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The foundations of wave mechanics

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Contents

		Page
	Introduction	1
§ 1.	The variational principles of dynamics.	2
§ 2.	The wave principles of mechanics.	6
§ 3.	Phase waves and quantum theory.	18
§ 4.	Further development of wave mechanics.	30
	Concluding remarks and summary.	35

Introduction

The fact that quantum theory, in the form that **N. Bohr** and **A. Sommerfeld** gave it, and despite its initial colossal successes, is not suitable for all quantum phenomena, was repeatedly stated in recent years, and ultimately by **N. Bohr** himself (¹). One believed that one would have to tread along a path that implied a radical departure from the methods of physics that had been used up till then and were described thoroughly by **M. Born**'s *Probleme der Atomdynamik* (²), namely, the creation of a "true discontinuum theory." Here, the foundations of an attempt shall be developed that has precisely that subject as its goal, namely, a direct incorporation of quantum theory into the methods of classical physics. The cornerstone of that was laid by **Louis de Broglie** in his *Thèse* (³), and one has **E. Schrödinger** "Quantisierung als Eigenwertproblem" (⁴) for its further, and uncommonly successful, development. In that way, wave concepts were employed that **Hamilton** had appealed to with great success in his own time in order to develop classical mechanics (⁵), but

^{(&}lt;sup>1</sup>) "Atomtheorie und Mechanik," Naturwissenschaften **14** (1926), 1. See also the conclusion of **M. Planck**, "Physikalische Gesetzlichkeit im Lichte neuerer Forschung," Naturwissenschaften **14** (1926), 257.

^{(&}lt;sup>2</sup>) Berlin, 1926. On its first orientation: **A. Landé**, "Neue Wege der Quantentheorie," Naturwissenschaften **14** (1926), 455.

^{(&}lt;sup>3</sup>) Masson et Cie., Paris, 1924. See also Ann. de Physique (10) **3** (1925), pp. 22.

^{(&}lt;sup>4</sup>) The basic ideas are found in the second communication, in particular, Ann. Phys. (Leipzig) (4) **79** (1926), pp. 489.

^{(&}lt;sup>5</sup>) Whittaker, Analytische Dynamik der Punkte und starren Körper, German by F. and K. Mittelsten-Scheid, Berlin, 1924, pp. 306.

later paid no attention to, as a rule. However, it is precisely that wave picture that seems to come to mind if we are to come to a better understanding of the enigma of quantum phenomena.

§ 1. – The variational principles of dynamics.

For the sake of simplicity, all considerations shall be based upon the motion of a mass particle in a static potential force field. However, the general considerations might be developed in a form that satisfies the principle of special relativity from the outset, since the transition to the formulas of **Newtonian** mechanics can be completed very simply in each case. For a velocity:

$$\mathfrak{v}=\frac{d\mathfrak{r}}{dt}\,,$$

the particle has a mass:

$$m = \frac{m_0}{\sqrt{1 - \frac{\mathfrak{v}^2}{c^2}}},$$

and the impulse:

$$\mathfrak{G}=m\mathfrak{v}$$

Under the assumptions that were made, the basic law of dynamics is written in the simple form:

$$\frac{d\mathfrak{G}}{dt} = -\operatorname{grad} V(x, y, z) \ .$$

Hamilton's wave picture is based upon the variational principles of dynamics. They are based upon the variational formula:

$$\delta \int \mathfrak{G} d\mathfrak{r} = \int_{A}^{A'} \left(\delta \mathfrak{G} \cdot d\mathfrak{r} + \mathfrak{G} \cdot d\delta \mathfrak{r} \right) = \left| \mathfrak{G} \cdot \delta \mathfrak{r} \right|_{A}^{A'} + \int_{A}^{A'} \left(\delta \mathfrak{G} \cdot d\mathfrak{r} + \mathfrak{G} \cdot \delta \mathfrak{r} \right) ,$$

which represents the difference between the values of:

for the two infinitely-differing integration paths AA' and BB' in Fig. 1, which include the limits.

The variational formula will reduce to a simpler form in the special case in which the initial integral between the limits A and A' extends over a trajectory that satisfies the basic equation of dynamics. Namely, if one gives the total energy:

$$W = mc^2 + V,$$

as a result of the formula:

$$m = m_0 \sqrt{1 + \frac{\mathfrak{G}^2}{m_0^2 c^2}}$$
,

which one arrives at by eliminating v from the expressions for *m* and \mathfrak{G} , the form:

$$W = m_0 c^2 \sqrt{1 + \frac{\mathfrak{G}^2}{m_0^2 c^2}} + V ,$$

then one will infer that:

$$\delta W = \frac{\mathfrak{G} \,\delta \mathfrak{G}}{m_0 \sqrt{1 + \frac{\mathfrak{G}^2}{m_0^2 \,c^2}}} + \operatorname{grad} V \,\delta \mathfrak{r} ,$$

or more simply:

$$\delta W = \mathfrak{v} \, \delta \mathfrak{G} + \operatorname{grad} V \, \delta \mathfrak{r}$$



Figure 1.

For a path that is traversed by a particle, from the laws of dynamics, one has:

$$\mathfrak{v} = \frac{d\mathfrak{r}}{dt}, \qquad \text{grad } V = -\frac{d\mathfrak{G}}{dt},$$

and in that case, one will infer that:

$$\delta W dt = \delta \mathfrak{G} \cdot d\mathfrak{r} - d\mathfrak{G} \cdot \delta \mathfrak{r},$$

and furthermore:

$$\int_{A}^{A'} \left(\delta \mathfrak{G} \cdot d\mathfrak{r} - d\mathfrak{G} \cdot \delta \mathfrak{r} \right) = \int_{t}^{t'} \delta W \, dt \,,$$

in which t and t' mean the times at which the particle will pass the point A (A', resp.) while traversing the path. Moreover, it will be written simply:

$$\left|\mathfrak{G}\cdot d\mathfrak{r}\right|_{A}^{A'}=\mathfrak{G}'\cdot\delta\mathfrak{r}'-\mathfrak{G}\cdot\delta\mathfrak{r},$$

in which, from now on, \mathfrak{G} and \mathfrak{G}' will mean the values of the impulses at the point *A* (*A'*, resp.), $\delta \mathfrak{r}$ is the displacement of the point *A* to *B*, and $\delta \mathfrak{r}'$ is the displacement from the point *A'* to *B'* in Fig.1. In that way, the variational formula will assume the form:

$$\delta \int_{A}^{A'} \mathfrak{G} d\mathfrak{r} = \mathfrak{G}' \cdot \delta \mathfrak{r}' - \mathfrak{G} \cdot \delta \mathfrak{r} + \int_{t}^{t'} \delta W dt , \qquad (1)$$

which will then be valid only for the case of a generally completely arbitrary variation of a dynamically possible initial path.

Under the assumption that:

$$\delta \mathfrak{r} = 0$$
, $\delta \mathfrak{r}' = 0$, $\delta W = 0$,

the variational formula will assume the especially-simple form:

$$\delta \int_{A}^{A'} \mathfrak{G} d\mathfrak{r} = 0 .$$

That condition equation states that the motion of a mass-particle is characterized by the fact that the quantity:

$$\int_{A}^{A'} \mathfrak{G} d\mathfrak{r},$$

which is referred to as the *action*, must possess an extreme value along the conceivable paths between the same initial and final point A and A', and only for those values of \mathfrak{G} that lead to one and the same value for the total energy W. That is the statement of **Maupertuis**'s principle of least action, which is the oldest variational principle of dynamics. If one substitutes:

$$d \mathfrak{r} = \mathfrak{v} dt$$

in the integral then the principle of least action can also be written in the form:

$$\delta \int_{t}^{t'} (\mathfrak{Gv}) dt = 0.$$

In **Newtonian** mechanics, and *only* then, that formula can be further simplified since one can write twice the *vis viva* for the integrand and then drop the factor 2 afterward. From the principle of special relativity, that relation will no longer exist, and one must leave the principle of least action in the form that was written down. Here, as before, the starting point A and end point A' of the path, as well as the total energy W are fixed during the variation, while the initial time t and final time t' can be varied during it. As one sees, the form of the principle of least action that was written first is theoretically simpler.

The conversion of the action integral that was just carried out can also be performed on the more general variational formula (1) already. One will then get:

$$\delta \int_{t}^{t'} (\mathfrak{G}\mathfrak{v}) dt = \mathfrak{G}' \cdot \delta\mathfrak{r}' - \mathfrak{G} \cdot \delta\mathfrak{r} + \int_{t}^{t'} \delta W dt .$$
 (1')

The last integral can be converted into:

$$\int_{t}^{t'} \delta W \, dt = \delta \int_{t}^{t'} W \, dt - W \, \delta(t'-t) \, ,$$

when one observes that the law of energy:

$$W = \text{const}$$

is valid for the initial path. As a result, the variational formula (1') can be converted into:

$$\delta \int_{t}^{t'} L dt = \mathfrak{G}' \cdot \delta \mathfrak{r}' - \mathfrak{G} \cdot \delta \mathfrak{r} + W \,\delta(t' - t) \,, \tag{2}$$

in which:

$$L = \mathfrak{G} \mathfrak{v} - W$$
,

and will be referred to as the **Lagrangian** function. In **Newtonian** mechanics, as one recognizes immediately, that expression will reduce to the difference between the *vis viva* and potential energy. However, from the principle of special relativity, one will find that:

$$L = mv^{2} - mc^{2} - V = -mc^{2}\left(1 - \frac{v^{2}}{c^{2}}\right) - V = -m_{0}c^{2}\sqrt{1 - \frac{v^{2}}{c^{2}}} - V$$

Under the assumption that:

$$\delta \mathfrak{r} = 0$$
, $\delta \mathfrak{r}' = 0$, $\delta (t' - t) = 0$

one will obtain the simple variational formula from (2):

$$\delta \int_{t}^{t'} L \, dt = 0 \, .$$

That is **Hamilton's** principle, which states that of all conceivable paths that are traversed between the same initial point A and the same end point A' in the same time interval t'-t, the motion that actually takes place will make the integral:

 $\int L dt$

an extremum. That variational principle will not change form when one increases the **Lagrangian** function by an arbitrary constant, e.g., by
$$m_0 c^2$$
, such that one will get:

$$L = m_0 c^2 \left(1 - \sqrt{1 - \frac{\mathfrak{v}^2}{c^2}} \right) - V \,.$$

In that form, the expression will produce the classical value precisely as an asymptotic formula for small \mathfrak{v} .

§ 2. – The wave principles of mechanics.

The principle of least action was presented by **Maupertuis** during the time of the corpuscular theory of light, and indeed in his search for a mechanical analogue of **Fermat**'s principle in optics. The latter says that the path of the light ray will be determined by:

$$\delta \int_{A}^{A'} \frac{ds}{u} = 0 ,$$

u = u(x, y, z)

in which:

means the speed of light, and the limit points A and A' of the integral are fixed under variation. The extreme value of the integral that appears is merely a function of the limits, and in that that way determines a function:

$$\int_{A(x,y,z)}^{A'(x',y',z')} \frac{ds}{u} = V(x, y, z, x', y', z') ,$$

which is the so-called *Hamilton* characteristic function of the optical medium in question, which is generally inhomogeneous. On the other hand, one will also find that for the extremum of the integral that appears:

$$\int_{A}^{A'} \frac{ds}{u} = t' - t \,,$$

i.e., it is the time interval that the light requires to propagate from A to A'. Both relations together will yield the equation:

$$V(x, y, z, x', y', z') = t' - t$$

That obviously represents nothing but **Huygens**'s elementary wave for any fixed generating point with the coordinates x, y, z. **Fermat**'s principle will then lead back to the wave theory of light with no further analysis.

Analogously, the principle of least action will define a function:

$$\int_{A(x,y,z)}^{A'(x',y',z')} \mathfrak{G} d\mathfrak{r} = S(x,y,z,x',y',z',W)$$

by way of the extreme value of the integral that appeared when it is written in the first form. The peculiarity of this case is the dependency on W, as well. If one once more converts the extreme value of the integral by means of:

$$\int_{A}^{A'} \mathfrak{G} d\mathfrak{r} = \int_{t}^{t'} (\mathfrak{G}\mathfrak{v}) dt,$$

and in this case t and t' mean the time points at which the point A(A', resp.) will be passed through during the actual motion of the mass-particle, then one can infer the equation:

$$S(x, y, z, x', y', z', W) = \int_{t}^{t'} (\mathfrak{Gv}) dt.$$
(3)

Here as well, the right-hand side will increase with increasing distance between the points A and A', and therefore the time interval t'-t will simultaneously increase, as well, such that equation (3) can also be interpreted as that of elementary waves.

However, the analogy with the optical case can be made even more intimate. Namely, one will further find from (1) that:

$$\delta S = \mathfrak{G}' \delta \mathfrak{r}' - \mathfrak{G} \delta \mathfrak{r} + \delta W (t' - t), \qquad (3')$$

when one observes that δW is considered to be constant under the transition to a neighboring extremal integral. On the other hand, differentiating the left-hand side of (3) will give:

$$\delta S = \operatorname{grad}' S \, \delta \mathfrak{r}' - \operatorname{grad} S \, \delta \mathfrak{r} + \frac{\partial S}{\partial W} \, \delta W$$

directly. Since the differentials $\delta \mathfrak{r}'$, $\delta \mathfrak{r}$, and δW are arbitrary and independent of each other, it will necessarily follow from a comparison with (3') that:

grad'
$$S = \mathfrak{G}', \text{ grad } S = -\mathfrak{G},$$

$$\frac{\partial S}{\partial W} = t' - t.$$
(4)

The left-hand side of the last equation once more represents a function of x, y, z, x', y', z', and W such that one has in (4) the equation of elementary waves in precisely the same form that was derived from **Fermat**'s principle.

Hamilton's principle:

$$\delta \int_{t}^{t'} L dt = 0$$

also defines a function:

$$S^{*}(x, y, z, x', y', z', t' - t) = \int_{t}^{t'} L dt$$
(5)

by the extreme value of the integral between fixed points and fixed times. One infers from that and the relation (2) that:

$$\delta S^* = \mathfrak{G}' \delta \mathfrak{r}' - \mathfrak{G} \delta \mathfrak{r} - W \delta (t' - t).$$
^(5')

Analogous to the previous argument, one concludes from this that:

$$\operatorname{grad}' S^* = \mathfrak{G}', \qquad \operatorname{grad} S^* = -\mathfrak{G},$$

and

and

$$\frac{\partial S^*}{\partial (t'-t)} = -W.$$
(6)

Equations (5) and (6) can also be interpreted as those of elementary waves. However, it will follow from (3) and (5) that:

$$S-S^* = \int_t^{t'} (\mathfrak{Gv}-L) dt = W(t'-t) .$$

That can also be written as:

$$S^* = S - W(t' - t).$$
 (7)

That relation will imply the functional dependency between W and t'-t for any two points A and A'.

Equations (3) and (3') shall now be applied to infinitely-small elementary waves. The time interval:

$$t'-t = \tau$$

is considered to be infinitely-small. For A' and A, one will then have the simple relations:

$$\mathfrak{r}' = \mathfrak{r} + \mathfrak{v} \tau,$$

$$\mathfrak{G}' = \mathfrak{G} + \frac{d\mathfrak{G}}{dt} \tau,$$

$$\delta \mathfrak{r}' = \delta \mathfrak{r} + \delta \mathfrak{v} \cdot \tau + \mathfrak{v} \delta \tau,$$

and furthermore:

$$\mathfrak{G}' \,\delta \mathfrak{r}' - \mathfrak{G} \,\,\delta \mathfrak{r} = \left(\frac{d\mathfrak{G}}{dt} \,\delta \mathfrak{r} + \mathfrak{G} \,\delta \mathfrak{v}\right) \tau + (\mathfrak{G} \mathfrak{v}) \,\delta \tau \,.$$

Thus, equation (3') is written:

$$\delta S = \left(\delta W + \frac{d\mathfrak{G}}{dt} \delta \mathfrak{r} + \mathfrak{G} \delta \mathfrak{v} \right) \tau + (\mathfrak{G}\mathfrak{v}) \delta \tau \,.$$

On the other hand, (3) can be written as:

$$S = (\mathfrak{G} \mathfrak{v}) \tau$$
,

and it follows from this that:

$$\delta S = (\mathfrak{v} \ \delta \mathfrak{G} + \mathfrak{G} \ \delta \mathfrak{v}) \ \tau + (\mathfrak{G} \ \mathfrak{v}) \ \delta \tau.$$

The identification of the two expressions for δS that were obtained will give:

$$\delta S = \mathfrak{v} \,\,\delta \mathfrak{G} - \frac{d\mathfrak{G}}{dt} \,\,\delta \mathfrak{r} \,.$$

In generalized coordinates, one writes:

$$\begin{split} \mathfrak{v} \ \delta \mathfrak{G} &= \sum \dot{q}_i \ \delta p_i \ , \\ \frac{d \mathfrak{G}}{dt} \ \delta \mathfrak{r} &= \sum \dot{p}_i \ \delta q_i \ , \end{split}$$

and one will get:

$$\delta W = \sum \dot{q}_i \, \delta p_i - \sum \dot{p}_i \, \delta q_i = \delta H \,,$$

in which:

$$H(p_1, p_2, p_3, q_1, q_2, q_3) = W$$

means the **Hamiltonian** function. It will necessarily follow from the expression that was obtained for the complete differential δH that:

$$\dot{q}_i = rac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -rac{\partial H}{\partial q_i}.$$

However, they represent nothing but **Hamilton**'s fundamental canonical equations of dynamics. They will necessarily lead one to a wave theory of mechanics, and it will therefore happen that they have the form of an infinitesimal contact transformation $(^1)$.

Equations (5) and (5) shall also be applied to equations (5) and (5') in the same way. (5') can next be written:

$$\delta S^* = \left(\frac{d\mathfrak{G}}{dt}\delta\mathfrak{r} + \mathfrak{G}\delta\mathfrak{v}\right)\tau + (\mathfrak{G}\mathfrak{v} - W)\delta\tau.$$

 $S^* = L \tau$,

From (5):

one finds that:

$$\delta S^* = \delta L \cdot \tau + L \cdot \delta \tau.$$

 $L = \mathfrak{G} \mathfrak{v} - W$,

When one considers the formula:

it will follow upon identifying the two expressions for δS^* that one has the relation:

$$\delta L = \frac{d\mathfrak{G}}{dt}\delta\mathfrak{r} + \mathfrak{G}\delta\mathfrak{v} \,.$$

In generalized coordinates, one must again set:

$$\frac{d\mathfrak{G}}{dt}\delta\mathfrak{r} = \sum \dot{p}_i \,\delta q_i$$

$$\mathfrak{G}_i \,\delta\mathfrak{n} = \sum \mathbf{n}_i \,\delta\dot{q}_i$$

and

$$\mathfrak{G}\,\,\delta\mathfrak{v}=\sum\,p_i\,\delta\dot{q}_i\,.$$

The formula for the complete differential can then be written:

$$\delta L = \sum \dot{p}_i \, \delta q_i + \sum p_i \, \delta \dot{q}_i \,,$$

which will necessarily imply that:

$$\dot{p}_i = \frac{\partial L}{\partial q_i}, \qquad p_i = \frac{\partial L}{\partial \dot{q}_i}.$$

One can also combine the system into equations of the form:

^{(&}lt;sup>1</sup>) **Whittaker**, *loc*, *cit*., pp. 323.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

Those are the **Lagrange** equations of the second kind in dynamics, which are ordinarily derived as the extremal conditions for **Hamilton**'s principle, although they likewise yield a wave theory of mechanics.

An entirely peculiar interpretation can be given for the wave equation (4). One starts by considering the wave motion that one obtains when one sets the left-hand side of (5) equal to an arbitrarily-chosen constant, so:

$$S^*(x, y, z, x', y', z', t'-t) = \text{const.}$$

If these wave surfaces, as opposed to the ones that were considered up to now, do not advance with the same velocity as the mass-particle then as a result of the previously-proved relation:

$$\mathfrak{G}' = \operatorname{grad}' S^*$$

will correctly give the impulse of the mass-particle at each place A' again. As a result of the relation (7), the wave equation is also represented by the equation:

$$S - W(t' - t) = \text{const.}$$

The phase of the wave equation is then constructed from:

$$\Phi = S - Wt' - \Theta ,$$

in which Θ has a constant value for a well-defined wave surface. However, for a given A and A', the waves will also vary with W such that an arbitrary propagation that is carried by the spreading of the wave will be given by a spectral formula:

$$\Psi = \int a(x, y, z, x', y', z', W) \sin\left(\frac{2\pi}{h}\Phi\right) dW,$$

in which *h* means a constant with the same dimension as *S*. Θ has that dimension, and in general one assumes that:

$$\Theta = \Theta(W)$$
.

The integration is carried out with A, A', and t' fixed. With the same auxiliary conditions, differentiating Φ will yield:

$$d\Phi = \frac{\partial \Phi}{\partial W} \, dW \,,$$

such that the integral can also be written:

$$\Psi = \int a \frac{\sin\left(\frac{2\pi}{h}\Phi\right)}{\frac{\partial\Phi}{\partial W}} d\Phi \ .$$

In particular, it will now be assumed that the phenomenon being treated shall occupy only a very thin spectral domain, so the amplitude *a* shall differ from zero noticeably only within the narrow limits of *W* to $W + \Delta W$ for the energy parameter. That will correspond to the limits Φ to Φ' , which are related by:

$$\Phi' = \Phi + \frac{\partial \Phi}{\partial W} \Delta W \, .$$

The "sin" in the integrand can be assumed to be the only rapidly-varying function in the domain of integration. One can then write:

$$\Psi = \frac{\overline{a}}{\frac{\overline{\partial \Phi}}{\overline{\partial W}}} \int_{\Phi}^{\Phi'} \sin\left(\frac{2\pi}{h}\Phi\right) d\Phi$$

to a good approximation. When the integral is evaluated, that will give:

$$-\frac{\cos\left(\frac{2\pi}{h}\Phi\right)}{\frac{2\pi}{h}} \bigg|_{\Phi}^{\Phi'} = \frac{2\sin\left(\frac{2\pi}{h}\frac{\Phi'-\Phi}{2}\right)\sin\left(\frac{2\pi}{h}\frac{\Phi'+\Phi}{2}\right)}{\frac{2\pi}{h}}.$$

From the above, one must introduce:

$$\Phi' - \Phi = \overline{\frac{\partial \Phi}{\partial W}} \Delta W$$

and one can further set:

$$\frac{1}{2}\left(\Phi'+\Phi\right)=\overline{\Phi},$$

to a sufficient approximation, such that one can write the result of the calculation as:

$$\Psi = \overline{a} \frac{\sin\left(\frac{\pi}{h} \frac{\overline{\partial \Phi}}{\partial W} \Delta W\right)}{\frac{\pi}{h} \frac{\overline{\partial \Phi}}{\partial W}} \sin\left(\frac{2\pi}{h} \overline{\Phi}\right).$$

It will then suffice to remark that for all functions that appear, one takes a mean value over the spectral domain. One can then drop the overbars from now on. One can then write the formulas that were obtained in the forms:

$$\Psi = P \sin\left(\frac{2\pi}{h}\Phi\right), \quad P = a \frac{\sin\left(\frac{\pi}{h}\frac{\partial\Phi}{\partial W}\Delta W\right)}{\frac{\pi}{h}\frac{\partial\Phi}{\partial W}}$$

That result states that the original carrier wave:

$$\psi = a \sin\left(\frac{2\pi}{h}\Phi\right)$$

is modulated. That effect comes about by the interference of all waves in the narrow spectral domain that have different energy parameters, which is the so-called wave-group.

Now *P*, as a function of:

$$\frac{\partial \Phi}{\partial W}$$

has a maximum of:

$$P = a \Delta W$$
 for $\frac{\partial \Phi}{\partial W} = 0$.

Its zeroes appear symmetrically:



Figure 2.

A decreasing sequence of extremal values with magnitudes:

$$|P| = \frac{a\Delta W}{(2n+1)\frac{\pi}{2}}$$
 for $\frac{\partial \Phi}{\partial W}\Delta W = (2n+1)\frac{\pi}{2}$

must be further noted. That behavior corresponds completely to that of the **Fraunhofer** diffraction phenomena behind a narrow slit. The values of Ψ that are given by the formula are illustrated graphically in Fig. 2.

The equation:

$$\frac{\partial \Phi}{\partial W} = \text{const}$$

then determines the advance of the modulation. Its speed of propagation is referred to as the *group velocity*, as opposed to the speed of propagation of the carrier wave that is determined by the equation:

$$\Phi = \text{const.},$$

which is referred to as the *phase velocity*. One then sees that in general the group velocity will be different from the phase velocity. When one makes the energy density of the waves proportional to the square of the amplitude, the group velocity will obviously represent the speed of propagation of the energy that is carried by the waves. In particular, the equation:

$$\frac{\partial \Phi}{\partial W} = 0$$

represents the motion of the energy maximum. Upon replacing Φ with its value, one can write the equation:

$$\frac{\partial S}{\partial W} - t' - \frac{d\Theta}{dW} = 0 .$$
$$\frac{d\Theta}{dW} = -t ,$$

and obtain the relation:

One can set:

which is precisely equation (4). One is already accustomed to regard a mass-point as an energy node in the theory of relativity. When considered from that viewpoint, the description of the motion of the mass-point by a wave-group will seem to be something that is quite natural.

 $\frac{\partial S}{\partial W} = t' - t \,,$

Equation (6) agrees with (4), as can be shown. Namely, one will get equation (7) by differentiating with respect to t'-t:

$$\frac{\partial S}{\partial (t'-t)} = \frac{\partial S}{\partial W} \frac{\partial W}{\partial (t'-t)} - (t'-t) \frac{\partial W}{\partial (t'-t)} - W,$$

or when written differently:

$$\frac{\partial S^*}{\partial (t'-t)} + W = \left[\frac{\partial S}{\partial W} - (t'-t)\right] \frac{\partial W}{\partial (t'-t)}.$$

The right-hand side of that equation vanishes as a result of (4), and one will get:

$$\frac{\partial S^*}{\partial (t'-t)} + W = 0 ,$$

which coincides with equation (6). The elementary waves that are represented by (4) and (6) are the same then.

The picture of a wave-group for the motion of the mass-particle leads immediately to an integration method for dynamics. The elementary phase waves that emanate from each point *A* allow one to ascertain the advance of an entirely-arbitrary initial phase wave with the help of **Huyghens**'s principle. Its envelope is composed of nothing but surface elements of elementary phase waves. The relation that is true for them:

$$\mathfrak{G}' = \operatorname{grad}' S$$

must also remain true for the envelope itself.

From now on, we shall work with those completely-general phases. Since no initial center A will generally exist for them, we can call the point that advances on the wave surface simply A now, when we previously called it A', i.e., the prime shall simply be dropped from the spatial coordinates that advance on the phase surfaces from now on. The formula that was just cited can then be written:

$$\mathfrak{G} = \operatorname{grad} S \,. \tag{8}$$

That says that the impulse can be derived from the gradient of a scalar potential that one must interpret as the wave function of the phase waves that were just considered. In generalized coordinates, one writes that relation as:

$$p_i = \frac{\partial S}{\partial q_i}.$$

Under the assumptions that were made, the p_i must fulfill the law of energy in the form:

$$H(p_1, p_2, p_3, q_1, q_2, q_3) = W.$$

It will then follow that the partial differential equation:

$$H\left(\frac{\partial S}{\partial q_1}, \frac{\partial S}{\partial q_2}, \frac{\partial S}{\partial q_3}, q_1, q_2, q_3\right) = W$$

exist for the wave-surface function *S*. One sees that this is the **Hamilton-Jacobi** differential equation of mechanics. Every complete solution of it:

$$S = S(q_1, q_2, q_3, \alpha_1, \alpha_2, \alpha_3)$$

with three arbitrary integration constants α_1 , α_2 , α_3 represents a possible phase-wave surface function. Upon substituting that integral, the partial differential equation must reduce to a relation between the constants that appear:

$$W(\alpha_1, \alpha_2, \alpha_3) = W$$
.

Conversely, one can also use that relation in order to introduce the energy parameter as one of three arbitrary integration constants.

The phase function:

$$\Phi = S - W t - \Theta (\alpha_1, \alpha_2, \alpha_3)$$

of that general phase wave includes the previously-considered elementary phase waves as a special case. Now, the three parameters α_1 , α_2 , α_3 can be varied arbitrarily, which amounts to the transition to families of waves with other energy parameters and other orientations. An entirely arbitrary propagation that is carried by the spreading of those waves will then be given by the more general spectral formula:

$$\Psi = \iiint a(q_1, q_2, q_3, \alpha_1, \alpha_2, \alpha_3) \sin\left(\frac{2\pi}{h}\Phi\right) d\alpha_1 d\alpha_2 d\alpha_3 .$$

If one restricts the spectrum to the narrow region $\Delta \alpha_1$, $\Delta \alpha_2$, $\Delta \alpha_3$ and makes the same assumptions for each of them concerning ΔW that were made before then one will get the approximate result:

$$\Psi = P \sin\left(\frac{2\pi}{h}\Phi\right),\,$$

with:

$$P = a \frac{\sin\left(\frac{\pi}{h}\frac{\partial\Phi}{\partial\alpha_{1}}\Delta\alpha_{1}\right)}{\frac{\pi}{h}\frac{\partial\Phi}{\partial\alpha_{1}}} \frac{\sin\left(\frac{\pi}{h}\frac{\partial\Phi}{\partial\alpha_{2}}\Delta\alpha_{2}\right)}{\frac{\pi}{h}\frac{\partial\Phi}{\partial\alpha_{2}}} \frac{\sin\left(\frac{\pi}{h}\frac{\partial\Phi}{\partial\alpha_{3}}\Delta\alpha_{3}\right)}{\frac{\pi}{h}\frac{\partial\Phi}{\partial\alpha_{3}}}$$

by step-wise integration when one performs the same calculations that one did before three times in succession. One again takes a mean value over the spectral region for all functions that appear. The formula will, in fact, represent a point-like concentration of energy at the location:

$$\frac{\partial \Phi}{\partial \alpha_1} = 0 , \qquad \frac{\partial \Phi}{\partial \alpha_1} = 0 , \qquad \frac{\partial \Phi}{\partial \alpha_1} = 0 .$$

Those three equations rigorously define the location of the energy maximum, quite independently of all simplifying assumptions and approximations. The type of energy drop from that location in all directions will probably depend upon the special assumptions that one makes.

The equations:

$$\frac{\partial \Phi}{\partial \alpha_i} = 0 ,$$

which determine the position of the energy node, can be written:

$$\frac{\partial S}{\partial \alpha_i} - \frac{\partial W}{\partial \alpha_i} t - \frac{\partial \Theta}{\partial \alpha_i} = 0$$
$$\frac{\partial S}{\partial \alpha_i} = \frac{\partial W}{\partial \alpha_i} t + \frac{\partial \Theta}{\partial \alpha_i}$$

or

after one introduces the formula for the phase function. One can introduce the notations:

$$\frac{\partial W}{\partial \alpha_i} = \omega_i , \qquad \frac{\partial \Theta}{\partial \alpha_i} = \beta_i .$$

When *W* is a given function of the α_i , the ω_i will be completely determined constants. By contrast, Θ is an arbitrary function of the α_i , so the β_i will be completely arbitrary constants. The equations of motion for the energy nodes will then assume the form:

$$\frac{\partial S}{\partial \alpha_1} = \omega_1 t + \beta_1 ,$$

$$\frac{\partial S}{\partial \alpha_2} = \omega_2 t + \beta_2 ,$$

$$\frac{\partial S}{\partial \alpha_3} = \omega_3 t + \beta_3 .$$

However, had one chosen the integration constants α_i from the outset in such a way that one of them is identical to the energy parameter, say:

$$\alpha_1 = W$$
,

then one will have:

$$\omega_1 - \frac{\partial W}{\partial \alpha_1} = 1$$
, $\omega_2 = \frac{\partial W}{\partial \alpha_2} = 0$, $\omega_3 = \frac{\partial W}{\partial \alpha_3} = 0$,

and the equations of motion can be written more simply:

$$\frac{\partial S}{\partial \alpha_1} = \beta_1 + t ,$$
$$\frac{\partial S}{\partial \alpha_2} = \beta_2 ,$$
$$\frac{\partial S}{\partial \alpha_3} = \beta_3 .$$

The equations obtained represent the **Hamilton-Jacobi** integration method for the dynamics of mass-points precisely when the **Hamiltonian** function does not include time explicitly. That result allows one to recognize the basic meaning that the concepts that were developed for phase waves would possess in the context of mass-particles. An idea that **E. Schrödinger** (¹) expressed about that seems to have been followed through analytically.

All of the wave equations of dynamics that were obtained (to the extent that they did not prove to be identities to begin with) have repeatedly produced the fundamental equations of dynamics in one form or another, such that all of those wave equations must be regarded as only different representations of one and the same dynamic of wave propagation. As **A. Sommerfeld** and **E. Schrödinger** (²) have pointed out, **Felix Klein** had tried in vain to bring to bear such quasi-optical considerations in mechanics again in connection with **Hamilton**. One believes that one can obviously do without such notions.

§ 3. – Phase waves and quantum theory.

Carrier waves of the form:

$$\psi = a \sin \frac{2\pi}{h} (S - W t - \Theta),$$

which are the so-called phase waves, account for the laws of advanced dynamics in a downright amazing way, as was shown in the foregoing. A narrow spectral region of such phase waves will create energy nodes by their interference that represent the mass-particle. **L. de Broglie** has rediscovered those phase waves by way of the important role that they play in quantum theory.

The spreading of phase wave surfaces is given by the equation:

$$S - W t = \text{const.}$$

^{(&}lt;sup>1</sup>) *Loc. cit.*, pp. 501.

^{(&}lt;sup>2</sup>) Loc. cit., pp. 490.

Upon differentiating that, one will get the differential equation:

grad
$$S d \mathfrak{r} - W dt = 0$$
,

and it further follows from (8) that:

$$\mathfrak{G} d \mathfrak{r} = W dt$$
.

The arc-length element for the orthogonal trajectories of the wave surfaces, viz., the *rays*, will be denoted by *ds*. One has:

$$\mathfrak{G} d \mathfrak{r} = | \mathfrak{G} | ds ,$$

and the differential equation of the spreading of the phase-wave surfaces can be further written:

$$|\mathfrak{G}| ds = W dt$$
.

 $\frac{ds}{dt} = u$

Now:

by specializing it to:

is the speed of propagation of the phase waves, or more briefly, the phase velocity. One finds from the differential equation above that:

$$u = \frac{W}{\mathfrak{G}} \tag{9}$$

for them. We likewise juxtapose the formula for the magnitude v of the velocity of the massparticle. One obtains it from **Hamilton**'s canonical equation:

$$\dot{q}_i = \frac{\partial H}{\partial p_i}$$
$$w = \frac{\partial W}{\partial |\mathcal{G}|}$$

for a coordinate that lies along the particle path. A comparison of that formula with (9) will allow one to see that one must generally expect that:

 $v \neq u$.

One will see the connection between u and v most simply when one substitutes the values of W and \mathfrak{G} in equation (9). One will then get the formula:

$$u = \frac{mc^2 + V}{mv},\tag{9'}$$

in which the dependency of the mass m on the velocity v of the particle must generally be considered. That connection will become especially simple in the absence of an external force field:

V=0.

since equation (9') will reduce to:

$$u = \frac{1}{v}$$

 $u = \frac{c^2}{c^2}$

for them. Now, a particle can only move with subluminal velocity, so one will always have:

v < c.

With that, the equation will necessarily imply that:

 $u>c\;,$

i.e., the phase waves move with superluminal velocity, but that will not mean that the principle of special relativity has been contradicted, since the energy transport of those waves does not result with the phase velocity u, but merely with the group velocity v. The equation can be written more symmetrically:

$$\frac{u}{c}\frac{v}{c} = 1.$$

One sees from this that the phase velocity exceeds the vacuum speed of light by the same ratio that the particle velocity falls short of it. For small particle velocities, that will then yield an enormous phase velocity. In general, for the cases that are given by formula (9), the ratios are not very different, since the one has:

$$V \ll mc^2$$

for the useful fields.

It further follows from equation (8) that the particle trajectories are identical to the paths of the rays of the phase waves. As a result, one can convert the action integral by way of:

$$\int_{A}^{A'} \mathfrak{G} d\mathfrak{r} = \int_{A}^{A'} |\mathfrak{G}| ds$$

for the extreme value. It further follows from (9) that:

$$|\mathfrak{G}| = \frac{W}{u},$$

such that one will ultimately get:

$$\int_{A}^{A'} \mathfrak{G} d\mathfrak{r} = W \int_{A}^{A'} \frac{ds}{u}$$

for the extremal value of the action integral, i.e., it will be identical to the extreme value of the integral in **Fermat**'s principle for the phase waves, up to the constant factor *W*. Thus, the principle of least action for the mass-particle and **Fermat**'s principle for the associated phase waves implies not only the same extremals as a result of the identity of trajectories and rays, but the extremal values of the integrals that appear will differ by only a constant factor. One can account for that when one identifies not only the extremal values in the two principles with each other, but the integrals themselves, up to a constant factor that can be identified with the energy by merely dimensional considerations. **L. de Broglie** deduced the properties of phase waves by four-dimensional considerations of a similar sort.

We shall now write the equation of the phase waves (i.e., the carrier waves) in the form:

$$\psi = a \sin 2\pi \left[\frac{S}{h} - \frac{W}{h} t - \frac{\Theta}{h} \right].$$

The factor of *t* inside of the square brackets has the meaning of the frequency of the phase waves:

$$v = \frac{W}{h}$$
,
and that will imply the relation:
 $W = h v$ (10)

for the energy parameter. Thus, the energy will be proportional to the frequency of the phase waves, and the proportionality factor will be that quantity that was introduced as a constant with the same dimension as S, so the dimension of an action. The relation (10) will then have precisely the same form that exists between the energy of an emitted ray and its frequency according to the **Bohr** frequency condition. It corresponds to that conception of quantum theory by which **M**. **Planck** first explained the energy distribution of the blackbody radiation, according to which a ray will appear in only an energy quantum of magnitude h v. That formal agreement is closely related to the facts that this quantum hypothesis can also be extended to the phase waves and that the constant that is denoted by h is identified with **Planck**'s quantum of energy, a mass-particle will then be coupled with a frequency, as an energy quantum, namely, the frequency of its phase waves. That is the line of reasoning by which **L. de Broglie** inferred the frequency of the phase waves.

The formula for the magnitude of the impulse that one will get from (9) when one applies (10) is:

$$|\mathfrak{G}| = \frac{hv}{u}$$

Now, the expression:

$$\frac{\nu}{u} = \frac{1}{\lambda}$$

means nothing but the wave number, i.e., the number of phase wavelengths that lie along a unit of length. One will then obtain the simple formula:

$$|\mathfrak{G}| = h\frac{1}{\lambda},\tag{11}$$

which says that the magnitude of the impulse is proportional to the wave number of the phase waves, in which the proportionality factor is likewise the **Planck** quantum of action.

We shall now state the special relationship that exists when the **Hamilton-Jacobi** partial differential equation is integrable by the method of separation of variables. That case will come about when one obtains a solution of the form:

$$S = \sum S_k(q_k, \alpha_k)$$

by introducing suitable variables. One will then find the expressions for the impulse from that:

$$p_i = rac{\partial \sum S_k}{\partial q_i} = rac{dS_i}{dq_i} \; .$$

The equation of phase waves is then written:

$$\psi = a \sin \frac{2\pi}{h} \left(\sum S_k - W t - \Theta \right) \, .$$

The advance of the phase wave surfaces along one of the coordinate lines – say, the q_i -line with the equation:

$$q_k = \text{const.}$$
 for $k \neq i$,

will result from the formula:

$$S_i - W t = \text{const.}$$

Differentiating that and observing the above will give the differential equation for that:

$$p_i \, dq_i - W \, dt = 0 \; .$$

One infers the formula for the phase velocity along the q_i -line from that:

$$u_i = \frac{dq_i}{dt} = \frac{W}{p_i}$$

Analogously to the above, it will follow that the impulse components are:

$$p_i=\frac{W}{u_i},$$

and it further follows from (10) that:

$$p_i=\frac{h\,\nu}{u_i}\,=\,h\frac{1}{\lambda_i}\,,$$

in which the reciprocal value of λ_i means the number of locations with equal phase that fall along a unit length of the q_i -line can briefly be referred to as the wave number along the q_i -line. It should probably be observed that when the phase surfaces are pierced by the q_i -lines transversally, that quantity will naturally be always smaller than the previous quantity that was referred to as the wave number, by abuse of terminology.

Whenever the integration method of separation of variables leads to one's goal, one will always be dealing with so-called constrained periodic systems in which the motions relative to each of the separation variables are periodic in their own right. Now, the resulting motion will also be periodic under the condition that the periods of the individual variables are mutually commensurable.

According to **Epstein** and **Schwarzschild**, the separation variables will serve to exhibit the **Sommerfeld** quantum conditions:

$$\oint p_i dq_i = n h$$
, n is a whole number.

The introduction of the expression for p_i that was obtained above will give:

$$\oint \frac{dq_i}{\lambda_i} = n , \qquad n \text{ is a whole number} \qquad (12)$$

after dividing through by h. The loop integral means integration over one period of the integration variables. However, the condition for the stationary orbits of quantum theory in the form (12) admits an uncommonly-simple interpretation. The integral on the left represents nothing but the number of periods of phase waves that fall along a closed q_i -loop. The condition then says that this can only be a whole number. It has the same form as the interference condition in **Fraunhofer** theory of the diffraction grating, or a resonance condition for the appearance of standing waves. In fact, the continuous train of phase waves will not be annihilated by interference only when the condition (12) is fulfilled.

Along q_i -lines that are not closed, what will generally result is merely a pendulum motion in the phase between fixed limits, and the loop will consist of merely a motion back and forth along a bounded segment. The phases waves that run back and forth must then create standing waves

with nodes and antinodes along such q_i -lines. By analogy with open and covered pipes, one must expect that in some situations, odd multiples of 1/2 can also appear in place of whole numbers in the quantum conditions. In the cases where the separation variables are not fixed uniquely, an even more complicated system of standing phase waves must appear.

Since, conversely, (12) will again imply the **Sommerfeld** quantum conditions, one can say that the conditions for the stationary orbits of quantum theory are nothing but the interference conditions for the phase waves in their entirely-familiar form. Thus, in the final analysis, the quantum mechanics of the **Bohr** atomic model is merely a natural consequence of the energy quantum hypothesis that is expressed in equation (10).

However, one must imagine that not only the motion of the electrons, but also that of gas molecules, is generated by phase waves. For the ideal gas, one has no attracting forces between the molecules that must be accounted for, such that one can infer the simple formula for the connection between the molecular velocity v and the phase velocity u:

$$u=\frac{c^2}{v}.$$

For:

$$v = 10^5 \text{ cm} / \text{sec},$$

which is the order of magnitude of the mean molecular velocities at normal temperatures, one can calculate the enormous value for the phase velocity:

$$u = 9 \cdot 10^{15} \text{ cm} / \text{ sec.}$$

For a mean value of the free path length of 10^{-5} cm, one can further infer that the mean time between two collisions of gas molecules is 10^{-10} sec. In that time, the phase wave will cover the immense distance:

$$9 \cdot 10^5 \, \text{cm} = 9 \, \text{km}$$
.

It will then be reflected back and forth many thousands of times between the walls of a gas tank whose linear dimensions have the order of magnitude of 1 m. The interior of the gas tank is suffused with the phase waves of the gas molecules that it includes in all directions. According to **L. de Broglie**, despite the collisions of gas molecules, the interference of the phase waves in its own right must always define a stationary state of oscillation, as had also been assumed to be in effect in the case of blackbody radiation to such great effect. Only the standing waves that are possible in the volume of the gas tank can be kept stationary.

In the case of:

V=0,

the **Hamilton-Jacobi** partial differential equation for *S* will yield a linear function of the **Cartesian** coordinates as a solution, such that one can assume that the phase waves are plane waves. The possible standing waves in a blackbody can be deduced for a special boundary, namely, a rectangular parallelepiped by a method that goes back to **Rayleigh** and **Jeans**. A rectangular

system of axes is laid at one corner and let the edges that run through it be as in Fig. 3. For the reflection at the reflecting sides of the cavity, the assumption will be made that the oscillation changes sign there. A plane wave whose wave normal subtends the angles α_1 , α_2 , and α_3 with the *X*, *Y*, *Z*-axes, resp., will then give:

$$\psi =$$

$$a\sin 2\pi \left(vt + \frac{x\cos\alpha_1 + y\cos\alpha_2 + z\cos\alpha_3}{\lambda} + \vartheta\right) - a\sin 2\pi \left(vt + \frac{-x\cos\alpha_1 + y\cos\alpha_2 + z\cos\alpha_3}{\lambda} + \vartheta\right)$$
$$= 2a\cos 2\pi \left(vt + \frac{y\cos\alpha_2 + z\cos\alpha_3}{\lambda} + \vartheta\right)\sin 2\pi \frac{x\cos\alpha_1}{\lambda}$$

by interfering with the wave that is reflected by the x = 0 plane.



Nodes $\psi = 0$ will appear on the parallel planes:

$$\frac{2x\cos\alpha_1}{\lambda} = n \qquad \text{(whole number).}$$

In particular, the plane of reflection is a nodal plane since:

$$x = 0$$
 for $n = 0$.

Since the opposite plane of the parallelepiped $x = l_1$ is likewise reflecting, it must also be a nodal plane, such that a condition equation of the form:

$$\frac{2l_1\cos\alpha_1}{\lambda}=n_1$$

will result, in which n_1 can be any arbitrary non-zero whole number. If the oscillation does not change sign under reflection of the wave then the same condition equation will exist, so only antinodes of oscillation will appear at the boundary surfaces in place of the nodes (¹). The condition equation that was just derived must be combined with the ones that are derived for the other two axis directions. We write out all of them in the form:

$$\frac{n_1}{l_1} = \frac{2}{\lambda} \cos \alpha_1, \qquad \frac{n_2}{l_2} = \frac{2}{\lambda} \cos \alpha_2, \qquad \frac{n_3}{l_3} = \frac{2}{\lambda} \cos \alpha_3.$$

Under reflection at a boundary plane, merely the sign of one of the direction cosines will change, but its magnitude will always remain the same. The combined action of all waves with direction cosines of equal absolute values will create a standing wave when the three condition equations above are fulfilled. The various standing waves will then be determined uniquely by triples of values:

$$n_1, n_2, n_3$$

of non-zero positive whole numbers. One eliminates the direction cosines by squaring and adding the three equations and obtains the relation:

$$\left(\frac{n_1}{l_1}\right)^2 + \left(\frac{n_2}{l_2}\right)^2 + \left(\frac{n_3}{l_3}\right)^2 = \left(\frac{2}{\lambda}\right)^2$$

for calculating the wave number:

of a standing wave.

The equations obtained can be given an intuitive geometric interpretation. One imagines constructing all points with the coordinates:

 $\frac{1}{\lambda}$

$$x = n_1 \frac{1}{l_1}, \qquad y = n_2 \frac{1}{l_2}, \qquad z = n_3 \frac{1}{l_3}$$

in an orthogonal coordinate system. One will then get a spatial lattice that takes the form of rectangular parallelepipeds. A net of planes that are normal to the z-axis is illustrated in Fig. 4. The columns of dots are spaced at a distance of $1 / l_1$ apart parallel to the y-axis, while the rows of points are spaced at a distance of $1 / l_2$ apart parallel to the x-axis. C represents any one of the points on the positive z-axis that is spaced at a distance of $1 / l_3$ apart from the coordinate origin

^{(&}lt;sup>1</sup>) A geometric derivation of that condition is found in the author's article in Phys. Zeit. **19** (1918), pp. 122.

and at each of which a plane of the net cuts through. The lattice points fill up merely the first octant of the axis-cross, and the bounding coordinate planes themselves are not laid through lattice points.



The positions of the lattice points determine a triple of non-zero positive whole numbers n_1 , n_2 , n_3 , and are uniquely associated uniquely with the eigenfrequencies of the cavity in question. The distance from any lattice point to the coordinate origin:

$$r = \sqrt{x^2 + y^2 + z^2} ,$$

as a result of the relation:

$$r=\frac{2}{\lambda},$$

will determine twice the wave number of the associated eigenfrequency. The ray that is drawn from the coordinate origin to the lattice point has direction cosines that are just the further determining data:

$$|\cos \alpha_1|$$
, $|\cos \alpha_2|$, $|\cos \alpha_3|$

for the associated eigenfrequencies.

A ball of radius:

$$r=rac{2}{\lambda}$$

about the coordinate origin includes all lattice points that represent eigenfrequencies with wave numbers $\leq 1 / \lambda$. It fills up a spherical octant of spatial volume:

$$J = \frac{1}{8} \frac{4\pi}{3} r^{3} = \frac{4\pi}{3} \left(\frac{1}{\lambda}\right)^{3}.$$

Each of the elementary rectangular parallelepipeds of the lattice has a spatial volume of:

$$i = \frac{1}{l_1} \frac{1}{l_2} \frac{1}{l_3}$$
.

The number of elementary parallelepipeds in the spherical octant above is calculated from an asymptotic formula that is valid for large r to be simply:

$$Z = \frac{J}{i} = l_1 l_2 l_3 \frac{4\pi}{3} \left(\frac{1}{\lambda}\right)^3.$$

That is also the number of lattice points that it includes since they appear in such spatial lattices in the same number as the elementary parallelepipeds. Namely, if one fixes the elementary parallelepipeds and displaces the lattice point:

$$n_1 = n_2 = n_3 = 1$$

a little bit towards the coordinate origin and all remaining lattice points parallel to it by the same path in the same direction then each of the elementary parallelepipeds will include one and only one lattice point. In particular, *Z* likewise represents the number of all those eigenfrequencies that are represented by the lattice points that are included, so the number of all eigenfrequencies with a wave number < $1 / \lambda$. By differentiation, one further concludes that:

$$dZ = l_1 l_2 l_3 4\pi \left(\frac{1}{\lambda}\right)^2 d\left(\frac{1}{\lambda}\right)$$

is the number of all eigenfrequencies with a wave number between the limits of:

$$\frac{1}{\lambda}$$
 and $\frac{1}{\lambda} + d\left(\frac{1}{\lambda}\right)$

What is remarkable about these asymptotic formulas is that the number of eigenfrequencies is simply proportional to only the volume of the cavity, while the special form of the rectangular parallelepipeds has no effect. More general considerations will also prove the validity of that law for an arbitrary form of the boundary (¹).

In particular, from (11), one must set:

^{(&}lt;sup>1</sup>) **H. Weyl**, Math. Ann. **71** (1912), pp. 411.

$$\frac{1}{\lambda} = \frac{\mathfrak{G}}{h}$$

for the phase waves, and in that way further get:

$$dZ = \frac{l_1 l_2 l_3 4\pi \mathfrak{G}^2 d |\mathfrak{G}|}{h^3}$$

The numerator of that fraction represents nothing but the phase volume:

$$d\Phi = l_1 l_2 l_3 4\pi \mathfrak{G}^2 d |\mathfrak{G}|$$

of the molecules in the volume $l_1 l_2 l_3$ with velocities in between the impulse limits:

$$|\mathfrak{G}|$$
 and $|\mathfrak{G}| + d |\mathfrak{G}|$.

The formula that is obtained says that only a number:

$$dZ = \frac{d\Phi}{h^3}$$

of possible molecular states can exist in that phase region. That formula was exhibited by **M**. **Planck** in a purely-formal way as the quantum prescription for the gas molecules and was employed successfully for the calculation of the chemical constants of gases. However, the picture of quantized molecular states seemed so adventuresome to many that one mostly preferred to calculate the chemical constants of gases by getting around the quantization of gas molecules. The discrete molecular states were obtained here along the route that **L. de Broglie** followed as a natural consequence of the fact that only a discrete number of stationary phase waves that have a fixed coupling to the molecular states would be possible in a cavity. Indeed, one can also appeal to formula (12) directly in order to treat that problem. The path that was taken was chosen only for the sake of greater intuitive appeal and in order to link up with the familiar pictures. In the final analysis, the quantization of gas molecules is therefore based upon nothing but the fundamental quantum formula (10).

The phase waves then allow one to treat the ideal gas statistically in the same way as blackbody radiation. Conversely, **Bose** (¹) treated blackbody radiation like an ideal gas with a statistical law that he discovered especially for that purpose, which would naturally have to be possible according to the intuitions that were developed. **A. Einstein** (²) further built up those ideas into a theory of degenerate gases and built the first bridges to the school of **L. de Broglie**. A lengthy series of

^{(&}lt;sup>1</sup>) "Plancks Gesetz und die Lichtquantenhypothese," translated and annotated by **A. Einstein**, Zeit. Phys. **26** (1924), pp. 178.

^{(&}lt;sup>2</sup>) "Quantentheorie des einatomigen idealen Gases," Berl. Ber. (1924), pp. 261; second treatise, *ibid*. (1925), pp. 3.

discussions followed from that (¹), and ultimately **E. Schrödinger** (²) adapted the problem completely to the language of phase waves and put it into a closed form.

§ 4. – Further development of wave mechanics.

The possibility of explaining the **Sommerfeld** quantum conditions and the quantization of ideal gases by the interference of phase waves that was indicated in the previous section also points to an improvement of that method. That is because just as in the realm of optics not all of the subtleties of interference and diffraction phenomena are implied by the **Fresnel** and **Fraunhofer** theories, but only when one integrates the differential equations of the electromagnetic theory of light, one must expect something similar to be true in a rigorous wave mechanics.

The fundamental wave equation of mechanics must be arranged such that they possess the phase waves as integrals. If one further assumes that they are closely analogous to optics by being linear and homogeneous of second order, which represents the simplest case among the possibilities that present themselves, then the solutions must admit the form:

$$\psi = f(S - Wt - \Theta),$$

in which f means an arbitrary function of the argument.

The wave surface function *S* is also not entirely arbitrary, except for the fact that it must fulfill the **Hamilton-Jacobi** partial differential equation. Namely, when one applies the div operation to both sides of the fundamental dynamical equation:

$$\frac{d\mathfrak{G}}{dt} = -\operatorname{grad} V$$

one will get:

$$\frac{d}{dt}(\operatorname{div}\mathfrak{G}) = 0$$

from known rules of calculation, and upon integration:

div
$$\mathfrak{G} = \text{const.}$$

along the paths, i.e., along the orthogonal trajectories of the family of surfaces:

S = const.

^{(&}lt;sup>1</sup>) **A. Einstein**, "Zur Quantentheorie des idealen Gases," Berl. Ber. (1925), pp. 18; **M. Planck**, "Zur Frage der Quantelung einatomiger Gases," Berl. Ber. (1925), pp. 49; **E. Schrödinger**, "Bemerkungen über die statistische Entropiedefinition beim idealen Gas," Berl. Ber. (1925), pp. 434; **M. Planck**, "Über die statistische Entropiedefinition," Berl. Ber. (1925), pp. 442; **E. Schrödinger**, "Die Entropiestufen des idealen einatomigen Gasmodelles," Berl. Ber. (1926), pp. 23.

^{(&}lt;sup>2</sup>) "Zur Einsteinschen Gastheorie," Phys. Zeit. **27** (1926), pp. 85.

As a result of (8), one can also say:

div grad
$$S = \text{const.}$$

Above and beyond that, one shall make the natural assumption that:

div grad S = 0.

Under those conditions, one will find that:

div grad
$$\psi = (\text{grad } S)^2 \cdot f''$$
,

which one can also write, as a result of (8):

div grad
$$\psi = \mathfrak{G}^2 \cdot f''$$

and one will have, with no further analysis:

$$\frac{\partial^2 \psi}{\partial t^2} = W^2 \cdot f''.$$

Eliminating f'' will produce the wave equation:

div grad
$$\psi - \frac{\mathfrak{G}^2}{W^2} \frac{\partial^2 \psi}{\partial t^2} = 0$$
, (13)

and from (9), it will take the usual form:

div grad
$$\psi - \frac{1}{u^2} \frac{\partial^2 \psi}{\partial t^2} = 0$$
.

Now, one must specialize the arbitrary function f in such a way that the wave function ψ will appear in the form that was employed:

$$\psi = a \sin \frac{2\pi}{h} (S - W t - \Theta) \; .$$

That will be possible only when one makes the assumption that a is independent of position, moreover. One concludes that:

$$\frac{\partial^2 \psi}{\partial t^2} = -\frac{4\pi^2}{h^2} W^2 \psi,$$

and the wave equation (13) will be converted into:

div grad
$$\psi + \frac{4\pi^2}{h^2} \mathfrak{G}^2 \psi = 0$$
. (14)

From (11), that is precisely the form that is familiar in optics:

div grad
$$\psi + \frac{4\pi^2}{\lambda^2}\psi = 0$$
.

In order to calculate the coefficients of equation (14) as functions of position, one appeals to the law of energy:

$$m_0 c^2 \sqrt{1 + \frac{\mathfrak{G}^2}{m_0^2 c^2} + V} = W.$$

$$\mathfrak{G}^2 = m_0^2 c^2 \left[\left(\frac{W - V}{m_0 c^2} \right)^2 - 1 \right].$$

- - -

If one calculates in the Newtonian approximation then the law of energy will read:

$$\frac{\mathfrak{G}^2}{2m} + V = E,$$

$$\mathfrak{G}^2 = 2m \left(E - V\right). \tag{15'}$$

C C

One finds from this that:

and one will find that:

One can also get that formula directly from (15) by making the substitution:

$$W=m_0\,c^2+E\,,$$

which is the expression in the first approximation.

E. Schrödinger actually used the wave equation that one obtains by substituting one of the expressions (15) and (15') into (14) as the fundamental equation of mechanics and referred to the dynamics thus founded as "undulatory mechanics." In particular, for the case of the hydrogen atom, one has the potential energy:

$$V=-\frac{e}{r^2},$$

and on the basis of **Newtonian** mechanics, one will get the following wave equation from (14) and (15'):

div grad
$$\psi + \frac{8\pi^2 m}{h^2} \left(E + \frac{e}{r^2} \right) \psi = 0$$
.

(15)

The integration of that equation will define the first of the problems of undulatory mechanics that **E. Schrödinger** communicated (¹). One seeks the finite, single-valued solutions of that wave equation. One can decompose ψ into the product of a spherical surface function and a function of only *r* that **E. Schrödinger** denoted by χ (*r*). The differential equation that is true for χ (*r*) will always give everywhere-finite solutions χ for positive *E* for which the **Kepler** orbits will extend to infinity. By contrast, in the case of a negative *E*, the **Kepler** orbits will go through finite points, so the differential equation will yield an everywhere-finite solution *c* for only discrete eigenvalues of the coefficients that are determined by:

$$-E = \frac{2\pi^2 m e^4}{h^2 l^2}, \qquad l \text{ is whole number.}$$

However, those are precisely the **Bohr** energy levels of the hydrogen atom, which gives the **Balmer** formula. The whole number l that appears in it is then the principal quantum number. The differential equation of the spherical surface functions also yields single-valued solutions for only the discrete eigenvalues of the coefficients that depend upon a whole number that is denoted by n. Now, n + 1 proves to be the analogue of the azimuthal quantum number here. The discussion of the solution will further show that the radial quantum number l - n - 1 determines the number of spherical modes of the wave system.

That first example already shows in the typical way how the undulatory mechanics arrives at quantum numbers. They are obtained in an entirely natural way by integrating the partial differential equation (14) as an oscillation and eigenvalue problem in mathematical physics (²). One finds the following simpler cases treated in the same way in **Schrödinger**'s second communication that was cited to begin with:

- 1. The **Planck** oscillator.
- 2. The rotator with a spatially-fixed axis.
- 3. The rigid rotator with a free axis.
- 4. The non-rigid rotator (two-atom molecule).

The next continuation treated the perturbation theory, with applications to the **Stark** effect of the **Balmer** lines $(^{3})$, in which the same method was built up further.

As one sees, for the time being, the only problems that were treated are the ones for which the older methods, with the **Sommerfeld** quantum conditions, proved to be successful. However, that result already justifies the statement that **L. de Broglie** (⁴) made before in regard to classical mechanics: "Our dynamics (including its **Einstein** form) falls behind optics. It is still in the arena of geometrical optics." The first step in the transition to his wave mechanics had already been taken by **Hamilton** almost one hundred years before, but in general one did not feel permitted to

^{(&}lt;sup>1</sup>) "Quantisierung als Eigenwertproblem (Erste Mitteilung)," Ann. Phys. (Leipzig) (4) **79** (1926), pp. 301.

^{(&}lt;sup>2</sup>) Courant and Hilbert, Methoden der math. Physik, volume one, Berlin, 1924, pp. 221, et seq.

^{(&}lt;sup>3</sup>) "Quantisierung als Eigenwertproblem (Dritte Mitteilung)," Ann. Phys. (Leipzig) (4) 80 (1926), pp. 437.

^{(&}lt;sup>4</sup>) *Loc. cit.*, pp. 61.

follow him. However, we now see that this sin of omission has been avenged since up to now, we have not been in a position to understand quantum phenomena. Based upon such considerations, **E. Schrödinger** aptly referred to classical mechanics as "geometrical mechanics," which was to be replaced with undulatory mechanics. Since the concepts of electron orbits and electron positions represent the analogues of light rays and light points in geometrical optics, they might belong to the domain of validity of geometrical mechanics. In the atomic interior, where the phase waves have the order of magnitude of the orbital dimensions, one can no longer actually speak of an electron orbit or an electron position, but only of the state of oscillation that is created by the phase waves. In that case, dynamics can also no longer be determined by the "ordinary differential equations" of classical mechanics for the position of the electron, but rather by a partial differential equation of the type of the wave equation (14). From that standpoint, geometrical mechanics was considered to be a limiting case of undulatory mechanics that would be attained, among other ways, asymptotically with increasing height of the energy level, which would make the energy seem to be increasingly concentrated around a certain moving location.

E. Schrödinger was of the opinion that equation (14) itself already determined the entire state of oscillation that appeared at all. Under that assumption, he calculated and plotted the state of oscillation for increasing energy level for the case of the linear **Planck** oscillator (¹). However, the energy concentration only takes place when one allows the collective action of a group of quantum states. **E. Schrödinger** was much inclined towards the picture of the atom that had been handed down, e.g., imagining several oscillations that are generally assumed to be stimulated simultaneously and the radiation would be the difference between the oscillations. However, for the time being, there exist hardly more than some scattered suggestions about those matters, and indeed in a new quantum theory, the first order of business would be to show that it leads to a correct calculation of the empirically-sharply-testable energy levels.

Another treatise by **E. Schrödinger** was entitled: "Über das Verhältnis der **Heisenberg-Born-Jordansche** Quantenmechanik zu der meinen" (²). That is the opinion of the "discontinuum theory" that was mentioned to begin with, which likewise places the problems of quantum mechanics on a new foundation. Despite the entirely different basic assumptions and methods, that new quantum theory still agrees with that of **E. Schrödinger** in terms of the present results, and indeed even when the two of them deviate from the older quantum theory, namely, in the appearance of odd multiples of 1/2 in place of the previous whole numbers in the case of the oscillator and the rotator. **E. Schrödinger** could point to the deep intrinsic connection between the two new quantum theories as the root of that agreement, from which he hoped to derive benefits for the further construction of undulatory mechanics.

What is most impressive about undulatory mechanics is the fact that it seems to be the natural development of a great physical legacy. Around a half century ago, electricity was brought under the same umbrella as optics by the advent of **Maxwell**'s theory. Wave mechanics also seemed to put dynamics on a common basis with optics. Should that not ultimately lead to mechanics, optics, and electricity flowing together harmoniously into an all-encompassing theory? That is because for a long time now even the theory of electricity no longer suffices in its present form, one has considered **Maxwell**'s theory to be the start of a line of development that should replace the older

^{(&}lt;sup>1</sup>) "Der stetige Übergang von der Mikro- zur Makro-mechanik," Naturwissenschaften **14** (1926), pp. 664.

^{(&}lt;sup>2</sup>) Ann. Phys. (Leipzig) (4) **79** (1926), pp. 734.

physics of matter with a new physics of fields. However, **H. A. Lorentz** has further developed the theory of electrons, in which the matter that electricity is composed of in the form of "electrons" is assigned a role that is just as fundamental as that of the electromagnetic field. It is only the phase waves that seem destined to pave the way to a true field theory of physics. When seen from that viewpoint, one would like to believe that wave mechanics can already be recognized to be a route to a promising and more-successful arena for the development of physics (¹).

It is still just as unclear as before how one might interpret the equation (10) for the energy quantum hypothesis. Moreover, one basically knows nothing at all about the physical meaning of the wave function ψ . Should it be treated as a scalar, or can ψ also be considered to have vector components? In the latter case, all types of polarization for the phase waves would be possible. Should one account for the electron rotation of **Goudsmit** and **Uhlenbeck** in that way? From those open questions, one sees that undulatory mechanics, in its present state, represents merely a beginning, but the results that have been achieved already make that beginning seem very auspicious (²).

Concluding remarks and summary.

The purpose of this work was to be methodically present the new development of mechanics that would have the goal of creating a unified statement of all dynamics, including quantum phenomena. For the sake of simplicity, the author has then confined himself to treating the onebody problem under the assumption that a potential that is independent of time exists, since the results of quantum theory, up to now, basically relate to that restriction.

The author appealed to **L. de Broglie**'s *Thèse* as the most important foundations of quantum theory, which was treated in § **3**, and the relevant work of **E. Schrödinger**, to which § **4** was mainly directed. Only **L. de Broglie**'s explanation of the quantum conditions by means of phase waves was represented and developed further here, which returned quantum theory to its oldest conception, namely, the energy quantum hypothesis. By contrast, **L. de Broglie**'s speculations about the light quantum hypothesis were not presented since they are not as definitive yet and tended to fall beyond the scope of the matters that were treated here. Concerning the works of **E. Schrödinger** that rigorously presented the theory of quantum phenomena, the author was content to merely refer to the main results since the original works are easily accessible to the German readers anyway. Therefore, their foundations and meanings were worked out all the more thoroughly.

^{(&}lt;sup>1</sup>) The same difficulties appeared in the general theory of relativity and were rectified in the same way. An attempt along those lines was made already in: **O. Klein**, "Quantentheorie und fünfdimensionale Relativitätstheorie," Zeit. Phys. **37** (1926), pp. 805.

⁽²⁾ Editor's remark: The author is grateful to **E. Schrödinger** for the friendly kindness of giving him a glimpse into the fourth communication of "Quantisierung als Eigenwertproblem" before its appearance in Annalen der Physik. The non-stationary quantum problem that is treated in it already appeals to the simple foundations that were communicated here. In regard to what was submitted, it should be remarked that, as of now, **E. Schrödinger** had arrived at entirely-concrete statements about the physical meaning of ψ that indeed have a somewhat-complicated nature.

In Whittaker's book Analytical Dynamics of Points and Rigid Bodies, one can read about how wave concepts have repeatedly had a decisive influence on the development of mechanics. **E.** Schrödinger had also already recognized the connection with the Hamilton-Jacobi integration method in the spirit of Hamilton. The author has devoted considerable effort to building that up further and implementing the wave methods for all of classical dynamics, as well, in a unified and consistent way, and that was treated in the first two sections. Since wave mechanics can be more than a mere working hypothesis only when one appeals to **Einstein**'s conception of the inertia of energy, the author has preferred to construct dynamics from the standpoint of the special principle of relativity from the outset. In my opinion, every modern presentation of mechanics must proceed in that way from now on.

Moreover, the author has taken special care to work through the older conception of quantum theory, which is characterized by mainly the names of **M. Planck**, **N. Bohr**, and **A. Sommerfeld**, as sharply as possible from the wave mechanics that an understanding of it will convey and to mention some possibilities for its further development. That is because, just as one still presently appeals to the optics of **Fresnel** and **Fraunhofer** whenever one strives for clarity and simplicity, so will the quantum rules of the older theory, which are supported by simple interference considerations, also continue to be used extensively in wave mechanics analogously. However, just as the manner by which the rigorous theory of diffraction alone might prove to be exact for the diffraction phenomena of optics, so might one also generally expect no more than the full truth in quantum mechanics to come from an undulatory mechanics.

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