i / dt)$:

$$\mathcal{L} = L(t, q^i, \dot{q}^i) + \sum_i \frac{\partial L}{\partial \dot{q}^i}\left(\frac{dq^i}{dt} - \dot{q}^i\right).$$

Now, the search for curves $\theta \equiv q^i = f^i(t)$, $\dot{q}^i = g^i(t)$ in that space that are extremals of $\int \mathcal{L}\, dt$ (and are parameterized by t, by definition) is obviously equivalent to the search for those of the extremals of P_0 that are parametrized by t. Indeed, one will obviously have $\int_\theta \mathcal{L}\, dt = \int_\theta \omega$ for such a curve θ. However, Königsberger [who actually stated a more-general result that would be valid for higher-order Lagrangians (see above)] gave no indication of the origin of his result. It seems that the equivalence of the problems P_1 and Q_1 is due to Hermann L. von Helmholtz ([7]), pps. 161 and 221, which is why the equivalence of the principles P_1 and P_0 deserves to be referred to by the name of the *Helmholtz-Cartan theorem*.

2. – Consider the space of (t, q^i) to be the space $J^0 = J^0(\mathbb{R}, \mathbb{R}^n)$, which is fibered over $\mathbb{R}$, of jets of order zero of maps from $\mathbb{R}$ into $\mathbb{R}^n$. The space $J^s = J^s(\mathbb{R}, \mathbb{R}^n)$ of jets of order s is endowed with coordinates $(t, q^i, q^i_{(1)}, \ldots, q^i_{(s)})$. L. Königsberger ([9]) considered a Lagrangian L of order s, $L : J^s \to \mathbb{R}$, and the variational problem $P_s = P_s(J^0, L)$ of order s in J^0 that is attached to the integral:

$$I_s(\gamma) = \int_\gamma L\left(t, q^i, \frac{dq^i}{dt}, \ldots, \frac{d^s q^i}{dt^s}\right) dt,$$

in which $\gamma : \mathbb{R} \to J^0$ is a section of $J^0 \to \mathbb{R}$. Upon identifying J^s with $\bar{J}^0 = J^0(\mathbb{R}, \mathbb{R}^{(s+1)\cdot n})$, he associated it with the problem $Q_s = \mathcal{P}_s(\bar{J}^0, \mathcal{L})$ of the same order s in $J^s = \bar{J}^0$ that is defined by the Lagrangian $\mathcal{L}$: $J_4(\bar{J}^0) \to \mathbb{R}$:

$$\mathcal{L} = L + \sum \frac{\partial L}{\partial q^i_{(1)}} \left(\frac{dq^i}{dt} - \dot{q}^i_{(1)}\, dt \right) + \cdots + \sum \frac{\partial L}{\partial q^i_{(4)}} \left(\frac{d^4 q^i}{dt^4} - q^i_{(4)} \right) .$$

Theorem 2.1 (Helmholtz-Königsberger):

Under the condition K_s that the $ns \times ns$ matrix $\left\| \dfrac{\partial^2 L}{\partial q^i_{(k)}\, \partial q^j_{(l)}} \right\|$ $(1 \leq k, l \leq s)$ *is non-singular, the problems P_s and Q_s will be equivalent. More precisely, the extremals of Q_s are subject to the equations* $\dfrac{d^k q^i}{dt^k} = q^i_{(k)}$ $(1 \leq k \leq s)$ *and to the Euler-Lagrange equations* $\dfrac{\delta L}{\delta q^i} = 0$ *of order* $2s$.

On the other hand, consider the following Pfaff form on the space $J^{2s-1} = J^{2s-1}(\mathbb{R}, \mathbb{R}^n)$ of jets of order $2s - 1$ ($q^i_{(0)} = q^i$):

$$\Omega = L\, dt + \sum \frac{\delta L}{\delta q^i_{(1)}} (dq^i_{(0)} - q^i_{(1)}\, dt) + \cdots + \sum \frac{\delta L}{\delta q^i_{(s)}} (dq^i_{(s-1)} - q^i_{(s)}\, dt) ,$$

$$\frac{\delta L}{\delta q^i_{(s)}} = \frac{\partial L}{\partial q^i_{(s)}} , \qquad \frac{\delta L}{\delta q^i_{(k)}} = \frac{\partial L}{\partial q^i_{(k)}} - \frac{d}{dt} \frac{\delta L}{\delta q^i_{(k+1)}} , \qquad k = s - 1, \ldots, 0 .$$

It corresponds to the variational problem P_0 of order zero $\delta \int \Omega = 0$ in J^{2s-1}.

Theorem 2.2. (generalized Helmholtz-Cartan theorem).

Under the condition C_s that the $(n \times n)$ matrix $\left\| \dfrac{\partial^2 L}{\partial q^i_{(s)}\, \partial q^j_{(s)}} \right\|$ *is non-singular, the problems P_s and P_0 will be equivalent, in the following sense: The extremals of P_0 are parameterized by t and verify the equations* $dq^i_{(k)} - q^i_{(k+1)}\, dt = 0$ $(0 \leq k \leq s)$, *as well as the Euler-Lagrange equations, and conversely.*

That statement resolves a problem that was posed in ([2]), pp. 152 for the simple integrals. Its extension to multiple integrals of higher order (which posed in *ibidem*) will be treated later.

One should note that for $s > 1$, there is no direct relationship between the integrals $\int \mathcal{L}\, dt$ and $\int \Omega$, and that the conditions K_s and C_s will coincide for only $s = 1$. *The condition C_s is S. Sternberg's* ([11]) ***regularity*** *condition, which is related to Gelfand-Dikii's* ([6]) ***normality*** *condition.*

That condition also ensures the local invertibility of the higher *Legendre transformation* in J^{2s-1}:

$$(t, q^i_{(0)}, \ldots, q^i_{(s-1)}, q^i_{(s)}, \ldots, q^i_{(2s-1)}) \to (t, q^i_{(0)}, \ldots, q^i_{(s-1)}, p_i^{(s-1)}, \ldots, p_i^{(0)}) \ ,$$

$$p_i^{(s-1)} = \frac{\partial L}{\partial q^i_{(s)}}, \qquad p_i^{(s-2)} = \frac{\delta L}{\delta q^i_{(s-1)}}, \qquad \ldots, \qquad p_i^{(0)} = \frac{\delta L}{\delta q^i_{(1)}} ,$$

which will give the form Ω the expression:

$$\Omega = \sum_{i=1}^{n} \sum_{k=0}^{s-1} p_i^{(k)} dq^i_{(k)} - H\,(t, q^i_{(0)}, \ldots, q^i_{(s-1)}, p_i^{(0)}, \ldots, p_i^{(s-1)})\, dt ,$$

and the differential system for the extremals will have the *canonical Hamiltonian form:*

$$dq^i_{(k)} = \frac{\partial H}{\partial p_i^{(k)}} dt , \qquad dp_i^{(k)} = - \frac{\partial H}{\partial q^i_{(k)}} dt ,$$

whose general solution can be deduced from a *complete integral* $S : J^{s-1} \times \mathbb{R}^{s \cdot n} \to \mathbb{R}$ (a notion due to Lagrange) of the *Hamilton-Jacobi partial differential equation:*

$$\frac{\partial S}{\partial t} + H\left(t, q^i_{(0)}, \ldots, q^i_{(s-1)}, \frac{\partial S}{\partial q^i_{(0)}}, \ldots, \frac{\partial S}{\partial q^i_{(s-1)}} \right) = 0 \ .$$

The condition C_s then ensures that the set of extremals constitutes a one-dimensional foliation in J^{s-1} (viz., an *extremal foliation*), that the form Ω will induce a contact structure on it, and that its differential $d\Omega$ will define a symplectic structure on the $2^{n \cdot s}$-manifold of leaves (i.e., extremals) [Theorem ?.1 in ([11])]. Up to now, one can see that some of the properties of this section are found to have been mentioned explicitly by C. Jacobi ([8]), although of course, in the language of his era. In addition, one has the following extension to the higher-order problems of Élie Cartan's *principle of the conservation of the energy-momentum:*

Theorem 2.3:

The extremal foliation of J^{s-1} admits the relative integral invariant Ω and the absolute integral invariant $d\,\Omega$. Conversely, any foliation of J^{s-1}, regardless of dimension, that admits one of those integral invariants will necessarily coincide with the extremal foliation.

3. – The Pfaff form Ω occurs naturally in the classical calculus of the variation Δ of the integral $\int L\,dt$ between two neighboring curves AB and $A'B'$ in the space $J^0 = (t, q^i)$, the first of which is the extremal: $\Delta = \delta \int L\,dt = [\Omega]_{AA'}^{BB'} = [\Omega]_{\delta A}^{\delta B}$. For $s = 1$, it appeared in Hamilton's work in 1834. That formula also gives an immediate proof of the direct property of the relative integral invariant (but not its converse).

The form Ω on J^{s-1} can also be characterized by the *Lepage congruences* ([10]):

1. $$\Omega \equiv L\,dt \bmod (I)\,,$$

2. $$d\,\Omega \equiv 0 \bmod (I),$$

in which I is the ideal of differential forms on J^{s-1} that is generated by $dq_{(0)}^i - q_{(1)}^i\,dt$, …, $dq_{(2s-2)}^i - q_{(2s-1)}^i\,dt$, which are congruences for which S. Sternberg gave an interpretation that was adapted to the space J^∞ of infinite-order jets in which they live [(11), Th. 2.1]. Although the first condition is imposed automatically, neither Lepage nor Sternberg gave a physical or geometric justification for the second one. Meanwhile, there is obviously some interest in deducing the form Ω from just the first condition, thanks to the following property:

Proposition 3.1:

Let $\bar{J}^{2s-1} = J^{2s-1} \times \mathbb{R}^{n\cdot s} \to J^{2s-1}$ be the fiber manifold of (point-like) 1-forms that satisfy the first Lepage congruence, which is a manifold that is canonically endowed with a semi-basic Pfaff form $\bar{\Omega}$. There exists one and only one section s : $J^{2s-1} \to \bar{J}^{2s-1}$ that contains all of the extremals of the variational problem $\bar{P}_0 \equiv \delta \int \bar{\Omega} = 0$ in $\bar{J}^{2s-1}$, which are naturally identified with the extremals of the problem P_s, and one will have $\Omega = \sigma^* \bar{\Omega}$.

N. B. – The second Lepage congruence means that $\delta \int_\theta \Omega = 0$ when Ω is integrated along the canonical lift $\theta = j^s\gamma : \mathbb{R} \to J^{2s-1}$ of a section $\gamma : \mathbb{R} \to J^0$, and one confines oneself to variations of θ that project to zero in J^0.

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12. In which one must obviously replace "I" with "J = ∫ ?" in the sixth line from the bottom.

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