# On the invariant form of the wave equations and the equations of motion for a charged mass-point $\left({ }^{1}\right)$. 

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#### Abstract

Schrödinger's wave equation will be written as an invariant Laplace equation, and the equations of motion will be written as those of a geodetic line in a five-dimensional space. The superfluous fifth coordinate parameter is closely related to the linear differential form of the electromagnetic potential.


In his still-unpublished paper, H. Mandel $\left({ }^{2}\right)$ appealed to the concept of fivedimensional space in order to consider gravitation and the electromagnetic field from a unified standpoint. The introduction of a fifth coordinate parameter seems to be wellsuited to the presentation of Schrödinger's wave equation, as well as the mechanical equations, in invariant form.

## 1. Special theory of relativity.

The Lagrange function for the motion of a charged mass-point is, with a notation that is easy to understand:

$$
\begin{equation*}
L=-m c^{2} \sqrt{1-\frac{\mathfrak{v}^{2}}{c^{2}}}+\frac{e}{c} \mathfrak{A} \cdot \mathfrak{v}-e \varphi \tag{1}
\end{equation*}
$$

and the corresponding Hamilton-Jacobi equation (H. P.) reads:

$$
\begin{equation*}
(\operatorname{grad} W)^{2}-\frac{1}{c^{2}}\left(\frac{\partial W}{\partial t}\right)^{2}-\frac{2 e}{c}\left(\mathfrak{A} \cdot \operatorname{grad} W+\frac{\varphi}{c} \frac{\partial W}{\partial t}\right)+m^{2} c^{2}+\frac{e^{2}}{c^{2}}\left(\mathrm{~A}^{2}-\varphi^{2}\right)=0 . \tag{2}
\end{equation*}
$$

[^0]In analogy with the Ansatz that was used in our previous paper $\left({ }^{1}\right)$, here we shall set:

$$
\begin{equation*}
\operatorname{grad} W=\frac{\operatorname{grad} \psi}{\frac{\partial \psi}{\partial p}}, \quad \frac{\partial W}{\partial t}=\frac{\frac{\partial \psi}{\partial t}}{\frac{\partial \psi}{\partial p}}, \tag{3}
\end{equation*}
$$

in which $p$ denotes a new parameter that has the dimension of a quantum of action. After multiplying by $\left(\frac{\partial \psi}{\partial p}\right)^{2}$, we will get a quadratic form:
$Q=$
$=(\operatorname{grad} \psi)^{2}-\frac{1}{c^{2}}\left(\frac{\partial \psi}{\partial t}\right)^{2}-\frac{2 e}{c} \frac{\partial \psi}{\partial p}\left(A \cdot \operatorname{grad} \psi+\frac{\varphi}{c} \frac{\partial \psi}{\partial t}\right)+\left[m^{2} c^{2}+\frac{e^{2}}{c^{2}}\left(\mathfrak{A}^{2}-\varphi^{2}\right)\right]\left(\frac{\partial \psi}{\partial p}\right)^{2}$.

We remark that the coefficients of the zeroth, first, and second powers of $\frac{\partial \psi}{\partial p}$ are fourdimensional invariants. Furthermore, the form $Q$ remains invariant when one sets:

$$
\left.\begin{array}{rl}
\mathfrak{A} & =\mathfrak{A}_{1}+\operatorname{grad} f, \\
\varphi & =\varphi_{1}-\frac{1}{c} \frac{\partial f}{\partial t},  \tag{5}\\
p & =p_{1}-\frac{e}{c} f,
\end{array}\right\}
$$

in which $f$ denotes an arbitrary function of the coordinates and time. The latter transformation will also leave the linear differential form:

$$
\begin{equation*}
d^{\prime} \Omega=\left(\mathfrak{A}_{x} d x+\mathfrak{A}_{y} d y+\mathfrak{A}_{z} d z\right)-\frac{e}{m c} \varphi d t+\frac{1}{m c} d p \tag{6}
\end{equation*}
$$

invariant $\left({ }^{2}\right)$.
We would now like to regard the form $Q$ as the square of the gradient of the function $\psi$ in five-dimensional space $\left(R_{5}\right)$ and seek the corresponding line element. We easily finds that:

$$
\begin{equation*}
d s^{2}=d x^{2}+d y^{2}+d z^{2}-c^{2} d t^{2}+\left(d^{\prime} \Omega\right)^{2} \tag{7}
\end{equation*}
$$

The Laplace equation in $R_{5}$ reads:

[^1]\[

$$
\begin{gather*}
\Delta \psi-\frac{1}{c^{2}} \frac{\partial^{2} \psi}{\partial t^{2}}-\frac{2 e}{c}\left(\mathfrak{A} \cdot \operatorname{grad} \frac{\partial \psi}{\partial p}+\frac{\varphi}{c} \frac{\partial^{2} \psi}{\partial t \partial p}\right)-\frac{e}{c} \frac{\partial \psi}{\partial p}\left(\operatorname{div} \mathfrak{A}+\frac{1}{c} \frac{\partial \varphi}{\partial t}\right) \\
+\left[m^{2} c^{2}+\frac{e^{2}}{c^{2}}\left(\mathfrak{A}^{2}-\varphi^{2}\right)\right] \frac{\partial^{2} \psi}{\partial p^{2}}=0 . \tag{8}
\end{gather*}
$$
\]

Just like (7) and (4), it remains invariant under a Lorentz transformation and the transformation (5).

Since the coefficients of equation (8) do not include the parameter $p$, we can make the dependency of the function $\psi$ on $p$ take the form of an exponential factor, and indeed, in order to get agreement with experiments, we must set $\left({ }^{1}\right)$ :

$$
\begin{equation*}
\psi=\psi_{0} e^{2 \pi i p / h} \tag{9}
\end{equation*}
$$

The equation for $\psi_{0}$ is invariant under Lorentz transformations, but not under the transformations (5). Namely, the meaning of the superfluous coordinate parameter $p$ seems to lie precisely in the fact that it brings about the invariance of the equations relative to the addition of an arbitrary gradient to the four-potential.

Here, let it be remarked that the coefficients of the equation for $\psi_{0}$ are generally complex.

Furthermore, if one assumes that these coefficients do not depend upon $t$ and sets:

$$
\begin{equation*}
\psi_{0}=e^{-\frac{2 \pi i}{h}\left(E+m c^{2}\right) t} \psi_{1} \tag{10}
\end{equation*}
$$

then one will get an equation for $\psi_{1}$ that is free of time and is identical to the generalization of Schrödinger's wave equation that we presented in our previous paper. Those values of $E$ for which a function $\psi_{1}$ can exist, which satisfy certain finiteness and continuity requirements, are the Bohr energy levels then. It follows from the considerations that one makes in that way that the addition of a gradient to the fourpotential can exert no influence on the energy levels. Namely, two functions $\psi_{1}$ and $\bar{\psi}_{1}$ that contain the vector potentials $\mathfrak{A}$ and $\overline{\mathfrak{A}}=\mathfrak{A}-\operatorname{grad} f$, would differ by only a factor $e^{-\frac{2 \pi i e}{c h} f}$ of absolute magnitude 1 , and as a result, they would have the same continuity properties (under very general assumptions about the function $f$ ).

## 2. General theory of relativity.

a. Wave equation. - For the line element in five-dimensional space, we assume that:

[^2]\[

\left.$$
\begin{array}{rl}
d s^{2} & =\sum_{i, k=1}^{5} \gamma_{i k} d x_{i} d x_{k}  \tag{11}\\
& =\sum_{i, k=1}^{5} g_{i k} d x_{i} d x_{k}+\frac{e^{2}}{m^{2}}\left(\sum_{i=1}^{5} q_{i} d x_{i}\right)^{2}
\end{array}
$$\right\}
\]

Here, the quantities $g_{i k}$ are the components of the Einstein fundamental tensor, and the quantities $q_{i}(i=1,2,3,4)$ are the components of the four-potentials, divided by $c^{2}$, so:

$$
\begin{equation*}
\sum_{i=1}^{4} q_{i} d x_{i}=\frac{1}{c^{2}}\left(\mathfrak{A}_{x} d x+\mathfrak{A}_{y} d y+\mathfrak{A}_{z} d z-\varphi c d t\right) \tag{12}
\end{equation*}
$$

in which the quantity $q_{5}$ is a constant, and $x_{5}$ is the superfluous coordinate parameter. All coefficients are real and independent of $x_{5}$.

The quantities $g_{i k}$ and $q_{i}$ depend upon only the fields, but not upon the composition of the mass-point. The latter will be represented by the factor $e^{2} / \mathrm{m}^{2}$. However, to abbreviate, we would like to introduce the quantities:

$$
\begin{equation*}
\frac{e}{m} q_{i}=a_{i} \quad(i=1,2,3,4,5) \tag{13}
\end{equation*}
$$

which depend upon $e / m$, and suggest the following abbreviation: The summation sign will be written out for the summation from 1 to 5 , while it will be omitted for the summation from 1 to 4 .

With those notations, we find that:

$$
\left.\begin{array}{l}
\gamma_{i k}=g_{i k}+a_{i} a_{k}, \quad g_{i 5}=0, \\
\gamma=\left\|\gamma_{i k}\right\|=a_{5}^{2} g, \\
\gamma^{i k}=g^{i k}, \\
\gamma^{5 k}=-\frac{1}{a_{5}} g^{i k} a_{i}=-\frac{a^{i}}{a_{5}}, \\
\gamma^{55}=\frac{1}{\left(a_{5}\right)^{2}} \cdot\left(1+a_{i} a^{i}\right) \tag{16}
\end{array}\right\} \quad(i, k=1,2,3,4,5), 1,2,3,
$$

The wave equation that corresponds to (8) reads:

$$
\begin{equation*}
\sum_{i, k=1}^{5} \frac{\partial}{\partial x_{i}}\left(\sqrt{-\gamma} \gamma^{i k} \frac{\partial \psi}{\partial x_{k}}\right)=0 \tag{17}
\end{equation*}
$$

or, when written out in detail:

$$
\begin{equation*}
\frac{1}{\sqrt{-g}} \frac{\partial}{\partial x_{i}}\left(\sqrt{-g} g^{i k} \frac{\partial \psi}{\partial x_{k}}\right)-\frac{2}{a_{5}} a^{i} \frac{\partial^{2} \psi}{\partial x_{i} \partial x_{5}}+\frac{1}{\left(a_{5}\right)^{2}}\left(1+a_{i} a^{i}\right) \frac{\partial^{2} \psi}{\partial x_{5}^{2}}=0 . \tag{18}
\end{equation*}
$$

If one finally introduces the function $\psi_{0}$ and the potential $q_{i}$ then that equation can be written:

$$
\begin{equation*}
\frac{1}{\sqrt{-g}} \frac{\partial}{\partial x_{i}}\left(\sqrt{-g} g^{i k} \frac{\partial \psi_{0}}{\partial x_{k}}\right)-\frac{4 \pi}{h} \sqrt{-1} c e q^{i} \frac{\partial \psi}{\partial x_{i}}-\frac{4 \pi^{2} c^{2}}{h^{2}}\left(m^{2}+e^{2} q_{i} q^{i}\right) \psi_{0}=0 \tag{19}
\end{equation*}
$$

b. Equations of motion. - We would now like to exhibit the equations of motion of a charged mass-point as those of a geodetic line in $R_{5}$.

To that end, we must first calculate the Christoffel brackets. We denote the fivedimension bracket symbols $\left\{\begin{array}{c}k l \\ r\end{array}\right\}_{5}$ and the four-dimensional ones by $\left\{\begin{array}{c}k l \\ r\end{array}\right\}_{4}$. We further introduce the covariant derivative of the four-potential:

$$
A_{l k}=\frac{\partial a_{l}}{\partial x_{k}}-\left\{\begin{array}{c}
k l  \tag{20}\\
r
\end{array}\right\}_{4} a_{r}
$$

and split the tensor $2 A_{i k}$ into its symmetric and antisymmetric parts:

$$
\left.\begin{array}{rl}
B_{l k} & =A_{l k}+A_{k l}  \tag{21}\\
M_{l k} & =A_{l k}-A_{k l}=\frac{\partial a_{l}}{\partial x_{k}}-\frac{\partial a_{k}}{\partial x_{l}}
\end{array}\right\}
$$

We will then have:

$$
\begin{align*}
& \left\{\begin{array}{c}
k l \\
r
\end{array}\right\}_{5}=\left\{\begin{array}{c}
k l \\
r
\end{array}\right\}_{4}+\frac{1}{2}\left(a_{k} g^{i r} M_{i l}+a_{l} g^{i r} M_{i k}\right), \\
& \left\{\begin{array}{c}
k l \\
5
\end{array}\right\}_{5}=\frac{1}{2 a_{5}} B_{l k}-\frac{1}{2 a_{5}}\left(a_{k} a^{i} M_{i l}+a_{l} a^{i} M_{i k}\right), \\
& \left\{\begin{array}{c}
k \\
5
\end{array}\right\}_{5}=-\frac{1}{2} a^{i} M_{i k},  \tag{22}\\
& \left\{\begin{array}{c}
5 \\
k
\end{array}\right\}_{5}=0, \\
& \left\{\begin{array}{c}
5 \\
5
\end{array}\right\}_{5}=0 .
\end{align*}
$$

The equations of the geodetic line in $R_{5}$ then read:

$$
\begin{align*}
& \frac{d^{2} x_{r}}{d s^{2}}+\left\{\begin{array}{c}
k l \\
r
\end{array}\right\}_{4} \frac{d x_{k}}{d s} \frac{d x_{l}}{d s}+\frac{d^{\prime} \Omega}{d s} \cdot g^{i r} M_{i l} \frac{d x_{l}}{d s}=0,  \tag{23}\\
& \frac{d^{2} x_{5}}{d s^{2}}+\frac{1}{2 a_{5}} B_{l k} \frac{d x_{k}}{d s} \frac{d x_{l}}{d s}-\frac{1}{a_{5}} \frac{d^{\prime} \Omega}{d s} a^{i} M_{i l} \frac{d x_{l}}{d s}=0 . \tag{24}
\end{align*}
$$

Here, as before, $d^{\prime} \Omega$ denotes the linear form:

$$
\begin{equation*}
d^{\prime} \Omega=a_{i} d x^{i}+a_{5} d x^{5} . \tag{25}
\end{equation*}
$$

If one multiplies the four equations (23) by $a_{r}$ and the fifth one (24) by $a_{5}$ and adds them then one will get an equation that one can write in the form:

$$
\begin{equation*}
\frac{d}{d s}\left(\frac{d^{\prime} \Omega}{d s}\right)=0 . \tag{26}
\end{equation*}
$$

One then has:

$$
\begin{equation*}
\frac{d^{\prime} \Omega}{d s}=\text { const. } \tag{27}
\end{equation*}
$$

If one multiplies (23) by $g_{r \alpha} \frac{d x_{\alpha}}{d s}$ and sums over $r$ and $\alpha$ then, due to the antisymmetry of $M_{i k}$, it will follow that:

$$
\begin{equation*}
\frac{d}{d s}\left(g_{r \alpha} \frac{d x_{r}}{d s} \frac{d x_{\alpha}}{d s}\right)=0 \tag{28}
\end{equation*}
$$

or, when one introduces the proper time $\tau$ through the formula:

$$
\begin{align*}
& g_{i k} d x_{i} d x_{k}=-c^{2} d \tau^{2}  \tag{29}\\
& \frac{d}{d s}\left(\frac{d \tau}{d s}\right)^{2}=0 \tag{30}
\end{align*}
$$

Equation (28) or (30) will be a consequence of (26) due to the fact that:

$$
\begin{equation*}
d s^{2}=-c^{2} d \tau^{2}+\left(d^{\prime} \Omega\right)^{2} \tag{31}
\end{equation*}
$$

From what was said, it will follow that equation (24) is a consequence of (23); we can then omit it. If we introduce the proper time into (23) as an independent variable then the fifth parameter will drop out completely. We affix the numeral 4 to the bracket as a subscript:

$$
\frac{d^{2} x_{r}}{d \tau^{2}}+\left\{\begin{array}{c}
k l  \tag{32}\\
r
\end{array}\right\} \frac{d x_{k}}{d \tau} \frac{d x_{l}}{d \tau}+\frac{d^{\prime} \Omega}{d \tau} g^{i r} M_{i l} \frac{d x_{l}}{d \tau}=0
$$

The last term on the left-hand side represents the Lorentz force. In the special theory of relativity, the first of these equations can be written:

$$
\begin{equation*}
m \frac{d}{d t} \frac{d x}{d \tau}+\frac{1}{c} \frac{d^{\prime} \Omega}{d \tau}\left[\frac{e}{c}\left(\dot{z} H_{y}-\dot{y} H_{z}+\frac{\partial \mathfrak{A}_{x}}{\partial t}\right)+e \frac{\partial \varphi}{\partial x}\right]=0 . \tag{33}
\end{equation*}
$$

In order to get agreement with experiments, the factor in the square brackets must have the value 1 . One will then have:

$$
\begin{equation*}
\frac{d^{\prime} \Omega}{d \tau}=c \tag{34}
\end{equation*}
$$

and

$$
\begin{equation*}
d s^{2}=0 \tag{35}
\end{equation*}
$$

The trajectories of the mass-point will then be null geodetic lines in five-dimensional space.

In order to get the Hamilton-Jacobi equation, we set the square of the fivedimensional gradient of a function $\psi$ equal to zero:

$$
\begin{equation*}
g^{i k} \frac{\partial \psi}{\partial x_{i}} \frac{\partial \psi}{\partial x_{k}}-\frac{2}{a_{5}} \frac{\partial \psi}{\partial x_{5}} a^{i} \frac{\partial \psi}{\partial x_{i}}+\left(1+a_{i} a^{i}\right)\left(\frac{1}{a_{5}} \frac{\partial \psi}{\partial x_{5}}\right)^{2}=0 . \tag{36}
\end{equation*}
$$

If we set:

$$
\begin{equation*}
m c a_{5} \frac{\frac{\partial \psi}{\partial x_{i}}}{\frac{\partial \psi}{\partial x_{5}}}=\frac{\partial W}{\partial x_{i}} \tag{37}
\end{equation*}
$$

here and introduce the potentials $q_{i}$ in place of $a_{i}$ then we will get an equation:

$$
\begin{equation*}
g^{i k} \frac{\partial W}{\partial x_{i}} \frac{\partial W}{\partial x_{k}}-2 e c q^{i} \frac{\partial W}{\partial x_{i}}+c^{2}\left(m^{2}+e^{2} q_{i} q^{i}\right)=0 \tag{38}
\end{equation*}
$$

that will represent a generalization of our equation (2) that served as our starting point.
Leningrad, Physical Institute of the University, 24 July 1926.


[^0]:    $\left({ }^{1}\right)$ The idea for this study arose from a conversation with Prof. V. Fréedericksz, whom I must also thank for much worthwhile advice.

    Added in proof: While this notice was going to print, the beautiful paper by Oskar Klein [Zeit. Phys. 37 (1926), 895] arrived in Leningrad, in which the author deduced some results that are basically identical to the ones in this notice. However, due to the importance of the results, it might be of interest to derive them in a different way (viz., a generalization of an Ansatz that I used in my earlier work).
    $\left(^{2}\right)$ The author was kind enough to make it possible for me to read his manuscript.

[^1]:    ${ }^{(1)}$ V. Fock, "Zur Schrödingerschen Wellenmechanik," Zeit. Phys. 38 (1926), 242.
    $\left(^{2}\right)$ The symbol $d$ 'shall suggest that $d^{\prime} \Omega$ is not a complete differential.

[^2]:    $\left({ }^{1}\right)$ The appearance of the parameter $p$ that is linked with the linear form in the experimental function might perhaps be connected with some relationships that E. Schrödinger pointed out [Zeit. Phys. 12 (1923), 13].

