"Graphische Veranschaulichung der De Broglie Phasenwellen in der fünfdimensionalen Welt von O. Klein," Zeit. Phys. **39** (1926), 495-498.

Graphical illustration of De Broglie's phase waves in O. Klein's five-dimensional universe ([1])

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With one figure

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De Broglie's conception of electrons as groups of phase waves is interpreted in **Klein**'s theory in the case of force-free motion and the connection is illustrated graphically.

§ 1. – According to **de Broglie**'s ideas ([2]), the motion of an electron "in reality" amounts to the propagation of groups of waves in a *dispersive* ether that lies in the ordinary *four*-dimensional universe. **Schrödinger** ([3]) then extended that wave theory of matter quite appreciably. **O. Klein** also arrived at similar thoughts independently. The most important difference is that for him the waves propagate in a *dispersion-less* medium that lies in a *five*-dimensional universe ([4]). The **de Broglie** waves are then the "traces" of those five-dimensional waves in ordinary space.

Here, we will consider that connection more closely in the case of force-free motion and illustrate it graphically.

§ 2. – According to **Klein**, in the case of the *force-free motion* of an electron, the five-dimensional phase waves are *plane* waves that first of all satisfy the *dispersion-less* wave equation:

$$\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}+m^2c^2\,\frac{\partial^2}{\partial x_0^2}\right]U = 0\,, \tag{1}$$

([1]) **O. Klein**, Zeit. Phys. **37** (1926), 895.
([2]) **L. de Broglie**, Ann. de Phys. (10) **3** (1925), 22.
([3]) **E. Schrödinger**, Ann. Phys. (Leipzig) **79** (1926), 361, 489, 734.
([4]) Which, as **Klein** assumed, is periodic in its fifth dimension with a period that is connected with **Planck**'s constant.

in which x_0 means the fifth dimension. Secondly, the x_0-direction has the prescribed period h.

We can then represent those phase or U-waves by:

$$U = u\, e^{\frac{2\pi i}{h}(h\nu t - px - qy - rz - x_0)}, \tag{2}$$

in which the periods $1 / \nu$, h / p, h / q, h / r in the t, x, y, z directions, resp., are coupled with each other by:

$$p^2 + q^2 + r^2 + m^2 c^2 = \left(\frac{h\nu}{c}\right)^2, \tag{3}$$

as a result of (1). Their "traces" in the ordinary universe ([1]) are also once more plane waves with the superluminal velocity:

$$\upsilon_{ph} = \frac{h\nu}{c\sqrt{p^2 + q^2 + r^2}}, \tag{4}$$

and therefore, from (3), with the *de Broglie dispersion law:*

$$\upsilon_{ph} = \frac{h\nu}{\sqrt{h^2 \nu^2 - m^2 c^4}}. \tag{5}$$

§ 3. – When we take just *one* of the *three* ordinary spatial dimensions x, y, z – say x – we can then illustrate the relationship graphically.

If we write ([2]):

$$\tau = c\,t, \qquad \xi_0 = \frac{x_0}{m\,c}$$

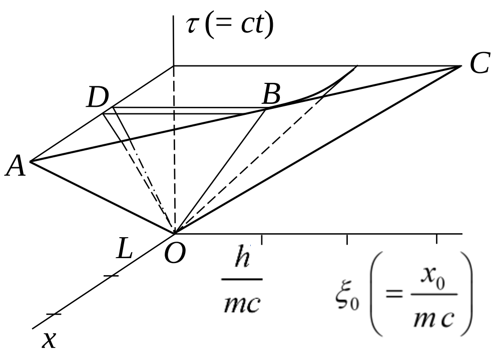

Figure 1.

then one will see, first of all, that the planes of equal phase in (x, τ, ξ_0)-space always define an angle of 45° with the τ-axis, and will thus be tangent to the cone $x^2 + \xi_0^2 - \tau^2 = 0$ at the origin and secondly, it repeats in the ξ_0-direction with a period of h / mc.

The periods in the x and τ-directions are then coupled to each other by (2). Both of them will also be established when one gives the tangent OB.

([1]) I. e., its intersection with an R_4 that is established by x_0 = const.

([2]) For protons, one must replace m with M everywhere. The yardstick in the ξ_0-direction is different for protons and electrons.

If we now *assert* that the electron moves like a *group* that is modelled on *U*-waves, in the sense of the ideas of **de Broglie**, **Schrödinger**, and **Klein**, then it will be clear that *OB* represents the five (three, here) dimensional world-line of the electron ([1]), because the phases of two neighboring *U*-waves will reinforce each other only along the line *OB*.

In (x, τ, ξ_0)-space, all electrons have *the same velocity* then ([2]). In the ordinary *four* (*two*, here) dimensional universe, the world-lines of the electron is represented by the projection *OD* of *OB* onto the $x0\tau$-plane. *All* velocities can then appear to be smaller than the speed of light. Now, one can further easily show that:

a) The quantities p, q, r, which have determined only the periods of the phase waves in the x, y, z direction, up to now, *also* means the impulse components of the electron now.

Proof:

The equation of *OD* is $(h\nu / c)\, x - p\, t = 0$, so the velocity of the electron in $x0\tau$-space is:

$$\upsilon_e = \frac{c\,p}{h\nu}, \tag{6}$$

and (3) will become:

$$p^2 + m^2c^2 = \left(\frac{h\nu}{c}\right)^2.$$

If $\beta = \upsilon_e / c$ then one will find from this that:

$$p = \frac{m\upsilon_e}{\sqrt{1-\beta^2}}.$$

b) In the ordinary *four* (*two*, here) dimensional space, the velocity of the electron is also once more the group velocity of the "traces" of the phase waves ([3]).

Proof:

The equation of *OA* is: $(h\nu / c)\ \tau - p\,x = 0$, so the velocity of the wave trace will be:

$$\upsilon_{ph} = \frac{h\nu}{c\,p} = \frac{h\nu}{\sqrt{h^2\nu^2 - m^2c^4}}, \tag{7}$$

([1]) Naturally, just as one does with phase waves, one must also think of *OB* are repeating in the ξ_0-direction with a period of h / mc.

([2]) In (x, t, x_0)-space, that will be a certain *velocity region* that is different for protons and electrons.

([3]) As one sees from the figure and (4), they will always have superluminal velocity.

in agreement with (4) and (5). Now, one must show that the known connection between group and phase velocity exists between υ_e and υ_{ph} :

$$\frac{1}{\upsilon_e} = \frac{d}{d\nu}\left(\frac{\nu}{\upsilon_{ph}}\right) .$$

§ 4. – When the electron moves with a certain period (e.g., in a ring or between two walls), the phase wave must also have that period then. Now, one sees from the figure and (3) that when a certain period L in the x-direction is given, in addition to the period h in the x_0-direction, *only a discrete number* of tangents *OB*, and therefore only certain velocities, can appear in *four* (*two*, here) dimensional space. In fact, if n is a whole number then one must have:

$$L = n\frac{h}{p}$$

or

$$p\,L = \int p\,dq = n\,h . \tag{8}$$

Since p also means the electron impulse, that determines the possible velocities. That is the ordinary quantum condition for this case. As one sees from the figure and (3), the period in time will also be determined then.

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