"Über die invariante Form der Wellen- und der Bewegungsgleichungen für einen geladenen Massenpunkt," Zeit. Phys. **39** (1926), 226-232.

On the invariant form of the wave equations and the equations of motion for a charged mass-point (¹).

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(Received on 30 July 1926)

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Abstract. – **Schrödinger**'s wave equation will be written as an invariant **Laplace** equation, and the equations of motion will be written as those of a geodetic line in a five-dimensional space. The superfluous fifth coordinate parameter is closely related to the linear differential form of the electromagnetic potential.

In his still-unpublished paper, **H. Mandel** (²) appealed to the concept of fivedimensional space in order to consider gravitation and the electromagnetic field from a unified standpoint. The introduction of a fifth coordinate parameter seems to be wellsuited to the presentation of **Schrödinger**'s wave equation, as well as the mechanical equations, in invariant form.

1. Special theory of relativity.

The **Lagrange** function for the motion of a charged mass-point is, with a notation that is easy to understand:

$$L = -mc^{2}\sqrt{1 - \frac{\mathfrak{v}^{2}}{c^{2}}} + \frac{e}{c}\mathfrak{A}\cdot\mathfrak{v} - e\,\varphi\,,\tag{1}$$

and the corresponding Hamilton-Jacobi equation (H. P.) reads:

$$(\operatorname{grad} W)^{2} - \frac{1}{c^{2}} \left(\frac{\partial W}{\partial t}\right)^{2} - \frac{2e}{c} \left(\mathfrak{A} \cdot \operatorname{grad} W + \frac{\varphi}{c} \frac{\partial W}{\partial t}\right) + m^{2} c^{2} + \frac{e^{2}}{c^{2}} (A^{2} - \varphi^{2}) = 0.$$
(2)

^{(&}lt;sup>1</sup>) The idea for this study arose from a conversation with Prof. **V. Fréedericksz**, whom I must also thank for much worthwhile advice.

Added in proof: While this notice was going to print, the beautiful paper by Oskar Klein [Zeit. Phys. **37** (1926), 895] arrived in Leningrad, in which the author deduced some results that are basically identical to the ones in this notice. However, due to the importance of the results, it might be of interest to derive them in a different way (viz., a generalization of an Ansatz that I used in my earlier work).

 $[\]binom{2}{2}$ The author was kind enough to make it possible for me to read his manuscript.

In analogy with the Ansatz that was used in our previous paper (¹), here we shall set:

grad
$$W = \frac{\text{grad}\,\psi}{\frac{\partial\,\psi}{\partial p}}, \qquad \qquad \frac{\partial W}{\partial t} = \frac{\frac{\partial\,\psi}{\partial t}}{\frac{\partial\,\psi}{\partial p}}, \qquad (3)$$

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in which p denotes a new parameter that has the dimension of a quantum of action. After multiplying by $\left(\frac{\partial \psi}{\partial p}\right)^2$, we will get a quadratic form:

$$Q = \left(\operatorname{grad} \psi\right)^{2} - \frac{1}{c^{2}} \left(\frac{\partial \psi}{\partial t}\right)^{2} - \frac{2e}{c} \frac{\partial \psi}{\partial p} \left(A \cdot \operatorname{grad} \psi + \frac{\varphi}{c} \frac{\partial \psi}{\partial t}\right) + \left[m^{2}c^{2} + \frac{e^{2}}{c^{2}}(\mathfrak{A}^{2} - \varphi^{2})\right] \left(\frac{\partial \psi}{\partial p}\right)^{2}.$$
 (4)

We remark that the coefficients of the zeroth, first, and second powers of $\frac{\partial \psi}{\partial p}$ are fourdimensional invariants. Furthermore, the form Q remains invariant when one sets:

$$\begin{aligned} \mathfrak{A} &= \mathfrak{A}_{1} + \operatorname{grad} f, \\ \varphi &= \varphi_{1} - \frac{1}{c} \frac{\partial f}{\partial t}, \\ p &= p_{1} - \frac{e}{c} f, \end{aligned}$$
 (5)

in which f denotes an arbitrary function of the coordinates and time. The latter transformation will also leave the linear differential form:

$$d'\Omega = (\mathfrak{A}_x \, dx + \mathfrak{A}_y \, dy + \mathfrak{A}_z \, dz) - \frac{e}{mc} \varphi \, dt + \frac{1}{mc} dp \tag{6}$$

invariant (²).

We would now like to regard the form Q as the square of the gradient of the function ψ in five-dimensional space (R_5) and seek the corresponding line element. We easily finds that:

$$ds^{2} = dx^{2} + dy^{2} + dz^{2} - c^{2} dt^{2} + (d'\Omega)^{2}.$$
 (7)

The **Laplace** equation in R_5 reads:

^{(&}lt;sup>1</sup>) **V. Fock**, "Zur Schrödingerschen Wellenmechanik," Zeit. Phys. 38 (1926), 242.

^{(&}lt;sup>2</sup>) The symbol d'shall suggest that $d'\Omega$ is not a complete differential.

$$\Delta \psi - \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2} - \frac{2e}{c} \left(\mathfrak{A} \cdot \operatorname{grad} \frac{\partial \psi}{\partial p} + \frac{\varphi}{c} \frac{\partial^2 \psi}{\partial t \, \partial p} \right) - \frac{e}{c} \frac{\partial \psi}{\partial p} \left(\operatorname{div} \mathfrak{A} + \frac{1}{c} \frac{\partial \varphi}{\partial t} \right) + \left[m^2 c^2 + \frac{e^2}{c^2} (\mathfrak{A}^2 - \varphi^2) \right] \frac{\partial^2 \psi}{\partial p^2} = 0.$$
(8)

Just like (7) and (4), it remains invariant under a Lorentz transformation and the transformation (5).

Since the coefficients of equation (8) do not include the parameter p, we can make the dependency of the function ψ on p take the form of an exponential factor, and indeed, in order to get agreement with experiments, we must set (¹):

$$\Psi = \Psi_0 e^{2\pi i p/h} \,. \tag{9}$$

The equation for ψ_0 is invariant under Lorentz transformations, but not under the transformations (5). Namely, the meaning of the superfluous coordinate parameter p seems to lie precisely in the fact that it brings about the invariance of the equations relative to the addition of an arbitrary gradient to the four-potential.

Here, let it be remarked that the coefficients of the equation for ψ_0 are generally complex.

Furthermore, if one assumes that these coefficients do not depend upon t and sets:

$$\psi_0 = e^{-\frac{2\pi i}{h}(E+mc^2)t}\psi_1$$
(10)

then one will get an equation for ψ_1 that is free of time and is identical to the generalization of **Schrödinger**'s wave equation that we presented in our previous paper. Those values of *E* for which a function ψ_1 can exist, which satisfy certain finiteness and continuity requirements, are the **Bohr** energy levels then. It follows from the considerations that one makes in that way that the addition of a gradient to the four-potential can exert no influence on the energy levels. Namely, two functions ψ_1 and $\overline{\psi}_1$ that contain the vector potentials \mathfrak{A} and $\overline{\mathfrak{A}} = \mathfrak{A} - \operatorname{grad} f$, would differ by only a factor $e^{\frac{2\pi i e}{ch}f}$ of absolute magnitude 1, and as a result, they would have the same continuity properties (under very general assumptions about the function *f*).

2. General theory of relativity.

a. Wave equation. – For the line element in five-dimensional space, we assume that:

^{(&}lt;sup>1</sup>) The appearance of the parameter p that is linked with the linear form in the experimental function might perhaps be connected with some relationships that **E. Schrödinger** pointed out [Zeit. Phys. 12 (1923), 13].

$$ds^{2} = \sum_{i,k=1}^{5} \gamma_{ik} dx_{i} dx_{k}$$

$$= \sum_{i,k=1}^{5} g_{ik} dx_{i} dx_{k} + \frac{e^{2}}{m^{2}} \left(\sum_{i=1}^{5} q_{i} dx_{i} \right)^{2}.$$
(11)

Here, the quantities g_{ik} are the components of the **Einstein** fundamental tensor, and the quantities q_i (i = 1, 2, 3, 4) are the components of the four-potentials, divided by c^2 , so:

$$\sum_{i=1}^{4} q_i \, dx_i = \frac{1}{c^2} (\mathfrak{A}_x \, dx + \mathfrak{A}_y \, dy + \mathfrak{A}_z \, dz - \varphi \, c \, dt), \tag{12}$$

in which the quantity q_5 is a constant, and x_5 is the superfluous coordinate parameter. All coefficients are real and independent of x_5 .

The quantities g_{ik} and q_i depend upon only the fields, but not upon the composition of the mass-point. The latter will be represented by the factor e^2 / m^2 . However, to abbreviate, we would like to introduce the quantities:

$$\frac{e}{m}q_i = a_i \qquad (i = 1, 2, 3, 4, 5), \tag{13}$$

which depend upon e / m, and suggest the following abbreviation: The summation sign will be written out for the summation from 1 to 5, while it will be omitted for the summation from 1 to 4.

With those notations, we find that:

$$\gamma_{ik} = g_{ik} + a_i a_k$$
, $g_{i5} = 0$, (14)
 $(i, k = 1, 2, 3, 4, 5)$

$$\gamma = \parallel \gamma_{ik} \parallel = a_5^2 g , \qquad (15)$$

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$$\gamma^{ik} = g^{ik},$$

$$\gamma^{5k} = -\frac{1}{a_5} g^{ik} a_i = -\frac{a^i}{a_5},$$

$$\gamma^{55} = \frac{1}{(a_5)^2} \cdot (1 + a_i a^i)$$

$$(i, k, l = 1, 2, 3, 4).$$
(16)

The wave equation that corresponds to (8) reads:

$$\sum_{i,k=1}^{5} \frac{\partial}{\partial x_{i}} \left(\sqrt{-\gamma} \, \gamma^{ik} \, \frac{\partial \psi}{\partial x_{k}} \right) = 0, \tag{17}$$

or, when written out in detail:

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x_i}\left(\sqrt{-g} g^{ik}\frac{\partial\psi}{\partial x_k}\right) - \frac{2}{a_5}a^i\frac{\partial^2\psi}{\partial x_i\partial x_5} + \frac{1}{(a_5)^2}(1+a_ia^i)\frac{\partial^2\psi}{\partial x_5^2} = 0.$$
(18)

If one finally introduces the function ψ_0 and the potential q_i then that equation can be written:

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x_i}\left(\sqrt{-g} g^{ik}\frac{\partial\psi_0}{\partial x_k}\right) - \frac{4\pi}{h}\sqrt{-1} c e q^i \frac{\partial\psi}{\partial x_i} - \frac{4\pi^2 c^2}{h^2}(m^2 + e^2 q_i q^i)\psi_0 = 0.$$
(19)

b. Equations of motion. – We would now like to exhibit the equations of motion of a charged mass-point as those of a geodetic line in R_5 .

To that end, we must first calculate the **Christoffel** brackets. We denote the fivedimension bracket symbols $\begin{cases} k \ l \\ r \\ s \end{cases}$ and the four-dimensional ones by $\begin{cases} k \ l \\ r \\ s \end{bmatrix}_4$. We further introduce the covariant derivative of the four-potential:

$$A_{lk} = \frac{\partial a_l}{\partial x_k} - \begin{cases} k \ l \\ r \end{cases}_4 a_r$$
(20)

and split the tensor $2A_{ik}$ into its symmetric and antisymmetric parts:

$$B_{lk} = A_{lk} + A_{kl},$$

$$M_{lk} = A_{lk} - A_{kl} = \frac{\partial a_l}{\partial x_k} - \frac{\partial a_k}{\partial x_l}.$$

$$(21)$$

We will then have:

The equations of the geodetic line in R_5 then read:

$$\frac{d^2 x_r}{ds^2} + \begin{cases} k \ l \\ r \end{cases}_4 \frac{dx_k}{ds} \frac{dx_l}{ds} + \frac{d'\Omega}{ds} \cdot g^{ir} M_{il} \frac{dx_l}{ds} = 0,$$
(23)

$$\frac{d^2 x_5}{ds^2} + \frac{1}{2a_5} B_{lk} \frac{dx_k}{ds} \frac{dx_l}{ds} - \frac{1}{a_5} \frac{d'\Omega}{ds} a^i M_{il} \frac{dx_l}{ds} = 0.$$
(24)

Here, as before, $d'\Omega$ denotes the linear form:

$$d'\Omega = a_i \, dx^i + a_5 \, dx^5 \,. \tag{25}$$

If one multiplies the four equations (23) by a_r and the fifth one (24) by a_5 and adds them then one will get an equation that one can write in the form:

$$\frac{d}{ds} \left(\frac{d'\Omega}{ds} \right) = 0.$$
(26)

One then has:

$$\frac{d'\Omega}{ds} = \text{const.}$$
(27)

If one multiplies (23) by $g_{r\alpha} \frac{dx_{\alpha}}{ds}$ and sums over *r* and α then, due to the antisymmetry of M_{ik} , it will follow that:

$$\frac{d}{ds}\left(g_{r\alpha}\frac{dx_r}{ds}\frac{dx_{\alpha}}{ds}\right) = 0,$$
(28)

or, when one introduces the proper time τ through the formula:

$$g_{ik} dx_i dx_k = -c^2 d\tau^2, \qquad (29)$$

$$\frac{d}{ds} \left(\frac{d\tau}{ds}\right)^2 = 0.$$
(30)

Equation (28) or (30) will be a consequence of (26) due to the fact that:

$$ds^{2} = -c^{2} d\tau^{2} + (d'\Omega)^{2}.$$
 (31)

From what was said, it will follow that equation (24) is a consequence of (23); we can then omit it. If we introduce the proper time into (23) as an independent variable then the fifth parameter will drop out completely. We affix the numeral 4 to the bracket as a subscript:

$$\frac{d^2 x_r}{d\tau^2} + \begin{cases} k \ l \\ r \end{cases} \frac{dx_k}{d\tau} \frac{dx_l}{d\tau} + \frac{d'\Omega}{d\tau} g^{ir} M_{il} \frac{dx_l}{d\tau} = 0.$$
(32)

The last term on the left-hand side represents the **Lorentz** force. In the special theory of relativity, the first of these equations can be written:

$$m\frac{d}{dt}\frac{dx}{d\tau} + \frac{1}{c}\frac{d'\Omega}{d\tau} \left[\frac{e}{c} \left(\dot{z}H_y - \dot{y}H_z + \frac{\partial\mathfrak{A}_x}{\partial t} \right) + e\frac{\partial\varphi}{\partial x} \right] = 0.$$
(33)

In order to get agreement with experiments, the factor in the square brackets must have the value 1. One will then have:

$$\frac{d'\Omega}{d\tau} = c \tag{34}$$

and

$$ds^2 = 0. (35)$$

The trajectories of the mass-point will then be null geodetic lines in five-dimensional space.

In order to get the **Hamilton-Jacobi** equation, we set the square of the fivedimensional gradient of a function ψ equal to zero:

$$g^{ik}\frac{\partial\psi}{\partial x_i}\frac{\partial\psi}{\partial x_k} - \frac{2}{a_5}\frac{\partial\psi}{\partial x_5}a^i\frac{\partial\psi}{\partial x_i} + (1+a_ia^i)\left(\frac{1}{a_5}\frac{\partial\psi}{\partial x_5}\right)^2 = 0.$$
(36)

If we set:

$$mc a_{5} \frac{\frac{\partial \psi}{\partial x_{i}}}{\frac{\partial \psi}{\partial x_{5}}} = \frac{\partial W}{\partial x_{i}}$$
(37)

here and introduce the potentials q_i in place of a_i then we will get an equation:

$$g^{ik} \frac{\partial W}{\partial x_i} \frac{\partial W}{\partial x_k} - 2e c q^i \frac{\partial W}{\partial x_i} + c^2 (m^2 + e^2 q_i q^i) = 0$$
(38)

that will represent a generalization of our equation (2) that served as our starting point.

Leningrad, Physical Institute of the University, 24 July 1926.