"Über den Begriff der Geschwindigkeit in der Diracschen Theorie des Elektrons," Zeit. f. Phys, **55** (1929), 127-140.

On the concept of velocity in the Dirac theory of the electron

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One and the same classical mechanical quantity – the velocity of the electron – corresponds to two different quantum-mechanical quantities in the Dirac theory, which one can refer to as the corpuscular and wave velocities of the electron.

1. According to Dirac [†], the wave equation for the electron reads:

$$(P_0 + \alpha_1 P_1 + \alpha_2 P_2 + \alpha_3 P_3 + \alpha_4 mc)\ \psi = 0, \tag{1}$$

in which the abbreviations have been introduced:

$$\left.\begin{aligned} &P_k = p_k + \frac{e}{c} A_k = \frac{h}{2\pi i}\frac{\partial}{\partial x_k} + \frac{e}{c} A_k \qquad (k = 0,1,2,3), \\ &x_0 = ct, \quad x_1 = x, \quad x_2 = y, \quad x_3 = z. \end{aligned}\right\} \tag{1*}$$

The charge of the electron is denoted by $-e$ here; A_1, A_2, A_3 are the components of the vector potential, and A_0 is the scalar potential.

Equation (1) can also be written in the form:

$$H\psi + \frac{h}{2\pi i}\frac{\partial \psi}{\partial t} = 0, \tag{2}$$

where H denotes the Hamiltonian operator:

$$H = e\,A_0 + c\,(\alpha_1 P_1 + \alpha_2 P_2 + \alpha_3 P_3 + \alpha_4 mc). \tag{3}$$

We would like to assume that the four-rowed matrices α_1, α_2, α_3, α_4 are Hermitian; the operator H is then self-adjoint.

The so-called equations of motion for an operator F read:

[†] P. A. M. Dirac, "The Quantum Theory of the Electron," Proc. Roy. Soc. London (A) **117**, 610, 1928; **118**, 351, 1928.

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \frac{2\pi i}{h}(HF - FH). \tag{4}$$

This equation expresses the following mathematical fact [*]: If one constructs the matrix $||F_{mn}||$ for an operator F with the aid of a complete system of solutions ψ_n then the derivative with respect to time of the matrix element:

$$F_{mn} = \int \bar{\psi}_m F \psi_n \, dx\, dy\, dz \tag{5}$$

is equal to the corresponding matrix element for the operator on the right-hand side of (4), i.e.:

$$\frac{dF_{mn}}{dt} = \left\{ \frac{\partial F_{mn}}{\partial t} + \frac{2\pi i}{h}(HF - FH) \right\}_{mn}. \tag{6}$$

2. One can now ask: Why is it that time plays a privileged role in our formulas, while the coordinates and time appear in a formally symmetric way in relativity theory? However, this concern may be easily eliminated.

The symmetry in regard to coordinates and time that arises in relativity theory is purely formal and is valid only insofar as one does not need to distinguish between real and imaginary quantities. However, in reality, time (more generally: the temporal variable) is distinguished by its sign from the spatial variables in the fundamental metric form, and plays an entirely special role.

We would like to discuss this state of affairs more closely by considering the eigenvalue problem for an operator F.

In formulas (2) to (6), we have chosen time to be the independent variable, and it correspondingly enters as only a parameter in all operators and their eigenfunctions. The basic domain for the operators is then all of infinite space, i.e., a certain domain of variation for the spatial variables; one must integrate over the basic domain in formula (5), for example. The eigenfunction must satisfy certain boundary conditions on the boundary of the basic domain. On the other hand, if one chooses – say – the coordinate x to be the independent variable then all of the operators are expressed in terms of y, z, t, in which x only enters as a parameter. A certain spacetime domain would then be regarded as the basic domain. Now, the boundary conditions for a spacetime domain have an entirely different character from those of a spatial domain; for instance, they might not be given over the entire boundary. One can no longer speak of an eigenvalue problem in this case. The difference corresponds to the difference between elliptic and hyperbolic differential equations.

We then see that the time plays a special role in relativity theory, as well as in relativistic quantum theory, that is essentially different from that of the spatial variables.

[*] On this, cf. V. Fock, "Über die Beziehung zwischen den Integralen der quantenmechanischen Bewegunsgleichungen und der Schrödingerschen Wellengleichung," ZS. f. Phys. **49**, 323, 1928. The developments in the cited paper are applicable to the Dirac wave equation with no further assumptions, as long as the operator H is self-adjoint.

Here, we would like to interject a brief remark on the classical concept of proper time. Classically, the proper time is defined as the arc length of the world-line of a mass point. However, the concept of the path of a mass point has no immediate sense in quantum theory, and a definition of proper time will thus be impossible [*].

3. We now turn to the equations of motion. Using formula (4), we compute the operator that corresponds to the derivative of the coordinate x_k with respect to time – i.e., the velocity dx_k / dt – and obtain:

$$\frac{dx_k}{dt} = c\alpha_k \qquad (k = 1, 2, 3). \tag{7}$$

It is well-known that according to the Dirac theory, the current density vector has the components:

$$j_k = ec\bar{\psi}\alpha_k\psi . \tag{8}$$

This formula is in complete accord with the interpretation of the operator $c\alpha_k$ as the representative of the three-dimensional (but not four-dimensional!) velocity. On the other hand, this interpretation is linked with an essential difficulty: The eigenvalues of this operator are $\pm c$. One thus arrives at the paradoxical conclusion that the measurement of any component of the velocity of an electron can yield only the values $\pm c$. Breit [**] sought to make this result physically understandable, which he had obtained in a different way [***]; however, his argument does not seem very convincing to the author.

4. We now consider the four-dimensional velocity, which classically has the components:

$$v_1 = \frac{\dot{x}}{\sqrt{1-\frac{v^2}{c^2}}},\ v_2 = \frac{\dot{y}}{\sqrt{1-\frac{v^2}{c^2}}},\ v_3 = \frac{\dot{z}}{\sqrt{1-\frac{v^2}{c^2}}},\ v_0 = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}. \tag{9}$$

In relativistic classical mechanics these quantities can be expressed by the components of the impulse p_k and the four-potential A_k:

$$v_k = \frac{1}{m}\left(p_k + \frac{e}{c}A_k \right) \qquad (k = 1, 2, 3), \tag{10}$$

$$v_0 = \frac{1}{m}\left(p_0 + \frac{e}{c}A_0 \right). \tag{11}$$

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[*] The assertion of Eddington ["The Charge of an Electron," Proc. Roy. Soc. London (A) **122**, 358, 1929] that one takes the derivative with respect to proper time in his formula (11) must obviously be based upon an error.

[**] G. Breit, "An Interpretation of Dirac's Theory of the Electron," Proc. Nat. Acad. Amer. **14**, 553, 1928.

[***] Breit interpreted α_4 as $\sqrt{1-v^2/c^2}$, which can scarcely be justified; on this, cf., our formula (16).

One further has the equations of motion (charge = – e):

$$\left.\begin{aligned} m\frac{dv_1}{dt} &= -\frac{e}{c}(\dot{y}\mathfrak{H}_z - \dot{z}\mathfrak{H}_y) - e\mathfrak{E}_x, \\ m\frac{dv_2}{dt} &= -\frac{e}{c}(\dot{z}\mathfrak{H}_x - \dot{x}\mathfrak{H}_z) - e\mathfrak{E}_y, \\ m\frac{dv_3}{dt} &= -\frac{e}{c}(\dot{x}\mathfrak{H}_y - \dot{y}\mathfrak{H}_x) - e\mathfrak{E}_z, \end{aligned}\right\} \tag{12}$$

$$mc^2\frac{dv_0}{dt} = e(\dot{x}\mathfrak{E}_x + \dot{y}\mathfrak{E}_y + \dot{z}\mathfrak{E}_z)\,. \tag{13}$$

The question now arises of whether we can regard the right-hand sides of (10) and (11) as the correct operator for the four-velocity when we understand p_k to mean the ordinary operator $\frac{h}{2\pi i}\frac{\partial}{\partial x_k}$ in them. It seems that one can give an affirmative response to this question. Firstly, the operators (10) have continuous spectra in the interval from –∞ to +∞, and the operator for v_0 * has an absolute value that is greater than 1 **. This range of values coincides with that of the classical velocity. Secondly, however, equation (4), when one sets F in it equal to $P_1 = mv_1$, $P_2 = mv_2$, $P_3 = mv_3$ and introduces the field quantities:

$$\left.\begin{aligned} \mathfrak{E}_x &= -\frac{1}{c}\frac{\partial A_1}{\partial t} + \frac{\partial A_0}{\partial x_1}, \text{etc.}, \\ \mathfrak{H}_x &= \frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}, \text{etc.}, \end{aligned}\right\} \tag{14}$$

yields:

$$\left.\begin{aligned} \frac{dP_1}{dt} &= -e(\alpha_2\mathfrak{H}_z - \alpha_3\mathfrak{H}_y) - e\mathfrak{E}_x, \\ \frac{dP_2}{dt} &= -e(\alpha_3\mathfrak{H}_x - \alpha_1\mathfrak{H}_z) - e\mathfrak{E}_y, \\ \frac{dP_3}{dt} &= -e(\alpha_1\mathfrak{H}_y - \alpha_2\mathfrak{H}_x) - e\mathfrak{E}_z. \end{aligned}\right\} \tag{15}$$

We obtain the operator for the time-component of the four-velocity when we eliminate P_0 from (11) with the help of the wave equation. We obtain:

$$\text{Operator } v_0 = G = \alpha_4 + \frac{1}{mc}(\alpha_1 P_1 + \alpha_2 P_2 + \alpha_3 P_3). \tag{16}$$

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* See formula (16) below.

** The proof is given in the appendix.

The introduction of (16) into (4) yields, after a brief computation:

$$mc^2 \frac{dG}{dt} = - ec\,(\alpha_1\,\mathfrak{E}_x + \alpha_2\,\mathfrak{E}_y + \alpha_3\,\mathfrak{E}_z). \tag{17}$$

The quantum-mechanical formulas (15) and (17) define a complete analogy with the classical formulas (12) and (13). The one goes into the other when one sets $P_k = m\,v_k$, $c\,\alpha_k = \dot{x}$, $G = v_0$.

The results that were obtained up to now seem to ultimately confirm that the correct quantum-mechanical operators are found for the three-dimensional, as well as the four-dimensional velocity.

5. Up to now, we have consistently assumed that the operators (7), on the one hand, and (10) and (11), on the other, correspond to three- (four-, resp.) dimensional representations of one and the same physical quantity, namely, the "velocity of the electron." However, it now appears that this assumption is in no way pertinent, and that one is dealing with different physical quantities here.

From classical mechanics, the three-dimensional velocity may be expressed in terms of the four-dimensional one, and vice versa. In particular, one has the relation:

$$1 - \frac{1}{c^2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = \frac{1}{v_0^2}. \tag{18}$$

On the other hand, one has the following rules for the association of operators with physical quantities:

a) The square of a quantity corresponds to the double application of the operator to this quantity.

b) The reciprocal value of a quantity corresponds to the inverse operator.

c) In case the operators of two quantities commute, the sum of the quantities corresponds to the sum of their operators.

Now, we have convinced ourselves that the quantity v_0 corresponds to the operator G. Correspondingly, from the rules a) and b), the right-hand side of (18) corresponds to the operator G^{-2}:

$$\frac{1}{v_0^2} \to G^{-2}. \tag{19}$$

We assume that the quantities correspond to the operators $c\alpha_k$ (k = 1, 2, 3), so their squares are all equal to c^2, and therefore obviously commute. From our rules of computation, it would then follow that the left-hand side of (18) corresponds to the operator – 2 (i.e., multiplication by – 2):

$$1 - \frac{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}{c^2} \to -2, \tag{20}$$

which is obviously absurd, and contradicts the relation (18).

We have thus proved that the operators $c\alpha_k$ and v_k correspond to different concepts. Which concepts can they be, in fact? There is hardly any doubt about the physical sense of the operator v_k : The associated physical quantity is the mechanical four-velocity of the electron as a "point charge," which we would like to refer to in the sequel as the *corpuscular velocity*. As far as the operators $c\alpha_k$ are concerned, they must be associated with a triple of quantities for which there is no sense in taking the sum of the squares of the components. Such a triple of quantities now forms the mutually perpendicular velocity components of a wave that spreads out in all directions with the light velocity c. Due to the close relationship between the operators $c\alpha_k$ and the ordinary velocity of the electron (e.g., the ones that enter into the equations of motion), it can only be the de Broglie wave that is associated with the electron, which, as is known from the relativistic Dirac equations, propagates with the velocity c and not with a superluminal velocity c^2 / v^2.

We have thus arrived at the conclusion that the difference between the operators for the velocity of an electron manifests the dual nature of the electron as a corpuscle and a wave.

6. From what we said, one should expect that one can also find an operator for the ordinary mechanical three-dimensional (corpuscular) velocity. From the classical analogy, this operator must have a continuous spectrum in the interval from $-c$ to $+c$, and satisfy equation (18) in the event that the squares of its components commute.

We would now like show that an operator with these properties is easy to give in the electrostatic case (absence of a magnetic field).

In classical mechanics, the three-dimensional velocity $\dot{x}_k$ can be expressed in terms of the four-dimensional one v_k as follows:

$$\dot{x}_k = \frac{v_k}{v_0} \qquad (k = 1, 2, 3). \tag{21}$$

In the electrostatic case, the operators v_k and v_0 commute, and we can carry over relation (21) to quantum theory with no further assumptions. If we denote the operator for $\dot{x}_k$ by V_k then we obtain:

$$V_k = \frac{1}{m} p_k G^{-1}. \tag{22}$$

In order to obtain the eigenvalues of the operators V_k, we obtain the equation for its eigenfunctions:

$$\frac{1}{m} p_k G^{-1} f = \lambda f. \tag{23}$$

From (23), it follows that:

$$\frac{1}{m} p_k f = \lambda G f,$$

and after a repeated application of the operator G:

$$c^2 p_k^2 f = \lambda^2 (p_1^2 + p_2^2 + p_3^2 + m^2 c^2) f.$$

For $k = 1$, this equation reads, when it is written out in more detail:

$$\left(1 - \frac{c^2}{\lambda^2}\right)\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} - \frac{4\pi^2 m^2 c^2}{h^2} f = 0. \tag{24}$$

This differential equation has a solution that is finite and continuous in all of space when the coefficient of $\partial^2 f / \partial x^2$ is negative. It follows from this that:

$$-c < \lambda < c, \tag{25}$$

i.e., the eigenvalues of the operator V_k define a continuous spectrum in the interval $(-c, +c)$.

One further effortlessly convinces oneself that the operators V_1, V_2, V_3 fulfill the relation:

$$G^2 \left\{1 - \frac{1}{c^2}(V_1^2 + V_2^2 + V_3^2)\right\} = 1, \tag{26}$$

which corresponds to the classical equation (18).

We thus see that in the electrostatic case the operators V_k have the desired properties and can be considered to be representatives of the three-dimensional corpuscular velocity of the electron. In the general case, the translation of formula (21) into the language of quantum theory is not entirely single-valued, but perhaps also not necessary, since the operators V_k find no immediate utility is quantum theory; our objective was only to show that the corpuscular and wave velocities of the electron have different quantum-mechanical operators.

7. However, there exists an analogy between the two operators $c\alpha_k$ and V_k that we would like to pursue.

The most important analogy consists in the behavior of the two operators under the correspondence principle. We assume that the energy operator H does not include the time, and we define a complete system of solutions:

$$\psi_k(x, y, z, t; E_n) \qquad (k = 1, 2, 3, 4)$$

of the Dirac equations (2), which are then likewise eigenfunctions of the energy operator. When we now construct the matrix for an operator with the help of these functions – e.g., for the coordinate x:

$$x_{mn} = \sum_{k=1}^{4} \int \bar{\psi}_k(E_m)\, x\, \psi_k(E_n)\, dx\, dy\, dz, \tag{27}$$

the matrix element x_{mn} must go to the corresponding term in the classical Fourier development of the coordinate x with respect to time after a certain passage to the limit as

a result of the correspondence principle. The passage to the limit consists in the following: The two quantum numbers m [*] and n must grow unbounded, and the Planck constant h goes to zero, and indeed, in such a way that the difference:

$$n - m = s$$

and the quantity:

$$n\,h = J$$

remain finite [**].

From equation (6), it now follows that:

$$\frac{dx_{mn}}{dt} = \sum_{k,l=1}^{4} c \int \bar{\psi}_k(E_m)(\alpha_1)_{kl}\psi_l(E_n)\,dx\,dy\,dz\,. \tag{28}$$

If we carry out the same passage to the limit here then the matrix element of the operator $c\alpha_1$ must agree in the limit with the derivative of the corresponding term of the Fourier development of the coordinate x; i.e., with the term in the development of the classical velocity.

We would now like to show that the matrix element of the operator V_1 tends to the same limiting value. To that end, we consider the difference:

$$B_k = V_k - c\,\alpha_k\,. \tag{29}$$

We have:

$$B_k = \frac{1}{m}G^{-1}(p_k - mc\,G\,\alpha_k) = \frac{1}{2mc}\,G^{-1}(\alpha_k H - H\,\alpha_k),$$

or:

$$B_k = \frac{ih}{4\pi mc}G^{-1}\dot{\alpha}_k\,, \tag{30}$$

where we have set:

$$\dot{\alpha}_k = \frac{2\pi i}{h}(H\,\alpha_k - \alpha_k\,H). \tag{31}$$

The matrix elements of the operator $\dot{a}_k$ are derivatives of those of α_k and remain finite under passage to the limit. Since the operator G^{-1} is uniformly restricted (i.e., for all h) the matrix elements of the operator $G^{-1}\dot{\alpha}_k$ also remain finite. However, the matrix elements of B_k include the quantity h as a factor, which tends to zero and thus itself tends to zero.

We have thus proved that the matrix elements of V_k coincide with those of $c\alpha_k$ in the limit; from this, it follows that the operator V_k likewise satisfies the correspondence principle.

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[*] For the sake of simplicity, it is assumed here that the energy depends upon only one quantum number.

[**] Cf., on this Carl Eckhart, "Die correspondenzmäßige Beziehung zwischen den Matrizen und den Fourierkoeffizienten des Wasserstoffproblem," ZS. f. Phys. **48**, 295, 1928.

A further analogy between $c\alpha_k$ and V_k consists in the fact that the four-velocity v_k will be expressed in terms of these operators in precisely the same way. Namely, we have, on the one hand:

$$v_k = \tfrac{1}{2}(G\,V_k + V_k\,G), \tag{32}$$

and, on the other hand:

$$v_k = \frac{c}{2}(G\,\alpha_k + \alpha_k\,G). \tag{33}$$

Furthermore, we have the relation:

$$V_k^2 = \frac{c}{2}(V_k\,\alpha_k + \alpha_k\,V_k). \tag{34}$$

such that the twice-applied operator V_k is equal to the "symmetrized" product of the operators V_k and $c\alpha_k$.

8. We can summarize the results of this investigation as follows:

One and the same classical mechanical quantity – viz., the velocity of the electron – corresponds to two different quantum-mechanical quantities in the Dirac theory that one can refer to as the corpuscular and wave velocities of the electron. The operators for the three-dimensional components of the wave velocity are $c\alpha_k$ (k = 1, 2, 3), where α_k are the four-rowed Dirac matrices. They have the point spectrum $\pm c$. The corpuscular velocity may be best described by its four-dimensional components. The operators for these components are linked with the operators for the impulse components by the same equations as the corresponding quantities in the classical theory. These operators have continuous spectra that fall within the range of values for the components of the classical four-dimensional velocity. In the electrostatic case, the corpuscular velocity may also be described in a three-dimensional way; the operators for the three-dimensional components have continuous spectra from $-c$ to $+c$, and satisfy the correspondence principle, just like those of the wave velocity. In the quantum-mechanical equations of motion for the electron, the corpuscular velocity enters into the expression for the acceleration, and thus has a mechanical interpretation; on the other hand, the components of the wave velocity enter as factors in the electromagnetic field quantities, and thus serve to describe the influence of the electromagnetic field on the electron.

Appendix.

The eigenvalue spectrum of the four-velocity.

1. We first consider one of the spatial components of the four-velocity – e.g., v_1. The differential equation for the eigenfunction of the associated operator is:

$$\frac{1}{m}\left(\frac{h}{2\pi i}\frac{\partial \psi}{\partial x} + \frac{e}{c}A_x \cdot \psi\right) = \lambda\,\psi. \tag{1a}$$

As is known, we can now normalize the vector potential by the addition of a gradient in such a way that precisely its *x*-component vanishes. (1a) then goes over to a differential equation with constant coefficients that has all real values λ for its spectrum. With that, it is proved that the operators for the three spatial components of the four-velocity have continuous spectra $-\infty$ to $+\infty$, which we would also like to show.

2. We now consider the temporal component v_0 of the four-velocity, and indeed, first in the case where no magnetic field is present, such that one can assume that the spatial components of the four-potential vanish. The associated operator is:

$$G = \alpha_4 + \frac{1}{mc}(\alpha_1 p_1 + \alpha_2 p_2 + \alpha_3 p_3). \tag{2a}$$

The equation for its eigenfunctions reads:

$$G\psi = \lambda \psi. \tag{3a}$$

However, the eigenfunctions also satisfy the differential equation:

$$G^2\psi = \lambda^2 \psi, \tag{4a}$$

which can be written in the form:

$$\psi = \frac{1}{k^2}\Delta\psi = \lambda^2 \psi, \tag{5a}$$

where Δ refers to the ordinary Laplace operator and, to abbreviate, we have set:

$$k = \frac{2\pi mc}{h}. \tag{6a}$$

As is known, equation (5a) has only one solution for $\lambda^2 \geq 1$ that is everywhere finite and continuous. From this, it follows that in the electrostatic case the operator for the temporal components of the four-velocity has a continuous spectrum, and its eigenvalues have a magnitude that is greater than or equal to 1.

It likewise follows from this that the inverse operators G^{-1} and G^{-2} are restricted.

Let it be remarked here that these inverse operators have (actual) kernels that are easy to give.

The kernel of G^{-2} delivers the solution formula for the equation:

$$F - \frac{1}{k^2}\Delta F = f, \tag{7a}$$

namely:

$$F(x, y, z) = \frac{k^2}{4\pi}\int \frac{e^{-kr}}{r} f(\xi, \eta, \zeta)\, d\xi\, d\eta\, d\zeta, \tag{8a}$$

where we have set:

$$r = \sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2}\ .$$

G^{-2} thus has the kernel:

$$K(x, y, z; \xi, \eta, \zeta) = \frac{k^2}{4\pi}\frac{e^{-kr}}{r}, \tag{9a}$$

and G^{-1} has the kernel:

$$\frac{k^2}{4\pi} G \frac{e^{-kr}}{r}, \tag{10a}$$

where the differentiations in G are taken with respect to the variables x, y, z.

3. We would now like to examine the spectrum of the operator G for the case of a constant magnetic field that is parallel to the z-axis. The vector potential in this case is:

$$A_x = -\tfrac{1}{2} H y, \quad A_y = \tfrac{1}{2} H x, \qquad A_z = 0. \tag{11a}$$

The operator G has the form:

$$G = \alpha_4 + \frac{1}{mc}\left[\alpha_1\left(p_x - \frac{e}{2c}Hy\right) + \alpha_2\left(p_y + \frac{e}{2c}Hx\right) + \alpha_3 p_z\right]. \tag{12a}$$

We must now choose the matrices α_1, α_2, α_3, α_4 in a particular way. We set:

$$\alpha_1 = \sigma_1, \qquad \alpha_2 = \rho_3\sigma_2, \qquad \alpha_3 = \sigma_3, \qquad \alpha_4 = \rho_2\sigma_2, \tag{13a}$$

in which ρ_k and σ_k are Dirac matrices, which deviates from the Dirac Ansatz [*].

We remark that the operator:

$$L = \rho_1\sigma_1 - \frac{p_z}{mc}\rho_3 \tag{14a}$$

commutes with G. The eigenfunctions of G, which satisfy the equation:

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[*] This choice suggests itself by the particularly simple and intuitive transformation properties of the associated ψ-functions. Under an arbitrary Lorentz transformation, the functions ψ_k transform by the formulas:

$$\psi'_1 = \alpha\,\psi_1 + \beta\,\psi_2\,, \qquad \psi'_3 = \bar{\alpha}\,\psi_3 + \bar{\beta}\,\psi_4\,,$$
$$\psi'_2 = \gamma\,\psi_1 + \delta\,\psi_2\,, \qquad \psi'_4 = \bar{\gamma}\,\psi_3 + \bar{\delta}\,\psi_4\,,$$

where $\alpha, \beta, \gamma, \delta$ are complex parameters that satisfy the condition:

$$\alpha\delta - \beta\gamma = 1.$$

In the special case of a rotation of the spatial coordinate system, these are the ordinary Cayley-Klein parameters.

$$G\ \psi = \lambda\ \psi, \tag{15a}$$

can then be subjected to the further condition:

$$L\psi \equiv \left(\rho_1\sigma_1 - \frac{p_z}{mc}\rho_3 \right)\psi\ = l\ \psi. \tag{16a}$$

If we apply the operator L once more then we find:

$$L^2\psi = \left(1 + \frac{p_z^2}{m^2c^2} \right)\psi\ = l^2\psi. \tag{17a}$$

If we understand p_z to mean the eigenvalue of the operator $\frac{h}{2\pi i}\frac{\partial}{\partial z}$ then the relation:

$$l^2 = 1 + \frac{p_z^2}{m^2c^2} \tag{18a}$$

follows from (17a).

The expression $\alpha_4 + \frac{p_z}{mc}\alpha_3$ that appears in G may be written in the form:

$$\alpha_4 + \frac{p_z}{mc}\alpha_3 = \rho_2\sigma_2 + \frac{p_z}{mc}\sigma_3 = -\rho_3\sigma_3\, L. \tag{19a}$$

If we introduce this expression into G and consider (16a) then we obtain the following equation from (15a):

$$\left[-l\rho_3\sigma_3 + \frac{1}{mc}(\sigma_1 P_x + \rho_3\sigma_2 P_y) \right]\psi\ = \lambda\ \psi, \tag{20a}$$

where we have set:

$$P_x = p_x - \frac{e}{2c}H \cdot y, \qquad P_y = p_y + \frac{e}{2c}H \cdot x \tag{21a}$$

to abbreviate. The first two equations of the system of equations (20a) contain only the functions ψ_1 and ψ_2, and read:

$$\left.\begin{aligned} -l\psi_1 + \frac{1}{mc}(P_x - iP_y)\psi_2 &= \lambda\psi_1, \\ -l\psi_2 + \frac{1}{mc}(P_x + iP_y)\psi_1 &= \lambda\psi_2. \end{aligned}\right\} \tag{22a}$$

If we eliminate the function ψ_1 from these equations then we find the differential equation for ψ_2:

$$\frac{1}{2m}(P_x^2 + P_y^2)\psi_2 = E\psi_2, \tag{23a}$$

where we have set:

$$E = \frac{mc^2}{2}(\lambda^2 - l^2) + \frac{eh}{4\pi mc}H, \tag{24a}$$

to abbreviate. However, (23a) is the ordinary Schrödinger equation for a constant magnetic field; its eigenvalues are known, namely:

$$E_n = (2n + 1)\, h\nu, \qquad \nu = \left|\frac{eH}{2mc}\right|, \tag{25a}$$

where n is a non-negative whole number. From (25a), (24a), and (18a), we ultimately find the expression for the eigenvalues of G^2:

$$\lambda^2 = 1 + \frac{p_z^2}{m^2c^2} + \frac{4nh\nu}{mc^2} \qquad (n = 0, 1, 2, \ldots). \tag{26a}$$

Since the latter two summands are positive or zero in λ^2, and p_z can assume all values, it is thus proved that the eigenvalues of the operator G define a continuous spectrum, and their magnitudes are greater than or equal to 1.

In the general case*, it is easy to show that the eigenvalues of the operator have absolute magnitudes that are greater than or equal to 1. Namely, if we let P denote the self-adjoint operator:

$$P = \alpha_1 P_1 + \alpha_2 P_2 + \alpha_3 P_3 \tag{27a}$$

with real eigenvalues p then the operator G^2 can be written in the form:

$$G^2 = 1 + \frac{1}{m^2c^2}P^2. \tag{28a}$$

Its eigenvalues are equal to:

$$\lambda^2 = 1 + \frac{p^2}{m^2c^2} \geq 1. \tag{29a}$$

Q.E.D.

Presumably, the operators G and P also have continuous spectra in the general case; however, the proof of this fact might be somewhat complicated.

* This passage has been added by the editor.