"Beiträge zur Analysis Situs. I. Aufsatz. Ein- und zweidimensionalen Manifigfaltigkeiten," Math. Ann. **32** (1888), 457-463.

Contributions to analysis situs.

Part I.

One and two-dimensional manifolds.

By

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(With three lithographed tables)

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Introduction.

In the investigations that are contained in the following pages (in this initial Part I), geometric, as well as analytical data, are treated with equal consideration in order to produce to a systematic development of those properties that manifolds possess in the sense of analysis situs, so they will refer exclusively to their classifying relationships.

In that study, one must distinguish absolute from relative properties according to whether a manifold possesses them intrinsically or only in conjunction with other manifolds (*). The absolute properties can also be referred to as the ones whose coincidence for two manifolds is necessary and sufficient for one to exhibit an invertible, single-valued, continuous relationship between *all* elements of the two manifolds (**).

(*) That distinction was probably first emphasized in the treatise of **Klein** "Ueber den Zusammenhang der Flächen," (Zweite Abhandlung), these Annals **9** (1875), pp. 478.

(**) Let it be stressed here that we will then be dealing with *continuous relationships between continuous manifolds*, i.e., with relationships under which neighboring elements go to neighboring elements. Hence, relationships like the **Cantor-Lüroth** maps of manifolds of differing dimensions to each other [in particular, cf., the article by **Cantor** "Ein Beitrag zur Manifaltigkeitslehre" in Bd. 85 of Crelle's Journal (1877), and also the notice in Göttinger Nachrichten (1879), and then the articles by **Lüroth** in the Erlanger Berichten (1878) and in the Math. Ann. **21** (1882), "Ueber eine eindeutige Entwicklung von Zahlen in eine unendliche Reihe"] are excluded from the outset for being *discontinuous*.

One should probably further observe that the agreement between the absolute properties of two manifolds will *no longer* be required when we demand only single-valued, invertible relations, *in general*, so when we allow singular elements. One can then confer **Klein**'s article that we shall mention further "Ueber den Zusammenhang der Flächen" (Erste Abhandlung), Math. Ann. **7** (1874), pp. 554, in which the appearance of fundamental points and associated fundamental curves is considered for the case of single-valued, invertible algebraic relationships between two surfaces.

For two dimensions, the *geometric representation of the absolute properties* is well-known in the theory of the connection of surfaces, as it was presented by **Riemann** and **Neumann** (*), as well as in the investigations that are connected with **Euler** and **L'Huilier** (**), and their extensions to more dimensions by **Listing** (***) and **Möbius** (†). Finally, **Betti** (††) gave formulas analogous to Riemann's for three and more dimensions, to which **Picard** (†††) returned in recent times.

The *geometric investigations into relative properties* include the work done regarding the forms and linking of planar and spatial curves by **Listing**, **Tait**, **F. Meyer**, **Simony** (‡), *et al.*

On that topic, see also **Klein**'s further two treatises "Ueber ein neuer Art Riemann'scher Flächen" in vols. VII and X of these Annals.

The theorem that for two-dimensional manifolds (i.e., surfaces), the coincidence of the soon-to-be-discussed "connection number" and the number of one-dimensional boundaries (i.e., boundary curves) on them is the only absolute property of manifolds (surfaces) that consist of pieces that is necessary and sufficient for the existence of an invertible single-valued relationship between all elements of those manifolds was probably first proved by **C. Jordan**. For that, cf., the treatise "Sur la déformation des surfaces" in Liouville's Journal (2) **11** (1866). For a more precise formulation, see also § 5, pp. 488 in the following Part II.

(*) **Riemann**, "Theorie der Abel'schen Functionen," (1857), *Werke*, pp. 84, *et seq.*, as well as "Fragment aus der Analysis situs," pp. 146, *et seq.*

Neumann, *Riemann's Theorie der Abel'schen Functionen*, 1st ed., 1865, pp. 291, *et seq.*, 2nd ed., 1884, pp. 146, *et seq.*

As far as the properties of special forms of "Riemann surfaces" are concerned, let us also mentioned the investigations of **Lüroth** and **Clebsch** in Bd. IV and VI of these Annals on the exhibition of certain canonical forms, and furthermore, the form of the "Riemann surface lying freely in space" that was first used by **Klein**, as well as the one that was given in the aforementioned treatise "neue Art von Riemann'schen Flächen."

Moreover, let a treatise of **Clifford** be mentioned here "On the canonical form and Dissection of a Riemann's surface," Proc. London Math. Soc. **8** (1877), as well as a recently-appeared booklet by **F. Hofmann** "Methodik der Stetigen Deformation von zweiblättrigen Riemann'schen Flächen," Halle a/S., Nebert's Verlag 1888, in which the conversion of such surfaces into canonical form was discussed in an especially intuitive way. Allow me to correct a citation in regard to one section of Hofmann's presentation (no. 8, pp. 27, *et seq.*). The concept of double-sided surfaces that was given there does *not* refer to the "new type of Riemann surface" that **Klein** gave in the aforementioned locations (Annallen VII and X), but rather to the essential-distinct conception of a double-sided Riemann surface as a symmetric surface that is connected with **Schwarz** and **Schottky**, such that one now deals with simple or so-called double surfaces. For that, cf., the investigations of **Schwarz** in Crelle's Journal **70** (1869) and **75** (1872) and the Berliner Monatsberichten for 1865. Furthermore, cf., **Schottky** in Crelle's J. **83** (1877), as well as the discussion by **Klein** in *Riemann's Theorie der algebraischen Functionen*, Leipzig, 1882, pp. 78. *et seq.*

In regard to the way of looking at things that is connected with Riemann, let us mention the treatise of **Lippich**, "Untersuchungen über den Zusammenhang der Flächen," Math. Ann. Bd. VII, **Schläfli's** article "Ueber die linearen Relationen zwischen den 2*p* Kreiswegen erster Art, etc." in Crelle's Journal **76** (1873), as well as the relevant section in **Clebsch-Gordan**'s *Theorie der Abel'schen Functionen*, 1866.

(**) As **Baltzer** has remarked (Monatsberichte der Berliner Akademie for 1861), **Euler**'s polyhedron theorem was contained already in a fragment by **Descartes**. **Poinsot** gave the extension of Euler's theorem to star-like polyhedra (multiple spherical covering) in 1809 in issue 10 (v. 5) of the Journal de l'école polyt., while the first exposition of the further "singular cases" of Euler's theorem goes back to **L'Huilier** [Mémoires de l'Acad. de St. Petersbourg (1811), Gergonne's Annals **3** (1812)].

(***) **Listing**, "Census räumlicher Complexe," Göttinger Abh. **10** (1861), as well as Göttinger Nachr. (1867).

(†) **Möbius**, *Werke*, Bd. II, and indeed: "Theorie der elementaren Verwandschaft," Sitz. Ges. Wiss. Leipzig **15** (1863), "Ueber die Bestimmun des Inhalts eines Polyeder," *ibidem*, **17** (1867), "Zur Theorie der Polyeder und der Elementarverscwandschaft," published for the first time as a supplement to his collected works.

(††) **Betti**, "Sopra gli spazi di un numero qualunque di dimensioni," Ann. di. Mat. (2) **4** (1870).

(†††) **Picard**, "Sur les fonctions hyperabéliennes," Liouville's J. de Math. (4) **1** (1885), pp. 106.

(‡) See: **Listing**, in particular, in the "Vorstudien zur Topologie," Göttinger Studien (1847).

Tait, in the great work "On knots," Trans. Roy. Soc. Edinburgh (1879), then (1884) and (1886), and in the Proceedings of the same Society (1876-79).

Analytical formulations of certain properties of the position of planar and spatial structures go back to **Gauss**, above all [the proof of the fundamental theorem of algebra in 1799 and 1849, definition and representation of the *curvatura integra* of surfaces, and then especially the potential theorems in his "Theoria attractionis…" (1813) and in the "Allgemeine Lehtsätze…" (1839) (*Werke*, Bd. V), and finally the analytical representation of the linking of two curves ([*]) (1833)], and furthermore, to **Cauchy** (the so-called Cauchy integral, his suggestion for a proof of the fundamental theorem of algebra). On the other hand, there is the theorem of **Sturm** in determining the number of real roots of an algebraic equation between given limits and the investigations that were connected with it by **Sylvester**, **Hermite**, **Jacobi**, **Brioschi**, *et al.* ([**]).

The work by **Kronecker** regarding **Sturm** functions and systems of functions of several variables ([***]) is connected with those investigations. His characteristic of a system of functions gave, on the one hand, the most general formulation of **Sturm**'s investigations for arbitrarily many variables that had been suggested by **Sylvester**, and on the other hand, established the connection

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A series of treatises by **Kirkmann** are connected with that, which are likewise in the aforementioned volumes of the Transactions and Proceedings, as well as by **Little** in the Transactions of the Connecticut Academy (1885).

Franz Meyer, in his inaugural dissertation "Anwendungen der Topologie auf die Gestalten der algebraischen Curven," (1878) and "Ueber algebraische Knoten" in the Proc. Roy. Soc. Edinburgh (1885-86).

Simony, "Neue Tatsachen aus dem Gebiete der Topologie," Math. Ann. Bd. XIX and XXIV, as well as a series of treatises in the Sitzungsberichte der Wiener Akad. in the years 1881, 82, 83, 87, and related works of **Koller** and **Schuster**, *ibidem*.

Analytical representations of space curves with knots were given by **Brill** (these Annals, Bd. XVIII, which assigns that property to the algebraic space curves of lowest – viz., fifth – order), and then by **Hoppe**, **Durége**, **Schlegel** [Grunert's Archiv, Bd. 64 and 65, Wiener Akademie-Berichte **82** (1880), Schlömilch's Zeitschrift, Bd. 28].

([*]) For the relationship between the integral for the linking of two curves that **Gauss** gave (*Werke*, Bd. V, pp. 605) to the theory of galvanic currents, one should see the remarks of **Schering** at the conclusion of the cited volume, as well as the dissertation by **Böddiker** that Schering suggested "Erweiterung der Gauss'schen Theorie der Verschlingungen." Furthermore, **Maxwell**'s *Treatise on Electricity and Magnetism*, 2nd ed., 1881, §§ 417-422, and the treatises of **Thomson** "On vortex motion" and "On vortex statics," Trans. Roy. Soc. Edinburgh **25** (1869), esp. **27** (1875).

Mention should also be made of the related physical investigations in which the appearance of surfaces and bodies of higher connection were called upon for the behavior of potential functions. In addition to the aforementioned treatises of **Thomson**, one should also see the remark of **Helmholtz** in the treatise "Ueber Integrale der hydrodynamischen Gleichungen, welche den Wirbelbewegungen entsprechen," Crelle **55** (1858), pp. 27, as well as **Riemann**'s lectures on "Schwere, Electricität und Magnetismus," 1876, § 83, *et seq.*; in the aforementioned works of **Maxwell**, articles 18-28, 100, 113, in the book by **Lamb**, *A treatise on the mathematical theory of the motion of fluids*, 1879, articles 53-59, 67, 120, and note B.

([**]) **Sturm**, in Bulletin der Férussac **11** (1829), pp. 419, and more thoroughly in the Mémoires présentés par divers savants, vol. VI, 1835.

Sylvester, in particular, in "On a theory of syzygetic relations of two rational integral functions," Phil. Trans. London, Part III (1883).

Hermite, "Sur l'extension du théorème de M. Sturm," C. R. Acad. Sci. Paris **35** (1852), pp. 52, **36** (1853), pp. 294, as well as **Jacobi**, Crelle's Journal Bd. 30, pp. 127, *et seq.* and Bd. 53, pp. 265, *et seq.* (*Werke*, Bd. III). See also the historical remarks of **Borchard**, likewise in Crelle's Journal, Bd. 53.

Brioschi, in particular, the two treatises in the Nouvelles Annales de Mathématique, **13** (1854) and **15** (1856), "Sur la fonctions de Sturm" and "Sur les séries qui donnent le nombre de racines réelles, etc."

Other literature that one can confer is the survey by **Hattendorf**: *Die Sturm'schen Functionen*, Hannover, 1874, and in particular, the similarly-titled treatises by **Kronecker.**

([***]) **Kronecker**, "Ueber Systeme von Functionen mehrer Varibeln," Montsberichte de Berliner Akad., pp. 159 and 688, and then "Ueber die verschiedenen Sturm'schen Reihen und ihre gegenseitigen Beziehungen," *ibidem*, (1873), pp. 117, and furthermore, "Ueber Sturm'schen Functionen," and "Ueber die Characteristik von Functionensystemem," *ibidem* (1878), pp. 95 and pp. 145, resp.

between those investigations to the other ones by **Gauss** and **Cauchy** with his representation of the characteristic as a certain integral. At the same time, the foundations for the analytical treatment of all questions of analysis situs was given by it.

The present treatise (*) links the geometric and analytical treatments of the *absolute characteristics of one and two-dimensional manifolds*. The path that is taken for doing that then leads further to certain *relative properties* that relate to systems of curves on arbitrary surfaces, as well as systems of surfaces in our space.

I. *The geometric derivation of a characteristic number* (which is treated in section I) will be connected with a purely formal presentation of those manifolds whose basic features will be presented here, just as they are for all dimensions.

We start from certain elementary structures and fix certain typical processes for converting them. We then:

a) assign the *elementary structure* with the characteristic number 1, and

b) the *conversion processes* that are initially employed to represent new elementary structures (annihilate the one that is present, respectively) can then distinguish between positive and negative operations, as well as denote the individual processes with the numbers + 1 and – 1 according to their influence on the manifolds that are produced. We shall preserve that enumeration of the conversion processes even when they produce *other* changes to a manifold than the aforementioned ones, and when we:

c) establish the enumeration of the successive operations that are applied to an elementary structure 1 by their sum, we will arrive at a *characteristic number* for the manifold that is produced.

The number that is obtained will take on its fundamental significance from the fact that we can recognize its *independence of the creation process that was established in each case* and can likewise recognize the property of *immediate additivity*, when we are dealing with an aggregate of manifolds of the same dimension. I would also like to emphasize here that ***no** difference exists between the simple and so-called double manifolds in the enumeration that is developed here for two dimensions* (for a more detailed discussion of that, confer, in particular, §§ 3 to 5 of Part Two), and that corresponds to the fact that the double manifolds, as such, cannot be defined by the characteristic, in general (**).

(*) The further development of two talks that were presented to the sächs. Gesellschaft der Wissenschaften zu Leipzig (July 1885 and February 1885), "Beiträge zur Analysis situs I and II."

(**) Double surfaces were probably first exhibited by **Möbius** in his "Theorie der Polyeder und der Elementarverwandschadt." Confer his *Werke*, Bd. II, pp. 484. According to remark by **Schläfli**, the extension of the enumeration of the connection to double surfaces was first carried out by **Klein**, Annalen Bd. VII, pp. 550, *et seq.* On that subject, one should also confer the aforementioned paper by **Klein** "Ueber Riemanns Theorie der algebraischer Functionen und ihrer Integrale," § 23, as well as the note "Ueber die conforme Abbildung von Flächen," Annalen XIX, pp. 159, and finally the dissertation by **Weichold** "Ueber symmetrische Riemann'schen Flächen und die Periodicitätsmoduln der zugehörigen Abel'schen Normalintegrale erster Gattung," Dresden 1883 (published in Schlömich's Zeitschrift, Bd. 28).

The characteristic number that is obtained is essentially no different from the well-known Riemann connection number and its modifications and deviates from it only by the direction in which one counts it, so to speak. I have discussed the reciprocal relationships thoroughly in § 4 (Part II), and also spoken there of the basis for why it did not seem unjustified to me to define the numbering that we encountered here in place of the Riemann-Neumann one by following the systematic development of the number that was given here and the analytical representation of it that shall be discussed shortly.

II. *In order to formulate the characteristic analytically* (which we shall treat in section II), one then deals with the question of also analytically formulating a basis for the enumeration of continuous creation processes for the manifolds that are given by equations and inequalities. To that end, we consider the gradual conversion of a manifold into a simply-infinite, continuous system of such manifolds. The characteristic will generally remain unchanged by it, but in particular, it will also directly create jumps (*Sprungstellen*) at which a change will take place in the sense of the aforementioned operations (in the positive or negative sense, resp.) The enumeration will then be produced directly by a summation over certain singular locations that are defined by equations and inequalities whose "point characters" refer to the sense of those operations – i.e., as a summation in the **Kronecker** sense over the point characters of certain locations that are defined by a system of functions. *Our geometrically-derived numbers directly prove to be **Kronecker** characteristics*, with the meaning that was developed in the aforementioned treatises, and in the present investigations into the connection numbers of curves and surfaces with the enumerations that one encounters in three-dimensional space, I believe that they define an especially intuitive example of that general theory of characteristics.

Moreover, the consideration of systems of manifolds at whose singular locations one encounters jumps in the characteristic of the enumeration will lead to remarkable *relations between such singular locations*, insofar as the enumeration proves to be independent of the special system that it is based upon. Special formulations of those relations (which are expressed here for arbitrary systems of curves in the plane and arbitrary surfaces, as well as for arbitrary systems of surfaces in our space) are known, and probably lead back to the aforementioned works of **Möbius**, above all, who virtually based his enumeration of surfaces upon certain distinguished systems of curves on them ([*]). In fact, the analytical side of **Kronecker**'s developments, as well as the geometrical side of those of **Möbius** were also what led me to the following investigations.

([*]) See the aforementioned treatises of **Möbius**, in particular, the "Theorie der elementaren Verwandschaft" (*Werke*, Bd. II, pp. 540, *et seq.* and 462, 465, *et seq.*). Furthermore, confer **Gauss**'s remarks about it in "Theorie des Erdmagnetismus," art. 12 (*Werke* Bd. V, pp. 134, *et seq.*), as well as a treatise by **Reech** "Démonstration d'une propriété générale des surfaces fermées," in v. 21 (issue 37) of the Journal de l'école Polytechnique (1858), as well as the treatises of **Poincaré**, "Sur les courbes définies par une équation différentielle," in the Journal de l'école Normale (3) **7** (1881), **8** (1882), (4) **1** (1885), and **2** (1886), where the relations that were described in § 11 of the present treatise were treated. Finally, confer a remark by **Klein** in "Riemann's Theorie," pp. 39. Relations concerning the type and number of singular locations of potential functions (on surfaces and in space) are also treated in the aforementioned physical investigations. On that topic, let us mention **Betti**'s considerations regarding the number of singular locations for a potential function on the simply-connected outer surface of a magnetic body (**Betti**, *Lehrbuch der Potentialtheorie*, German version by **Franz Meyer**, Stuttgart, 1885, pp. 334, *et seq*.)