# On the invariant theory of systems of Pfaff equations 

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The invariant theory of a single Pfaff equation has been dealt with for some time now, but almost everything remains to be done for systems of Pfaff equations. A small first step into that topic shall be made in what follows.

In § 1, I shall recall the connection that exists between the systems:

$$
\begin{equation*}
\sum_{i=1}^{n} \alpha_{\mu i}\left(x_{1}, \ldots, x_{n}\right) d x_{i}=0 \quad(\mu=1, \ldots, m) \tag{1}
\end{equation*}
$$

of $m$ independent Pfaff equations and systems:

$$
\begin{equation*}
A_{k} f=\sum_{i=1}^{n} \beta_{k i}\left(x_{1}, \ldots, x_{n}\right) \frac{\partial f}{\partial x_{i}}=0 \quad(k=1, \ldots, n-m) \tag{2}
\end{equation*}
$$

of $n-m$ independent linear first-order partial differential equations. Although that connection has been know for some time, I still believe that it must be briefly recalled in order to ease the understanding of the remaining paragraphs.

In § 2 and § 3, I shall develop two different methods for deriving new systems of Pfaff equations from a given one that are invariantly linked with the original system. Those two methods are new. However, in particular, I would like to draw attention to the simple, but important theorem $\mathbf{4}$ on pp. 8, which is the basis for the method in $\S 3$.

Finally, in § 4, I will give a complete invariant theory of systems of two independent Pfaff equations in four variables.

## § 1.

One can interpret any system (1) of Pfaff equations in two different ways:
First of all, one can regard it as a system of differential equations, and one correspondingly poses the problem of determining all systems of equations in the variables $x_{1}, \ldots, x_{n}$ that fulfill (1); that is, all systems of equations:

$$
\Phi_{1}\left(x_{1}, \ldots, x_{n}\right)=0, \quad \ldots, \quad \Phi_{q}\left(x_{1}, \ldots, x_{n}\right)=0
$$

that are arranged in such a way that the equations (1) will be true by means of:

$$
\Phi_{1}=0, \ldots, \Phi_{q}=0, \quad d \Phi_{1}=0, \ldots, d \Phi_{q}=0
$$

However, in the second place, one can also regard the quantities $d x_{1}, \ldots, d x_{n}$ in (1) as the infinitely-small increments that an infinitesimal transformation:

$$
X f=\xi_{1}\left(x_{1}, \ldots, x_{n}\right) \frac{\partial f}{\partial x_{1}}+\ldots+\xi_{n}\left(x_{1}, \ldots, x_{n}\right) \frac{\partial f}{\partial x_{n}}
$$

of the variables $x_{1}, \ldots, x_{n}$ experiences. From that standpoint, (1) defines a family of infinitely many infinitesimal transformations, namely, the totality of all infinitesimal transformations $X f$ that satisfy the $m$ equations:

$$
\begin{equation*}
\sum_{i=1}^{n} \alpha_{\mu i}\left(x_{1}, \ldots, x_{n}\right) \xi_{i}=0 \quad(\mu=1, \ldots, m) \tag{3}
\end{equation*}
$$

identically. The next problem would then be: Determine all systems of equations in $x_{1}$, $\ldots, x_{n}$ that admit all of those infinitesimal transformations; in general, that problem will be different from the one above.

Now, let:

$$
\xi_{1}=\xi_{k 1}\left(x_{1}, \ldots, x_{n}\right), \quad \ldots, \quad \xi_{n}=\xi_{k n}\left(x_{1}, \ldots, x_{n}\right) \quad(k=1, \ldots, n-m)
$$

be $n-m$ such systems of solutions of equations (3) such that between the $n-m$ infinitesimal transformations:

$$
X_{k} f=\sum_{i=1}^{n} \xi_{k i}\left(x_{1}, \ldots, x_{n}\right) \frac{\partial f}{\partial x_{i}} \quad(k=1, \ldots, n-m),
$$

no identity of the form:

$$
\xi_{1}\left(x_{1}, \ldots, x_{n}\right) \cdot X_{1} f+\ldots+\xi_{n-m}\left(x_{1}, \ldots, x_{n}\right) \cdot X_{n-m} f=0
$$

exists, in which do not all vanish; the expression:

$$
\begin{equation*}
\chi_{1}\left(x_{1}, \ldots, x_{n}\right) \cdot X_{1} f+\ldots+\chi_{n-m}\left(x_{1}, \ldots, x_{n}\right) \cdot X_{n-m} f \tag{4}
\end{equation*}
$$

with the $n-m$ arbitrary functions $\chi_{1}, \ldots, \chi_{n-m}$ then represents the totality of all infinitesimal transformations that defines the system (1) in the basis of the second viewpoint. The expression (4) then seems to be only a different notation for the system of Pfaff equations (1) for the viewpoint in question. That explains the fact that the system (1) and the expression (4) are invariantly linked with each other: Each of them is determined uniquely by it and conversely, and the relation in question between the two
will remain preserved when one introduces new independent variables in place of $x_{1}, \ldots$, $x_{n}$. In particular, it is clear that any transformation:

$$
x_{i}^{\prime}=F_{i}\left(x_{1}, \ldots, x_{n}\right) \quad(i=1, \ldots, n)
$$

that leaves the system (1) invariant will also take the totality of all infinitesimal transformations (4) to itself, and conversely.

If we set all expressions of the form (4) equal to zero then we will obtain the system of $n-m$ independent linear partial differential equations:

$$
X_{1} f=0, \ldots, X_{n \rightarrow m} f=0,
$$

which are naturally likewise coupled invariantly with the system (1).
An invertible single-valued correspondence exists between systems of $m$ independent Pfaff equations (1) and systems of $n-m$ linear partial differential equations (2): Every system of the one kind is associated with a system of the other kind in a single-valued and invertible way.

One obviously obtains the system (2) that corresponds to a given system (1) by setting all $(m+1)$-rowed determinants in the matrix:

$$
\left|\begin{array}{cccc}
\frac{\partial f}{\partial x_{1}} & \cdots & \cdots & \frac{\partial f}{\partial x_{n}} \\
\alpha_{11} & \cdots & \cdots & \alpha_{1 n} \\
\cdots & \cdots & \cdots & \cdots \\
\alpha_{m 1} & \cdots & \cdots & \alpha_{m n}
\end{array}\right|
$$

equal to zero. On the other hand, one will get the system (1) that corresponds to a given system (2) by setting all $(n-m+1)$-rowed determinants of the matrix:

$$
\left|\begin{array}{cccc}
d x_{1} & \cdots & \cdots & d x_{n} \\
\beta_{11} & \cdots & \cdots & \beta_{1 n} \\
\cdots & \cdots & \cdots & \cdots \\
\beta_{n-m, 1} & \cdots & \cdots & \beta_{n-m, n}
\end{array}\right|
$$

equal to zero. If a system (1) is given in solved form:

$$
\begin{equation*}
d x_{\mu}-\sum_{k=1}^{n-m} \mathfrak{a}_{m+k, \mu} d x_{m+k}=0 \quad(\mu=1, \ldots, m) \tag{1'}
\end{equation*}
$$

then one can also write down the corresponding system (2) in solved form directly; it reads:

$$
\mathfrak{A}_{m+k} f=\frac{\partial f}{\partial x_{m+k}}+\sum_{\mu=1}^{m} \mathfrak{a}_{m+k, \mu} \frac{\partial f}{\partial x_{\mu}}=0 \quad(k=1, \ldots, n-m) .
$$

Those remarks shall suffice for what follows.

## § 4.

The aforementioned two possible viewpoints for considering a system of Pfaff equations will lead us very easily to a new system of Pfaff equations that is invariantly coupled with the original one.

Let a system of $m$ independent Pfaff equations be given in solved form:

$$
\begin{equation*}
\Delta_{\mu}=d x_{\mu}-\sum_{k=1}^{n-m} \mathfrak{a}_{m+k, \mu} d x_{m+k}=0 \quad(\mu=1, \ldots, m) \tag{5}
\end{equation*}
$$

If we interpret the $d x_{1}, \ldots, d x_{n}$ in them as infinitely-small increments that the $x_{1}, \ldots, x_{n}$ take on under an infinitesimal transformation then, as we saw above, (5) will define infinitely many infinitesimal transformations whose general symbol is coupled with:

$$
\begin{equation*}
W f=\chi_{1}\left(x_{1}, \ldots, x_{n}\right) \cdot \mathfrak{A}_{m+1} f+\ldots+\chi_{n}\left(x_{1}, \ldots, x_{n}\right) \cdot \mathfrak{A}_{n} f, \tag{6}
\end{equation*}
$$

in which the $\mathfrak{A}_{m+k} f$ possess the form that was given in (2).
The family of infinitesimal transformations (6) is invariantly linked with the system (5). If we then imagine that all infinitesimal transformations (6) have been performed on the system (3) then we will necessarily obtain a new system of Pfaff equations that are invariantly linked with the system (5).

Upon performing the infinitesimal transformation $W f$, the system (5) will go to:

$$
\begin{equation*}
\Delta_{\mu}+\delta t \cdot W \Delta_{\mu}=0 \quad(\mu=1, \ldots, m) \tag{7}
\end{equation*}
$$

in which $\delta t$ denotes an infinitely-small quantity, and $W \Delta_{\mu}$ possesses the value $\left({ }^{1}\right)$ :

$$
W \Delta_{\mu}=d\left(W \Delta_{\mu}\right)-\sum_{k=1}^{n-m} W \mathfrak{a}_{m+k, \mu} d x_{m+k}-\sum_{k=1}^{n-m} \mathfrak{a}_{m+k, \mu} \cdot d\left(W x_{m+k}\right) .
$$

However, that will imply that:

$$
W \Delta_{\mu}=\sum_{k=1}^{n-m} \chi_{m+k}(x) \cdot \mathfrak{A}_{m+k} \Delta_{\mu}+\sum_{k=1}^{n-m} d \chi_{m+k}\left\{\mathfrak{A}_{m+k} x_{\mu}-\sum_{j=1}^{n-m} \mathfrak{a}_{m+j, \mu} \cdot \mathfrak{A}_{m+k} x_{m+k}\right\}
$$

so one has:

[^0]$$
W \Delta_{\mu}=\sum_{k=1}^{n-m} \chi_{m+k}(x) \cdot \mathfrak{A}_{m+k} \Delta_{\mu}
$$

Therefore, if we consider $\chi_{m+1}, \ldots, \chi_{n}$ to be arbitrary functions of their arguments then we will see that equations (7) can be replaced with the following ones:
(7') $\quad \Delta_{1}=0, \ldots, \Delta_{m}=0, \quad \mathfrak{A}_{m+k} \Delta_{1}=0, \ldots, \mathfrak{A}_{m+k} \Delta_{m}=0, \quad(k=1, \ldots, n-m)$.

In order to actually exhibit the system of equations (7'), we remark that for every function $f$ of $x_{1}, \ldots, x_{n}$, the equation:

$$
\begin{equation*}
d f=\sum_{\pi=1}^{m} \frac{\partial f}{\partial x_{\pi}} \Delta_{\pi}+\sum_{j=1}^{n-m} \mathfrak{A}_{m+j} f \cdot d x_{m+j} \tag{8}
\end{equation*}
$$

exists identically. With the help of that identity, we get:

$$
\left\{\begin{align*}
\mathfrak{A}_{m+k} \Delta_{\mu} & =d \mathfrak{a}_{m+k}-\sum_{j=1}^{n-m} \mathfrak{A}_{m+k} \cdot \mathfrak{a}_{m+j, \mu} \cdot d x_{m+j}  \tag{9}\\
& =\sum_{j=1}^{n-m}\left(\mathfrak{A}_{m+j} \cdot \mathfrak{a}_{m+k, \mu}-\mathfrak{A}_{m+k} \cdot \mathfrak{a}_{m+j, \mu}\right) d x_{m+j}+\sum_{\pi=1}^{m} \frac{\partial \mathfrak{a}_{m+j, \mu}}{\partial x_{\pi}} \Delta_{\pi}
\end{align*}\right.
$$

and with the use of the abbreviation:

$$
\begin{equation*}
\mathfrak{A}_{m+j} \mathfrak{A}_{m+k} f-\mathfrak{A}_{m+k} \mathfrak{A}_{m+j} f=\mathfrak{B}_{j k} f, \tag{10}
\end{equation*}
$$

it can also be written:

$$
\begin{equation*}
\mathfrak{A}_{m+k} \Delta_{\mu}=\sum_{j=1}^{n-m} \mathfrak{B}_{j k} x_{\mu} \cdot d x_{m+j}+\sum_{\pi=1}^{m} \frac{\partial \mathfrak{a}_{m+k, \mu}}{\partial x_{\pi}} \Delta_{\pi} . \tag{9'}
\end{equation*}
$$

Therefore, the system of equations (7) will have the form:

$$
\begin{equation*}
\Delta_{\mu}=0, \quad \sum_{j=1}^{n-m} \mathfrak{B}_{j k} x_{\mu} \cdot d x_{m+j}=0 \quad(\mu=1, \ldots, m ; k=1, \ldots, n-m) \tag{11}
\end{equation*}
$$

and we will then have:

## Theorem 1:

If:

$$
\begin{equation*}
\Delta_{\mu}=d x_{\mu}-\sum_{k=1}^{n-m} \mathfrak{a}_{m+k, \mu} d x_{m+k}=0 \quad(\mu=1, \ldots, m) \tag{5}
\end{equation*}
$$

is a system of $m$ independent Pfaff equations in the variables $x_{1}, \ldots, x_{n}$, and if:

$$
\mathfrak{A}_{m+k} f=\frac{\partial f}{\partial x_{m+k}}+\sum_{\mu=1}^{m} \mathfrak{a}_{m+k, \mu} \frac{\partial f}{\partial x_{\mu}}=0 \quad(k=1, \ldots, n-m)
$$

is the associated system of $n-m$ independent linear partial differential equations then the system of Pfaff equations:

$$
\begin{equation*}
\Delta_{\mu}=0, \quad \mathfrak{A}_{m+k} \Delta_{\mu}=0 \quad(\mu=1, \ldots, m ; k=1, \ldots, n-m), \tag{7'}
\end{equation*}
$$

or when written out more thoroughly, the system:

$$
\left\{\begin{array}{l}
d x_{\mu}-\sum_{k=1}^{n-m} \mathfrak{a}_{m+k, \mu} d x_{m+k}=0,  \tag{12}\\
\sum_{k=1}^{n-m}\left\{\mathfrak{A}_{m+k} \mathfrak{a}_{m+j, \mu}-\mathfrak{A}_{m+j} \mathfrak{a}_{m+k, \mu}\right\} d x_{m+k}=0 \quad(\mu=1, \ldots, m ; j=1, \ldots, n-m),
\end{array}\right.
$$

is invariantly coupled with the system (5).
In some situations, the system of equations (12) will coincide with the system (5) itself, namely, when the expressions $\mathfrak{B}_{k j} f$ all vanish identically. In those cases, the system of equations (5) admit all infinitesimal transformations (6), so it will be integrable without restriction, and the equations:

$$
\mathfrak{A}_{m+1} f=0, \quad \ldots, \quad \mathfrak{A}_{n} f=0,
$$

define a complete ( $n-m$ )-parameter system.
Naturally, the system (12) can also be interpreted in such a way that it defines a family of infinitesimal transformations. The infinitesimal transformations of that family are then distinguished by the fact that they all leave the system of equations (5) invariant.

In fact, should the infinitesimal transformation:

$$
X f=\sum_{i=1}^{n} \xi_{i}\left(x_{1}, \ldots, x_{n}\right) \frac{\partial f}{\partial x_{i}}
$$

leave the system (5) invariant, it would be necessary and sufficient for the $m$ expressions $X \Delta_{\mu}$ to all vanish by means of (5). Now, if the infinitesimal transformations $X f$ that are all defined by (12) have the form:

$$
X f=\sum_{k=1}^{n-m} \xi_{m+k} \mathfrak{A}_{m+k} f,
$$

in which the $\xi_{m+k}$ fulfill the equations:

$$
\begin{equation*}
\sum_{k=1}^{n-m}\left\{\mathfrak{A}_{m+k} \mathfrak{a}_{m+j, \mu}-\mathfrak{A}_{m+j} \mathfrak{a}_{m+k, \mu}\right\} \xi_{m+k}=0 \quad(\mu=1, \ldots, m ; j=1, \ldots, n-m) \tag{13}
\end{equation*}
$$

identically, then one will then have:

$$
X \Delta_{\mu}=\sum_{k=1}^{n-m} \xi_{m+k} \mathfrak{A}_{m+k} \Delta_{\mu},
$$

or, with the use of formula (9) and the identities (13):

$$
X \Delta_{\mu}=\sum_{k=1}^{n-m} \sum_{\pi=1}^{m} \xi_{m+k} \frac{\partial \mathfrak{a}_{m+k, \mu}}{\partial x_{\pi}} \cdot \Delta_{\pi},
$$

which actually vanishes, due to (5).
As a result of that, one will have:

## Theorem 2:

If:

$$
\begin{equation*}
\Delta_{\mu}=d x_{\mu}-\sum_{k=1}^{n-m} \mathfrak{a}_{m+k, \mu} d x_{m+k}=0 \quad(\mu=1, \ldots, m) \tag{5}
\end{equation*}
$$

is a system of $m$ independent Pfaff equations in the variables $x_{1}, \ldots, x_{n}$, and if:

$$
\mathfrak{A}_{m+k} f=\frac{\partial f}{\partial x_{m+k}}+\sum_{\mu=1}^{m} \mathfrak{a}_{m+k, \mu} \frac{\partial f}{\partial x_{\mu}}=0 \quad(k=1, \ldots, n-m)
$$

is the associated system of $n-m$ independent linear partial differential equations then the system (5) will admit all transformations:

$$
X f=\sum_{i=1}^{n} \xi_{i}\left(x_{1}, \ldots, x_{n}\right) \frac{\partial f}{\partial x_{i}},
$$

in which $x_{1}, \ldots, x_{n}$ experience infinitely-small increments $d x_{1}, \ldots, d x_{n}$ such that the equations:

$$
\Delta_{\mu}=0, \mathfrak{A}_{m+k} \Delta_{\mu}=0 \quad(\mu=1, \ldots, m ; k=1, \ldots, n-m)
$$

are true identically, or what amounts to the same thing, it admits all infinitesimal transformations $X f$ that fulfill the equations:

$$
\left\{\begin{array}{l}
\xi_{\mu}-\sum_{k=1}^{n-m} \mathfrak{a}_{m+k} \xi_{m+k}=0,  \tag{11}\\
\sum_{k=1}^{n-m}\left(\mathfrak{A}_{m+k} \mathfrak{a}_{m+j, \mu}-\mathfrak{A}_{m+j} \mathfrak{a}_{m+k, \mu}\right) \xi_{m+k}=0 \quad(\mu=1, \ldots, m ; j=1, \ldots, n-m)
\end{array}\right.
$$

identically.

Obviously, the aforementioned infinitesimal transformations can also leave the system (12) invariant, which is coupled invariantly with (5), so the former is integrable without restriction:

## Theorem 3:

The system of Pfaff equations (12) always integrable without restriction.

## § 3.

In this paragraph, we shall develop another method for finding a new system of Pfaff equations from a given one of the same kind that is invariantly coupled with the original one. The method in question is based upon the following important theorem, which has, however, remained unnoticed up to now:

## Theorem 4:

The system of $n-m$ independent linear partial differential equations:

$$
\begin{equation*}
A_{k} f=\sum_{i=1}^{n} \beta_{k i}\left(x_{1}, \ldots, x_{n}\right) \frac{\partial f}{\partial x_{i}}=0 \quad(k=1, \ldots, n-m) \tag{15}
\end{equation*}
$$

is invariantly coupled with the system of equations:

$$
\begin{equation*}
A_{k} f=0, \quad A_{k} A_{j} f-A_{j} A_{k} f=\left(A_{k} A_{j}\right)=0 \quad(k, j=1, \ldots, n-m) . \tag{16}
\end{equation*}
$$

That theorem says two things:
Firstly: The system (16) is determined by the system (15) independently of the form upon which one bases (15). Hence, if:

$$
B_{k} f=\sum_{j=1}^{n-m} \psi_{k j}\left(x_{1}, \ldots, x_{n}\right) A_{j} f=0 \quad(k=1, \ldots, n-m)
$$

is any other form for (15) then the system of equations:

$$
B_{k} f=0, \quad B_{k} B_{j} f-B_{j} B_{k} f=0 \quad(k, j=1, \ldots, n-m)
$$

will be equivalent to (16).
Secondly: When one introduces the new variables:

$$
x_{i}^{\prime}=F_{i}\left(x_{1}, \ldots, x_{n}\right) \quad(i=1, \ldots, n),
$$

$A_{k} f$ will be converted into $C_{k} f$, and therefore (15) will be converted into:

$$
C_{1} f=0, \ldots, C_{n-m} f=0,
$$

so the system (16) will keep the same form in the $x^{\prime}$ :

$$
C_{k} f=0, \quad C_{k} C_{j} f-C_{j} C_{k} f=0 \quad(k, j=1, \ldots, n-m) .
$$

The first statement is immediately clear, while the second one follows from the known theorem that the expression: $A_{k} A_{j} f-A_{j} A_{k} f$ goes to the $C_{k} C_{j} f-C_{j} C_{k} f$ when one introduces the new variables $x^{\prime}$ (cf., Theorie der Transformationsgruppen, I, pp. 84, Theorem 2).

Theorem 4 will also be true when the $n-m$ equations (15) are not mutuallyindependent, so as a result, it can be applied directly to the system (16). In that way, one will then see that the system:

$$
\left\{\begin{array}{cl}
A_{k} f=0, & A_{k} A_{j} f-A_{j} A_{k} f=0,  \tag{17}\\
\left(\left(A_{h}\left(A_{k} A_{j}\right)\right)=0,\right. & \left(\left(A_{h} A_{l}\right)\left(A_{k} A_{j}\right)\right)=0 \quad(k, j, h, l=1, \ldots, n-m)
\end{array}\right.
$$

is invariantly coupled with (16). However, that explains the fact that is also has that relationship to the original system (15).

One can, in turn, apply Theorem 4 to (17), and so on.
One might expressly state a theorem here that is contained implicitly in Theorem 4 :

## Theorem 5:

If the system of $n-m$ linear partial differential equations:

$$
\begin{equation*}
A_{k} f=\sum_{i=1}^{n} \beta_{k i}\left(x_{1}, \ldots, x_{n}\right) \frac{\partial f}{\partial x_{i}}=0 \quad(k=1, \ldots, n-m) \tag{15}
\end{equation*}
$$

is invariant under the transformation:

$$
x_{i}^{\prime}=F_{i}\left(x_{1}, \ldots, x_{n}\right) \quad(i=1, \ldots, n)
$$

then the same will also be true for the system of equations:

$$
\begin{equation*}
A_{k} f=0, \quad A_{k} A_{j} f-A_{j} A_{k} f=\left(A_{k} A_{j}\right)=0 \quad(k, j=1, \ldots, n-m) . \tag{16}
\end{equation*}
$$

The system (16) in this theorem can be replaced with (17), and so on.
The two theorems $\mathbf{4}$ and 5 can now be adapted to systems of Pfaff equations with no further discussion. Indeed, from § 1, a system of Pfaff equations is determined uniquely by (15) and (16), and that explains the fact that a system of Pfaff equations belongs to
(16) that is invariantly coupled with the one that belongs to (15). For more convenience, we then imagine that equations (15) are given in the solved form:

$$
\begin{equation*}
\mathfrak{A}_{m+k} f=\frac{\partial f}{\partial x_{m+k}}+\sum_{\mu=1}^{n} \mathfrak{a}_{k i}\left(x_{1}, \ldots, x_{n}\right) \frac{\partial f}{\partial x_{\mu}}=0 \quad(k=1, \ldots, n-m), \tag{15'}
\end{equation*}
$$

so we will obtain the following theorem from theorem 4 :

## Theorem 6:

The system of $m$ independent Pfaff equations:

$$
\begin{equation*}
d x_{\mu}-\sum_{k=1}^{n-m} \mathfrak{a}_{m+k, \mu}\left(x_{1}, \ldots, x_{n}\right) d x_{m+k}=0 \quad(\mu=1, \ldots, m) \tag{18}
\end{equation*}
$$

is coupled invariantly with the system of Pfaff equations that corresponds to the system of linear partial differential equations:

$$
\left\{\begin{array}{l}
\mathfrak{A}_{m+k} f=\frac{\partial f}{\partial x_{m+k}}+\sum_{\mu=1}^{m} \mathfrak{a}_{m+k, \mu} \frac{\partial f}{\partial x_{\mu}}=0,  \tag{16'}\\
\left(\mathfrak{A}_{m+k} \mathfrak{A}_{m+j}\right)=\sum_{\mu=1}^{m}\left(\mathfrak{A}_{m+k} \mathfrak{a}_{m+k, \mu}-\mathfrak{A}_{m+k} \mathfrak{a}_{m+k, \mu}\right) \frac{\partial f}{\partial x_{\mu}}=0 \quad(k, j=1, \ldots, n-m),
\end{array}\right.
$$

and it follows from Theorem 5 that:

## Theorem 7:

If the system of Pfaff equations (18) remains invariant under the transformation:

$$
x_{i}^{\prime}=F_{i}\left(x_{1}, \ldots, x_{n}\right) \quad(i=1, \ldots, n)
$$

then the same thing will also be true for the system of Pfaff equations that corresponds to the system (16').

When the system (18) is integrable without restriction, all $\left(\mathfrak{A}_{m+k}, \mathfrak{A}_{m+j}\right)$ will vanish identically, so the system of Pfaff equations that corresponds to (16') will coincide with (18), and Theorem 6 will not yield anything new. On the other hand, if the system (18) is not integrable without restriction then it can happen that ( $16^{\prime}$ ) includes precisely $n$ mutually-independent equations. In that case, the system of Pfaff equations that belongs to (16') will collapse to $0=0$, and Theorem $\mathbf{6}$ will likewise say nothing new.

We will obtain a truly new system of Pfaff equations that is invariantly coupled with (18) from Theorem 6 iff (18) is not integrable without restriction and (16') includes less than $n$ mutually-independent equations. If those conditions are fulfilled and the
mutually-independent equations in ( $16^{\prime}$ ) do not define a complete system then we can once more apply Theorem 6 to the system of Pfaff equations that belongs to (16) and then possibly obtain a new system of Pfaff equations that is invariantly coupled with (18). In some situations, we can find a whole series of such systems.

If $m=2$ and (18) is not integrable without restriction then (16') will include three mutually-independent equations. We can then state the theorem:

## Theorem 8:

Every system:

$$
\begin{equation*}
d x_{\mu}-\mathfrak{a}_{n-1, \mu} d x_{n-1}-\mathfrak{a}_{n \mu} d x_{n}=0 \quad(\mu=1, \ldots, n-2) \tag{19}
\end{equation*}
$$

of $n-2$ independent Pfaff equations in $n>3$ variables $x_{1}, \ldots, x_{n}$ is invariantly coupled with a system of $n-3$ independent equations of that kind. The same thing will be obtained by setting all four-rowed determinants of the matrix:

$$
\left|\begin{array}{lllll}
d x_{1} & \cdots & d x_{n-2} & d x_{n-1} & d x_{n}  \tag{20}\\
\mathfrak{a}_{n-1,1} & \cdots & \mathfrak{a}_{n-1, n-2} & 1 & 0 \\
\mathfrak{a}_{n, 1} & \cdots & \mathfrak{a}_{n, n-2} & 0 & 1 \\
\mathfrak{A}_{n-1} \mathfrak{a}_{n, 1}-\mathfrak{A}_{n} \mathfrak{a}_{n-1,1} & \cdots & \mathfrak{A}_{n-1} \mathfrak{a}_{n, 2}-\mathfrak{A}_{n} \mathfrak{a}_{n-1,2} & 0 & 0
\end{array}\right|
$$

equal to zero, in which the symbols $\mathfrak{A}_{n-1} f$ and $\mathfrak{A}_{n} f$ possess the form:

$$
\mathfrak{A}_{n-2-k} f=\frac{\partial f}{\partial x_{n-2+k}}+\sum_{\mu=1}^{n-2} \mathfrak{a}_{n-2+k, \mu} \frac{\partial f}{\partial x_{\mu}} \quad(k=1,2) .
$$

Similar theorems are true for $m=n-3, n>6$, for $m=n-4, n>10$, and so on.
One can make a remarkable application of Theorem 8 .
Page determined all primitive groups of four-fold extended space in his dissertation (American Journal, v. 10, pp. 293-346). His proof encountered special difficulties due to the fact that certain groups in that space were not primitive. The ones in question are transitive, and behave as follows, in addition: If one fixes a point in general position under such a group then a plane pencil of $\infty^{1}$ directions in the projective space of $\infty^{3}$ directions that go through the point will remain invariant, but not a plane bundle of $\infty^{2}$ directions.

On the basis of Theorem 8, one can easily prove that the groups that are defined in that way are all primitive.

In fact, it is clear that any of the groups in $R_{4}$ that we speak of will leave a system of two independent Pfaff systems in four variables invariant, but not a single Pfaff equation in those variables. Now, if that system is not integrable without restriction then, from Theorem 8, there must be a Pfaff equation that remains invariant under the group, but that is a contradiction. As a result, the system of Pfaff equations considered must be
integrable without restriction and possess $\infty^{2}$ doubly-extended integral manifolds that fill up the four-fold extended space precisely once and determine a decomposition of it. Naturally, that decomposition remains invariant under the group, and the group will therefore be imprimitive.

The rather long-winded calculations by means of which Page proved the imprimitivity of the group in question are avoided by the argument that was just presented. Nonetheless, the service that Page performed remains undiminished, since on the one hand, he was the first to determine all primitive groups of the four-fold extended space, and on the other hand, his treatise yielded very important contributions to the determination of all transitive groups on that space.

## § 4.

The theorems that were obtained allow us to develop the invariant theory of systems of two independent Pfaff equations:

$$
\begin{equation*}
\sum_{i=1}^{4} \lambda_{i \mu}\left(x_{1}, \ldots, x_{n}\right) d x_{i}=0 \quad(\mu=1,2) \tag{21}
\end{equation*}
$$

in four variables completely.
We can address the case in which the system (21) is integrable very easily.
Namely, if it is integrable without restriction, and if $u_{1}\left(x_{1}, \ldots, x_{4}\right), u_{2}\left(x_{1}, \ldots, x_{4}\right)$ are two independent integral functions of that system then if we introduce two suitable functions $u_{3}, u_{4}$ of the $x$ as independent variables, along with $u_{1}, u_{2}$, then we will get the simple form for it:

$$
d u_{1}=0, \quad d u_{2}=0
$$

On the other hand, it is indeed integrable, but not without restriction, and if $v_{1}\left(x_{1}, \ldots\right.$, $x_{4}$ ) is an integral function the we introduce new variables $v_{1}, \ldots, v_{4}$ and get:

$$
\begin{equation*}
d v_{1}=0, \quad \sum_{i=2}^{4} \sigma_{i}\left(v_{1}, \ldots, v_{4}\right) d v_{i}=0 . \tag{22}
\end{equation*}
$$

In these equations, if we consider $v_{1}$ to be a constant then:

$$
\sigma_{2} d v_{2}+\sigma_{3} d v_{3}+\sigma_{4} d v_{4}=0
$$

will be a non-integrable Pfaff equation in the variables $v_{2}, v_{3}, v_{4}$, and from the theory of the Pfaff problem, they can then be brought into the form $d u_{2}-u_{3} d u_{4}=0$ by a transformation of the form:

$$
u_{i}=\psi_{i}\left(v_{1}, v_{2}, v_{3}, v_{4}\right) \quad(i=2,3,4) .
$$

Finally, if we set $v_{1}=u_{1}$ and introduce $u_{1}, u_{2}, u_{3}, u_{4}$ in (22) as new variables then that will give:

$$
d u_{1}=0, \quad d u_{2}-u_{3} d u_{4}=0
$$

The system (21) can then be brought into that form under the assumptions that were made.

The interesting case in which the system (21) is not integrable at all still remains. We would now like to treat it.

For the sake of convenience, we think of the system (21) as being given in solved form:

$$
\left\{\begin{array}{l}
\Delta_{1}=d x_{1}-\alpha_{3}\left(x_{1}, \ldots, x_{4}\right) d x_{3}-\alpha_{4}\left(x_{1}, \ldots, x_{4}\right) d x_{4}=0,  \tag{23}\\
\Delta_{2}=d x_{2}-\beta_{3}\left(x_{1}, \ldots, x_{4}\right) d x_{3}-\beta_{4}\left(x_{1}, \ldots, x_{4}\right) d x_{4}=0 .
\end{array}\right.
$$

Since it is not integrable, there will then be no functions $\rho_{1}$ and $\rho_{2}$ of $x_{1}, \ldots, x_{4}$ such that the expression $\rho_{1} \Delta_{1}+\rho_{2} \Delta_{2}$ is a complete differential.

Our first task is to exhibit the system of linear partial differential equations that is associated with (23). It reads:

$$
\left\{\begin{array}{l}
A_{3} f=\frac{\partial f}{\partial x_{3}}+\alpha_{3} \frac{\partial f}{\partial x_{1}}+\beta_{3} \frac{\partial f}{\partial x_{2}}=0  \tag{24}\\
A_{4} f=\frac{\partial f}{\partial x_{4}}+\alpha_{4} \frac{\partial f}{\partial x_{1}}+\beta_{4} \frac{\partial f}{\partial x_{2}}=0
\end{array}\right.
$$

and it is certainly not a complete two-parameter system under the assumptions that were made. It follows from this that the expression:

$$
A_{3} A_{4}-A_{4} A_{3} f=B f=\left(A_{3} \alpha_{4}-A_{3} \alpha_{4}\right) \frac{\partial f}{\partial x_{1}}+\left(A_{3} \beta_{4}-A_{3} \beta_{4}\right) \frac{\partial f}{\partial x_{2}}
$$

does not vanish identically, and that the three equations:

$$
\begin{equation*}
A_{3} f=0, \quad A_{4} f=0, \quad B f=0 \tag{24}
\end{equation*}
$$

are mutually independent.
The system ( $24^{\prime}$ ) now corresponds to a Pfaff equation that is coupled invariantly with the system of Pfaff equations (23) by Theorem 6, pp. 10. It possesses the form:

$$
\left|\begin{array}{cccc}
d x_{1} & d x_{2} & d x_{3} & d x_{4} \\
\alpha_{3} & \beta_{3} & 1 & 0 \\
\alpha_{4} & \beta_{4} & 0 & 1 \\
A_{3} \alpha_{4}-A_{4} \alpha_{3} & A_{3} \beta_{4}-A_{4} \beta_{3} & 0 & 0
\end{array}\right|=0
$$

or

$$
\begin{equation*}
\Delta=\left(A_{3} \beta_{4}-A_{4} \beta_{3}\right) \Delta_{1}-\left(A_{3} \alpha_{4}-A_{4} \alpha_{3}\right) \Delta_{2}=0 \tag{25}
\end{equation*}
$$

which can also be written as:

$$
\Delta=B x_{2} \cdot \Delta_{1}-B x_{1} \cdot \Delta_{2}=0,
$$

so it is obviously not integrable. We conclude from this that of the two equations:

$$
\begin{equation*}
A_{3} B f-B A_{3} f=C_{3} f=0, \quad A_{4} B f-B A_{4} f=C_{4} f=0, \tag{26}
\end{equation*}
$$

in any event, one of them is independent of $B f=0$, so the determinants:

$$
\left|\begin{array}{cc}
B x_{1} & B x_{3}  \tag{27}\\
C_{3} x_{1} & C_{3} x_{2}
\end{array}\right|, \quad\left|\begin{array}{cc}
B x_{1} & B x_{3} \\
C_{4} x_{1} & C_{4} x_{2}
\end{array}\right|
$$

do not both vanish identically in any case.
We shall now apply Theorem 1, pp. 5 to the Pfaff equation (25'). That is, we form the system of Pfaff equations:

$$
\Delta=0, \quad A_{3} \Delta=0, \quad A_{4} \Delta=0, \quad B \Delta=0,
$$

which is invariantly coupled with the equation (25'), and therefore also with the original system (23), from the theorem in question.

In order to actually exhibit equations (28), we next point out that for every function $f$ of $x_{1}, \ldots, x_{4}$, we have the identity:

$$
\begin{equation*}
d f=\frac{\partial f}{\partial x_{1}} \Delta_{1}+\frac{\partial f}{\partial x_{2}} \Delta_{2}+A_{3} f \cdot d x_{3}+A_{4} f \cdot d x_{4} . \tag{29}
\end{equation*}
$$

By means of it, one gets (cf., pp. 5):

$$
\begin{aligned}
A_{3} \Delta_{1} & =d \alpha_{3}-A_{3} \alpha_{3} \cdot d x_{3}-A_{3} \alpha_{3} \cdot d x_{3} \\
& =\frac{\partial \alpha_{3}}{\partial x_{1}} \cdot \Delta_{1}+\frac{\partial \alpha_{3}}{\partial x_{2}} \cdot \Delta_{2}-B x_{1} \cdot d x_{4},
\end{aligned}
$$

and

$$
A_{3} \Delta_{2}=\frac{\partial \beta_{3}}{\partial x_{1}} \cdot \Delta_{1}+\frac{\partial \beta_{3}}{\partial x_{2}} \cdot \Delta_{2}-B x_{2} \cdot d x_{4},
$$

so one will have:
$A_{3} \Delta=A_{3} B x_{2} \cdot \Delta_{1}-A_{3} B x_{1} \cdot \Delta_{2}+\left(\frac{\partial \alpha_{3}}{\partial x_{1}} \cdot B x_{2}-\frac{\partial \beta_{3}}{\partial x_{1}} \cdot B x_{1}\right) \Delta_{1}+\left(\frac{\partial \alpha_{3}}{\partial x_{2}} \cdot B x_{2}-\frac{\partial \beta_{3}}{\partial x_{2}} \cdot B x_{1}\right) \Delta_{2}$,
which can be written as follows:

$$
A_{3} \Delta=\left(A_{3} B x_{2}-B A_{3} x_{2}\right) \Delta_{1}-\left(A_{3} B x_{1}-B A_{3} x_{1}\right) \Delta_{2}+\left(\frac{\partial \alpha_{3}}{\partial x_{1}}+\frac{\partial \beta_{3}}{\partial x_{2}}\right) \Delta
$$

such that we get:

$$
A_{3} \Delta=C_{3} x_{2} \cdot \Delta_{1}-C_{3} x_{1} \cdot \Delta_{2}+\left(\frac{\partial \alpha_{3}}{\partial x_{1}}+\frac{\partial \beta_{3}}{\partial x_{2}}\right) \Delta
$$

and likewise:

$$
A_{4} \Delta=C_{4} x_{2} \cdot \Delta_{1}-C_{4} x_{1} \cdot \Delta_{2}+\left(\frac{\partial \alpha_{4}}{\partial x_{1}}+\frac{\partial \beta_{4}}{\partial x_{2}}\right) \Delta .
$$

Now since the determinants (27) do not vanish, as we said above, we see immediately that the system of three equations:

$$
\Delta=0, \quad A_{3} \Delta=0, \quad A_{4} \Delta=0
$$

is equivalent to the system of equations $\Delta_{1}=0, \Delta_{2}=0$.
Moreover, all that remains is to calculate the equation $B \Delta=0$. We find that:

$$
\begin{aligned}
B \Delta & =d\left(B x_{1}\right)-B \alpha_{3} \cdot d x_{3}-B \alpha_{4} \cdot d x_{4} \\
& =d\left(B x_{1}\right)-B A_{3} x_{1} \cdot d x_{3}-B A_{4} x_{1} \cdot d x_{4}
\end{aligned}
$$

which yields:

$$
B \Delta_{1}=C_{3} x_{1} \cdot d x_{3}+C_{4} x_{1} \cdot d x_{4}+\sigma_{1} \cdot \Delta_{1}+\sigma_{2} \cdot \Delta_{2}
$$

with the use of the identity (29). We likewise get:

$$
B \Delta_{2}=C_{3} x_{2} \cdot d x_{3}+C_{4} x_{2} \cdot d x_{4}+\tau_{1} \cdot \Delta_{1}+\tau_{2} \cdot \Delta_{2},
$$

so
$B \Delta=\left(B x_{2} \cdot C_{3} x_{1}-B x_{1} \cdot C_{3} x_{2}\right) d x_{3}+\left(B x_{2} \cdot C_{4} x_{1}-B x_{1} \cdot C_{4} x_{2}\right) d x_{4}+\omega_{1} \cdot \Delta_{1}+\omega_{2} \cdot \Delta_{2}$.
We see from this that (28) can be replaced with the three independent equations:

$$
\left\{\begin{array}{c}
d x_{1}-\alpha_{3} d x_{3}-\alpha_{4} d x_{4}=0, \quad d x_{2}-\beta_{3} d x_{3}-\beta_{4} d x_{4}=0,  \tag{30}\\
\left(B x_{1} \cdot C_{3} x_{2}-B x_{2} \cdot C_{3} x_{1}\right) d x_{3}+\left(B x_{1} \cdot C_{4} x_{2}-B x_{2} \cdot C_{4} x_{1}\right) d x_{4}=0 .
\end{array}\right.
$$

This system (30) is invariantly coupled with the Pfaff equation (25'), as well as with the original system of equations (23).

If one interprets $d x_{1}, \ldots, d x_{4}$ in the known way as the infinitely-small increment that an infinitesimal transformation gives to the variables $x_{1}, \ldots, x_{4}$ then (30) will determine a family of infinitesimal transformations whose general symbol can be easily given. Namely, if one sets:

$$
D f=\left|\begin{array}{cc}
B x_{1} & B x_{3}  \tag{31}\\
C_{4} x_{1} & C_{4} x_{2}
\end{array}\right| \cdot A_{3} f-\left|\begin{array}{cc}
B x_{1} & B x_{3} \\
C_{3} x_{1} & C_{3} x_{2}
\end{array}\right| \cdot A_{4} f,
$$

and one understands $\chi$ to mean an arbitrary function of $x_{1}, \ldots, x_{4}$ then the general symbol under consideration will read simply $\chi\left(x_{1}, \ldots, x_{4}\right) \cdot D f$. However, the system of equations (30) is only another form of (28), and $A_{3} f=0, A_{4} f=0, B f=0$ is the system of linear partial differential equations that belong to the Pfaff equation (25'), so from Theorem 2, pp. 7, the Pfaff equation (25') will remain invariant under all infinitesimal transformations of the form $\chi\left(x_{1}, \ldots, x_{4}\right) \cdot D f$.

We now introduce new independent variables $y_{1}, \ldots, y_{4}$ in place of $x_{1}, \ldots, x_{4}$ that are chosen in such a way that the infinitesimal transformation $D f$ will assume the form $\partial f /$ $\partial y_{1}$. Obviously, the system (30) will assume the form:

$$
d y_{1}=0, \quad d y_{2}=0, \quad d y_{3}=0
$$

in those new variables, and the Pfaff equation (25') will assume this form:

$$
\beta_{1} d y_{1}+\beta_{2} d y_{2}+\beta_{3} d y_{3}=0
$$

in which are free of $y_{1}$, since ( $25^{\prime \prime}$ ) must clearly admit all infinitesimal transformations of the form $\psi\left(y_{1}, \ldots, y_{4}\right) \partial f / \partial y_{4}$. However, since the Pfaff equation (25") is not integrable, we can imagine the variables $y_{1}, y_{2}, y_{3}$ as being chosen in particular such that (25") takes the simple form:

$$
d y_{2}-y_{3} d y_{1}=0 .
$$

The system of equations (23) likewise takes on a new form in our new variables. That form will be necessarily free of $d y_{4}$, since (23) is contained in the system ( $30^{\prime}$ ), and the latter assumes the form (30), which is free of $d y_{4}$. Moreover, the new form of (23) must include the equation ( $25^{\prime \prime \prime}$ ) in any case, since (23) includes equation ( $25^{\prime}$ ), so it will take the form:

$$
\begin{equation*}
d y_{2}-y_{3} d y_{1}=0, \quad d y_{3}-\omega\left(y_{1}, \ldots, y_{4}\right) d y_{1}=0 \tag{23'}
\end{equation*}
$$

If we combine the results up to now of the foregoing paragraphs then we will get the remarkable:

## Theorem 9:

Any system of two independent Pfaff equations:

$$
\begin{equation*}
\sum_{i=1}^{4} \lambda_{i \mu}\left(x_{1}, \ldots, x_{4}\right) d x_{i}=0 \quad(\mu=1,2) \tag{32}
\end{equation*}
$$

infour variables can be brought into one of the three forms:

$$
\begin{gather*}
d y_{1}=0, \quad d y_{2}=0 \\
d y_{2}-y_{3} d y_{1}=0, \quad d y_{4}=0, \\
d y_{2}-y_{3} d y_{1}=0, \quad d y_{3}-y_{4} d y_{1}=0, \tag{32}
\end{gather*}
$$

by the introduction of new independent variables $y_{1}, \ldots, y_{4}$, according to whether it is integrable without restriction, with restriction, or not at all, respectively.

It is then easy to give the invariant properties that an arbitrary system (32) of two independent Pfaff equations possesses under all transformations of the four variables $x_{1}$, $\ldots, x_{4}$. Namely, if $U_{1} f=0, U_{2} f=0$ is the system of linear partial differential equations that belongs to (32) then the invariant properties in question will be nothing but the following two numbers: Firstly, the number of mutually-independent equations among the equations:

$$
U_{1} f=0, \quad U_{2} f=0, \quad\left(U_{1} U_{2}\right)=0,
$$

and secondly, the number of mutually-independent equations among:

$$
U_{1} f=0, \quad U_{2} f=0, \quad\left(U_{1} U_{2}\right)=0, \quad\left(U_{1}\left(U_{1} U_{2}\right)\right)=0, \quad\left(U_{2}\left(U_{1} U_{2}\right)\right)=0
$$

All groups of four-fold extended space that leave a non-integrable system (32) invariant can also be characterized in a very simple way on the basis of Theorem 9 .

If a group in the variables $x_{1}, \ldots, x_{4}$ is given that leaves a non-integrable system (32) invariant then we imagine introducing new independent variables $x, y, y^{\prime}, y^{\prime \prime}$ such that (32) will take on the form:

$$
\begin{equation*}
d y-y^{\prime} d x=0, \quad d y-y^{\prime \prime} d x=0 \tag{33}
\end{equation*}
$$

Naturally, from the original group, a group in $x, y, y^{\prime}, y^{\prime \prime}$ will leave (33) invariant. Now, it emerges from the investigations of A. V. Bäcklund in Mathematische Annalen, Bd. IX, pp. 297, et seq., that any transformation in $x, y, y^{\prime}, y^{\prime \prime}$ that leaves (33) invariant will arise from a contact transformation:

$$
x_{1}=X\left(x, y, y^{\prime}\right), \quad y_{1}=Y\left(x, y, y^{\prime}\right), \quad y_{1}^{\prime}=P\left(x, y, y^{\prime}\right)
$$

of the plane $x, y$ by extension. As a result, one has:

## Theorem 10:

Any transformation group of the four-fold extended space that leaves a nonintegrable system of two independent Pfaff equations invariant will be similar to a group that arises from a group of contact transformations of the plane $x, y$ in such a way that
one extends the transformations of the latter by adding the second differential quotient of $y$ with respect to $x$.


[^0]:    ( ${ }^{1}$ ) Cf., Theorie der Transformationsgruppen, Part One, written by Sophus Lie in collaboration with Engel, 1888, Teubner, Leipzig, and esp. pp. 529, 530.

