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The second variation of a simple integral

(First Communication)

by

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Ever since **Clebsch** introduced the study of the second variation for the relative maxima-minima of simple integrals with his treatise ([1]): "Über die Reduction der zweiten Variation auf ihre einfachste Form," that topic has found renewed interest only as a result of a treatise by **A. Mayer** ([2]), insofar as he established **Clebsch**'s main result in a simpler and clearer way. That is because the work of **Scheefer** ([3]), who promised to fill in a gap in the theory and "shed some new light on the foundations of the theory and the intrinsic meaning of the criteria of the maximum and minimum," has by no means fulfilled that promise since when one follows through the allegedly new train of thought from which that treatise starts, one will find that it includes too many gaps and suffers such flaws that it is almost worthless as a result. I shall forego the task of giving a more rigorous basis for that assertion here since I have indicated such a thing for the Jacobi case in a previous work ([4]), and I will show how that basic idea is to be implemented exactly for the general case in a later one. However, in the interests of historical truth, I would like to already mention here that it was by no means new at the time of publication of **Scheefer**'s treatise, but it had already

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([1]) J. für Math., Bd. 55.
([2]) *Ibid.*, Bd. 69.
([3]) Math. Ann. Bd. 25 and 26.
([4]) These Sitzungsberichte, Bd. XCVII, Abt. II. Unfortunately, that treatise is marred by a vast number of printing errors.

been thoroughly exploited some years before by **Weierstrass** in his lectures on the calculus of variations ([1]), and it was also suggested by **Hesse** ([2]).

In the present article, I seek to arrive at the criteria for the permanence of the sign of the second variation along the trail that **Clebsch** blazed of transforming the latter. It leads to that objective in a simpler and more-natural way and gives a clear insight into all prevailing relationships. However, in order to commit to that quest, if would first be necessary to discuss the integration of a canonical system of first-order differential equations in order to make a sufficiently-clear presentation of the derivation of its integrasl and to find sufficiently-general conditions under which those integrals would be differentiable with respect to the integration constants or parameters. The application of the rules thus-found to the differential equations that originate in the first variation likewise point to the necessity (to which **Scheefer** had also referred, but in a more peripheral way) of distinguishing between the two cases of whether *every* condition equation does or does not also include derivatives of the desired functions.

However, as will be shown, that separation, which will first become clear here, is not restricted to this special question, but permeates the entire study of the second variation. It will be likewise required in the treatment of the second-order system of linear equations that is already posed at the onset of one's investigation of the second variation and that I call the "accessory system of equations." In that way, the application of the theorems about the canonical system of differential equations will imply the validity of the assumption that was made up to now that integrating the differential equations of the first variation will also imply the solution of the accessory system of equations. The attempt to appeal to that system of equations in order to transform the second variation will shift a first-order bilinear differential expression to the foreground that already appeared implicitly in **Clebsch** and which will vanish for certain systems of solutions to the accessory system of equations that I call conjugate. Its properties yield the transformation that **Clebsch** found in a natural way, and I feel it would be appropriate to take up that derivation because it seems to me that it flows directly from the two sources that are known to us up to now, namely, the property of the accessory system of equations can be referred to briefly by an expression that is customary in the theory of linear differential equations: It is self-adjoint. The significant role that is thereby conferred to the conjugate systems will give one a good reason to examine the systems of solutions to the accessory system of equations more precisely. The properties of a certain determinant whose elements are the aforementioned bilinear differential expressions guide one to a distinguished class of fundamental systems of solutions in which every two terms are associated with each other pair-wise and which I call "involutory" fundamental systems. With their help, it will be possible to probe deeper into the nature of conjugation and solve a series of relevant

([1]) I have a transcript of those lectures that were presented in Summer semester of 1879. Moreover, I would not in any way like to use the remark above in order to justify the suspicion that **Scheefer** had borrowed his basic ideas from **Weierstrass**'s lectures. Rather, I infer the conclusion that he did not know of the lectures from the fact that he published such an inchoate work. That is because if **Weierstrass** did not go beyond the simplest case (so the one where the function includes merely one dependent variable and only its first derivative) in them, and as is clear from a remark in the lectures in the Summer semester of 1884, he also could not go beyond it, then one might direct one's attention to several complex and delicate questions that recur in the greater complication of the general problem. Had he known about them, **Scheefer** would have therefore either been saved from committing many errors or he would have postponed publication. I am pleased to take this opportunity to gratefully acknowledge the fact that **Weierstrass**' lectures offered me much encouragement to pursue my own work on the second variation in the intended sense.

([2]) J. für Math., Bd. 54.

problems that include, among others, the general form of the conjugate systems that **Clebsch** ([1]) found before by means of partial differential equations. A proper class of conjugate systems will be singled out by the transformation that **Mayer** ([2]) employed before. Each point of the domain of integration is associated with a group of such systems whose determinants differ by only a constant factor.

Those systems are crucial in the criteria for the permanence of the sign of the second variation, but its use for that purpose first became possible when I succeeded in discovering a less-obvious formula that represented a connection between the determinants of two conjugate systems, their first derivatives, and the quadratic form in the transformed second variation. With its help, one can now prove rigorously what had remained an assumption up to now, namely, that a logical adaptation of the theorems that are true for the relative maxima and minima in regard to the sign of the second variation that had been established for some time now in the simplest Jacobi case is still true. The service that the formula contributes to these investigations might seem to be the key to the entire study of the second variation, but the justification for that statement will emerge much better in the later work since the formula also makes it possible to answer the question of the extent to which the conditions for the permanence of the sign of the second variation that are found here, which yield merely necessary conditions for the occurrence of a maximum or minimum, are also sufficient, and to then exhibit that fact ([3]).

The investigations in this treatise are, in part, carried out under assumptions that are more general than usual, but on the other hand, restrictions will again be made that would be inadmissible when one would not like to give the results the pretense of greater generality than would be justified by the developments. On that basis, in the present work, the assumption will also be made consistently that the ordinates along the curves that are obtained from the first variation are single-valued functions of the abscissa in the entire domain of integration, such that the given curve integral will then reduce to an ordinary definite integral. In a later work that will appear next, the results that are obtained under that restriction will be freed of it.

I have divided the treatise into three parts on largely superficial grounds. The brief preface to each part will give some information about its contents.

In this first part, what will be discussed initially is the integration of a canonical system of first-order differential equations and the differentiation of its integrals with respect to integration constants and parameters, as well as the continuation of a integral element, and the theorems thus-obtained will then be applied to the differential equations of the first variation. As a preliminary to that, the problem of transformation will be treated for the case in which no condition equations are present, and the case in which condition equation includes derivatives of the desired functions can then be addressed. The accessory system of differential equations will be examined for that case,

([1]) J. für Math., Bd. 56.
([2]) *Loc. cit.*
([3]) In the meantime, I convinced myself that the second variation of multiple integrals can also be treated with the method that is developed here.

and the transformation of the second variation into the form that **Clebsch** gave, which is called the *reduced* for, will be performed.

I.

1. – Some general properties of systems of real differential equations will be used in the following developments, and it would not be inappropriate then to discuss the integration of such systems in more detail here, but I shall defer that to a later occasion and confine myself to only what is immediately necessary. **Cauchy**'s general method for exhibiting a system of integrals (or more correctly, an element in it) of the system of differential equations:

$$\frac{dx_1}{dt} = f_1(t, x_1, \ldots, x_n), \qquad \ldots, \qquad \frac{dx_n}{dt} = f_n(t, x_1, \ldots, x_n)$$

that fulfill the given initial conditions was given a very convenient form by **Picard** ([1]), and I would like to derive the necessary theorems from it.

The assumptions that are made here about the n functions $f_1, \ldots, f_n$ shall not be the most general ones under which those theorems will still be well-defined, but merely the following ones: They shall be single-valued, finite, and continuous in a certain neighborhood of a location t_0, $x_1^0, \ldots, x_n^0$ of radius ρ about t and radius r about $x_1, \ldots, x_n$, along with their first differential quotients with respect to $x_1, \ldots, x_n$, whose existence will be assumed, which is ultimately understood to mean that they have finite upper and lower limits. One can then successively exhibit an element of the system of integrals in the following way, in which $x_1, \ldots, x_n$ assume the values $x_1^0, \ldots, x_n^0$, respectively, at t_0. The first approximation $x_1^1, \ldots, x_n^1$ is given by:

$$x_1^1 - x_1^0 = \int_{t_0}^t f_1(t, x_1^0, \ldots, x_n^0)\,dt, \ \ldots, \ x_n^1 - x_n^0 = \int_{t_0}^t f_n(t, x_1^0, \ldots, x_n^0)\,dt,$$

the second one $x_1^2, \ldots, x_n^2$ by:

$$x_1^2 - x_1^0 = \int_{t_0}^t f_1(t, x_1^1, \ldots, x_n^1)\,dt, \ \ldots, \ x_n^2 - x_n^0 = \int_{t_0}^t f_n(t, x_1^1, \ldots, x_n^1)\,dt,$$

and the m^{th} one $x_1^m, \ldots, x_n^m$ by:

([1]) Journal de Mathématiques (1890).

$$x_1^m - x_1^0 = \int_{t_0}^t f_1(t, x_1^{m-1}, \ldots, x_n^{m-1})\,dt\,, \ldots,\; x_n^m - x_n^0 = \int_{t_0}^t f_n(t, x_1^{m-1}, \ldots, x_n^{m-1})\,dt\,.$$

If one continually takes the upper limit t of the integral to be sufficiently close to t_0 then the t, x_1^m, …, x_n^m will always fulfill the necessary condition for the step-wise calculation to fall within the neighborhood of t_0, x_1^0, …, x_n^0. If the absolute values of $f_1(t, x_1, \ldots, x_n)$, …, $f_n(t, x_1, \ldots, x_n)$ remain smaller than the number M in that neighborhood then it would suffice that t is taken to be small enough that $x_k^0 - M(t - t_0)$ will lie in it, as well as $x_k^0 + M(t - t_0)$ for $k = 1, 2, \ldots, n$, which will be achieved when $|t - t_0|$ is taken to be smaller than the smaller of the two quantities ρ and r / M. If one introduces the notation $x_k^m - x_k^{m-1} = X_k^m$ and chooses M such that one will also have $|\partial f_k / \partial x_l| < M$ in that neighborhood then from the mean value theorem, one will have:

$$\begin{aligned} X_k^m &= \int_{t_0}^t [f_k(t, x_1^{m-1}, \ldots, x_n^{m-1}) - f_k(t, x_1^{m-2}, \ldots, x_n^{m-2})]\;dt \\ &< M\int_{t_0}^t [|x_1^{m-1} - x_1^{m-2}| + \ldots + |x_n^{m-1} - x_n^{m-2}|]\;dt \\ &< M\int_{t_0}^t \left(\sum_{k=1}^m |X_k^{m-1}|\right) dt \end{aligned}$$

for $m > 1$. For $m = 1$, one gets:

$$|X_k^1| < M\,|t - t_0|$$

directly, from which it will then follow that:

$$|X_k^m| < \frac{n^{m-1} M^m |t - t_0|^m}{m!} < \frac{[n M |t - t_0|]^m}{m!}.$$

Thus, each of the series:

$$x_k = x_k^0 + X_k^1 + X_k^2 + \cdots \qquad (k = 1, 2, \ldots, n),$$

which collectively define an element of a system of integrals in a neighborhood of the location t_0, x_1^0, …, x_n^0 with a radius equal to the smaller of the two quantities ρ and r / M, will converge absolutely and uniformly. Obviously, the system thus-obtained is also the only one that will assume the values x_1^0, …, x_n^0, respectively, at t_0 under the conditions that were imposed.

2. – Since the first derivatives of the functions $f_1(t, x_1, \ldots, x_n)$, …, $f_n(t, x_1, \ldots, x_n)$ are continuous in *all* variables in the given region, the X_k^0, X_k^1, …, X_k^m, …, so each term in the series that represents x_k, will also be continuous in the integration constants x_1^0, x_2^0, …, x_n^0. Therefore, since it would emerge from the foregoing inequality that the series for x_k will converge *uniformly* at x_1^0, …, x_n^0 and in a neighborhood of it, x_k will also be a continuous function of those quantities. If one combines the previous conditions with the further one that the $f_1(t, x_1, …, x_n)$, …, $f_n(t, x_1, …, x_n)$ must also possess second differential quotients with respect to $x_1, x_2, …, x_n$ in a neighborhood of t_0, x_1^0, …, x_n^0 and that they are all continuous with respect to all variables in it then the integrals that were obtained in (1) will also be differentiable with respect to the integration constants x_1^0, x_2^0, …, x_n^0.

Since differentiation under the integral sign is permissible as a consequence of the assumptions that were made, one will next get:

$$\left|\frac{\partial x_k^m}{\partial x_i^0}\right| < e^{nM|t-t_0|} < e^{nM\delta}$$

from successive calculations when one takes $|t - t_0| < \delta$, in which δ is the smaller of the two numbers ρ and r / M.

If one denotes:

$$\frac{\partial f_k(t, x_1, \ldots, x_n)}{\partial x_\lambda} \qquad \text{and} \qquad \frac{\partial^2 f_k(t, x_1, \ldots, x_n)}{\partial x_\lambda\, \partial x_\mu}$$

by

$$f_{k,\lambda}(t, x_1, \ldots, x_n) \qquad \text{and} \qquad f_{k,\lambda,\mu}(t, x_1, \ldots, x_n),$$

respectively, for the sake of brevity, and considers the fact that:

$$x_k^{m-1} = x_k^{m-2} + X_k^{m-1},$$

then that will give:

$$\frac{\partial X_k^m}{\partial x_i^0} = \int_{t_0}^t \sum_{\lambda=1}^n \left\{[f_{k,\lambda}(t, x_1^{m-1}, \ldots, x_n^{m-1}) - f_{k,\lambda}(t, x_1^{m-2}, \ldots, x_n^{m-2})]\frac{\partial x_\lambda^{m-2}}{\partial x_i^0} + f_{k,\lambda}(t, x_1^{m-1}, \ldots, x_n^{m-1})\frac{\partial X_\lambda^{m-1}}{\partial x_i^0}\right\} dt$$

$$= \int_{t_0}^t \sum_\lambda \left[\sum_\mu f_{k,\lambda,\mu}(t, x_1^{m-2} + \theta X_1^{m-1}, \ldots, x_n^{m-2} + \theta X_1^{m-1})\frac{\partial x_\lambda^{m-2}}{\partial x_i^0} X_\mu^{m-1} - f_{k,\lambda}(t, x_1^{m-1}, \ldots, x_n^{m-1})\frac{\partial X_\lambda^{m-1}}{\partial x_i^0}\right] dt\ .$$

Now, if M is chosen to be large enough that it is also greater than the second derivatives of the $f_1, f_2, …, f_n$ with respect to the $x_1, x_2, …, x_n$ in a neighborhood of the location t_0, x_1^0, …, x_n^0 and one lets M' denote a number that is greater than:

$$M \qquad \text{and} \qquad n\,M\,e^{nM\delta}$$

then one will find that:

$$\left|\frac{\partial X_k^m}{\partial x_i^0}\right| < M'\int_{t_0}^{t}\sum_{\lambda=1}^{n}\left(\left|\frac{\partial X_k^{m-1}}{\partial x_i^0}\right| + X_\lambda^{m-1}\right)dt,$$

which is a formula that is true for $m > 1$. For $m = 1$, that will give:

$$\left|\frac{\partial X_k^m}{\partial x_i^0}\right| < M' \mid t - t_0 \mid$$

directly, and finally, by means of the formula above:

$$\left|\frac{\partial X_k^m}{\partial x_i^0}\right| < M' \mid t - t_0 \mid \frac{[n M' \mid t - t_0 \mid]^m}{m!},$$

from which it will follow that the series:

$$x_k^0 + X_k^1 + X_k^2 + \cdots$$

is term-wise differentiable with respect to x_i^0 since the series of differential quotients of those terms converges uniformly.

One then gets the theorem:

If the first derivatives with respect to $x_1, x_2, \ldots, x_n$ *of the functions* $f_1 (t, x_1, \ldots, x_n), \ldots, f_n (t, x_1, \ldots, x_n)$ *in the differential equations:*

$$\frac{dx_1}{dt} = f_1 (t, x_1, \ldots, x_n), \qquad \ldots, \qquad \frac{dx_n}{dt} = f_n (t, x_1, \ldots, x_n)$$

are all single-valued, finite, and continuous in all of the variables at the location t_0, $x_1^0, \ldots, x_n^0$ *and in a neighborhood of it then there will be one and only one system of integrals whose terms assume the values* $x_1^0, \ldots, x_n^0$, *respectively, at* t_0 *. Those terms are continuous in the* $x_1^0, \ldots, x_n^0$ *in a neighborhood of it and also certainly possess differential quotients with respect to them when the second derivatives of the* $f_1, f_2, \ldots, f_n$ *with respect to the* $x_1, x_2, \ldots, x_n$ *exist at the location* t_0, $x_1^0, \ldots, x_n^0$, *and are likewise single-valued, finite, and continuous with respect to* $t, x_2, \ldots, x_n$ *in a neighborhood of* t_0, $x_1^0, \ldots, x_n^0$ ([1]).

Obviously, under those assumptions, the integrals will also have second and third derivatives with respect to t that are single-valued, finite, and continuous in the aforementioned neighborhood.

([1]) I have already presented these theorems in several lectures at the beginning of this decade.

Since the series of differential quotients of the terms in each series above will also converge uniformly with respect to t at the location t_0, one will have:

$$\frac{\partial x_i}{\partial x_k^0} = 0 \qquad \text{for} \qquad i \neq k \quad \text{and} \qquad \frac{\partial x_k}{\partial x_k^0} = 1$$

at that location.

If the neighborhood of t_0, x_1^0, …, x_n^0 in which f_1, f_2, …, f_n are single-valued, finite, and continuous in the t, x_1, …, x_n, along with their first derivatives with respect to x_1, x_2, …, x_n, has a radius of ρ with respect to t and r with respect to x_1, x_2, …, x_n, and if the absolute values of the functions and their first derivatives are not greater than M in that neighborhood, moreover, then one will get the element of the system of integrals for a neighborhood of t_0 that is smaller than the smaller of the quantities ρ and r / M. If ρ', r', and M' denote the analogous quantities relative to the functions of their first and second derivatives with respect to the x_1, x_2, …, x_n then the terms in the system of integrals will certainly be differentiable with respect to x_1^0, x_2^0, …, x_n^0 for the region $| t - t_0 | < \rho'$ and r' / M'.

One sees immediately from the derivation that the last part of the theorem can be extended in the case where the $f_1, f_2, …, f_n$ possess differential quotients with respect to the x_1, x_2, …, x_n up to order $(m + 1)$ and they are single-valued, finite, and continuous for all t, x_1, x_2, …, x_n for which $| t - t_0 | < \rho$, $| x_1 - x_1^0 | < r$, …, $| x_n - x_n^0 | < r$. If M is a number that is not smaller than the absolute magnitudes of the functions and their aforementioned derivatives in that neighborhood then the terms in the system of integrals thus-found will possess derivatives up to order m with respect to x_1^0, x_2^0, …, x_n^0 for all that are likewise $< \rho$ and r / M.

The considerations that led to the remark about the differentiation of the terms in the integral element that was obtained with respect to the constants x_1^0, x_2^0, …, x_n^0 can also be applied to the case in which the functions $f_1, f_2, …, f_n$ include a parameter μ, and that will lead to the following realization:

If $f_1, f_2, …, f_n$, along with all of their first derivatives with respect to x_1, x_2, …, x_n, μ (whose existence is then assumed) are single-valued, finite, and continuous with respect to t, x_1, x_2, …, x_n, μ at the location t_0, x_1^0, x_2^0, …, x_n^0, μ_0, and if the equality of the mixed differential quotients of the $f_1, f_2, …, f_n$ with respect to x_1, x_2, …, x_n is true for μ then the terms in the integral element will also be differentiable with respect to μ at μ_0.

If r is the radius of the cited neighborhood relative to t, and r is its radius relative to x_1, x_2, …, x_n, μ, and if M is not smaller than the absolute values of the aforementioned functions and their derivatives, moreover, then the differentiation of the integral will certainly be permissible for all t that satisfy the condition $| t - t_0 | < \rho$ and r / M.

That theorem can be extended immediately to the case in which the $f_1, f_2, …, f_n$ admit repeated differentiations with respect to the μ, and one will then get:

If the $f_1, f_2, \ldots, f_n$ have mixed differential quotients of first order with respect to the $x_1, x_2, \ldots, x_n$ and up to order m with respect to μ, and if they are single-valued, finite, and continuous with respect to $t, x_1, x_2, \ldots, x_n, \mu$ at t_0, x_1^0, x_2^0, …, x_n^0, μ_0, and in its neighborhood then the integrals will certainly possess differential quotients with respect to μ up to order m at μ_0 for all t in a neighborhood of t_0 whose radius is smaller than ρ' and r'/M', when ρ', r', M' mean the quantities analogous to ρ, r, M, resp.

3. – If one applies the concept of continuation that comes from the theory of functions of complex variables to the element that is obtained when one assumes that the center of continuation is in the domain of the element, and one constructs the associated integral element from its functional values then the domain of the new element can possibly go beyond the domain of the original one. By repeating that process, one will obtain a system of functions for a line segment t_0T that is greater than or equal to the domain of the initial element that satisfy the differential equations everywhere in t_0T, are single-valued, and are independent of the choice of continuation center since the functional values at each point of the segment admit only one integral element. Those integral elements that are associated with the individual locations of t_0T are equivalent to the construction of the functions, insofar as the entire system of functions that is a system of integrals van be exhibited in terms of each of them.

If the continuation is carried out on the basis of the conditions that were given (2) then every integral element in t_0T will be differentiable with respect to its initial values, and therefore also its immediate continuation. Since each successive immediate continuation is given, one will gain the insight that:

"The system of integrals is differentiable with respect to the initial values of each of its elements along the entire segment t_0T."

If the differential equations also include parameters and one preserves the conditions that were given in (2) under which an integral element proved to be differentiable with respect to the individual parameters in the continuation then one will once more see that the system of integrals will be differentiable with respect to the parameters along the entire segment t_0T, when the segment over which the initial elements is continued is denoted by t_0T.

Of all the locations that would be obtained by continuation from an initial element, the limits t_0 and T are the closest ones at which the conditions for the formation of an element that were given in (1) or (2) will no longer be fulfilled. However, they must not be by any means the most extreme limit points that can be reached in that way since it would still be possible that locations would exist in a neighborhood of one of them at which the associated integral element would be valid for greater than the smallest interval that was given in (1) and (2) and would thus go beyond the limit point. However, the possibility would still exist that a continuation beyond one or both limit points does not indeed exist, but that a new single-valued system of integrals begins there. The first one that was found would then be only one branch of a multi-valued system of integrals. It is the latter circumstance, which does not come to light in the integration process that is applied, that makes a modification of that process especially desirable.

A hint as to how to do that is given by a discussion of the simple case:

$$F(x, y) = 0\,,$$

when F is finite and continuous in a certain region of x and y, along with its first two partial derivatives F'_x and F'_y.

The theorem then exists:

If x_0, y_0 is a location in that region at which $F(x, y)$ vanishes and F'_y is non-zero there then there will exist one and only one function of x that is single-valued and continuous in a neighborhood of x_0 that possesses a differential quotient with respect to x at x_0, assumes the value y_0 there, and makes $F(x, y)$ identically zero for that neighborhood of x when it is substituted for y.

If one now continues the element that is found in the context of the equation $F(x, y) = 0$ then one will get a function of x that is single-valued, continuous, and differentiable in an interval ab and can substitute for y. $F(x, y)$ is made identically zero in that interval. $F(x, y)$, F'_x, and F'_y have the same properties that they have at the point x_0, y_0 inside of the interval, but they will no longer have them at a and b. Nevertheless, a and b must still not be the most-extreme reachable limits, and the function that is found must also not be the only branch that is derivable from the initial element. One sees the latter immediately when the continuation is impossible at one of the limit points a and b since F'_y will be zero there, while $F(x, y)$ and F'_x will remain finite and continuous. If F'_y is not zero then x can be represented as a single-valued continuous function of y in a neighborhood of the y in question, and it is therefore a multi-valued function of x in the neighborhood of the limit points in question, *inter alia* (German: i. A.). It will then, in fact, define a new single-valued function of x, namely, a second branch that satisfies the equation for y.

The drawbacks that were discussed can be avoided when one seeks to represent x and y as single-valued functions of a third variable. What proves to be very suitable for that purpose is the variable s that is defined by:

$$s = \int_{x_0}^{x} \sqrt{1 + y'^2}\, dx\,,$$

in which $y' = dy / dx$. Namely, with that choice, not only is s a single-valued function of x, but also, conversely, x is a single-valued function of s. Moreover, the integral will exist only as long as y' is not infinite of order first or higher in the integration interval (including the limits), but as one easily sees, it will certainly always exist when y exists, but does not become infinite along with y', and remains continuous.

The two differential equations:

$$\frac{dx}{ds} = \frac{F'_x}{\sqrt{F'^2_x + F'^2_y}}\,, \qquad \frac{dy}{ds} = \frac{F'_y}{\sqrt{F'^2_x + F'^2_y}}$$

will now enter in place of the previous equation.

If one associates the value s_0 to the given x_0, y_0 and defines the integral from those initial values using (1) then one can derive two single-valued functions of s that satisfy the two differential equations from that in the previously-explained way by continuing x and y. However, those continuations will now go beyond the locations where only one of the functions and F'_x and F'_y vanish, as well as beyond the one where both of them indeed become simultaneously zero, but dy / dx and dy / ds assume finite limiting values. The functions thus-obtained satisfy the equation $F(x, y) = 0$, and in that way the locations that were previously excluded from the representation will now be included. One will see that they satisfy the equation when one substitutes them in $F(x, y)$ since one will get:

$$\frac{dF}{ds} = \frac{\partial F}{\partial y}\frac{dy}{ds} + \frac{\partial F}{\partial x}\frac{dx}{ds} = 0 ,$$

so F = const.

However, since $F(x, y)$ is a continuous function of both variables x and y for all values of x and y that come under consideration, and each of them are also continuous functions of s, F will also be a continuous function of s. Therefore, the constant will have the same value for the entire interval of s, and since it is zero for x_0, it will be zero over the entire interval.

In order to make that argument useful for the integration of the system of differential equations in (1), one can start from the remark that the system of integrals establishes a curve in the manifold $x, x_1, \ldots, x_n$, in which x is written in place of t for the sake of uniformity, and then seek to represent them as functions of the arc-length, which is defined analytically by the equation:

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dx_1}{dx}\right)^2 + \cdots + \left(\frac{dx_n}{dx}\right)^2} .$$

Instead of the original system, one will now integrate the new one:

$$\frac{dx}{ds} = \frac{1}{\sqrt{1 + f_1^2 + \cdots + f_n^2}} ,$$

$$\frac{dx_1}{ds} = \frac{f_1}{\sqrt{1 + f_1^2 + \cdots + f_n^2}} ,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\frac{dx_n}{ds} = \frac{1}{\sqrt{1 + f_1^2 + \cdots + f_n^2}} .$$

The integrals of that system extend beyond locations that were inaccessible in the previous representation. It will no longer be an obstacle for the continuation when only one of the functions $f_1, f_2, \ldots, f_n$ becomes infinite at a location, or even when that happens with several of them, as long as the quotients above possess finite limiting values.

It is also clear that when one recalls the conditions in (2) for the continuation, the system of integrals that is obtained will be differentiable with respect to the initial values of any element for the entire range of s, and also with respect to any possible parameters, when one observes the sufficient conditions that relate to them that were given in (2).

If s is expressible as a single-valued function of a new variable t then one can naturally introduce that new variable in place of s, as well. One can obtain the original system of n integrals from the system of integrals of the $(n + 1)$ differential equations when one again expresses s in terms of x and substitutes that in the equations for $x_1, x_2, \ldots, x_n$.

That can always be arranged as long as dx / ds possesses only isolated zeroes, i.e., when a neighborhood of finite radius exists around each zero in which no other zero lies, and the lower limit of that radius is not zero. The interval can then be subdivided into a finite number of pieces in such a way that dx / ds will not vanish inside of each of them.

Each of those individual pieces will then correspond to an interval of x that is associated with a system of integrals. The validity of the system is restricted to that interval, but when two of those intervals for x have a common limit point and overlap completely, the two associated systems of integrals can be considered to be continuations, and will then be combined into a system of integrals that extends over both intervals for x.

All of those systems of integrals have the properties of the system of integrals that was defined by means of the variable s. If it were differentiable with respect to the initial values of an element or a parameter then that derived system of integrals would be differentiable with respect to the initial values of any one of its elements or with respect to the parameter.

One can introduce a new variable t in place of s, except that s must be a single-valued function of t and grow continuously from the starting point along the entire arc of the curve when t runs through a certain interval monotonically. In that way, one will see that one can treat the following system of differential equations:

$$\frac{dx_1}{dt} = f_1(x, x_1, \ldots, x_n)\,\frac{dx}{dt}, \qquad \ldots, \qquad \frac{dx_n}{dt} = f_n(x, x_1, \ldots, x_n)\,\frac{dx}{dt},$$

in which the choice of dx / dt remains open, in place of the original system.

No use of the remarks that were made here in no. **3** will indeed be made in the preset treatise, but only in the following ones, but they were already introduced here for the sake of describing the connection between the concepts in nos. **1** and **2**.

II.

1. – As is known, one can give the most-general problem in the calculus of variations for one independent variable the form:

Determine the variables $y_1, y_2, \ldots, y_n$ in such a way that they satisfy the m equations:

$$\varphi_1 = 0, \qquad \varphi_2 = 0, \qquad \ldots, \qquad \varphi_m = 0, \tag{1}$$

where $n > m$, and make the integral:

$$\int_a^b f(x, y_1, y_1', y_2, y_2', \ldots, y_n, y_n')\, dt$$

a maximum or minimum.

As was pointed out before in the Introduction, that integral is not an ordinary definite integral, and the limits that are placed on it actually indicate only that the $y_1, y_2, \ldots, y_n$ are again subject to certain conditions for those values of x. With no loss of generality, it is known that one can restrict oneself to the case ([1]) in which the $y_1, y_2, \ldots, y_n$ assume given values at a and b.

The function $f(x, y_1, y_1', y_2, y_2', \ldots, y_n, y_n')$ can be assumed to be very general insofar as it is by no means necessary for the following considerations that it should be analytic, but merely that its first, second, and third differential quotients with respect to $x, y_1, \ldots, y_n, y_1', \ldots, y_n'$ should exist in the domain of integration and remain finite there. The same thing should be true for the functions $\varphi_1, \varphi_2, \ldots, \varphi_m$.

According to the **Lagrange** process, which I shall regard as valid here, one must consider the integral:

$$J = \int_a^b F\, dx$$

in place of the original one, in which one has:

$$F = f + \lambda_1\, \varphi_1 + \ldots + \lambda_m\, \varphi_m ,$$

and the $\lambda_1, \lambda_2, \ldots, \lambda_m$ are still-undetermined functions of x that are to be calculated from the conditions on the problem.

If one now varies the function F by setting $y_1, y_2, \ldots, y_n$ equal to $y_1 + \eta_1, y_2 + \eta_2, \ldots, y_n + \eta_n$, and giving the η the very special form:

$$\eta_1 = \varepsilon\, u_1, \quad \eta_2 = \varepsilon\, u_2, \quad \ldots, \qquad \eta_n = \varepsilon\, u_n,$$

in which the $u_1, u_2, \ldots, u_n$ are functions of x that are free of ε, and their first derivatives all remain finite in the domain of integration, and ε is a differential quantity, then the development of $F(x, y_1 + \eta_1, y_1' + \eta_1', \ldots, y_n' + \eta_n')$ up to terms of order three will show that the $y_1, y_2, \ldots, y_n$ also have to satisfy the n differential equations:

$$\frac{\partial F}{\partial y_i} - \frac{d}{dx}\left(\frac{\partial F}{\partial y_i'}\right) = 0 \qquad (i = 1, 2, \ldots, n).$$

The $(n + m)$ quantities $y_1, y_2, \ldots, y_n, , \lambda_1, \lambda_2, \ldots, \lambda_m$ are then to be determined from the $(n + m)$ equations:

([1]) **Mayer**, *loc. cit.*

$$\left.\begin{array}{l}\dfrac{\partial F}{\partial y_i}-\dfrac{d}{dx}\left(\dfrac{\partial F}{\partial y_i'}\right)=0 \quad (i=1,2,\ldots,n), \\ \varphi_1=0, \quad \varphi_2=0, \quad \ldots, \quad \varphi_m=0.\end{array}\right\} \tag{2}$$

When one introduces new variables y_1', y_2', …, y_n' by way of the equations:

$$\frac{dy_1}{dx}=y_1', \quad \frac{dy_2}{dx}=y_2', \ldots, \quad \frac{dy_n}{dx}=y_n'$$

that system will become linear with respect to the quantities:

$$y_1', y_2', \ldots, y_n'; y_1, y_2, \ldots, y_n; \lambda_1, \lambda_2, \ldots, \lambda_m,$$

and if one calculates the:

$$\frac{dy_1'}{dx}, \frac{dy_2'}{dx}, \ldots, \frac{dy_n'}{dx}; \lambda_1', \lambda_2', \ldots, \lambda_m'$$

from the enlarged, but now linear, system then it will be put into the canonical form that the considerations of Section I were based upon. In the treatment of the second variation, it will be assumed that it has already been integrated, and by means of the remarks in I, it is now possible to give a somewhat-clearer picture of the structure of that system of integrals. However, it would then be necessary to take into account the behavior of the m condition equations (1) in that and distinguish between the two cases in which either each equation includes the first derivatives of the y_1, y_2, …, y_n, in addition to the latter variables, or equations are also present in which no derivatives occur. Naturally, in that way, it is assumed that a system of conditions equations of the first kind cannot be transformed into one of the second kind.

2. – In order to give the system (2) the canonical form in the first case, one must solve the system of $(n + m)$ equations:

$$\frac{\partial F}{\partial y_i}-\frac{d}{dx}\left(\frac{\partial F}{\partial y_i'}\right)=0 \qquad (i = 1, 2, \ldots, n),$$

$$\sum_{k=1}^{n}\frac{d\varphi_i}{dx}=0 \qquad (i = 1, 2, \ldots, m)$$

for the y_1'', y_2'', …, y_n'', λ_1', λ_2', …, λ_m', and then add the n equations:

$$\frac{dy_1}{dx}=y_1', \quad \frac{dy_2}{dx}=y_2', \ldots, \quad \frac{dy_n}{dx}=y_n'.$$

That representation will be possible at every location x in the domain of integration where the determinant of the system:

$$\begin{vmatrix} \dfrac{\partial^2 F}{\partial y_1' \partial y_1'} & \cdots & \dfrac{\partial^2 F}{\partial y_1' \partial y_n'} & \dfrac{\partial \varphi_1}{\partial y_1'} & \cdots & \dfrac{\partial \varphi_m}{\partial y_1'} \\ \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \in \\ \dfrac{\partial^2 F}{\partial y_n' \partial y_1'} & \cdots & \dfrac{\partial^2 F}{\partial y_n' \partial y_n'} & \dfrac{\partial \varphi_1}{\partial y_n'} & \cdots & \dfrac{\partial \varphi_m}{\partial y_n'} \\ \dfrac{\partial \varphi_1}{\partial y_1'} & \cdots & \dfrac{\partial \varphi_1}{\partial y_n'} & 0 & \cdots & 0 \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ \dfrac{\partial \varphi_m}{\partial y_1'} & \cdots & \dfrac{\partial \varphi_m}{\partial y_n'} & 0 & \cdots & 0 \end{vmatrix}$$

does not vanish. At such a location, not all determinants of degree m in the matrix:

$$\begin{vmatrix} \dfrac{\partial \varphi_1}{\partial y_1'} & \dfrac{\partial \varphi_1}{\partial y_2'} & \cdots & \dfrac{\partial \varphi_1}{\partial y_n'} \\ \vdots & \vdots & \cdots & \vdots \\ \dfrac{\partial \varphi_m}{\partial y_1'} & \dfrac{\partial \varphi_m}{\partial y_2'} & \cdots & \dfrac{\partial \varphi_m}{\partial y_n'} \end{vmatrix}$$

can be zero. If one now assumes that:

$$\frac{\partial(\varphi_1, \varphi_2, \ldots, \varphi_m)}{\partial(y_1', y_2', \ldots, y_m')} \neq 0$$

then from a known theorem, equations (1) will allow one to represent y_1', y_2', …, y_n' as single-valued, continuous, and differentiable functions of the x, y_1, …, y_n, y_1', y_2', …, y_n' in a neighborhood of the location in question such that the values of y_1', y_2', …, y_m' will then be given already by the values of y_1, …, y_n, y_{m+1}', …, y_n' at the location x by means of the m condition equations (1). One can then define an integral element whose initial values are the values that:

$$y_1, y_2, \ldots, y_n, \; y_{m+1}', \; y_{m+2}', \ldots, \; y_n', \; \lambda_1, \lambda_2, \ldots, \lambda_m$$

possess at that location x and think of the system as being obtained by continuation according to the suggestions in I.3. However, it is also clear that locations x where the determinant above is non-zero must exist since it cannot vanish identically in any interval, however small. That is because if that were the case then the number of integration constants would be less than $2n$, which is incompatible with the conditions at the limits.

It will now be assumed expressly that the determinant (3) *does not vanish anywhere in the domain of integration*. That assumption, which will be extended in a later work, cannot be avoided as long one would like to consider x to be the independent variable, which will be the case in the present work, i.e., one then assumes that the $y_1, y_2, \ldots, y_n$ are represented as single-valued functions of the abscissa x in the entire domain of integration such that the given curve integral will become an ordinary definite integral. *Those assumptions are also tacitly at the basis for all published investigations up to now, to the extent that they are exact*.

It then follows that one can imagine that the system of integrals in (2) arises by continuation from an initial element, say, the one at the initial point a. Now, since the canonical system of differential equations in (2) also fulfills the conditions that were given in I.2, the terms in the system of integrals can be differentiated with respect to the initial values of $y_1, y_2, \ldots, y_n, y'_{m+1}, y'_{m+2}, \ldots, y'_n, \lambda_1, \lambda_2, \ldots, \lambda_m$ at a, which can be considered to be integration constants, and might be denoted by:

$$c_1, c_2, \ldots, c_{2n},$$

respectively.

3. – Of the m condition equations:

$$\varphi_1 = 0 , \varphi_2 = 0 , \ldots, \varphi_m = 0 ,$$

it will no longer be true that each of them also includes first derivatives of the $y_1, y_2, \ldots, y_n$, but no derivatives will enter into μ of them now, say:

$$\varphi_1 = 0 , \varphi_2 = 0 , \ldots, \varphi_\mu = 0 ,$$

while each of the remaining $(m - \mu)$ equations:

$$\varphi_{\mu+1} = 0 , \varphi_{\mu+2} = 0 , \ldots, \varphi_m = 0$$

also possess first derivatives, in which it is obviously assumed that an equation of the first type cannot be derived from them.

The treatment of the system requires that it should also be represented in canonical form in this case since the known theorems and developments were derived for such systems. Now, the given system in this case can be converted into an equivalent canonical one to the same extent as the foregoing one (2), but one can also exhibit a canonical system here that each of its systems of integrals satisfies, as before. One will get it when one uses the system of first-order differential equations:

$$\left.\begin{array}{ll}
\dfrac{\partial F}{\partial y_i} - \dfrac{d}{dx}\left(\dfrac{\partial F}{\partial y_i'}\right) = 0 & i = 1, 2, \ldots, n, \\[2ex]
\dfrac{d^2}{dx^2}\varphi_i(x, y_1, y_2, \ldots, y_n) = 0 & i = 1, 2, \ldots, \mu, \\[2ex]
\dfrac{d}{dx}\varphi_i(x, y_1, y_2, \ldots, y_n, y_1', \ldots, y_n') = 0 & i = \mu + 1, \ldots, m \\[2ex]
y_1' = \dfrac{dy_1}{dx}, \quad y_2' = \dfrac{dy_2}{dx}, \quad \ldots, \quad y_n' = \dfrac{dy_n}{dx} &
\end{array}\right\} \tag{3}$$

to calculate the quantities:

$$\frac{dy_1'}{dx}, \ldots, \frac{dy_n'}{dx}, \qquad \lambda_1, \ldots, \lambda_\mu, \qquad \frac{d\lambda_{\mu+1}}{dx}, \ldots, \frac{d\lambda_m}{dx},$$

and adds the equations of the last group above to the ones that were thus obtained. Now, the determination of those quantities will be possible at any location where the determinant:

$$\begin{vmatrix}
\dfrac{\partial^2 F}{\partial y_1' \partial y_1'} & \cdots & \dfrac{\partial^2 F}{\partial y_1' \partial y_n'} & \dfrac{\partial \varphi_1}{\partial y_1} & \cdots & \dfrac{\partial \varphi_\mu}{\partial y_1} & \dfrac{\partial \varphi_{\mu+1}}{\partial y_1'} & \cdots & \dfrac{\partial \varphi_m}{\partial y_1'} \\
\vdots & \ddots & \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\
\dfrac{\partial^2 F}{\partial y_n' \partial y_1'} & \cdots & \dfrac{\partial^2 F}{\partial y_n' \partial y_n'} & \dfrac{\partial \varphi_1}{\partial y_n} & \cdots & \dfrac{\partial \varphi_\mu}{\partial y_n} & \dfrac{\partial \varphi_{\mu+1}}{\partial y_m'} & \cdots & \dfrac{\partial \varphi_m}{\partial y_n'} \\
\dfrac{\partial \varphi_1}{\partial y_1} & \cdots & \dfrac{\partial \varphi_1}{\partial y_n} & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\
\dfrac{\partial \varphi_\mu}{\partial y_1} & \cdots & \dfrac{\partial \varphi_\mu}{\partial y_n} & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\dfrac{\partial \varphi_1}{\partial y_1'} & \cdots & \dfrac{\partial \varphi_1}{\partial y_n'} & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\
\dfrac{\partial \varphi_m}{\partial y_1'} & \cdots & \dfrac{\partial \varphi_m}{\partial y_n'} & 0 & \cdots & 0 & 0 & \cdots & 0
\end{vmatrix} \tag{3*}$$

does not vanish, and in order to achieve the general result in this third case, the assumption that the determinant above, which cannot vanish identically in any interval, no matter how small, is nowhere zero in in the domain of integration will become unavoidable.

The canonical system that is obtained is a mixed one since it consists of a canonical system of $2n + m - \mu$ first-order differential equations and μ equations that express $\lambda_1, \lambda_2, \ldots, \lambda_\mu$ in terms of the remaining quantities such that they will also be obtained immediately by solving the canonical system.

Now, any system of solutions of the canonical system of equations will satisfy the equations:

$$\frac{\partial F}{\partial y_i} - \frac{d}{dx}\left(\frac{\partial F}{\partial y_i'}\right) = 0\,, \qquad i = 1, 2, \ldots, n,$$

$$\frac{dy_1}{dx} = y_1', \qquad \frac{dy_2}{dx} = y_2', \qquad \ldots, \qquad \frac{dy_n}{dx} = y_m'$$

$$\varphi_i\,(x, y_1, y_2, \ldots, y_n) = a_i\,x + b_i, \qquad i = 1, 2, \ldots, \mu,$$

$$\varphi_i\,(x, y_1, \ldots, y_n, y_1', \ldots, y_n') = c_i\,, \qquad i = \mu + 1, \ldots, n,$$

in which the a_i, b_i, c_i mean constants.

One will then get all solutions of the original system of equations when one looks for the canonical system for which the constants are zero. That will be true if and only if the solutions at one (and therefore each) location in the domain of integration satisfy the equations:

$$\left.\begin{array}{ll} \varphi_i(x, y_1, y_2, \ldots, y_n) = 0, \quad \dfrac{d}{dx}\varphi_i(x, y_1, y_2, \ldots, y_n) = 0, & i = 1, 2, \ldots, \mu, \\ \varphi_i(x, y_1, \ldots, y_n, y_1', \ldots, y_n') = 0, & i = \mu + 1, \ldots, m. \end{array}\right\} \tag{4}$$

The desired system of integrals will then be obtained when one chooses the initial values of y_1, y_2, $\ldots$, y_n, y_1', y_2', $\ldots$, y_n' in this equations suitably.

One easily sees that this can always be arranged because as a result of the assumption that the determinant (3*) above is non-zero at every location x of the domain of integration, none of the determinants of degree m in the matrix:

$$\left|\begin{array}{cccccc} \dfrac{\partial\varphi_1}{\partial y_1} & \cdots & \dfrac{\partial\varphi_\mu}{\partial y_1} & \dfrac{\partial\varphi_{\mu+1}}{\partial y_1'} & \cdots & \dfrac{\partial\varphi_m}{\partial y_1'} \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ \dfrac{\partial\varphi_1}{\partial y_n} & \cdots & \dfrac{\partial\varphi_\mu}{\partial y_n} & \dfrac{\partial\varphi_{\mu+1}}{\partial y_n'} & \cdots & \dfrac{\partial\varphi_m}{\partial y_n'} \end{array}\right|$$

can be zero, either. If, say:

$$\begin{vmatrix} \dfrac{\partial\varphi_1}{\partial y_1} & \cdots & \dfrac{\partial\varphi_\mu}{\partial y_1} & \dfrac{\partial\varphi_{\mu+1}}{\partial y_1'} & \cdots & \dfrac{\partial\varphi_m}{\partial y_1'} \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ \dfrac{\partial\varphi_1}{\partial y_m} & \cdots & \dfrac{\partial\varphi_\mu}{\partial y_m} & \dfrac{\partial\varphi_{\mu+1}}{\partial y_m'} & \cdots & \dfrac{\partial\varphi_m}{\partial y_m'} \end{vmatrix}$$

is non-zero then not all determinants of degree m in the matrix:

$$\begin{vmatrix} \dfrac{\partial\varphi_1}{\partial y_1} & \dfrac{\partial\varphi_1}{\partial y_2} & \cdots & \dfrac{\partial\varphi_1}{\partial y_m} \\ \dfrac{\partial\varphi_2}{\partial y_1} & \dfrac{\partial\varphi_2}{\partial y_2} & \cdots & \dfrac{\partial\varphi_2}{\partial y_m} \\ \vdots & \vdots & \cdots & \vdots \\ \dfrac{\partial\varphi_\mu}{\partial y_1} & \dfrac{\partial\varphi_\mu}{\partial y_2} & \cdots & \dfrac{\partial\varphi_\mu}{\partial y_m} \end{vmatrix}$$

can vanish either.

If, say:

$$\frac{\partial(\varphi_1,\ldots,\varphi_\mu)}{\partial(y_1,\ldots,y_\mu)}$$

is a determinant in it that does not vanish at x then from the theorem that has been referred to many times before, when the values of $y_1, y_2, \ldots, y_n$ are given at the location x that satisfy the μ equations:

$$\varphi_1 = 0\ , \ \varphi_2 = 0\ , \ \ldots, \ \varphi_\mu = 0\ ,$$

one can exhibit μ functions of x, $y_{\mu+1}$, …, y_n that are single-valued, continuous, differentiable functions of the variables x, $y_{\mu+1}$, …, y_n in a certain neighborhood of the location x, $y_{\mu+1}$, …, y_n, and will fulfill the equations identically inside of that neighborhood when they are substituted for $y_1, y_2, \ldots, y_\mu$.

If one denotes:

$$\Phi_i = \frac{d}{dx}\varphi_i(y_1, y_2, \ldots, y_n)\ ,$$

for brevity, when $i < \mu + 1$ then one will see immediately that the determinant above (6) will be identical to the functional determinant:

$$\frac{\partial(\Phi_1,\ldots,\Phi_\mu,\varphi_1,\ldots,\varphi_m)}{\partial(y_1',\ldots,y_\mu',y_{\mu+1}',\ldots,y_m')}\ .$$

From the same theorem, one can then represent the $y'_1, y'_2, \ldots, y'_m$ as functions of $x, y_1, \ldots, y_n$, $y'_1, y'_2, \ldots, y'_n$ by way of the system of equations:

$$\Phi_1 = 0 , \ldots, \Phi_\mu = 0 , \quad \varphi_{\mu+1} = 0 , \ldots, \varphi_m = 0 ,$$

when the values of the $y'_1, y'_2, \ldots, y'_n$ that satisfy those equations are also associated with the location x.

One can then get the:

$$y_1, y_2, \ldots, y_n, y'_1, y'_2, \ldots, y'_n$$

as functions of the variables $x, y_{\mu+1}, \ldots, y_n, y'_{m+1}, \ldots, y'_n$ from the equations (5).

If one applies those considerations to the starting point a, at which the values of $y_1, y_2, \ldots, y_n$, $y'_1, y'_2, \ldots, y'_n$ are assumed to be known since the equations of the first variation are assumed to have been solved already in the treatment of the second variation, then μ of the $y_1, y_2, \ldots, y_n$, their differential quotients, and $m - \mu$ of the $y'_1, y'_2, \ldots, y'_n$ can be regarded as single-valued functions of the remaining quantities by means of the m condition equations. If, say, the assumptions that were made for the location x hold true at a then the $y_{\mu+1}, \ldots, y_n, y'_1, \ldots, y'_n$, and x can be regarded as the independent variables among them, and merely the initial values of those quantities will also seem to be associated with the point $x = a$ arbitrarily, but the remaining ones will be determined by it. If one combines those initial values of the $y_1, \ldots, y_n, y'_1, \ldots, y'_n$, of which only the $y_{\mu+1}, \ldots,$ $y_n, y'_{m+1}, \ldots, y'_n$ can prove to be arbitrary, as was just discussed, with the values that $\lambda_{\mu+1}, \ldots, \lambda_m$ have at a then one can construct an integral element of the canonical system of differential equations that is included in the system of equations (4) from them. The remaining m equations of the system of equations will then give the associated values of the $\lambda_1, \lambda_2, \ldots, \lambda_\mu$. By continuing that element and parallel to the condition equations, then, from (I.1), one will get a system of solutions to the canonical system of equations that extends over ab, but since the conditions equations (5) are fulfilled, it will also satisfy the original system of equations.

The values of:

$$y'_{m+1}, \ldots, y'_n, y_{\mu+1}, \ldots, y_n, \lambda_{\mu+1}, \ldots, \lambda_m$$

at a are then considered to be integration constants in that system, which might be denoted by:

$$c_1, \ldots, c_{n-m}, c_{n-m+1}, \ldots, c_{2n-m+\mu}, \ldots, c_{2n-\mu},$$

respectively. Since the conditions in (II.2) are fulfilled in the canonical system of differential equations as a result of the assumptions that were made in (II.1), and since the determinant (3*) is nowhere zero in the domain of integration, the integrals $y_1, y_2, \ldots, y_n, y'_1, y'_2, \ldots, y'_n, \lambda_1, \lambda_2, \ldots, \lambda_m$ can also be differentiated with respect to those constants.

Even in that case, as in the foregoing one, in the present treatise, is it assumed that:

1. The $y_1, y_2, \ldots, y_n$ are single-valued functions of the variable x for all values of it from a to b, which has the consequence that the given curve integral can be regarded as an ordinary definite one, and

2. The determinant (3^*) is nowhere zero in the interval ab.

III.

Although the foregoing remarks about the differential equations of the first variation were very incomplete and aphoristic, they can nonetheless suffice for the treatment of the second variation $\delta^2 J$ of the integral. For the variations $\eta_1, \eta_2, \ldots, \eta_n$ that were assumed a while ago, it is given by:

$$\delta^2 J = \int_a^b \left(\frac{\partial}{\partial y_1}\eta_1 + \cdots + \frac{\partial}{\partial y_n}\eta_n, \frac{\partial}{\partial y_1'}\eta_1' + \cdots + \frac{\partial}{\partial y_n'}\eta_n'\right)^2 F\,dx = \int_a^b \Omega(\eta, \eta')\,dx, \tag{1}$$

when one sets:

$$\Omega(\eta, \eta') = \left(\frac{\partial}{\partial y_1}\eta_1 + \cdots + \frac{\partial}{\partial y_n}\eta_n, \frac{\partial}{\partial y_1'}\eta_1' + \cdots + \frac{\partial}{\partial y_n'}\eta_n'\right)^2 F.$$

It follows from the fact that:

$$\begin{aligned} 2\Omega(\eta, \eta') &= \sum_{k=1}^n \left(\frac{\partial \Omega(\eta, \eta')}{\partial \eta_k}\eta_k + \frac{\partial \Omega(\eta, \eta')}{\partial \eta_k'}\eta_k'\right) \\ &= \sum_{k=1}^n \left(\frac{\partial \Omega(\eta, \eta')}{\partial \eta_k} - \frac{d}{dx}\frac{\partial \Omega(\eta, \eta')}{\partial \eta_k'}\right)\eta_k + \frac{d}{dx}\left(\sum_{k=1}^n \frac{\partial \Omega(\eta, \eta')}{\partial \eta_k'}\eta_k\right), \end{aligned}$$

since the η_k vanish at the limits a and b, that:

$$\delta^2 J = \tfrac{1}{2}\sum_{k=1}^n \int_a^b \eta_k \left(\frac{\partial \Omega(\eta, \eta')}{\partial \eta_k} - \frac{d}{dx}\frac{\partial \Omega(\eta, \eta')}{\partial \eta_k'}\right)dx, \tag{2}$$

in which one then has:

$$\frac{\partial \Omega(\eta, \eta')}{\partial \eta_k} - \frac{d}{dx}\frac{\partial \Omega(\eta, \eta')}{\partial \eta_k'} = \sum_{k=1}^n \left[\frac{\partial^2 F}{\partial y_k\,\partial y_i}\eta_i + \frac{\partial^2 F}{\partial y_k\,\partial y_i'}\eta_i' - \frac{d}{dx}\left(\frac{\partial^2 F}{\partial y_k'\,\partial y_i}\eta_i + \frac{\partial^2 F}{\partial y_k'\,\partial y_i'}\eta_i'\right)\right]. \tag{3}$$

As a preliminary, the case of the absolute maximum and minimum shall be examined, so no condition equations exist between the desired functions $y_1, y_2, \ldots, y_n$ then, under the assumption that all authors tacitly make that the $y_1, y_2, \ldots, y_n$ are single-valued functions of x for all values of it in the interval ab. The above form for:

$$\frac{\partial \Omega(\eta,\eta')}{\partial \eta_k} - \frac{d}{dx}\frac{\partial \Omega(\eta,\eta')}{\partial \eta_k'}$$

suggests an immediate connection with the differential equations of the first variation:

$$\frac{\partial F}{\partial y_i} - \frac{d}{dx}\frac{\partial F}{\partial y_i'} = 0\,, \qquad i = 1, 2, \dots, n,$$

in which $F = f$ here, that will become immediately obvious when one makes the $y_1, y_2, \dots, y_n$ differentiable with respect to a parameter c and assumes that the equations are identically with respect to that parameter. The integration constants can serve as such parameters in their own right, as the following remark will clarify. If one completes the n differential equations of the first variation (4) to a linear system of $2n$ equations by adding the n equations:

$$\frac{dy_1}{dx} \equiv y_1', \qquad \frac{dy_2}{dx} \equiv y_2', \qquad \dots, \qquad \frac{dy_n}{dx} \equiv y_n'$$

then the assumptions that were made in (II.2) will be valid for it, and the $2n$ initial values at a will be taken to be the integration constants. If one imagines substituting the integrals into the system of differential equations then the individual equations will be fulfilled identically with respect to the integration constants, and since from (II.2), the integrals are differentiable with respect to those constants, moreover, when c denotes one such constant, one will get the $2n$ equations:

$$\sum_{k=1}^{n}\left[\frac{\partial^2 F}{\partial y_i\,\partial y_k}\frac{\partial y_k}{\partial c} + \frac{\partial^2 F}{\partial y_i\,\partial y_k'}\left(\frac{\partial y_k}{\partial c}\right)' - \frac{d}{dx}\left(\frac{\partial^2 F}{\partial y_i'\,\partial y_k}\frac{\partial y_k}{\partial c} + \frac{\partial^2 F}{\partial y_i'\,\partial y_k'}\left(\frac{\partial y_k}{\partial c}\right)'\right)\right] = 0 \qquad (i = 1, 2, \dots, n),$$

$$\frac{d}{dx}\left(\frac{\partial y_1}{\partial c}\right) = \frac{\partial y_1'}{\partial c}, \qquad \frac{d}{dx}\left(\frac{\partial y_2}{\partial c}\right) = \frac{\partial y_2'}{\partial c}, \qquad \dots, \qquad \frac{d}{dx}\left(\frac{\partial y_n}{\partial c}\right) = \frac{\partial y_n'}{\partial c}\,.$$

The system of $2n$ linear differential equations for z:

$$\left.\begin{aligned} \psi_i(z) = \sum_{k=1}^{n}\left[\frac{\partial^2 F}{\partial y_i\,\partial y_k} z_k + \frac{\partial^2 F}{\partial y_i\,\partial y_k'} z_k' - \frac{d}{dx}\left(\frac{\partial^2 F}{\partial y_i'\,\partial y_k} z_k + \frac{\partial^2 F}{\partial y_i'\,\partial y_k'} z_k'\right)\right] = 0 \quad (i = 1, 2, \dots, n), \\ \frac{dz_1}{dx} = z_1', \quad \frac{dz_2}{dx} = z_2', \quad \cdots, \quad \frac{dz_n}{dx} = z_n', \end{aligned}\right\} \qquad (5)$$

and under the assumption that was made in III that the determinant:

$$\begin{vmatrix} \frac{\partial^2 F}{\partial y_1' \partial y_1'} & \cdots & \frac{\partial^2 F}{\partial y_1' \partial y_n'} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 F}{\partial y_n' \partial y_1'} & \cdots & \frac{\partial^2 F}{\partial y_n' \partial y_n'} \end{vmatrix}$$

does not vanish anywhere in the integration interval, it can be represented in canonical form everywhere there, so it will have the system of $2n$ integrals:

$$\begin{matrix} \left(\frac{\partial y_1}{\partial c_1}\right)', & \left(\frac{\partial y_2}{\partial c_1}\right)', \ldots, & \left(\frac{\partial y_1}{\partial c_1}\right)', & \frac{\partial y_1}{\partial c_1}, \ldots, & \frac{\partial y_n}{\partial c_1}, \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \left(\frac{\partial y_1}{\partial c_{2n}}\right)', & \left(\frac{\partial y_2}{\partial c_{2n}}\right)', \ldots, & \left(\frac{\partial y_n}{\partial c_{2n}}\right)', & \frac{\partial y_1}{\partial c_{2n}}, \ldots, & \frac{\partial y_n}{\partial c_{2n}}. \end{matrix}$$

Now, since the convention above makes $c_1, \ldots, c_n, c_{n+1}, \ldots, c_{2n}$ equal to the values of $y_1', \ldots, y_n', y_1, \ldots, y_n$, respectively, at a, and since the representation of the integrals in I.1 shows that the differential quotients:

$$\frac{\partial y_1'}{\partial c_1}, \ldots, \frac{\partial y_n'}{\partial c_n}, \frac{\partial y_1}{\partial c_{n+1}}, \ldots, \frac{\partial y_n}{\partial c_{2n}}$$

will all be equal to 1 at a, while all remaining ones vanish, the determinant:

$$\begin{matrix} \frac{\partial y_1'}{\partial c_1}, \ldots, & \frac{\partial y_n'}{\partial c_n}, & \frac{\partial y_1}{\partial c_1}, \ldots, & \frac{\partial y_n}{\partial c_1}, \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial y_1'}{\partial c_{2n}}, \ldots, & \frac{\partial y_n'}{\partial c_{2n}}, & \frac{\partial y_1}{\partial c_{2n}}, \ldots, & \frac{\partial y_n}{\partial c_{2n}} \end{matrix}$$

will equal 1 at a and will therefore not vanish identically in the integration interval. Therefore, the system of integrals that was given above for the system of linear differential equations will be linearly independent of each other and thus define a fundamental system.

If one now considers the fact that:

$$\psi_i(z) = \frac{1}{2}\left[\frac{\partial \Omega(z, z')}{\partial z_i} - \frac{d}{dx}\frac{\partial \Omega(z, z')}{\partial z_k'}\right]$$

then one will get:

$$\delta^2 J = \sum_{i=1}^{n} \int_a^b \eta_i \, \psi_i(\eta)\, dx .$$

IV.

That expression for $\psi_i\,(z)$ has some consequences that follow from some properties that are intrinsic to second-order linear differential expressions and should therefore be derived beforehand.

Since $\Omega(z, z')$ is a quadratic form in $z_1, z_2, \ldots, z_n, z'_1, z'_2, \ldots, z'_n$, it will follow from a known property of such things that when $2n$ other quantities are denoted by $u_1, u_2, \ldots, u_n, u'_1, u'_2, \ldots, u'_n$, in which the prime should again mean differentiation, one will have:

$$\sum_{k=1}^{n}\left(u_k \frac{\partial \Omega(z,z')}{\partial z_k} + u'_k \frac{\partial \Omega(z,z')}{\partial z'_k}\right) = \sum_{k=1}^{n}\left(z_k \frac{\partial \Omega(u,u')}{\partial u_k} + z'_k \frac{\partial \Omega(u,u')}{\partial u'_k}\right)$$

or

$$\sum_{k=1}^{n} u_k \left[\frac{\partial \Omega(z,z')}{\partial z_k} - \frac{d}{dx}\frac{\partial \Omega(z,z')}{\partial z'_k}\right] + \frac{d}{dx}\sum_{k=1}^{n} u_k \frac{\partial \Omega(z,z')}{\partial z'_k}$$
$$= \sum_{k=1}^{n} z_k \left[\frac{\partial \Omega(u,u')}{\partial u_k} - \frac{d}{dx}\frac{\partial \Omega(u,u')}{\partial u'_k}\right] + \frac{d}{dx}\sum_{k=1}^{n} z_k \frac{\partial \Omega(u,u')}{\partial u'_k} .$$

Therefore:

$$\sum_{k=1}^{n} [u_k \psi_k(z) - z_k\, \psi_k(u)] = \frac{1}{2}\frac{d}{dx}\sum_{k=1}^{n}\left[z_k \frac{\partial \Omega(u,u')}{\partial u_k} - u_k \frac{\partial \Omega(z,z')}{\partial z'_k}\right] .$$

However, one has:

$$\frac{\partial \Omega(u,u')}{\partial u'_k} = \sum_{k=1}^{n}\left(\frac{\partial^2 \Omega(u,u')}{\partial u'_k\, \partial u'_i} u'_i + \frac{\partial^2 \Omega(u,u')}{\partial u'_k\, \partial u_i} u_i\right)$$

and

$$\frac{\partial \Omega(z,z')}{\partial z'_k} = \sum_{i=1}^{n}\left(\frac{\partial^2 \Omega(z,z')}{\partial z'_k\, \partial z'_i} z'_i + \frac{\partial^2 \Omega(z,z')}{\partial z'_k\, \partial z_i} z_i\right) .$$

One will then have:

$$\sum_{k=1}^{n} [u_k \psi_k(z) - z_k\, \psi_k(u)] = \frac{d}{dx}\sum_{i,k}\frac{1}{2}\frac{\partial^2 \Omega(u,u')}{\partial u'_i\, \partial u'_k}(u'_i\, z_k - u_i\, z'_k) + \frac{d}{dx}\sum_{i,k}\frac{1}{2}\frac{\partial^2 \Omega(u,u')}{\partial u'_k\, \partial u_i}(u_i\, z_k - u_i\, z_k) .$$

If one denotes the first-order bilinear differential expression:

$$\sum_{i,k}\frac{1}{2}\left[\frac{\partial^2\,\Omega(u,u')}{\partial u_i'\,\partial u_k'}(u_i'\,z_k - u_i\,z_k') + \frac{\partial^2\,\Omega(u,u')}{\partial u_k'\,\partial u_i}(u_i\,z_k - u_i\,z_k)\right]$$
$$= \sum_{i,k}\left[\frac{\partial^2 F}{\partial y_i'\,\partial y_k'}(u_i'\,z_k - u_i\,z_k') + \frac{\partial^2 F}{\partial y_k'\,\partial y_i}(u_i\,z_k - u_i\,z_k)\right]$$

by $\psi(z, u)$, for brevity, then the formula above will assume the form:

$$\sum_{k=1}^{n}[u_k\psi_k(z) - z_k\,\psi_k(u)] = \frac{d}{dx}\psi(z,u)\ .$$

The differential expression $\psi(z, u)$ is alternating, so $\psi(z, z) = 0$.
If:

$$z_1, z_2, \ldots, z_n\ ,$$
$$u_1, u_2, \ldots, u_n$$

are two linearly-independent systems of integrals of the system of differential equations (III.5) $\psi_k(z) = 0$ then it will follow that:

$$\psi(z, u) = \text{const.},$$

in which the constant will possess the same value in the entire integration interval due to the continuity of $\psi(z, u)$.

If all of the z and u vanish at the location x, moreover, then that constant will be zero. Two linearly-independent systems of integrals for which the constant above is zero shall be called *conjugate to each other*. Now, it is clear that n linearly-independent systems of integrals will exist at each location x in the integration interval whose individual terms will vanish at that location. Every two of those n systems are conjugate to each other. Now, every group of at least n linearly-independent systems of integrals that are so arranged that every two of them are conjugate to each other shall be briefly called a *conjugate system*. It is not difficult to define such conjugate systems since the $n\,(n-1)\,/\,2$ bilinear equations the exist between the constants when one expresses each of the systems in terms of the same fundamental system can be easily solved in succession.

Now, before we go into the precise investigation of these conjugate systems, we shall derive the transformation of the second variation of the integral that **Clebsch** gave, for which the foregoing remarks should just suffice.

V.

In order to employ the formula (1) that was obtained for the transformation of the second variation of the integral:

$$\delta^2 J = \sum_{i=1}^{n}\int_a^b \eta_i\,\psi_i(\eta)\,dx\,,$$

one sets:

$$\eta_k = \sum_{\lambda=1}^{n} \rho_\lambda\, z_k^\lambda ,$$

in which:

$$z_1^\lambda,\ z_2^\lambda,\ \ldots,\ z_n^\lambda \qquad\qquad (\lambda = 1, 2, \ldots, n)$$

means a system of integrals of (III.5) that will be denoted briefly by z^λ. One will then have:

$$\sum_{k=1}^{n} \eta_k\, \psi_k(\eta) = \sum_{k=1}^{n} \psi_k(\eta) \sum_{\lambda=1}^{n} \rho_\lambda\, z_k^\lambda = \sum_{\lambda 1}^{n} \rho_\lambda \sum_{k=1}^{n} \psi_k(\eta)\, z_k^\lambda .$$

On the other hand, one has:

$$\sum_{k=1}^{n} z_k^\lambda\, \psi_k(\eta) = \frac{d}{dx}\psi(\eta, z^\lambda) ,$$

in which:

$$\psi(\eta, z^\lambda) = \sum_{i,k} \left\{ \frac{\partial^2 F}{\partial y_i' \,\partial y_k'} [\eta_i (z_k^\lambda)' - \eta_k'\, z_i^\lambda] + \frac{\partial^2 F}{\partial y_k' \,\partial y_i} [\eta_i\, z_k^\lambda - \eta_k\, z_i^\lambda] \right\} .$$

Therefore, when one multiplies by ρ_λ and sums over λ, one will have:

$$\sum_{\lambda,k} \rho_\lambda\, z_k^\lambda\, \psi_k(\eta) = \sum_{k=1}^{n} \eta_k\, \psi_k(\eta) = \sum_{\lambda=1}^{n} \rho_\lambda \frac{d}{dx} \psi(\eta, z^\lambda) = \frac{d}{dx}\left[\sum_{\lambda=1}^{n} \rho_\lambda\, \psi(\eta, z^\lambda) \right] - \sum_{\lambda=1}^{n} \rho_\lambda'\, \psi(\eta, z^\lambda) . \quad (1)$$

However, when one sets $\psi(\eta, z^\lambda)$ equal to its value in (IV.2), one will have:

$$\sum_{\lambda=1}^{n} \rho_\lambda'\, \psi(\eta, z^\lambda) = \sum_{i,k,\lambda} \frac{\partial^2 F}{\partial y_i' \,\partial y_k'} \rho_\lambda' (\eta_i\, (z_k^\lambda)' - \eta_i'\, z_k^\lambda) + \sum_{i,k,\lambda} \frac{\partial^2 F}{\partial y_k' \,\partial y_i} \rho_\lambda' (\eta_i\, z_k^\lambda - \eta_k\, z_i^\lambda) .$$

Moreover, one has:

$$\begin{aligned} \rho_\lambda' (\eta_i\, (z_k^\lambda)' - \eta_i'\, z_k^\lambda) &= \rho_\lambda' \left[(z_k^\lambda)' \sum_{\mu=1}^{n} \rho_\mu\, z_i^\lambda - z_i^\lambda \sum_{\mu=1}^{n} (\rho_\mu\, (z_k^\lambda)' + \rho_\mu'\, z_k^\lambda) \right] \\ &= \rho_\lambda' \left[\sum_{\mu=1}^{n} \rho_\mu\, (z_i^\mu (z_k^\lambda)' - z_i^\lambda ((z_k^\mu)') - z_i^\lambda \sum_{\mu=1}^{n} \rho_\mu'\, z_k^\mu \right] , \end{aligned}$$

and one will then have:

$$\sum_{\lambda=1}^{n} \rho_\lambda'\, \psi(\eta, z^\lambda) = \sum_{i,k,\lambda} \rho_\lambda'\, \rho_\mu \left\{ \frac{\partial^2 F}{\partial y_i' \,\partial y_k'} (z_i^\mu\, (z_k^\lambda)' - z_i^\lambda\, (z_k^\mu)') + \frac{\partial^2 F}{\partial y_i' \,\partial y_k} (z_i^\mu\, z_k^\lambda - z_i^\lambda z_k^\mu) \right\}$$

$$- \sum_{i,k,\lambda,\mu} \frac{\partial^2 F}{\partial y_i' \partial y_k'} \rho_\lambda' \rho_\mu z_i^\lambda z_k^\mu$$

$$= \sum_{k,\lambda} \psi(z^\mu, z^\lambda) \rho_\lambda' \rho_\mu - \sum_{i,k} \frac{\partial^2 F}{\partial y_i' \partial y_k'} \sum_{\lambda=1}^n \rho_\lambda' z_i^\lambda \sum_{\mu=1}^n \rho_\lambda' z_k^\mu .$$

If one substitutes that expression in formula (1) above then when one further sets:

$$\zeta_i = \sum_{\lambda=1}^n \rho_\lambda' z_i^\lambda ,$$

one will get:

$$\sum_{k=1}^n \eta_k \psi_k(\eta) = \frac{d}{dx} \sum_{\lambda=1}^n \rho_\lambda \psi(\eta, z^\lambda) - \sum_{\lambda,\mu} \psi(z^\mu, z^\lambda) \rho_\lambda' \rho_\mu + \sum_{i,k} \frac{\partial^2 F}{\partial y_i' \partial y_k'} \zeta_i \zeta_k ,$$

in which:

$$\psi(z^\mu, z^\lambda) = C_{\lambda\mu}$$

is a constant. Therefore, if all n systems of z^λ are conjugate then that constant will be zero, and the expression above will simplify to:

$$\sum_{k=1}^n \eta_k \psi_k(\eta) = \frac{d}{dx} \sum_{\lambda=1}^n \rho_\lambda \psi(\eta, z^\lambda) + \sum_{i,k} \frac{\partial^2 F}{\partial y_i' \partial y_k'} \zeta_i \zeta_k .$$

The ρ are then to be determined from the equations:

$$\eta_k = \sum_{\lambda=1}^n \rho_\lambda z_k^\lambda , \qquad k = 1, 2, \ldots, n,$$

and the:

$$\zeta_k = \sum_{\lambda=1}^n \rho_\lambda' z_k^\lambda ,$$

are determined from:

$$\eta_k' = \zeta_k + \sum_{\lambda=1}^n \rho_\lambda (z_i^\lambda)'$$

and

$$\eta_k = \sum_{\lambda=1}^n \rho_\lambda z_i^\lambda , \qquad k = 1, 2, \ldots, n.$$

That will then give:

$$\zeta_k \sum \pm z_1' z_2' \cdots z_n^n = \begin{vmatrix} \eta_k' & (z_k')' & \cdots & (z_k^n)' \\ \eta_1 & z_1' & \cdots & z_1^n \\ \vdots & \vdots & \ddots & \vdots \\ \eta_n & z_n' & \cdots & z_n^n \end{vmatrix} . \tag{2}$$

Therefore, if the determinant $\sum \pm z_1' z_2' \cdots z_n^n$ is non-vanishing over the entire interval *ab*, including the limits, then the quantities ρ and ζ can be determined. Since the ρ will vanish at the limits *a* and *b*, the second variation can be given the form:

$$\delta^2 J = \int_a^b \sum_{k=1}^n \eta_k \, \psi_k(\eta) \, dx = \int_a^b \sum_{i,k} \frac{\partial^2 F}{\partial y_i' \, \partial y_k'} \zeta_i \, \zeta_k . \tag{3}$$

VI.

That transformation is then based essentially upon the assumption that a conjugate system of *n* systems of integrals of equations (III.5) exists, and its determinant does not vanish either at the limits of the integral or inside of it. In order to find the conditions for the existence of such a thing, it is initially necessary to consider more closely the bilinear expression that was obtained in (V.1):

$$\psi(u, z) = \sum_{i,k} \left[\frac{\partial^2 F}{\partial y_i' \, \partial y_k'} (z_i' \, u_k - z_k \, u_i') + \frac{\partial^2 F}{\partial y_i' \, \partial y_k} (z_i \, u_k - z_k \, u_i) \right] ,$$

which is first-order in the *z*.

If one sets:

$$\frac{\partial^2 F}{\partial y_i' \, \partial y_k'} = a_{ik} \quad \text{and} \quad \frac{\partial^2 F}{\partial y_i' \, \partial y_k} = b_{ik} ,$$

for brevity, then it will assume the form:

$$\psi(u, z) = \sum_{i=1}^n z_i' \sum_{k=1}^n a_{ik} u_k - \sum_{k=1}^n z_k \sum_{i=1}^n [b_{ik} u_i' + (a_{ik} - b_{ik}) \, u_i] .$$

Now, if:

$$\begin{array}{l} u_1^1, \, u_2^1, \, \ldots, \, u_n^1 , \\ u_1^2, \, u_2^2, \, \ldots, \, u_n^2 , \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ u_1^n, \, u_2^n, \, \ldots, \, u_n^n \end{array}$$

are n linearly-independent systems of integrals of equations (III.5) then one can define n new first-order differential expressions that are linear in the z:

$$\omega_\mu(z) = w_1^\mu \, \psi(z, u^1) + w_2^\mu \, \psi(z, u^2) + \cdots + w_n^\mu \, \psi(z, u^n)$$

with the n^2 undetermined multipliers w_1^μ, w_2^μ, …, w_n^μ ($\mu = 1, 2, …, n$). It will then arise that:

$$\begin{aligned}
\omega_\mu(z) &= \sum_{\lambda=1}^{n} w_\lambda^\mu \, \psi(z, u^\lambda) \\
&= \sum_{\lambda=1}^{n} w_\lambda^\mu \left[\sum_{i=1}^{n} z_i' \sum_{k=1}^{n} a_{ik} u_k^\lambda - \sum_{k=1}^{n} z_k \sum_{i=1}^{n} (b_{ik} (u_i^\lambda)' + (a_{ik} - b_{ik}) u_i^\lambda \right] \\
&= \sum_{i=1}^{n} z_i' \sum_{k=1}^{n} a_{ik} \sum_{\lambda=1}^{n} u_k^\lambda \, w_\lambda^\mu - \sum_{k=1}^{n} z_k \sum_{\lambda, i} [b_{ik} (u_i^\lambda)' + (a_{ik} - b_{ik}) u_i^\lambda] \, w_\lambda^\mu .
\end{aligned}$$

In that expression, one now decrees that the n multipliers w_1^μ, w_2^μ, …, w_n^μ must be such that the coefficients of z_1', …, $z_{\mu-1}'$, $z_{\mu+1}'$, …, z_n' must vanish, and the coefficient of z_μ' must be equal to a given constant C_μ. For that purpose, one must calculate the w^μ from the equations:

$$\sum_{k=1}^{n} a_{ik} \sum_{\lambda=1}^{n} u_k^\lambda \, w_\lambda^\mu = 0 \qquad (i = 1, 2, …, \mu - 1, \mu + 1, …, n),$$

$$\sum_{k=1}^{n} a_{\mu k} \sum_{\lambda=1}^{n} u_k^\lambda \, w_\lambda^\mu = C_\mu ,$$

and when one sets:

$$\sum_{\lambda=1}^{n} u_k^\lambda \, w_\lambda^\mu = W_k,$$

that will go to:

$$\sum_{k=1}^{n} a_{ik} \, W_k = 0 \qquad (i = 1, 2, …, \mu - 1, \mu + 1, …, n),$$

$$\sum_{k=1}^{n} a_{\mu k} \, W_k = C_\mu .$$

Since the determinant $A = \sum \pm a_{11} \, a_{22} \cdots a_{nn}$ does not vanish anywhere in the entire integration interval, by assumption, those equations will then yield the W_k, and from that, one will again get the w_1^μ, w_2^μ, …, w_n^μ at every location x where $U = \sum \pm u_1^1 \, u_1^2 \cdots u_n^n$ is also non-zero. If one lets A_{ik}

denote the subdeterminant of a_{ik} in A and lets U_k^λ denote the subdeterminant of u_k^λ in U then one will find that:

$$w_\lambda^\mu = C_\mu \frac{\sum_{k=1}^{n} a_{\mu k} U_k^\lambda}{AU}.$$

As a result of that determination of $w_1^\mu, w_2^\mu, \ldots, w_n^\mu$, $\omega_\mu(z)$ will now take the form:

$$\omega_\mu(z) = C_\mu z'_\mu - \sum_{k=1}^{n} c_k^\mu z_k = \sum_{\lambda=1}^{n} \psi(z, u^\lambda) w_\lambda^\mu = \frac{C_\mu}{AU} \sum_{\lambda=1}^{n} \sum_{k=1}^{m} \psi(z, u^\lambda) A_{\mu k} U_\lambda^\mu .$$

If the system $u^1, u^2, \ldots, u^n$ defines a conjugate then one will always have:

$$\psi(u^\nu, u^\lambda) = 0$$

for any λ and ν.

Therefore, the n first-order differential equations in z:

$$\omega_1(z) = 0, \qquad \omega_2(z) = 0, \qquad \ldots, \qquad \omega_n(z) = 0$$

will be satisfied by those n linearly-independent systems. They will then represent a fundamental system for those equations, and one will then have:

$$\omega_\mu(z) = \frac{C_\mu}{AU} \sum_{\lambda=1}^{n} \sum_{k=1}^{m} \psi(z, u^\lambda) A_{\mu k} U_\lambda^\mu$$
$$= \frac{C_\mu}{U} \begin{vmatrix} z'_\mu & z_1 & z_2 & \cdots & z_n \\ (u_\mu^1)' & u_1^1 & u_2^1 & \cdots & u_n^1 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ (u_\mu^n)' & u_1^n & u_2^n & \cdots & u_n^n \end{vmatrix},$$

which will then give:

$$\begin{vmatrix} z'_\mu & z_1 & z_2 & \cdots & z_n \\ (u_\mu^1)' & u_1^1 & u_2^1 & \cdots & u_n^1 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ (u_\mu^n)' & u_1^n & u_2^n & \cdots & u_n^n \end{vmatrix} = \sum_{\lambda=1}^{n} \sum_{k=1}^{m} \frac{A_{\mu k}}{A} \psi(z, u^\lambda) U_k^\lambda = \frac{U\,\omega_\mu(z)}{C_\mu} . \tag{3}$$

That formula represents an identity that can also be verified immediately, and will then guide one to the knowledge that:

If a system of integrals $v_1, v_2, \ldots, v_n$ of the differential equations (III.5), $\psi_k(z) = 0$ is conjugate to each of the n systems $u^1, u^2, \ldots, u^n$ above, which define conjugate systems in their own right, then $\psi(v, u^\lambda) = 0$ (for $\lambda = 1, 2, \ldots, n$), and it will therefore satisfy the system of equations $\omega_1(z) = 0$, $\omega_2(z) = 0, \ldots, \omega_n(z) = 0$, so it is linearly-dependent upon the system $u^1, u^2, \ldots, u^n$.

Therefore:

A conjugate system to the systems of integrals of the linear differential equations (III.5) *will include only n linearly-independent systems of integrals.*

If a system of integrals is conjugate to n mutually linearly independent systems of integrals of a conjugate system then it will depend upon them linearly.

The determinant that was obtained in (3) shows a strong similarity to the expression for ζ_k (V.2) and suggests that one might examine that analogy in deeper detail. To that end, however, it is first necessary to derive a more-general determinant relation.

VII.

If, in the determinant:

$$\chi_\mu = \begin{vmatrix} z'_\mu & (u_\mu^1)' & \cdots & (u_\mu^n)' \\ z_1 & u_1^1 & \cdots & u_1^n \\ \vdots & \vdots & \cdots & \vdots \\ z_n & u_n^1 & \cdots & u_n^n \end{vmatrix},$$

one takes the adjoint subdeterminant of:

$$\begin{vmatrix} z'_\mu & (u_\mu^k)' \\ z_\lambda & u_\lambda^k \end{vmatrix}$$

then one will get, from a known formula:

$$(-1)^{\lambda+k} \chi_\mu U_\lambda^k = \begin{vmatrix} z'_\mu & \cdots & (u_\mu^{k-1})' & (u_\mu^{k+1})' & \cdots & (u_\mu^n)' \\ z_1 & \cdots & u_1^{k-1} & u_1^{k+1} & \cdots & u_1^n \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ z_{\lambda-1} & \cdots & u_{\lambda-1}^{k-1} & u_{\lambda-1}^{k+1} & \cdots & u_{\lambda-1}^n \\ z_{\lambda+1} & \cdots & u_{\lambda+1}^{k-1} & u_{\lambda+1}^{k+1} & \cdots & u_{\lambda+1}^n \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ z_n & \cdots & u_n^1 & u_n^{k+1} & \cdots & u_n^n \end{vmatrix}$$

$$- \begin{vmatrix} (u_\mu^1)' & \cdots & (u_\mu^n)' \\ u_1^1 & \cdots & u_1^n \\ \vdots & \cdots & \vdots \\ u_{\lambda-1}^1 & \cdots & u_{\lambda-1}^n \\ u_{\lambda+1}^1 & \cdots & u_{\lambda+1}^n \\ \vdots & \cdots & \vdots \\ u_n^1 & \cdots & u_n^{k+1} \end{vmatrix} \begin{vmatrix} z_1 & u_1^1 & \cdots & u_1^{k-1} & u_1^{k+1} & \cdots & u_1^n \\ z_2 & u_2^1 & \cdots & u_{\lambda-1}^{k-1} & u_{\lambda-1}^{k+1} & \cdots & u_{\lambda-1}^n \\ \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ z_n & u_n^1 & \cdots & u_n^1 & u_n^{k+1} & \cdots & u_n^n \end{vmatrix},$$

or when one denotes the determinant that is formed from n systems:

$$\begin{matrix} v_1^1, & v_2^1, & \cdots & v_n^1, \\ v_1^2, & v_2^2, & \cdots & v_n^2, \\ \vdots & \vdots & \ddots & \vdots \\ v_1^n, & v_2^n, & \cdots & v_n^n \end{matrix}$$

by $\Delta(v^1, v^2, \ldots, v^n)$:

$$(-1)^{\lambda+k} \chi_\mu U_\lambda^k = \Delta(u^1, u^2, \ldots, u^n) \begin{vmatrix} z'_\mu & \cdots & (u_\mu^{k-1})' & (u_\mu^{k+1})' & \cdots & (u_\mu^n)' \\ z_1 & \cdots & u_1^{k-1} & u_1^{k+1} & \cdots & u_1^n \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ z_{\lambda-1} & \cdots & u_{\lambda-1}^{k-1} & u_{\lambda-1}^{k+1} & \cdots & u_{\lambda-1}^n \\ z_{\lambda+1} & \cdots & u_{\lambda+1}^{k-1} & u_{\lambda+1}^{k+1} & \cdots & u_{\lambda+1}^n \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ z_n & \cdots & u_n^1 & u_n^{k+1} & \cdots & u_n^n \end{vmatrix}$$

$$- \Delta(z, u^1, \ldots, u^{k-1}, u^{k+1}, \ldots, u^n) \begin{vmatrix} (u_\mu^1)' & \cdots & (u_\mu^n)' \\ u_1^1 & \cdots & u_1^n \\ \vdots & \cdots & \vdots \\ u_{\lambda-1}^1 & \cdots & u_{\lambda-1}^n \\ u_{\lambda+1}^1 & \cdots & u_{\lambda+1}^n \\ \vdots & \cdots & \vdots \\ u_n^1 & \cdots & u_n^{k+1} \end{vmatrix}.$$

If one takes $\lambda = \mu$ in that then one will get:

$$(-1)^{k-1}\chi_\mu U_\mu^k = \Delta(u^1,u^2,\ldots,u^n)\begin{vmatrix} z_1 & \cdots & u_1^{k-1} & u_1^{k+1} & \cdots & u_1^n \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ z_{\mu-1} & \cdots & u_{\mu-1}^{k-1} & u_{\mu-1}^{k+1} & \cdots & u_{\mu-1}^n \\ z'_\mu & \cdots & (u_\mu^{k-1})' & (u_\mu^{k+1})' & \cdots & (u_\mu^n)' \\ z_{\mu+1} & \cdots & u_{\mu+1}^{k-1} & u_{\mu+1}^{k+1} & \cdots & u_{\mu+1}^n \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ z_n & \cdots & u_n^1 & u_n^{k+1} & \cdots & u_n^n \end{vmatrix}$$

$$- \Delta(z,u^1,\ldots,u^{k-1},u^{k+1},\ldots,u^n)\begin{vmatrix} u_1^1 & \cdots & u_1^n \\ \vdots & \cdots & \vdots \\ u_{\mu-1}^1 & \cdots & u_{\mu-1}^n \\ (u_\mu^1)' & \cdots & (u_\mu^n)' \\ u_{\mu+1}^1 & \cdots & u_{\mu+1}^n \\ \vdots & \cdots & \vdots \\ u_n^1 & \cdots & u_n^{k+1} \end{vmatrix} .$$

If one then sums over μ from 1 to n then that will ultimately give:

$$(-1)^{k-1}\sum_{\mu=1}^n U_\mu^k \chi_\mu = \Delta(u^1,u^2,\ldots,u^n)\,\Delta'(z,u^1,\ldots,u^{k-1},u^{k+1},\ldots,u^n)$$
$$- \Delta'(u^1,u^2,\ldots,u^n)\,\Delta(z,u^1,\ldots,u^{k-1},u^{k+1},\ldots,u^n) ,$$

in which $\Delta' = d\,\Delta / dt$.

VIII.

If the system u^1, u^2, …, u^n defines a conjugate system then one can relate that formula to (VI.3) because one will then have:

$$\chi_\mu(z) = \frac{\Delta(u^1,u^2,\ldots,u^n)\,\omega_\mu(z)}{C_\mu} = \frac{1}{A}\sum_{\lambda=1}^n\sum_{\nu=1}^n \psi(z,u^\lambda)\,A_{\mu\nu}\,U_\nu^\lambda ,$$

and one will then get the identity:

$$\Delta(u^1,u^2,\ldots,u^n)\,\Delta'(z,u^1,\ldots,u^{k-1},u^{k+1},\ldots,u^n) - \Delta'(u^1,u^2,\ldots,u^n)\,\Delta(z,u^1,\ldots,u^{k-1},u^{k+1},\ldots,u^n)$$

$$= \frac{(-1)^{k-1}}{A} \sum_{\lambda,\mu,\nu} \psi(z, u^{\lambda})\, A_{\mu\nu} U_{\mu}^{k} U_{\nu}^{\lambda} . \tag{1}$$

If one now chooses the systems v_1, v_2, …, v_n, which is briefly denoted by v, such that it is conjugate to each of the systems u^1, …, u^{k-1}, u^{k+1}, …, u^n, but not to u^k, then one will have:

$$\psi(v, u^1) = \ldots = \psi(v, u^{k-1}) = \psi(v, u^{k+1}) = \ldots = \psi(v, u^n) = 0 ,$$

while $\psi(v, u^k) \neq 0$. The formula above will then simplify to:

$$\Delta(u^1, u^2, \ldots, u^n)\, \Delta'(z, u^1, \ldots, u^{k-1}, u^{k+1}, \ldots, u^n) - \Delta'(u^1, u^2, \ldots, u^n)\, \Delta(z, u^1, \ldots, u^{k-1}, u^{k+1}, \ldots, u^n)$$

$$= \frac{(-1)^{k-1}}{A} \psi(v, u^k) \sum_{\mu,\nu} A_{\mu\nu} U_{\mu}^{k} U_{\nu}^{k} , \tag{2}$$

and the expression for χ_μ will go to:

$$\chi_\mu(v) = \frac{1}{A} \psi(v, u^k) \sum_{\nu=1}^{n} A_{\mu\nu} U_{\nu}^{k} . \tag{3}$$

If one multiplies the latter expression by:

$$\chi_\lambda(v) = \frac{1}{A} \psi(v, u^k) \sum_{\tau=1}^{n} A_{\lambda\tau} U_{\tau}^{k}$$

then that will give:

$$\chi_\lambda(v)\, \chi_\mu(v) = \frac{\psi(v, u^k)^2}{A^2} \sum_{\nu,\tau} A_{\mu\nu} A_{\lambda\tau} U_{\nu}^{k} U_{\tau}^{k} ,$$

and from that:

$$\begin{aligned} \sum_{\lambda,\mu} a_{\mu\lambda}\, \chi_\lambda(v)\, \chi_\mu(v) &= \frac{\psi(v, u^k)^2}{A^2} \sum_{\lambda,\mu,\nu,\tau} A_{\mu\nu} A_{\lambda\tau}\, a_{\mu\lambda}\, U_{\nu}^{k} U_{\tau}^{k} \\ &= \frac{\psi(v, u^k)^2}{A^2} \sum_{\lambda,\nu,\tau} A_{\lambda\tau} U_{\nu}^{k} U_{\tau}^{k} \sum_{\mu=1}^{n} A_{\mu\nu}\, a_{\mu\nu} . \end{aligned}$$

Now since $\sum_{\mu=1}^{n} A_{\mu\nu}\, a_{\mu\nu}$ is equal to A for $\nu = \lambda$ and to zero for $\nu \neq \lambda$, one will have:

$$\sum_{\lambda,\mu} a_{\mu\lambda}\, \chi_\lambda(v)\, \chi_\mu(v) = \frac{\psi(v, u^k)^2}{A} \sum_{\lambda,\tau} A_{\lambda\tau} U_{\lambda}^{k} U_{\tau}^{k} .$$

If one compares that formula with the previous one (2) then one will get:

$$\Delta(u^1, u^2, \ldots, u^n)\,\Delta'(z, u^1, \ldots, u^{k-1}, u^{k+1}, \ldots, u^n) - \Delta'(u^1, u^2, \ldots, u^n)\,\Delta(z, u^1, \ldots, u^{k-1}, u^{k+1}, \ldots, u^n)$$

$$= \frac{(-1)^{k-1}}{\psi(v, u^k)} \sum_{\lambda,\mu} a_{\mu\lambda}\,\chi_\lambda(v)\,\chi_\mu(v), \tag{4}$$

in which $\psi(v, u^k)$ is a non-zero constant, and the quadratic form on the right-hand side appeared already in the transformation of the second variation of the integral (V.3).

IX.

1. – Since the formulas that were developed here for the case of an absolute maximum and minimum also revert to the case of relative maximum and minimum, we shall overlook the task of inferring the consequences that flow from them, and we might initially derive them for the case in which the following m condition equations:

$$\varphi_1 = 0, \qquad \varphi_2 = 0, \qquad \ldots, \qquad \varphi_m = 0$$

exist between the desired functions $y_1, y_2, \ldots, y_n$, each of which also includes first derivatives of $y_1, y_2, \ldots, y_n$.

The second variation will again have the form:

$$\delta^2 J = \tfrac{1}{2}\int_a^b \sum_{k=1}^{n}\left(\frac{\partial\Omega(\eta,\eta')}{\partial\eta_k} - \frac{d}{dx}\frac{\partial\Omega(\eta,\eta')}{\partial\eta'_k}\right)\eta_k\,dx, \tag{1}$$

in which $\Omega(\eta, \eta')$ has the same meaning as before, but one no longer has:

$$F = f + \lambda_1\,\varphi_1 + \lambda_2\,\varphi_2 + \ldots + \lambda_m\,\varphi_m,$$

which will also include the m functions of x: $\lambda_1, \lambda_2, \ldots, \lambda_m$ from now on, which depend upon the integration constants that are obtained by integrating the differential equations of the first variation:

$$\frac{\partial F}{\partial y_i} - \frac{d}{dx}\frac{\partial F}{\partial y'_i} = 0 \qquad\qquad i = 1, 2, \ldots, n. \tag{2}$$

As a result of that, the connection that exists between the equations that can be derived from them by differentiation with respect to an integration constant and:

$$\frac{\partial\Omega(\eta,\eta')}{\partial\eta_k} - \frac{d}{dx}\frac{\partial\Omega(\eta,\eta')}{\partial\eta'_k}$$

will also no longer be the same, but will stillbe similar to it, as shall be shown.

The same assumptions that were made in (II.1) shall be made for the system of first-order differential equations:

$$\frac{\partial F}{\partial y_i} - \frac{d}{dx}\frac{\partial F}{\partial y'_i} = 0 \qquad i = 1, 2, \ldots, n,$$

$$\frac{dy_1}{dx} = y'_1, \qquad \frac{dy_2}{dx} = y'_2, \qquad \ldots, \qquad \frac{dy_n}{dx} = y'_n,$$

$$\varphi_1 = 0,\ \varphi_2 = 0,\ \ldots,\ \varphi_m = 0$$

between the $2n + m$ functions:

$$y_1, y_2, \ldots, y_n,\ y'_1,\ y'_2, \ldots,\ y'_n,\ \lambda_1, \lambda_2, \ldots, \lambda_m,$$

and the integration constants c_1, c_2, …, c_{2n} shall also be chosen in the way that was given there (II.2). If one imagines integrating the system and substituting the integrals in it then one will obtain equations that are differentiable identities in the integration constants. The differentiation with respect to any of those integration constants c will then give:

$$\sum_{k=1}^{n}\left[\frac{\partial^2 F}{\partial y_i\,\partial y_k}\frac{\partial y_k}{\partial c} + \frac{\partial^2 F}{\partial y_i\,\partial y'_k}\frac{\partial y'_k}{\partial c} - \frac{d}{dx}\left(\frac{\partial^2 F}{\partial y'_i\,\partial y_k}\frac{\partial y_k}{\partial c} + \frac{\partial^2 F}{\partial y'_i\,\partial y'_k}\frac{\partial y'_k}{\partial c}\right)\right] + \sum_{k=1}^{n}\left(\frac{\partial\varphi_k}{\partial y_i}\frac{\partial\lambda_k}{\partial c} - \frac{d}{dx}\frac{\partial\varphi_k}{\partial y'_i}\frac{\partial\lambda_k}{\partial c}\right)$$

$$= 0 \qquad (i = 1, 2, \ldots, n),$$

$$\frac{d}{dx}\frac{\partial y_i}{\partial c} = \frac{\partial y'_i}{\partial c} \qquad (i = 1, 2, \ldots, n),$$

$$\sum_{k=1}^{n}\left(\frac{\partial\varphi_i}{\partial y_k}\frac{\partial y_k}{\partial c} + \frac{\partial\varphi_i}{\partial y'_k}\frac{\partial y_k}{\partial c}\right) = 0 \qquad (i = 1, 2, \ldots, m),$$

from which it is clear that:

$$\frac{\partial y_1}{\partial c}, \frac{\partial y_2}{\partial c}, \ldots, \frac{\partial y_n}{\partial c}; \frac{\partial\lambda_1}{\partial c}, \frac{\partial\lambda_2}{\partial c}, \ldots, \frac{\partial\lambda_n}{\partial c}$$

is a system of particular integrals of the system of first-order linear differential equations:

$$\left.\begin{array}{c}\sum_{k=1}^{n}\left[\frac{\partial^2 F}{\partial y_i\,\partial y_k}z_k+\frac{\partial^2 F}{\partial y_i\,\partial y_k'}z_k'-\frac{d}{dx}\left(\frac{\partial^2 F}{\partial y_i'\,\partial y_k}z_k+\frac{\partial^2 F}{\partial y_i'\,\partial y_k'}z_k'\right)\right]+\sum_{k=1}^{n}\left[\frac{\partial\varphi_k}{\partial y_i}r_k-\frac{d}{dx}\left(\frac{\partial\varphi_k}{\partial y_i'}r_k\right)\right]=0,\\ i=1,2,\ldots,n,\\ \frac{dz_i}{dx}=z_i', \qquad\qquad i=1,2,\ldots,n,\\ \sum_{k=1}^{n}\left(\frac{\partial\varphi_i}{\partial y_k}z_k+\frac{\partial\varphi_k}{\partial y_i'}z_k'\right)=0, \quad i=1,2,\ldots,m,\end{array}\right\} \tag{3}$$

between the $z_1, z_2, \ldots, z_n, r_1, r_2, \ldots, z_m$, since they will be satisfied when one takes:

$$z_k = \frac{\partial y_k}{\partial c}, \qquad r_k = \frac{\partial \lambda_k}{\partial c}.$$

Since c can be taken to be equal to $c_1, c_2, \ldots, c_{2n}$, one can obtain $2n$ particular systems of the system of linear first-order differential equations above (3) in that way, *which shall be called the accessory system of linear differential equations*.

One can next give that system (3) the canonical form when one combines its first n equations with the m other ones:

$$\sum_{k=1}^{n}\frac{d}{dx}\left(\frac{\partial\varphi_k}{\partial y_i'}z_k'+\frac{\partial\varphi_i}{\partial y_k}z_k\right)=0$$

and infers:

$$\frac{dz_1'}{dx},\ \frac{dz_2'}{dx},\ \ldots,\ \frac{dz_n'}{dx},\ \frac{dr_1}{dx},\ \frac{dr_2}{dx},\ \ldots,\ \frac{dr_m}{dx}$$

from them, which is always possible since, by the assumption in (II.2), the determinant of the coefficients of those unknowns in the system of $n + m$ equations is non-zero.

Since the remaining coefficients (II.a) that enter into the differential equations are finite and continuous in the domain of integration (including the limits), there will be no singular locations of the canonical system of differential equations anywhere in it, which will be an important convention in the further analysis.

However, that canonical system is by no means equivalent to the original system (3) since any system of integrals of the latter will probably satisfy the former, but not conversely. That is because the system of integrals of the former will no longer satisfy all of the equations:

$$\sum_{k=1}^{n}\left(\frac{\partial\varphi_k}{\partial y_i'}z_k'+\frac{\partial\varphi_i}{\partial y_k}z_k\right)=0 \qquad (i = 1, 2, \ldots, m),$$

but

$$\sum_{k=1}^{n}\left(\frac{\partial\varphi_k}{\partial y_i'}z_k'+\frac{\partial\varphi_i}{\partial y_k}z_k\right)=C_i \qquad (i=1,2,\ldots,m),$$

in which C_i mean constants. Thus, in order to find the system of integrals of (3) from the system of integrals of the canonical system, one must look for the ones among them for which those constants are zero, so since those constants have the same value in the entire domain of integration, due to the assumption of continuity, the ones that make the left-hand sides equal to zero at some location. One can then exhibit such systems of integrals when one chooses the initial values in such a way that this condition is fulfilled. In the present case, however, one is given $2n$ such systems of integrals from the outset:

$$\left(\frac{\partial y_1}{\partial c_i}\right)',\left(\frac{\partial y_2}{\partial c_i}\right)',\ldots,\left(\frac{\partial y_n}{\partial c_i}\right)';\ \frac{\partial y_1}{\partial c_i},\frac{\partial y_2}{\partial c_i},\ldots,\frac{\partial y_n}{\partial c_i};\quad \frac{\partial \lambda_1}{\partial c_i},\frac{\partial \lambda_2}{\partial c_i},\ldots,\frac{\partial \lambda_n}{\partial c_i}\qquad (i=1,2,\ldots,2n),$$

and that will next raise the question of whether they are also linearly independent.

2. – In order to answer it, one takes $2n+m$ systems of integrals of the canonical system of differential equations:

$$\left.\begin{array}{l}(z_1^1)'\cdots(z_n^1)',z_1^1\cdots z_n^1,r_1^1,\ldots,r_m^1\\(z_1^2)'\cdots(z_n^2)',z_1^2\cdots z_n^2,r_1^2,\ldots,r_m^2\\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\end{array}\right\}\tag{4}$$

One multiplies each row the determinant D of that system of integrals by the determinant of degree $(2n+m)$:

$$\begin{vmatrix}
\frac{\partial\varphi_1}{\partial y_1'} & \cdots & \frac{\partial\varphi_1}{\partial y_m'}; & \frac{\partial\varphi_1}{\partial y_{m+1}'}, & \frac{\partial\varphi_1}{\partial y_{m+2}'} & \cdots & \frac{\partial\varphi_n}{\partial y_n'}; & \frac{\partial\varphi_1}{\partial y_1} & \cdots & \frac{\partial\varphi_1}{\partial y_n}; & 0 & \cdots & 0\\
\vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots\\
\frac{\partial\varphi_m}{\partial y_1'} & \cdots & \frac{\partial\varphi_m}{\partial y_m'}; & \frac{\partial\varphi_m}{\partial y_{m+1}'}, & \frac{\partial\varphi_m}{\partial y_{m+2}'} & \cdots & \frac{\partial\varphi_m}{\partial y_n'}; & \frac{\partial\varphi_m}{\partial y_1} & \cdots & \frac{\partial\varphi_m}{\partial y_n}; & 0 & \cdots & 0\\
0 & \cdots & 0; & 1, & 0 & \cdots & 0; & 0 & \cdots & 0; & 0 & \cdots & 0\\
0 & \cdots & 0; & 0, & 1 & \cdots & 0; & 0 & \cdots & 0; & 0 & \cdots & 0\\
\vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots\\
0 & \cdots & 0; & 0, & 0 & \cdots & \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & 1
\end{vmatrix}=\frac{\partial(\varphi_1,\ldots,\varphi_m)}{\partial(y_1',\ldots,y_m')}$$

whose structure is seen to be obvious from the notation. The product of the two has the form:

$$\begin{vmatrix} C_1^1 & \cdots & C_m^1, & (z_{m+1}^1)' & \cdots & (z_n^1)', & z_1^1 & \cdots & z_n^1, & r_1^1 & \cdots & r_m^1 \\ C_1^2 & \cdots & C_m^2, & (z_{m+1}^2)' & \cdots & (z_n^2)', & z_n^2 & \cdots & z_n^2, & r_1^2 & \cdots & r_m^2 \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ C_1^{2n} & \cdots & C_m^{2n}, & (z_{m+1}^{2n})' & \ldots & (z_n^{2n})', & z_1^{2n} & \cdots & z_n^{2n}, & r_1^{2n} & \cdots & r_m^{2n} \\ C_1^{2n+1} & \cdots & C_m^{2n+1}, & (z_{m+1}^{2n+1})' & \ldots & (z_n^{2n+1})', & z_1^{2n+1} & \cdots & z_n^{2n+1}, & r_1^{2n+1} & \cdots & r_m^{2n+1} \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \end{vmatrix} = \mathfrak{D} ,$$

in which the C_k^i mean constants.

If the $(2n + m)$ systems of particular integrals of the canonical system are linearly-independent then D will be finite and non-zero in the entire integration interval since no singular point of the system of differential equations can exist in it.

From (II.2), at least one determinant of degree m in the matrix:

$$\begin{vmatrix} \dfrac{\partial \varphi_1}{\partial y_1'} & \dfrac{\partial \varphi_1}{\partial y_2'} & \cdots & \dfrac{\partial \varphi_1}{\partial y_n'} \\ \vdots & \vdots & \cdots & \vdots \\ \dfrac{\partial \varphi_m}{\partial y_1'} & \dfrac{\partial \varphi_m}{\partial y_2'} & \cdots & \dfrac{\partial \varphi_m}{\partial y_n'} \end{vmatrix} \tag{5}$$

must be non-zero at the arbitrary location x in the integration interval.

If, say:

$$\frac{\partial(\varphi_1, \varphi_2, \ldots, \varphi_m)}{\partial(y_1', y_2', \ldots, y_m')}$$

is non-zero in it then the product of it an F will also be non-zero since the determinant $\mathfrak{D}$ is non-zero at that location. Therefore:

Among $2n + m$ *linearly-independent systems of integrals of the canonical system of equations, at most* $2n$ *of them will possess nothing but vanishing constants.*

It will then follow from this apogogically (†) that:

If the associated constants vanish for each of $2n + 1$ *systems of particular integrals of the canonical system of equations then that system will be linearly independent.*

However, it is also easy to find a way of characterizing whether those systems are linearly independent or not.

If that system is:

(†) Translator: i.e., by *reductio ad absurdum*.

$$\left.\begin{array}{ccccccccc} (z_1^1)' & \cdots & (z_n^1)', & z_1^1 & \cdots & z_n^1, & r_1^1 & \cdots & r_m^1 \\ (z_1^2)' & \cdots & (z_n^2)', & z_1^2 & \cdots & z_n^2, & r_1^2 & \cdots & r_m^2 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ (z_1^{2n})' & \cdots & (z_n^{2n})', & z_1^{2n} & \cdots & z_n^{2n}, & r_1^{2n} & \cdots & r_m^{2n} \end{array}\right\} \tag{6}$$

then the constants in the first $2n$ rows of $\mathfrak{D}$ will vanish, and one will have:

$$\mathfrak{D} = C \cdot \Delta$$

when Δ denotes the determinant:

$$\Delta = \begin{vmatrix} (z_{m+1}^1)' & \cdots & (z_n^1)', & z_1^1 & \cdots & z_n^1, & r_1^1 & \cdots & r_m^1 \\ (z_{m+1}^2)' & \cdots & (z_n^2)', & z_1^2 & \cdots & z_n^2, & r_1^2 & \cdots & r_m^2 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ (z_{m+1}^{2n})' & \cdots & (z_n^{2n})', & z_1^{2n} & \cdots & z_n^{2n}, & r_1^{2n} & \cdots & r_m^{2n} \end{vmatrix}$$

and C denotes the determinant:

$$\begin{vmatrix} C_1^{2n+1} & \cdots & C_m^{2n+1} \\ \vdots & \cdots & \vdots \\ C_1^{2n+m} & \cdots & C_m^{2n+m} \end{vmatrix}.$$

However, one can assign arbitrary values to the C_i^k in the latter since one can choose the initial values of system z_1', …, z_n', z_1, …, z_n such that the equations:

$$\sum_{k=1}^{n}\left(\frac{\partial \varphi_i}{\partial y_k} z_k + \frac{\partial \varphi_i}{\partial y_k'} z_k'\right) = C_i \qquad (i = 1, 2, \ldots, m)$$

will be fulfilled by the given values C_1, C_2, …, C_m, and one can then arrange that C is non-zero. Δ and $\mathfrak{D}$ will then be simultaneously zero or simultaneously non-zero.

Therefore, if the $2n$ systems of integrals are not linearly independent then:

$$\mathfrak{D} = D\frac{\partial(\varphi_1, \ldots, \varphi_m)}{\partial(y_1', \ldots, y_m')}$$

will vanish over the entire integration interval since D will be everywhere zero in it, and therefore so will Δ.

If the latter vanishes at the location x in the integration interval then so will $\mathfrak{D}$, and therefore, when:

$$\frac{\partial(\varphi_1, \varphi_2, \ldots, \varphi_m)}{\partial(y'_1, y'_2, \ldots, y'_m)}$$

is non-zero at that location, so is D. The $2n + m$ systems of integrals (4) will no longer be linearly independent then, but since C is non-zero, that will restrict the dependency on the $2n$ system of integrals, as one will see from the fact that one assumed the opposite.

Now, one can define $\binom{n}{m}$ determinants of degree m from the matrix (5), and also define $\binom{n}{m}$ associated determinants Δ of degree $(2n)$, all but one of which will be obtained when one replaces the upper indices in its first $(n - m)$ columns with all combinations, without repetition, of the $(n - m)^{\text{th}}$ class of the first n numbers. Now, since at least one of the determinants of degree m in the matrix (5) is non-zero at every location in the integration interval, that will yield the theorems:

If the $2n$ *systems of integrals* (6) *of the accessory system are not linearly independent then each of the* $\binom{n}{m}$ *determinants* $\mathfrak{D}$ *will vanish at each location of the integral interval.*

If all determinants Δ *vanish at a location of the integration interval then the* $2n$ *systems of integrals* (6) *will not be linearly independent.*

By contrast, if Δ is non-zero at any location in the domain of integration then the associated $\mathfrak{D}$, and therefore D, as well, will be non-zero at that location. Therefore, the $2n$ systems of integrals will be mutually linearly independent, and one thus arrives at the result:

If the associated constants vanish for $2n$ *systems of integral of the canonical system of equations then they will always be linearly independent when one of the determinants* Δ *does not vanish at any location in the integration interval.*

A system of $2n$ *such systems of integrals shall be called a* ***fundamental system*** *of the accessory system of linear differential equations. As one easily sees, it will possess the essential properties that distinguish the fundamental system of the canonical system of first-order linear differential equations.*

If one now sets:

$$z_k^i = \frac{\partial y_k}{\partial c_i}, \qquad r_k^i = \frac{\partial \lambda_k}{\partial c_i}$$

then from the meaning of the $c_1, c_2, \ldots, c_{2n}$ in II, all of the elements of Δ above will vanish for $x = a$, with the exception of the ones in the principal diagonal, which will be equal to 1. One will then get the theorem:

Under the assumptions that were made in (II.2), *the accessory system of linear differential equations* (3) *will have the* 2n *linearly-independent systems of integrals:*

$$z_1^i = \frac{\partial y_1}{\partial c_i}, \ldots, \; z_n^i = \frac{\partial y_n}{\partial c_i}, \; r_1^i = \frac{\partial \lambda_1}{\partial c_i}, \ldots, \; r_m^i = \frac{\partial \lambda_m}{\partial c_i}, \qquad i = 1, 2, \ldots, n,$$

from which all other solutions can be obtained by linear combinations.

X.

If one sets the first expression in (IX.3) equal to:

$$\sum_{k=1}^{n}\left[\frac{\partial^2 F}{\partial y_i\,\partial y_k} z_k + \frac{\partial^2 F}{\partial y_i\,\partial y_k'} z_k' - \frac{d}{dx}\left(\frac{\partial^2 F}{\partial y_i'\,\partial y_k} z_k + \frac{\partial^2 F}{\partial y_i'\,\partial y_k'} z_k'\right)\right] + \sum_{k=1}^{n}\left[\frac{\partial \varphi_k}{\partial y_i} r_k - \frac{d}{dx}\left(\frac{\partial \varphi_k}{\partial y_i'} r_k\right)\right] = \psi_i(z, r) \quad (1)$$

then one can also write:

$$\psi_i(z, r) = \frac{1}{2}\left[\frac{\partial \Omega(z, z')}{\partial z_i} - \frac{d}{dx}\frac{\partial \Omega(z, z')}{\partial z_i'}\right] + \sum_{k=1}^{m}\left[\frac{\partial \varphi_k}{\partial y_i} r_k - \frac{d}{dx}\left(\frac{\partial \varphi_k}{\partial y_i} r_k\right)\right]. \qquad (1^*)$$

If one introduces the notation:

$$\bar{\varphi}_k(z) = \sum_{i=1}^{n}\left(\frac{\partial \varphi_k}{\partial y_i} z_k + \frac{\partial \varphi_k}{\partial y_i'} z_k'\right)$$

then one will get the following expression for the second variation (IX.1):

$$\begin{aligned}
\delta^2 J &= \int_a^b \left[\sum_{i=1}^{n} \eta_i \psi_i(\eta, r) - \sum_{i=1}^{m} r_i \bar{\varphi}_i(\eta) + \frac{d}{dx}\sum_{i=1}^{m}\left(\frac{\partial \varphi_k}{\partial y_i'} r_i \eta_i\right)\right] dx \\
&= \int_a^b \left[\sum_{i=1}^{n} \eta_i \psi_i(\eta, r) + \sum_{i=1}^{m} r_i \bar{\varphi}_i(\eta)\right] dx \\
&= \int_a^b \sum_{i=1}^{n} \eta_i \psi_i(\eta, r)\, dx, \qquad (2)
\end{aligned}$$

since $\bar{\varphi}_i(\eta) = 0$. The r that appear in that are arbitrary quantities.

In order to further transform the expression that was obtained, it will next be necessary to examine the $\psi_i(z, r)$.

If u and ρ are another system of quantities then one will have, analogously:

$$\psi_i\,(u,\,\rho) = \frac{1}{2}\left[\frac{\partial\Omega(u,u')}{\partial u_i} - \frac{d}{dx}\frac{\partial\Omega(u,u')}{\partial u_i'}\right] + \sum_{k=1}^{m}\left[\frac{\partial\varphi_k}{\partial y_i}\rho_k - \frac{d}{dx}\left(\frac{\partial\varphi_k}{\partial y_i}\rho_k\right)\right],$$

and therefore:

$$\sum_{i=1}^{n}[u_i\,\psi_i(z,r) - z_i\,\psi_i(u,\rho)]$$

$$= \frac{1}{2}\sum_{i=1}^{n}\left[\left(u_i\,\frac{\partial\Omega(z,z')}{\partial z_i} - z_i\,\frac{\partial\Omega(u,u')}{\partial u_i}\right) + z_i\,\frac{d}{dx}\frac{\partial\Omega(u,u')}{\partial u_i'} - u_i\,\frac{d}{dx}\frac{\partial\Omega(z,z')}{\partial z_i'}\right]$$

$$+ \sum_{i=1}^{n}\sum_{k=1}^{m}\left[\frac{\partial\varphi_k}{\partial y_i}(u_i\,\rho_k - z_i\,\rho_k) - u_i\,\frac{d}{dx}\left(\frac{\partial\varphi_k}{\partial y_i'}r_k\right) + z_i\,\frac{d}{dx}\left(\frac{\partial\varphi_k}{\partial y_i'}\rho_k\right)\right]$$

$$= \frac{1}{2}\sum_{i=1}^{n}\left[\left(z_i'\,\frac{\partial\Omega(u,u')}{\partial u_i'} - u_i'\,\frac{\partial\Omega(z,z')}{\partial z_i'}\right) + z_i\,\frac{d}{dx}\frac{\partial\Omega(u,u')}{\partial u_i'} - u_1\,\frac{\partial\Omega(z,z')}{\partial z_i'}\right]$$

$$+ \sum_{k=1}^{m}\left[r_k\sum_{i=1}^{n}\left(\frac{\partial\varphi_k}{\partial y_i}u_i + \frac{\partial\varphi_k}{\partial y_i'}u_i'\right) - \sum_{i=1}^{n}\frac{d}{dx}\left(\frac{\partial\varphi_k}{\partial y_i'}r_k\,u_i\right)\right.$$

$$\left. - \rho_k\sum_{i=1}^{n}\frac{d}{dx}\left(\frac{\partial\varphi_k}{\partial y_i}z_i + \frac{\partial\varphi_k}{\partial y_i'}z_i'\right) + \sum_{i=1}^{n}\frac{d}{dx}\left(\frac{\partial\varphi_k}{\partial y_i'}\rho_k\,z_i\right)\right]$$

$$= \tfrac{1}{2}\sum_{i,k}\frac{d}{dx}\left[\frac{\partial^2\,\Omega(u,u')}{\partial u_i'\,\partial u_k'}(z_i\,u_k' - u_i\,z_k') + \frac{\partial^2\,\Omega(u,u')}{\partial u_i'\,\partial u_k}(z_i\,u_k - u_i\,z_k)\right]$$

$$+ \sum_{k=1}^{m}\left[r_k\sum_{i=1}^{n}\left(\frac{\partial\varphi_k}{\partial y_i}u_i + \frac{\partial\varphi_k}{\partial y_i'}u_i'\right) - \rho_k\sum_{i=1}^{n}\left(\frac{\partial\varphi_k}{\partial y_i}z_i + \frac{\partial\varphi_k}{\partial y_i'}z_i'\right)\right]$$

$$-\frac{d}{dx}\sum_{k=1}^{m}\sum_{i=1}^{n}\frac{\partial\varphi_k}{\partial y_i'}(r_k\,u_i - \rho_k\,z_i)\,. \tag{3}$$

If one employs the notation that was used before:

$$\sum_{i=1}^{n}\left(\frac{\partial\varphi_k}{\partial y_i}z_i + \frac{\partial\varphi_k}{\partial y_i'}z_i'\right) = \bar{\varphi}_k(z) \tag{4}$$

then it will follow from equation (3) that:

$$\sum_{i=1}^{n}u_i\,\psi_i(z,r) + \sum_{k=1}^{m}\rho_k\,\bar{\varphi}_k(z) - \left[\sum_{i=1}^{n}z_i\,\psi_i(u,\rho) + \sum_{k=1}^{m}r_k\,\bar{\varphi}_k(u)\right]$$

$$= \frac{d}{dx}\left\{\sum_{i,k=1}^{n}\tfrac{1}{2}\left[\frac{\partial^2\Omega(u,u')}{\partial u_i'\,\partial u_k'}(z_i\,u_k' - u_i\,z_k') + \frac{\partial^2\Omega(u,u')}{\partial u_i'\,\partial u_k}(z_i\,u_k - u_i\,z_k)\right] - \sum_{k=1}^{m}\sum_{i=1}^{n}\frac{\partial\varphi_k}{\partial y_i'}(r_k\,u_i - \rho_k\,z_i)\right\}$$

$$= \frac{d}{dx}\,\psi(z, r\,;\,u, \rho)\,, \tag{5}$$

when one introduces the notation:

$$\sum_{k=1}^{m}\sum_{i=1}^{n}\tfrac{1}{2}\left[\frac{\partial^2\Omega(u,u')}{\partial u_i'\,\partial u_k'}(z_i\,u_k' - u_i\,z_k') + \frac{\partial^2\Omega(u,u')}{\partial u_i'\,\partial u_k}(z_i\,u_k - u_i\,z_k)\right] - \sum_{k=1}^{m}\sum_{i=1}^{n}\frac{\partial\varphi_k}{\partial y_i'}(r_k\,u_i - \rho_k\,z_i)$$

$$= \psi(z, r\,;\,u, \rho)\,. \tag{6}$$

That expression has the property that:

$$\psi(z, r\,;\,u, \rho) = -\,\psi(u, \rho\,;\,z, r)$$

and

$$\psi(z, r\,;\,z, r) = 0\,.$$

If:

$$u_1, u_2, \ldots, u_n\,;\,\rho_1, \rho_2, \ldots, \rho_m$$

is a system of integrals of (IX.3), so:

$$\psi_i(u, r) = 0 \qquad i = 1, 2, \ldots, n,$$
$$\bar{\varphi}_i(u) = 0 \qquad i = 1, 2, \ldots, m,$$

then the relation above will reduce to:

$$\sum_{i=1}^{n} u_i\,\psi_i(z,r) + \sum_{k=1}^{m}\rho_k\,\bar{\varphi}_k(z) = \frac{d}{dx}\,\psi(z, r\,;\,u, \rho)\,.$$

Moreover, if:

$$z_1, z_2, \ldots, z_n\,,\,r_1, r_2, \ldots, r_m$$

is also a system of integrals of (IX.3) such that one also has:

$$\left.\begin{aligned}\psi_i(z,r) &= 0 \quad (i = 1, 2, \ldots, n),\\ \bar{\varphi}(z) &= 0 \quad (i = 1, 2, \ldots, m),\end{aligned}\right\} \tag{7}$$

then:

$$\frac{d}{dx}\,\psi(z, r\,;\,u, \rho) = 0\,,$$

so

$$\psi(z, r\,;\,u, \rho) = C\,,$$

i.e., a constant that has the same value over the entire integration integral.

Of particular importance are two linearly-independent systems of integral of the accessory system of linear differential equations for which that constant is zero. They shall be called *conjugate* to each other. It is clear that the above two linearly-independent systems of integrals will be conjugate to each other when their terms $z_1, z_2, \ldots, z_n$; $u_1, u_2, \ldots, u_n$ vanish at the same location of the domain of integration, and that in that way one can construct n linearly-independent systems of integrals, any two of which are conjugate to each other. A group of at least n linearly-independent systems of integrals in which any two of them are conjugate to each other shall again be called a conjugate system. There is nothing difficult about defining such conjugate systems from n terms, so ones to which the aforementioned also belong, according to the procedure that was suggested in IV. **Clebsch** gave a different one in the cited treatise in volume 55.

Those remarks will suffice to exhibit the transformation of the second variation that was proposed. Let:

$$z_1^\lambda,\; z_2^\lambda,\; \ldots,\; z_n^\lambda;\; r_1^\lambda,\; r_2^\lambda,\; \ldots,\; r_m^\lambda \qquad (\lambda = 1, 2, \ldots, n)$$

be n linearly-independent particular system of integrals of the accessory system of linear differential equations, in which the ones that were written out:

$$z_1^\lambda,\; z_2^\lambda,\; \ldots,\; z_n^\lambda;\; r_1^\lambda,\; r_2^\lambda,\; \ldots,\; r_m^\lambda$$

will be denoted by z^λ, r^λ, briefly.

If one then sets:

$$\eta_k = \sum_{\lambda=1}^{n} \rho_\lambda\, z_k^\lambda \qquad k = 1, 2, \ldots, n,$$

in which the $\rho_1, \rho_2, \ldots, \rho_m$ mean undetermined multipliers, and one denotes the result of substituting the values of $\rho_1, \rho_2, \ldots, \rho_m$ that this gives in $\sum_{\lambda=1}^{n} \rho_\lambda\, z_k^\lambda$ by r_k, such that:

$$r_k, = \sum_{\lambda=1}^{n} \rho_\lambda\, r_k^\lambda$$

then one will have:

$$\sum_{k=1}^{n} \eta_k\, \psi_k(\eta, r) = \sum_{\lambda=1}^{n} \rho_\lambda \sum_{k=1}^{n} z_k^\lambda\, \psi_k(\eta, r)$$

$$\sum_{k=1}^{m} r_k\, \bar{\varphi}_k(\eta) = \sum_{\lambda=1}^{n} \rho_\lambda \sum_{k=1}^{m} r_k^\lambda\, \bar{\varphi}_k(\eta) \;,$$

so

$$\sum_{k=1}^{n} \eta_k\, \psi_k(\eta, r) - \sum_{k=1}^{m} r_k\, \bar{\varphi}_k(\eta) = \sum_{\lambda=1}^{n} \rho_\lambda \left[\sum_{k=1}^{n} z_k^\lambda\, \psi_k(\eta, r) - \sum_{k=1}^{m} r_k^\lambda\, \bar{\varphi}_k(\eta) \right]$$

$$= \sum_{\lambda=1}^{n} \rho_\lambda \frac{d}{dx} \psi(\eta, r; z^\lambda, r^\lambda)$$

$$= \frac{d}{dx}\left[\sum_{\lambda=1}^{n} \rho_\lambda \; \psi(\eta, r; z^\lambda, r^\lambda)\right] - \sum_{\lambda=1}^{n} \rho'_\lambda \; \psi(\eta, r; z^\lambda, r^\lambda) \;,$$

or also, since $\bar{\varphi}_k(\eta) = 0$:

$$\sum_{i=1}^{n} \eta_i \, \psi_i(\eta, r) = \frac{d}{dx}\left[\sum_{\lambda=1}^{n} \rho_\lambda \; \psi(\eta, r; z^\lambda, r^\lambda)\right] - \sum_{\lambda=1}^{n} \rho'_\lambda \; \psi(\eta, r; z^\lambda, r^\lambda) \,. \tag{8}$$

However, from (6), one has:

$$\sum_{\lambda=1}^{n} \rho'_\lambda \; \psi(\eta, r; z^\lambda, r^\lambda) = \sum_{\lambda=1}^{n} \rho'_\lambda \left\{ \sum_{i,k} \tfrac{1}{2} \left[\frac{\partial^2 \Omega(\eta, \eta')}{\partial \eta'_i \, \partial \eta'_k} (\eta_k \, (z_i^\lambda)' - \eta'_i \, z_k^\lambda) + \frac{\partial^2 \Omega(\eta, \eta')}{\partial \eta'_i \, \partial \eta_k} (\eta_k \, z_i^\lambda - \eta_i \, z_k^\lambda) \right] \right.$$
$$\left. - \sum_{i=1}^{m} \sum_{k=1}^{n} \frac{\partial \varphi_i}{\partial y'_k} (z_k^\lambda \, r^i - \eta_k \, r_i^\lambda) \right\} \,. \tag{9}$$

If one substitutes:

$$\eta_k \, (z_i^\lambda)' - z_k^\lambda \, \eta'_i = \sum_{\mu=1}^{n} \rho_\mu [z_k^\mu \, (z_i^\lambda)' - z_k^\lambda \, (z_i^\mu)'] - z_k^\lambda \sum_{\mu=1}^{n} \rho'_\mu \, z_i^\mu \,,$$

$$\eta_k \, z_i^\lambda - \eta_i \, z_k^\lambda = \sum_{\mu=1}^{n} \rho_\mu (z_i^\lambda \, z_k^\mu - z_i^\mu \, z_k^\lambda) \,,$$

$$z_k^\lambda \, r_i - \eta_k \, r_i^\lambda = \sum_{\mu=1}^{n} \rho_\mu (z_k^\lambda \, r_i^\mu - r_i^\lambda \, z_k^\mu)$$

in that, in analogy with V, then the expression (9) will go to:

$$\sum_{\lambda=1}^{n} \rho'_\lambda \; \psi(\eta, r; z^\lambda, r^\lambda) =$$

$$\sum_{\lambda,\mu=1}^{n} \rho'_\lambda \, \rho_\mu \left\{ \sum_{i,k=1}^{m} \tfrac{1}{2} \left[\frac{\partial^2 \Omega(\eta, \eta')}{\partial \eta'_i \, \partial \eta'_k} (z_k^\mu \, (z_i^\lambda)' - (z_i^\mu)' \, z_k^\lambda) + \frac{\partial^2 \Omega(\eta, \eta')}{\partial \eta'_i \, \partial \eta_k} (z_i^\lambda \, z_k^\mu - z_i^\mu \, z_k^\lambda) \right] \right.$$
$$\left. - \sum_{i=1}^{m} \sum_{k=1}^{n} \frac{\partial \varphi_i}{\partial y'_k} (r_i^\mu \, z_k^\lambda - r_i^\lambda \, z_k^\mu) \right\} - \sum_{i,k=1}^{n} \frac{1}{2} \frac{\partial^2 \Omega}{\partial \eta'_i \, \partial \eta'_k} \sum_{\lambda=1}^{n} \rho'_\lambda \, z_i^\lambda \sum_{\mu=1}^{n} \rho'_\mu \, z_i^\mu$$

$$= \sum_{\lambda,\mu=1}^{n} \rho'_\lambda \, \rho_\mu \, \psi(z^\mu, r^\mu; z^\lambda, r^\lambda) - \sum_{i,k} \frac{1}{2} \frac{\partial^2 \Omega(\eta, \eta')}{\partial \eta'_i \, \partial \eta'_k} \, \zeta_i \, \zeta_k \,,$$

when one sets:

$$\zeta_i = \sum_{\lambda=1}^{n} \rho'_\lambda \, z_i^\lambda .$$

Therefore, one will ultimately get from (8):

$$\sum_{i=1}^{n} \eta_i \, \psi_i(\eta, r)$$
$$= \sum_{i,k} \frac{1}{2} \frac{\partial^2 \Omega(\eta,\eta')}{\partial \eta'_i \, \partial \eta'_k} \zeta_i \, \zeta_k - \sum_{\lambda,\mu=1}^{n} \rho'_\lambda \, \rho_\mu \, \psi(z^\mu, r^\mu; z^\lambda, r^\lambda) + \frac{d}{dx}\left[\sum_{\lambda=1}^{n} \rho_\lambda \, \psi(\eta, r; z^\lambda, r^\lambda)\right], \tag{8*}$$

in which the $\psi(z^\mu, r^\mu; z^\lambda, r^\lambda) = C_{\lambda\mu}$ are constants. They will be zero when one employs an *n*-term conjugate system for the conversion. Formula (8*) above will then simplify to:

$$\sum_{i=1}^{n} \eta_i \, \psi_i(\eta, r) = \sum_{i,k=1}^{n} \frac{1}{2} \frac{\partial^2 \Omega(\eta,\eta')}{\partial \eta'_i \, \partial \eta'_k} \zeta_i \, \zeta_k + \frac{d}{dx}\left[\sum_{\lambda=1}^{n} \rho_\lambda \, \psi(\eta, r; z^\lambda, r^\lambda)\right]$$
$$= \sum_{i,k=1}^{n} \frac{\partial^2 F}{\partial y'_i \, \partial y'_k} \zeta_i \, \zeta_k + \frac{d}{dx}\left[\sum_{\lambda=1}^{n} \rho_\lambda \, \psi(\eta, r; z^\lambda, r^\lambda)\right], \tag{10}$$

so from (V.2), the following equation will exist for ζ_k :

$$\zeta_i \, \Delta(z^1, z^2, \ldots, z^n) = \begin{vmatrix} \eta'_k & (z_k^1)' & \cdots & (z_k^n)' \\ \eta_1 & z_1^1 & \cdots & z_1^n \\ \vdots & \vdots & \ddots & \vdots \\ \eta_n & z_1^n & \cdots & z_n^n \end{vmatrix},$$

when $\sum \pm z_1^1 \, z_2^2 \cdots z_n^n$ is denoted by $\Delta(z^1, z^2, \ldots, z^n)$. However, those quantities are not mutually independent because since:

$$\bar{\varphi}_i(\eta) = 0 \qquad (i = 1, 2, \ldots, m),$$

one will have:

$$\bar{\varphi}_i\left(\sum_{\lambda=1}^{n} \rho_\lambda \, z^\lambda\right) = 0$$

or

$$\sum_{\lambda=1}^{n} \rho_\lambda \, \bar{\varphi}_i(z^\lambda) + \sum_{k=1}^{n} \frac{\partial \varphi_i}{\partial y'_i} \sum_{\lambda=1}^{n} \rho'_\lambda \, z_k^\lambda = 0 \qquad (i = 1, 2, \ldots, m).$$

Due to the fact that:

$$\bar{\varphi}_i(z^\lambda) = 0 \qquad (i = 1, 2, \ldots, m),$$

one will have the relation:

$$\sum_{k=1}^{n} \frac{\partial \varphi_i}{\partial y_i'} \zeta_k = 0 \qquad (i = 1, 2, \ldots, m).$$

In order to determine the quantities $\rho_1, \rho_2, \ldots, \rho_n$, one appeals to the n equations:

$$\eta_k = \sum_{\lambda=1}^{n} \rho_\lambda \, z_k^\lambda \qquad (k = 1, 2, \ldots, n),$$

which will then possess the same determinant $\Delta(z^1, z^2, \ldots, z^n)$ that was obtained in the calculation of the $\zeta_1, \zeta_2, \ldots, \zeta_n$. Therefore, one can calculate the two systems of quantities from the associated system of equations if and only if that determinant does not vanish in the entire interval ab, including the limits a and b.

Under that assumption, from (10), one can given the second variation (2) the form ([1]):

$$\delta^2 J = \int_a^b \sum_{i,k=1}^{n} \frac{\partial^2 F}{\partial y_i' \, \partial y_k'} \zeta_i \, \zeta_k \tag{11}$$

since the $\rho_1, \rho_2, \ldots, \rho_n$ are zero at the limits, and in which the $\zeta_1, \zeta_2, \ldots, \zeta_n$ satisfy the m equations:

$$\sum_{k=1}^{n} \frac{\partial \varphi_i}{\partial y_i'} \zeta_k = 0 \qquad (i = 1, 2, \ldots, m).$$

This form of the second variation shall be called its *reduced form*.

____

([1]) **Clebsch**, J. für Math., Bd. 55.