THE

GEOMETRY OF VORTEX FIELDS

BASED ON THE AUTHOR'S BOOK ON MAXWELL'S THEORY OF ELECTRICITY AND ITS EXTENSIONS

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FOREWORD

This treatise defines a continuation of my book on Maxwell's theory of electricity, which appeared almost three ago with the same publishers. At the time, I had promised to publish a second volume, but later on, I had to renounce that comprehensive plan, since the intervening change of my official position made the demands on my time and efforts too compelling in other directions. On those grounds, I have confined myself to working out that section of the planned second volume that was to treat the theory of vector functions, which I assume to be a more necessary extension of my previous book than the other one.

I was led to that appraisal by various circumstances, namely, by the discussions of my *Maxw. Theorie* in the trade journals. Except for some individual exceptions that are not worth going into, I can, with some satisfaction, assert that my work generally found the recognition that I had hoped for. Meanwhile, complaints that were directed against my presentation of vector analysis were repeated rather often. On those grounds alone, I must assume that they were not unjustified. However, I can also reassure myself of that fact as a result of a lecture that I gave on the theory of electricity in the previous summer semester that was essentially based upon my book. I found that my previous work had left me with a sense of incompleteness, since many considerations that one would have expected along the way were deferred to the second volume. I hope to be able to remedy that inconvenience with the publication of the present book.

Other reasons that have further compelled me to place special weight on just that part of the discussion are discussed in more detail in § **1**. Here, I would only like to remark that I was unavoidably induced to consider the advantage that a detailed study of the purely-geometric properties of physical fields might impart by a mistake that I myself had previously made and which then pointed to an even broader sphere of ideas. On pp. 214 of my previous book, I said, in regard to the shielding of the magnetic field of a rectilinear electrical current by surrounding it with a steel tube: "In contrast to that, an absolutely magnetically-hard sheath would allow no force lines to cross through it into the atmosphere." Meanwhile, that conclusion was justified only under the assumption that was made there that the sheath separated the external space from the current-carrying conductor completely (i.e., with no gaps, no matter how small). By contrast, it would lose all validity for a tube that surrounded only a certain length of the wire. Namely, one can imagine two circular paths of integration, one of which links the tube, while the other links the free wire at some distance from the ends of the tube. The line integral of $\mathfrak{H}$ must be just as large for both paths on geometric grounds, no matter what material the tube might be made of. In order to see that, one imagines that each circle is cut at some location and the four endpoints are pair-wise coupled by two integration paths that lie next to each other in the air. In that way, one will get a new closed integration path for which the line integral must necessarily be equal to zero, since air can be regarded as a magnetically-soft body, and the integration path will then link small vortex filaments in the field. The contributions of the aforementioned adjacent connecting segments then cancel each other in the line integral, so the contributions of the two circles must be equally large. That is, the field must be just as large everywhere immediately outside of the tube as it would be if the tube were not present at all, even if it were absolutely magnetically-hard in the sense that I used. In other words, that result can be expressed by saying that the magnetization wave must propagate from the ends of the tube over the sheath of the tube in precisely the same way that it would if the steel had been replaced with air, even though it cannot penetrate magnetically-hard materials.

At this point, I would like to mention a misgiving that Beck raised against my developments in his treatise on magnetic hardness [Wied. Ann. **59** (1896), pp. 89]. That misgiving was based upon a misunderstanding about the terms that I had used. When I speak of the "propagation" that must take place on geometric grounds, as I did just now, I am not thinking of a process that plays out in time. Rather, I am using the expression "propagate" here only in the same sense that one speaks of the propagation of the hydraulic pressure in a system of connected tubes. That is, I am imagining that the instantaneous state of the field is given, and that I then proceed, step-by-step, throughout the entire region and then deduce what I expect to find later from what I had previously discovered at a certain point along the way. The law of the temporal change in the total field that Beck would have liked to have seen in place of my considerations was expressed by the two main equations. Of course, the argument became much more encompassing with their introduction. Had I known a method of avoiding the difficulties in integration that arose in that way, I would have preferred such a procedure from the outset. However, as long as I have not succeeded in arriving at a useful result in that way, any summary consideration whose logical justification might prove to be just as incisive as any mathematical formula could also perform a very useful service.

At this point, I would not like to leave it unmentioned that A. Kohn found a result in his study of the shielding effect of a steel tube [Wied. Ann. **58** (1896), pp. 527] that contradicts that of Beck, as well as the foregoing discussion, to some extent. From what I have heard about the absolutely trustworthy level of care with which that experiment was carried out, I can have no doubt that he was dealing with a well-observed fact in it. This is not the place for discussing the various possible ways of explaining that contradiction. At the moment, the study of magnetism is in such an unfinished state that it will probably require an even greater effort before clarity can be established in all directions. The best support for investigations of that kind, however, would be defined by a geometry of the field that is free from all physical hypotheses, in any event.

In recent times, some weight has been placed upon the problem of distinguishing between directed quantities that possess a polar character and the ones that possess an axial character. Wiechert has even introduced a special name for those quantities: He called the former "vectors" and the latter "rotors." As long as one can be certain that the field quantities possess that character in reality, nothing will prevent such a classification. No one would dispute that there exists a distinction between a translation and a rotation in kinematics or between a single force and a force-couple in mechanics or between velocity and vorticity in hydraulics (a distinction that is similar to the one between real and imaginary numbers, moreover) that can be felicitously expressed by the known terminology. One will also concede that a distinction of the same or similar kind must be assumed to exist between electric and magnetic fields. By contrast, for the time being, I consider it to be entirely hypothetical for one to assign both of those roles to the field quantities in any way, even to this day. It might very well be the case that their meanings will be inverted later. However, even when one overlooks that fact, making a more precise convention in regard to the physical meaning of a directed quantity has no place in the general geometric theory of fields, and all the more so because the same laws are true for the "rotor fields" as for the usual vector fields. On that basis I shall not go into that classification (which would seem to be getting quite popular now) in this book.

I have borrowed completely from my previous book in the writing of this book in its style of presentation,. As for its paradigm, I have, above all, appealed to Maxwell's treatise "Ueber Faraday's Kraftlinien" (German translation by Boltzmann in Ostwald's Klassikerausgabe, 1895) (†). As much as possible, I have also endeavored to remain understandable to the reader who is not familiar with my previous book. Many repetitions will then be unavoidable, but I hope that they will not be to the detriment of this presentation.

A mathematician, in the strict sense of the word, would perhaps be better qualified to present such a "geometry" or "function theory" than myself in many respects. Without a doubt, he would, at least, be better inclined to address the currently-customary demands on the rigor of the presentation, and he would also have many occasions to link up with relevant mathematical investigations that have already been worked out, but which still have not found their way into the community of people who are interested in only the applications of mathematics, in which I count myself. On the other hand, history teaches us that the most fruitful suggestions of mathematics have always pointed to concrete physical problems that necessarily required a mathematical formulation that would be suited to them. Obviously, that adaptation of mathematical form to the questions that prove to be necessary or convenient in physics has not yet reached its conclusion. Until that happens, the mathematician can hardly do without the assistance of occasional collaborators in the neighboring disciplines.

Munich, in December 1896.

A. Föppl.

(†) Translator: "On Faraday's lines of force," Trans. Camb. Phil. Soc. **10** (1864), 27-83.

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CHAPTER ONE

DEPICTING VECTOR FUNCTIONS. VORTEX-FREE FIELDS.

§ 1. – Defining the fields.

The most important concept in the Faraday-Maxwell theory of electricity is the concept of a physical field. One understands that to mean a region inside of which each point is assigned a uniquely-determined physical state of some type. One can distinguish various types of field according to the type of physical state that one actually has in mind. In this book, I shall nonetheless leave the question of the specific type of field completely open and concern myself with only the general geometric properties that all physical fields of certain classes have in common, which might also be their special origin.

I will generally organize fields into classes. However, the bases for the classification will be of a purely geometric kind and will have nothing at all to do with the physical meaning that one ascribes to the field in a special case of application. For that reason, I have given this volume the title of *The Geometry of Vortex Fields,* which is, of course, somewhat narrower in scope than the fields of other classes. However, I feel that it is important to strongly emphasize that the lectures that I have compiled here are independent of all physical hypotheses and can therefore lay claim to rigorous mathematical validity. On the other hand, I would also like to refer to the fact that the treatment of vortex fields plays the principal role in this work.

At this point, I would like to mention what induced me to take up this endeavor. Maxwell's theory of electricity is no more distinguished than any other physical theory in the absence of any special hypotheses whose justifications one can meanwhile still argue about. Only further experimentation will show which of its hypotheses must be ultimately retained and which ones must be dropped or altered. However, in the course of the further developments that Maxwell's theory experienced since the time of its foundation, the mathematical methods for investigating such problems have gradually experienced a not-unappreciable degree of completion that has had nothing at all to do with the hypotheses of the physical theory. Therefore, it would be worthwhile to separate those components of the theory and discuss them after they have been liberated from the others. Even the opponents of Maxwell's theory will have to agree that they need to understand these lectures so that they might not make any logical error in the derivation of the theorems that they must prove.

In the absence of any further conditions, I will constantly assume that the fields that I shall treat here do not extend to infinity, and I shall only go into detail about what one understands that assumption to mean at suitable places later on. Meanwhile, I will first point out that gravitation will be excluded from the sphere of fields under investigation by that restricting assumption. Moreover, I shall assume that the fields are continuous everywhere and do not become infinite; I shall allow discontinuities and infinitely-large field values as only limiting cases at best.

Naturally, one is also free to drop those assumptions and to see how the conclusions can then be generalized. However, I shall not address that here, since I would not like to go further in the presentation than would be necessary for the applications that one could propose for the lectures that are professed. The fact that one can also get along in the theory of electricity without considering discontinuities with the help of the principle of continuity in the transitions is probably unknown at present, in general.

The simplifications that one achieves by those assumptions are so significant that one cannot avoid them merely to attain a state of completeness that has almost no value in practice, except that one might perhaps object to the exclusion of gravitational fields. However, the study of gravitational fields seems to me to be still so far removed from the ultimate formulation that it will probably take on later that for the time being it would not be worthwhile to include those fields in the general considerations. All the same, one can probably still question whether gravitational fields actually extend to infinity. Indeed, in recent times, many strong objections have been made in regard to whether Newton's law of attraction is strictly valid at infinite distances that can hardly be ignored out of hand. In any event, the next problem in theoretical physics then consists of the study of electric and magnetic fields, which are accessible to experiment. Later on, the deeper insight that would be gained by that will also have value in its own right in the theory of gravitation.

In many cases, it is possible to describe completely the physical state that actually exists at each location in the field by giving a single number. Such fields shall be referred to as *scalar fields*. In other cases, directed quantities are required in order to characterize the state of a field. Those fields are called *vector fields*, and they will be discussed predominantly in this book. There are also fields for which one directed quantity will still not suffice to describe the state at each location completely. In the most general case, one will describe, e.g., the "stress state" in a medium (perhaps an elastic body) completely by either three directed quantities or nine numbers. In ordinary elasticity theory, those nine state numbers can be reduced to six, since no external forces can act in such a way as to rotate each volume element. However, one must generally keep all nine state components for the stress state that Maxwell has devised in order to explain the ponderomotive forces in magnetic fields.

It seems that the case in which the field state can be adequately described by only three directed quantities (in which one naturally ignores all incidental facts that have nothing to do with the fields that are actually considered) is the most general one that occurs in nature at all. Such a field probably relates to an ordinary vector field in the same way that a vector field relates to a scalar field. One can aptly refer to it as a *hyper-vector field*.

Finally, I shall point out that one occasionally has to deal with quantities that possess a double direction, and therefore a directed quantity with no definite sense of direction. The simplest example of that is the tension in a wire or the longitudinal stress in a rod. In order to make it clearer whether one is dealing with an elongation or a compression, it does not suffice to affix a single arrow to the line of action of the stress. It is necessary for one to give two directions, one of which can refer to, e.g., the side of the cross-section on which lies the part of the rod upon which the force from the other side acts, which carries the second arrow. Instead of that, it will also suffice to provide the line of action with a sign by which one can distinguish tension from compression.

The torsion in a rod also belongs to that category. It can result in such a way that the lines that are parallel to the axis of the rod appear to be deformed into right-handed or left-handed screws. One cannot distinguish between the two types of screws by the addition of a single arrow either, since a right-handed screw looks the same from each direction. However, adding a sign will also lead to the same objective here.

However, in all of those cases, one is basically dealing with only a hyper-vector field of an especially simple composition. I suspect that one will have to include the gravitational field in that class some time later.

§ 2. – Vector functions.

Analytically speaking, the theory of fields is nothing but the theory of functions of directed quantities. In a certain sense, it defines an extension of the ordinary theory of functions to the case in which the independent variables can have an arbitrary direction and magnitude in not only the plane, but also in triply-extended spaces.

If one chooses an arbitrary origin in a vector field from which the radius vector $\mathfrak{r}$ can be defined, and one lets $\mathfrak{v}$ denote the directed quantity that gives the state of the field at the location $\mathfrak{r}$ then the vector function:

$$\mathfrak{v} = f(\mathfrak{r}) \tag{1}$$

will define the analytical representation of the field. Of course, only special types of vector functions will come under consideration for us, namely, the ones that are everywhere single-valued, continuous, and finite and vanish at infinity.

A scalar field can also be represented by a function of $\mathfrak{r}$, except that the function must only be of the kind that leads to a scalar value. By contrast, a hyper-vector field corresponds to a vector function of two independent variables. In addition to the radius vector, a unit normal $\mathfrak{N}$ appears in it that points to the surface for which one would like to assign, e.g., the magnitude of the pressure when one is dealing with the stress state in a medium. The function that represents a hyper-vector field is always linear relative $\mathfrak{N}$, moreover. The concept of linear dependency will be defined more precisely in what follows.

If one would like to appeal to the coordinate method then one could also replace eq. (1) with the component equations:

$$v_1 = f_1(x, y, z), \qquad v_2 = f_2(x, y, z), \qquad v_3 = f_3(x, y, z)\,. \tag{2}$$

In many cases, one will achieve one's goal most simply in that way. However, in any case, one might always consider equations (2) to be only a substitute for eq. (1), which is all that matters. Namely, one must never lose sight of the fact that we will always be dealing with just the properties of the function f collectively, and not with the properties of the functions f_1, f_2, f_3, which are introduced into equations (2) as mere auxiliary concepts.

Things are different for a scalar field. It is basically irrelevant whether the scalar field quantity V is represented by the vector equation:

$$V = \varphi(\mathfrak{r}) \tag{3}$$

or by the coordinate equation:

$$V = \varphi (x, y, z), \tag{3}$$

since $\mathfrak{r}$ can always be regarded as the geometric sum of its components along three axis directions.

For that reason, the geometry of scalar fields was also developed long before the geometry of vector fields. Namely, at its basis, ordinary potential theory is nothing but the geometry of scalar fields. Of course, it also reaches into the realm of vector fields, and originally it was even invented expressly for the purpose of being able to treat the properties of certain vector fields analytically in the simplest-possible way. Ordinary potential theory reduces vector fields to scalar fields or derives those vector fields from the scalar fields. Of course, one must then grapple with the complication that this reduction is not always possible, and one does not therefore succeed in embracing the problem of vector field in its fill scope. In the ambition to get by with ordinary potential theory in all cases that might pertain to, e.g., the theory of electricity, one will often need to appeal to the most peculiar devices. In that way, one would probably be forced by the demands of the moment to be content with a particular problem, but only touch upon the subsequent definition of the field concept that is required for a fruitful development of the ideas.

§ 3. – Depicting vector functions.

We now know that our main problem consists of examining the general properties of the vector functions of *one* independent variable that are introduced by eq. (1), and we would now like to look for the most suitable means by which we can achieve that goal. The study of functions of one real scalar variable points to a direction for doing that. We recall the way that one can clarify the sense of the Ansatz $y = f(x)$ by which one first introduces the concept of a function into mathematics. In order to do that, one draws an abscissa axis that will carry the value of x, puts the y directly above it as the ordinate, and infers the properties of the function that it defines from the form of the curve that is obtained. It takes little effort to see that, e.g., $dy / dx = 0$ can correspond to either its maximum or minimum value or also to an inflection point of the curve, and that conversely the evolution of the curve can be inferred by calculation, even by a beginner. It might be that many mathematicians today pursue a different path when they would like to introduce their audience to the concept of a function for the first time. However, I do not believe that they have succeeded, and I assume that people will almost never correctly visualize the concept of a function who did not first understand the geometric picture that the curve provides.

A function of two variables $z = f(x, y)$ will likewise be depicted by a surface on which we can easily study the properties of the function. Of course, once the concept of a function is established that broadly, one can do without pictures for a larger number of variables. Meanwhile, it is possible to envision any scalar field for three independent variables if one wishes to appeal to one's intuition. Perhaps one imagines that each point in space with the coordinates x, y, z is assigned a temperature, which will then give the value of the function $f(x, y, z)$. Naturally, that picture is only purely formal, and we must beware of confusing any experimental law that is perhaps known to the theory of heat with the picture and the conclusions that we would like to infer with its help. Perhaps we can also raise the objection to those conclusions that they depend upon whether we possess a physically precise representation that would apply to temperature. For example, we can construct surfaces of equal temperature and calculate the curvature ratios, the temperature gradient along various directions, etc., without the results to which we arrive in that way depending in any way upon the special type of chosen geometric depiction of the function. In fact, with those surfaces, we deal with only the visible organization in our pictures and thoughts and not at all with actual physical bodies that we would like to construct a thermometer from in order to observe the temperature distributions in them. Indeed, it is in precisely that way that we must carefully distinguish between the curve that corresponds to the function $y = f(x)$ and the physical trace by which we draw it in order to stimulate our imagination.

We must also pursue precisely the same path in order to clarify the sense of eq. (1) – i.e., the concept of a vector function. I have thoroughly recalled the means of visualization that is employed as the starting point in mathematics in order to arouse the desire in the reader to test the applicability that is proved there to the general case that we are dealing with here. In fact, we would have to grope about tediously in the dark if we wished to study the general properties of vector functions without some kind of intuition, while the use of that expedient would cast a bright enough light upon our path that we could find our way around with no difficulty.

The geometric picture that is best suited to the representation of the function of the function $\mathfrak{v} = f(\mathfrak{r})$ is the hydrodynamical one. It accomplishes as much for vector functions as curves do for scalar functions. In it, we once more replace the function itself with a field, but it is a field that our imagination provides with no further discussion. To that end, we imagine that all of space is filled with an incompressible fluid, but not perhaps with water or any other fluid bodies, but with a fluid that has entirely arbitrary properties and is contrived especially for the purpose of not needing to be subject to the laws of mechanics, of which we will demand only that it must be fluid, moreover; i.e., that the velocity at neighboring locations can vary in an arbitrary way. It is only a consequence of the restriction that we have imposed upon our investigation from the outset that this variability of the velocity is always assumed to be continuous here, but otherwise entirely arbitrary. We think of the magnitude and direction of the velocity at each location in space as being chosen in such a way that the dependent variable $\mathfrak{v}$ in equation (1) will represent that velocity when we establish a suitable unit of measurement, while the independent variable $\mathfrak{r}$ will naturally refer to the radius vector by which the location in the field is given, as before.

Obviously, any vector function of one independent variable can be represented geometrically by that hydrodynamical construction of the most general type. On the other hand, the current in a volume of water would not suffice for that purpose. The continuity condition for it would stand in the way of that, while we can easily skip over that obstacle for our fictitious fluid.

If one considers a volume element $dx\,dy\,dz$ and one calculates how much more flows into its six sides than flows out when the velocity $\mathfrak{v}$ is given by eq. (1) or the components of $\mathfrak{v}$ are given by equations (2) then one will get:

$$\left(\frac{\partial v_1}{\partial x}+\frac{\partial v_2}{\partial y}+\frac{\partial v_3}{\partial z}\right) dx\, dy\, dz\ .$$

We shall employ the notation div $\mathfrak{v}$, which is completely independent of the choice of coordinate system for the expression in parentheses, which is likewise free of that choice of reference. For an ordinary incompressible fluid, it would be physically impossible for more to flow out of a given spatial region that is continually filled with fluid than into it, or conversely; i.e., div $\mathfrak{v}$ must necessarily be equal to zero. We can then depict only very special functions by means of a stream of water or a hydrodynamical construction in the strict sense, namely, the ones for which div $\mathfrak{v}$ is zero in all of space. However, the ideal fluid that we imagine is not subject to any such restrictions. Nothing prevents us from assuming that fluid is always created at certain places in space, but fluid is continually annihilated at others. We would like to refer to locations of the first kind as *sources*, while those of the second kind are *sinks*, or also *negative sources*. At the same time, we shall also reserve the word "source" for the more general sense that can simultaneously encompass positive sources, properly speaking, as well as negative ones. We shall call the totality of all sources that belong to a given vector function $\mathfrak{v} = f(\mathfrak{r})$ its *system of sources*.

If one thinks of a source of finite productivity as being concentrated at a point then infinitely large velocities will appear in the immediate neighborhood of that point, which is easy to see. Such a velocity distribution can only come under consideration for us as a limiting case. We must then think of the sources as being distributed in space in such a way that a source falls within the aforementioned volume element *dx dy dz* when its productivity is defined by the excess of the outflow over the inflow. The productivity of that source will therefore be infinitely small of order three, like the volume element itself, and will be proportional to it. When expressed per unit volume, the productivity, which might be expressed by *q*, will be:

$$q = \frac{\partial v_1}{\partial x}+\frac{\partial v_2}{\partial y}+\frac{\partial v_3}{\partial z} = \text{div}\, \mathfrak{v}. \tag{5}$$

§ 4. – General properties of a system of sources.

We think of the field as being bounded within any closed region. Due to the incompressibility condition, we can find how much more fluid flows out through the surface of the region than into it from the productivity of all sources that are included in that region. The equation that expresses that is known by the name of *Gauss's theorem* for the special case in which $\mathfrak{v} = f(\mathfrak{r})$ represents a force field that is subject to Newton's law of gravity. Meanwhile, as we see, it is true more generally, since it gives a property of all vector functions. If we let *df* denote the boundary surface, let $\mathfrak{N}_a$ the outward-pointing unit normal, let $\mathfrak{v}\ \mathfrak{N}_a$ denote the scalar or inner product of those two quantities –

i.e., the product of their absolute values and the cosine of the angle that their directions subtend – and let $d\tau$ denote a volume element of the region then we will have:

$$\int \mathfrak{v}\mathfrak{N}_a\, df = \int \operatorname{div} \mathfrak{v}\, d\tau = \int q\, d\tau . \tag{6}$$

That equation can also be applied to all of infinite space. We would like to deal with only those fields that do not extend to infinity, and we can now determine the meaning of that assumption more precisely. To that end, we imagine a ball whose radius increases continually, while its center remains at any location in the field at which $\mathfrak{v}$ has a finite value. When the radius of the ball becomes sufficiently large, $\mathfrak{v}$ must be only infinitely small outside of the ball, and the error that we commit when we neglect the field outside the ball completely must converge to zero when the radius of the ball increases to infinity. Naturally, that is due to the fact that the amount of fluid that flows through the surface of the ball will itself go to zero in the limit. The vector $\mathfrak{v}$ must then become infinitely small of order three for infinitely increasing $\mathfrak{r}$, in general.

With regard to that restricting assumption that we have imposed upon our investigation, we will get from eq. (6) that:

$$\int^{\infty} q\, d\tau = 0, \tag{7}$$

if the symbol ∞ indicates an integration over all of infinite space. The sum of all positive sources must then be just as large as the sum of all sinks. In the older theory of electricity, one always assumed that the algebraic sum of the electric and magnetic masses over the entire field would have to be zero, and we have now convinced ourselves that this is a necessary consequence of the property of electric and magnetic fields that they do not extend to infinity, which is inferred from observation.

Furthermore, we would like to establish the value of the spatial integral:

$$\mathfrak{F} = \int^{\infty} \mathfrak{v}\, d\tau . \tag{8}$$

It plays a role for vector functions that is similar to the role that the integral $\int_{-\infty}^{+\infty} y\, dx$ plays for scalar functions; one can refer to it as the *field sum*. The summation that is prescribed by the integral sign is a geometric one.

Initially, let us assume that the field has no sources, so div $\mathfrak{v}$ is zero everywhere. We think of lines being drawn through all points of the field that point in the direction of $\mathfrak{v}$ everywhere. From the general assumption that is at the basis for our investigation, those streamlines cannot reach to infinity. Moreover, they can have no end points, since no sources should be present, and they also cannot wind around any finite point infinitely many times, since that would contradict the assumption of continuity, so they must be lines that close on themselves.

If one extends the integration in eq. (8) over an isolated closed current wire whose outer surface is composed of nothing but streamlines then one will get the value zero, since $\mathfrak{v}\,d\tau$ can then be set equal to $f\,v\,d\mathfrak{s}$ when one understands f to mean the cross-section of the wire, v to mean the absolute value of $\mathfrak{v}$, and $d\mathfrak{s}$ to mean an element of centerline of the wire. However, the product $f\,v$ is constant for all cross-sections, since no sources are present, and no current flows through the surface of the wire, while the geometric sum of all line elements is equal to zero for a closed curve.

However, all of space will be composed of such current wires here, and we conclude from this that the field sum for a source-free field is equal to zero. We can then overlook the currents that close upon themselves completely in the calculation of the field sum.

Furthermore, let it now be assumed that only two point-like sources $+ Q$ and $- Q$ are present in the field. All of the streamlines that emanate from $+ Q$ must then terminate at $- Q$. If the magnitude and direction of the distance that is calculated from the source to the sink is denoted by $\mathfrak{u}$ and the integration is denoted in the same way as before and initially extended over an isolated current wire then one will get $f\,v\,\mathfrak{u}$ and therefore, since the sum of all $f\,v$ yields the productivity Q of the source, one will get:

$$\mathfrak{F} = \int^{\infty} \mathfrak{v}\,d\tau = Q\,\mathfrak{u}. \tag{9}$$

That result can be easily adapted to the case of an arbitrary system of sources. One chooses an origin in the field from which one defines the radius vector $\mathfrak{r}$. For every source $q\,d\tau$ in the volume element $d\tau$, one assumes that there is a source $- q\,d\tau$ at the origin with the opposite sign. One can apply eq. (9) to the system of sources that consists of that pair. If one then sums the given system of sources over all $q\,d\tau$ then one will get:

$$\mathfrak{F} = -\int^{\infty} q\,\mathfrak{r}\,d\tau. \tag{10}$$

From eq. (7), the sum of the sources $- q\,d\tau$ at the origin will, in fact, vanish, and one will get the sum over the entire field by superposing all source-pairs $q\,d\tau$ and $- q\,d\tau$.

The choice of point from which one defines the $\mathfrak{r}$ in that equation is irrelevant, moreover, since if one were to choose another point such that the radius vector from the first origin to the new one was $\mathfrak{r}_0$ then one would get:

$$\int^{\infty} q\,(\mathfrak{r}+\mathfrak{r}_0)\,d\tau = \int^{\infty} q\,\mathfrak{r}\,d\tau + \mathfrak{r}_0\int^{\infty} q\,d\tau = \int^{\infty} q\,\mathfrak{r}\,d\tau,$$

which is a result that is obvious from eq. (10).

In the derivation of eq. (10), I anticipated the theorem of the admissibility of superposition that will be established more rigorously later on.

§ 5. – Vortex-free fields.

We draw an arbitrary curve in the field, denote an element of the curve by $d\tau$, and define the scalar product $\mathfrak{v}\, d\,\mathfrak{s}$ (or *inner product*, with Grassmann's terminology), which is therefore the product of the absolute values of $\mathfrak{v}$ and $d\,\mathfrak{s}$ with the cosine of the angle that the two directions subtend. The algebraic sum of those products for any given path of integration will be called the *line integral of the vector* $\mathfrak{v}$. If that line integral vanishes for any arbitrary closed (i.e., returning to its starting point) integration path then the field will be called *vortex-free*. A vector function that represents the field analytically shall also be called *vortex-free* in that case. If the equation:

$$\int_A^A \mathfrak{v}\, d\mathfrak{s} = 0 \tag{11}$$

is not fulfilled for every closed curve in the entire field, but only for the closed curves that lie inside of a simply-connected region with boundary inside of the field then it shall be said that the field in that region, or the vector function inside of that region for the independent variables, is vortex-free. One sees here how the concepts and methods of the ordinary theory of functions can recur in an extended formulation.

In a region for which eq. (11) is fulfilled, every integration path between two given points A and B will lead to the same value for the line integral; i.e., when $d\,\mathfrak{s}$ and $d\,\mathfrak{s}'$ denote the elements of two different curves that go from A to B, one will have:

$$\int_B^A \mathfrak{v}\, d\mathfrak{s} = \int_B^A \mathfrak{v}\, d\mathfrak{s}', \tag{12}$$

since, from eq. (11), one has:

$$\int_A^B \mathfrak{v}\, d\mathfrak{s} + \int_B^A \mathfrak{v}\, d\mathfrak{s}' = 0.$$

One chooses an arbitrary origin O in such a region and sets:

$$V_A = V_0 + \int_A^0 \mathfrak{v}\, d\mathfrak{s} \tag{13}$$

for a second point A in the region.

The quantity V_A is determined up to an arbitrary constant V_0 in that way. When the field is a force field, one calls V_A the *potential at the point A*, and therefore V_0 is the potential at the point O. If the field under investigation is defined to be a velocity field in the first place, as it would be in actual hydrodynamical investigations, then V will be called the *velocity potential*. In our case, the field or the vector function $\mathfrak{v}$ is constructed only hydrodynamically, while the true physical meaning of $\mathfrak{v}$, and even that of $\mathfrak{r}$, can be left open. In order to have a concise terminology, however, we would also like to refer to the quantity that is defined by eq. (13) in the most general case as the *potential* of the field, or also as the potential that belongs to the vector function $\mathfrak{v} = f(\mathfrak{r})$ inside of the given region.

The constant V_0 in eq. (13) is entirely arbitrary, in general. However, when the entire field (and not just isolated regions in it, as has been assumed up to now) is vortex-free, we would like to think of the constant V_0 as always being determined by the condition that V must be equal to zero at infinity; i.e., we set:

$$V = \int_A^\infty \mathfrak{v}\, d\mathfrak{s} \tag{14}$$

in that case. It remains entirely irrelevant where we would like to place the end point at infinity in that determination, since the function vanishes everywhere there. Since $d\mathfrak{s} = d\mathfrak{r}$, if we recall eq. (1) then eq. (14) can also be written:

$$V = \int_{\mathfrak{r}}^\infty f(\mathfrak{r})\, d\mathfrak{r}\,, \tag{15}$$

and in that way, any vortex-free vector function is associated with a unique function that yields a scalar field and can be referred to as an *integral* of the given function, in some sense. In fact, it has just the form $\int_x^\infty f(x)\,dx$ that one uses in ordinary analysis. However, one must observe that vector functions will admit integrals of different kinds. One of them already appeared in the context of the "field sum," and we will address other integrals in detail later on.

The scalar field V is called the *potential field* of the given vector field. Every vector field – including ones that are not vortex-free – was already associated with one scalar field by eq. (5), namely, the source field q, which was derived from the vector field by a differential operation. Our next problem consists of revealing the further connections between the three fields $\mathfrak{v}$, V, q.

§ 6. – Deriving the vector field from the potential field.

Let a scalar field or a single-valued scalar function of $\mathfrak{r}$ be given by eq. (3):

$$V = f(\mathfrak{r}).$$

We imagine that surfaces are constructed by associating points that have the same value of V. If we proceed from any point in the field along the infinitely-small segment $d\mathfrak{r}$ then V will change, in general. The change will be zero only when either V is constant in the vicinity of the chosen point or when the direction of $d\mathfrak{r}$ falls in a tangent plane to one of the aforementioned equipotential surfaces. Under the assumption that V decreases when we proceed along $d\,\mathfrak{r}$, we would like to refer to the change dV that takes place along the length of $d\,\mathfrak{r}$ as the *gradient* in V in that direction. The gradient takes its largest value in the direction of a normal to the equipotential surface. We denote the drop in that direction by $\mathfrak{v}$, so $\mathfrak{v}$ will be determined as a directed quantity, and we set:

$$\mathfrak{v} = -\nabla V. \tag{16}$$

The symbol of the ∇ operator is defined completely in that way. The fact that a minus sign was chosen is explained by the fact that a differential quotient (and we are obviously dealing with a spatial differential here) is said to be positive when the associated quantity increases, whereas here we would like to determine the gradient, and therefore the decrease in the quantity V.

In order to obtain the differential dV from the spatial differential quotient ∇V, it will suffice to multiply the latter by the length of $d\,\mathfrak{r}$, in the event that $d\,\mathfrak{r}$ falls in the direction of a normal to an equipotential surface. In other cases, it comes down to only that component of $d\,\mathfrak{r}$ that falls in the direction of the normal; i.e., we must multiply the absolute value of ∇V by the projection of $d\,\mathfrak{r}$ onto the normal or the direction of ∇V. In any case, we then get the differential dV that belongs to a change $d\,\mathfrak{r}$ in the independent variables by inner geometric multiplication of ∇V and $d\,\mathfrak{r}$, or:

$$dV = d\,\mathfrak{r}\cdot\nabla V. \tag{17}$$

Any reference to a coordinate system would be entirely superfluous in this consideration; however, one does not need to reject the use of such a thing completely. One should then refer to the fact that the three rectangular components of $\mathfrak{v} = -\nabla V$ give the gradient in the three coordinate directions, so ∇V can be represented in the form of a geometric sum:

$$\nabla V = \mathfrak{i}\frac{\partial V}{\partial x} + \mathfrak{j}\frac{\partial V}{\partial y} + \mathfrak{k}\frac{\partial V}{\partial z}. \tag{18}$$

One understands $\mathfrak{i}$, $\mathfrak{j}$, $\mathfrak{k}$ in this to mean unit vectors in the three coordinate directions.

If one substitutes that value of ∇V in eq. (17) and also replaces $d\,\mathfrak{r}$ with its three components $\mathfrak{i}\,dx$, $\mathfrak{j}\,dy$, $\mathfrak{k}\,dz$ in that equation then one will get:

$$dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz \tag{19}$$

when one multiplies things out, which is then a self-explanatory result.

The gradient field (we call it that because of the way that it was created) that is derived from any scalar field V according to eq. (16) is always vortex-free. Namely, it follows from eq. (17) that:

$$dV = -\,\mathfrak{v}\,d\,\mathfrak{r} \qquad \text{and therefore} \qquad V_2 - V_1 = -\int_1^2 \mathfrak{v}\,d\mathfrak{r}\,. \tag{20}$$

Since V is supposed to be a single-valued function, the line integral will always be zero when we extend it over any closed curve.

Here, we have defined the connection between $\mathfrak{v}$ and V in a different way than what we did in the previous paragraph. There, we started from a vortex-free vector field $\mathfrak{v}$ and arrived at the potential field V by a certain type of integration. Here, we conversely start from the scalar field V and derive $\mathfrak{v}$ from it by means of the differential operator $-\nabla$. However, eq. (20) agrees with eq. (13) in the previous paragraph. The one operation is then the inverse of the other, and it is irrelevant whether we define the connection between $\mathfrak{v}$ and V in one way or the other.

§ 7. – Deriving a vector field from a source field.

Any vector field can be determined from the source field that it is associated with using eq. (5). We invert that problem and look for the vector field that belongs to a given system of sources. Meanwhile, the problem is still not determined uniquely in that form. We will see that directly when we imagine that the system of sources for a vector field is found by a differential operation, so we have to perform an integration here, in which a certain arbitrariness will always remain that can only be eliminated by introducing special conditions.

I will next prove that the problem will be determined uniquely when we subject the vector field that we seek to the condition that it should be vortex-free in all of space. To that end, I consider two vortex-free vector field (or vector functions) $\mathfrak{v}_1$ and $\mathfrak{v}_2$, which I assume to belong to the same system of sources. We shall then have:

$$\operatorname{div} \mathfrak{v}_1 = \operatorname{div} \mathfrak{v}_2\,, \qquad \int_A^A \mathfrak{v}_1\,d\mathfrak{s} = 0, \quad \int_A^A \mathfrak{v}_2\,d\mathfrak{s} = 0 \tag{21}$$

in all of space, and in addition both functions should naturally fulfill the condition that we are assuming of all functions that we deal with here.

I can derive a new field $\mathfrak{v}$ from both fields by setting $\mathfrak{v}$ equal to either the geometric sum or the geometric difference of $\mathfrak{v}_1$ and $\mathfrak{v}_2$ at each location. Both cases come under consideration for the further developments, and I would therefore like to address them together by setting:

$$\mathfrak{v} = \mathfrak{v}_1 \pm \mathfrak{v}_2\,, \tag{22}$$

in which the upper or lower sign can be taken at will. It follows directly from the definition of div in eq. (5) that:

$$\text{div}\,(\mathfrak{v}_1 \pm \mathfrak{v}_2) = \text{div}\,\mathfrak{v}_1 \pm \text{div}\,\mathfrak{v}_2\,.$$

It likewise follows from the concept of the scalar or inner product that:

$$(\mathfrak{v}_1 \pm \mathfrak{v}_2)\,d\,\mathfrak{s} = \mathfrak{v}_1\,d\,\mathfrak{s} \pm \mathfrak{v}_2\,d\,\mathfrak{s}\,.$$

If one chooses the lower sign everywhere then one will get:

$$\text{div}\,\mathfrak{v} = 0 \qquad \text{and} \qquad \int_A^A \mathfrak{v}\,d\mathfrak{s} = 0 \tag{23}$$

for the difference field $\mathfrak{v} = \mathfrak{v}_1 - \mathfrak{v}_2$, from eq. (21).

The difference field $\mathfrak{v}$ is then likewise vortex-free and source-free in all of space. However, such a field must necessarily vanish everywhere, since if it were not zero everywhere then we could lay streamlines through all points of the field that would point in the direction of $\mathfrak{v}$ everywhere. In those sub-regions where $\mathfrak{v}$ might be zero, we could extend those lines arbitrarily, except that they should not be continued to infinity in that way. If we now pursue such a streamline then it must either close on itself or it must have an endpoint at a finite point, since it cannot extend to infinity, because we have assumed that none of the fields that we deal with go to infinity. Endpoints would be possible only at places with sources, and they are excluded here. However, closed streamlines also cannot occur, since the field $\mathfrak{v}$ is likewise vortex-free. $\mathfrak{v}$ must then be zero everywhere; i.e., $\mathfrak{v}_1$ and $\mathfrak{v}_2$ must be identical to each other. There is, in fact, only one solution to the problem of determining a vortex-free vector function for a given system of sources. We shall prove that a solution always exists by actually constructing it.

If one chooses the upper sign in eq. (22) then equations (23) will go to:

$$q = q_1 + q_2 \qquad \text{and} \qquad \int_A^A \mathfrak{v}\,d\mathfrak{s} = 0, \tag{24}$$

in which one employs the abbreviation q for the intensity of the source div $\mathfrak{v}$.

The field that results from $\mathfrak{v}_1 + \mathfrak{v}_2$ is also vortex-free then, and its sources are obtained from an algebraic summation over the sources of the component fields. I shall therefore divide the given system of sources q into two or more parts, then look for the vortex-free fields $\mathfrak{v}_1$, $\mathfrak{v}_2$, etc. that belong to those parts, and then construct the resultant field $\mathfrak{v}$ by geometric summation. It will then be first of all vortex-free, and secondly it will belong to the system of sources q that was given originally. I will therefore find one solution to the problem of determining the vortex-free field $\mathfrak{v}$ that belongs to q, and it will likewise be the only solution, which would follow from the foregoing considerations.

With that, we have foreseen the precise path that we have embarked upon. I shall next separate from the system of sources q only the triply infinitely-small source $q\,d\tau$ that belongs to a volume element $d\tau$ and determine what it contributes to the field $\mathfrak{v}$ when taken by itself. An amount of fluid $q\ d\tau$ goes through the spherical surface of radius a whose center lies at $d\tau$, and since the geometric space that is the only one being treated here behaves the same in all directions, the current at all locations on the spherical surface must point in the direction of the radius $\mathfrak{a}$ and have the same magnitude. Instead of appealing to that symmetry property of space, I can also prove that it would be impossible for the current to be vortex-free in any other case. The flow velocity is therefore equal to $q\ d\tau\ /\ 4\pi\ a^2$ at a distance of a, and in order to also represent its direction, I set:

$$d\,\mathfrak{v} = \frac{q\,d\tau\,\mathfrak{a}}{4\pi a^3}. \tag{25}$$

That contribution $d\,\mathfrak{v}$ to the total vector field is infinitely-small of order three at a finite distance from $d\tau$. However, it is also infinitely-small of order one on the surface of $d\tau$, and we therefore do not need to investigate how the current is distributed over the that surface itself, since the contribution $d\,\mathfrak{v}$, even at that location, vanishes in comparison to the one that arises from the sources that lie at a finite distance.

We now get the solution for the total system of sources from eq. (25) by performing an integration over all of space, so:

$$\mathfrak{v} = \frac{1}{4\pi}\int^{\infty}\frac{q\,\mathfrak{a}}{a^3}\,d\tau. \tag{26}$$

In that derivation, I shall, for the moment, drop the assumption that the field should not extend to infinity, since the field that belongs to the isolated source $q\ d\tau$ actually does extend to infinity. However, since there is only a single solution to the problem, it is irrelevant how I derive it. The fact that the value that is given in eq. (26) fulfills all conditions emerges from the derivation with no further analysis.

§ 8. – Deriving a potential field from a source field.

Ordinary potential theory seeks to avoid the use of a vector function like the one that occurs in eq. (26). It prefers to solve such problems with the help of scalar fields. That is easy for vortex-free fields. It would then be only necessary to calculate $\mathfrak{v}$ from the potential field V in the same way as before. In that way, one will also know the vector field $\mathfrak{v}$ indirectly, since it emerges from V by performing the differential operator $-\nabla$.

One next easily finds that:

$$V = \int_A^{\infty}\mathfrak{v}\,d\mathfrak{r} = \int_A^{\infty}\mathfrak{v}_1\,d\mathfrak{r} + \int_A^{\infty}\mathfrak{v}_2\,d\mathfrak{r} = V_1 + V_2 \tag{27}$$

in a manner that is similar to what was done in the previous paragraphs in regard to the resulting fields. Therefore, V can also be found by summing over all contributions that arise from the isolated sources $q\,d\tau$. The current $d\,\mathfrak{v}$ has the value that is given by eq. (25) for the individual source $q\,d\tau$. One forms the line integral from that for a integration path that goes to infinity in a radial direction and obtains:

$$dV = \int_a^\infty \frac{q\,d\tau}{4\pi a^2}\,da = \frac{q\,d\tau}{4\pi a}. \tag{28}$$

As before, the a in that means the distance to the reference point at which one would like to calculate V from the volume element $d\tau$. One will get the potential that arises from the total system of sources from the last equation by performing an integration over all volume elements $d\tau$ in all of space, and therefore:

$$V = \frac{1}{4\pi}\int^\infty \frac{q\,d\tau}{a}. \tag{29}$$

V is calculated uniquely in that way when the system of sources q is given. Conversely, q could previously be calculated when V was given. Of course, a detour through the vector function $\mathfrak{v}$ that belongs to the two functions was necessary in that. Namely, one got $\mathfrak{v}$ from V by way of the differential operator $-\nabla$, and q further emerged from that by performing a second differential operator, namely, the operator div. One will then have:

$$\mathfrak{v} = -\nabla V, \qquad q = \operatorname{div}\mathfrak{v} = -\operatorname{div}\nabla V.$$

However, it is preferable to combine the two differential operators into a single one that leads from V to q directly. That will be immediately possible when we introduce an operator ∇^2 that is defined by the Ansatz:

$$\operatorname{div}\nabla V = \nabla^2 V, \qquad \text{and therefore} \qquad q = -\nabla^2 V. \tag{30}$$

When one calculates with rectangular coordinates, one will get the following prescription for the direct action of the operator ∇^2:

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} \tag{31}$$

when one combines equations (5) and (18). One can therefore also set:

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = -q \tag{32}$$

in place of eq. (30). Up to a missing factor of 4π, that is the Laplace-Poisson differential equation. Many formulas from ordinary potential theory differ by that factor that I have derived here and which I have yet to derive. That is based upon the fact that in that theory, one examines, from the outset, force fields that obey the law of the inverse-square of the distance between masses, and one defines the unit of mass to be the one that makes a unit mass exert a unit force on another unit mass at a unit distance. The sources play the role of mass here, and the intrinsic measure of the magnitude of a source is its productivity. In the general theory of vector functions, one has absolutely no right to deviate from that measure arbitrarily. Meanwhile, one can easily rewrite all formulas later on in such a way that they coincide with those of potential theory by setting q equal to $4\pi m$ everywhere, if m means the specific mass.

We have now answered all of our main questions that can be posed in regard to the connection between the vortex-free vector function $\mathfrak{v}$ and the two scalar functions V and q that it is associated with. Meanwhile, we might finally touch upon yet another question that has less significance for the geometry of fields, but which nonetheless arouses some interest due to the differing interpretations that one might give to it.

Let the field $\mathfrak{v}$ be vortex-free; V will then be a uniquely-determined quantity. One gives a boundary to a region in the field that can also be multiply connected. If V is equally large everywhere on the boundary surface V and no sources exist inside of it then $\mathfrak{v}$ must be equal to zero everywhere inside of the region, since closed streamlines are impossible, by assumption, and streamlines cannot terminate in the interior since there are no sources. However, no streamlines can enter the region from the outside either, since otherwise V would differ at the two intersection points with the boundary surface by the amount of the line integral of $\mathfrak{v}$, which contradicts the assumption. In fact, $\mathfrak{v}$ must then be equal to zero everywhere in the interior.

One further concludes from this that $\mathfrak{v}$ is determined uniquely inside the region as long as V is given arbitrarily on the boundary surface, along with the sources in its interior, since two fields $\mathfrak{v}_1$ and $\mathfrak{v}_2$ that correspond to those conditions would then be possible, so the difference field $\mathfrak{v} = \mathfrak{v}_1 - \mathfrak{v}_2$ would have to lead to a constant potential on the boundary surface (namely, the zero potential), and no sources would belong to $\mathfrak{v}$ inside the space. From what was proved before, one would then need to have $\mathfrak{v} = 0$, and therefore $\mathfrak{v}_1 = \mathfrak{v}_2$, inside the space. In that style of proof, it is essential that $\mathfrak{v}$ should be vortex-free in all of space. However, that is also, in fact, the only case for which the use of the auxiliary function V is important. There is therefore no point in considering any other cases.

The theorem that was just proved goes by the name of *Dirichlet's principle*.

§ 9. – Defining vorticity.

In § **5**, only a characterization of a vortex-free field was given. In general, we can then say only that a vortex exists in a field when the line integral of $\mathfrak{v}$ does not vanish for any closed curve. However, we still have to address the problem of determining the concept of a vortex itself more rigorously.

It would be simplest to write down a defining equation and avoid any discussion of why one refers to precisely that quantity as "vorticity" or why one employs it as a measure of the vorticity that is present in the field. However, the intuitive character of our considerations would be damaged by such a process, and that is something that we must strive for above all else if we are to arrive at a correct understanding of things.

On those grounds, I shall first give the word "vortex" a bourgeois meaning. Everyone knows intuitively about the approximately circular motions that the water in a river exhibits at many places, such as under a bridge abutment or a similar obstacle, and which one refers to as "vorticity" in the original sense of the word. Meanwhile, only the motions that close upon themselves are essential in one's conception of the word, but not the special form of the path, when we are accustomed to representing them as circular if the first approximation, as well. It would not lead to any change in the concept if that circular motion were joined to another one that perhaps led the whole vortex downriver, which is something that we usually observe.

As long as we are concerned with only pure vorticial motions, we can then refer to them as motions along closed paths and be consistent with the customary parlance. However, since pure vorticial motions can also be mixed with other ones, we must look for a more definitive way of characterizing whether a given velocity distribution does or does not possess a vorticial component. For a pure vorticial motion, one imagines selecting a closed streamline and forming the line integral $\int \mathfrak{v}_1 \, d\mathfrak{s}$ of the velocity for it, which is denoted by $\mathfrak{v}_1$ here. That integral will be composed of nothing but positive contributions when the sense of traversal along the curve is chosen to agree with the direction of $\mathfrak{v}_1$. In any event, it will then deviate from zero. If yet another field component $\mathfrak{v}_2$ appears that might originate from sources then $\int \mathfrak{v}_2 \, d\mathfrak{s}$ will be equal to zero for every path of integration. The integral $\int \mathfrak{v} \, d\mathfrak{s}$ for the total motion $\mathfrak{v} = \mathfrak{v}_1 + \mathfrak{v}_2$ is therefore just as large as it is for the vorticial part $\mathfrak{v}_1$. That explains why a motion for which $\int \mathfrak{v} \, d\mathfrak{s}$ vanishes for every closed path of integration is referred to as *vortex-free*. The demand that one must consider every closed curve to be a possible path of integration is justified by the fact that every such curve is conceivably a possible streamline for a pure vortex line that might possibly be included as a component of the given velocity distribution.

Naturally, that demand cannot be met in practice. One is not in a position to actually calculate the integral $\int \mathfrak{v} \, d\mathfrak{s}$ for all possible paths of integration. One then replaces the given condition with another one that is equivalent to it and which can be easily derived from it. In it, one makes use of the theorem that every vortex-free motion can be derived from a potential. Even if the motion is not vortex-free in all of space, but only in individual sub-regions, it will at least be possible to derive it from a potential inside of those sub-regions, as we saw before. We can think of such sub-regions as infinitely small when we are only concerned with examining whether the motion can be considered to be vortex-free at a certain location. It is precisely by that device that we will arrive at a simplification of our way of characterizing a vortex.

If one denotes the potential from which one can derive the function $\mathfrak{v}$ inside of a region that is perhaps thought of as infinitely small by V, as before, in the event that the motion in the region is vortex-free then one will have the following expressions for the components of $\mathfrak{v}$:

$$v_1 = -\frac{\partial V}{\partial x}, \quad v_2 = -\frac{\partial V}{\partial y}, \quad v_3 = -\frac{\partial V}{\partial z}, \tag{33}$$

from which the condition equations for the differential quotients of the v will emerge:

$$\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z} = 0, \qquad \frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x} = 0, \qquad \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} = 0. \tag{34}$$

We are free to fuse these three condition equations into a single one in such a way that we can regard the differences on the left-hand side as the components of a directed quantity $\mathfrak{w}$. We must then state that $\mathfrak{w}$ must be equal to zero everywhere that the motion should be vortex-free, and conversely, since we know that equations (34) define not only a necessary condition for the possibility of the Ansatz (33), but also a sufficient one.

A quantity upon whose vanishing the nonexistence of a vortex at a certain location depends is obviously itself suitable to serve as a measure of the vorticity at that location. One can only doubt that there are no other quantities for which the same thing is true and which might be even better suited to measure vorticity on other grounds. Obviously, one can also employ any multiple of $\mathfrak{w}$ as a measure of vorticity, and in fact, Helmholtz chose one-half of $\mathfrak{w}$ for that purpose in his own hydrodynamical investigations. In that way, he came to compare the motion of water with the motion of a rigid body. Meanwhile, one cannot deny that this convention is undermined by a certain arbitrariness. I therefore think that in the general theory of vector functions that are supposed to encompass all physical fields, it is better to skip the introduction of such an arbitrary factor and to introduce the quantity $\mathfrak{w}$ itself as a measure of the vorticity. Otherwise, our formulas would be pointlessly burdened with a basically irrelevant factor.

We then set:

$$\mathfrak{w} = \mathfrak{i}\left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right) + \mathfrak{j}\left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right) + \mathfrak{k}\left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right), \tag{35}$$

or, since one can write that more clearly in the form of a determinant:

$$\mathfrak{w} = \begin{vmatrix} \mathfrak{i} & \mathfrak{j} & \mathfrak{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix}. \tag{36}$$

In order to give a brief notation to the operation on the vector function $\mathfrak{v} = f(\mathfrak{r})$ that is thus written down precisely (which is entirely independent of the coordinate system, as one easily convinces oneself), we further set:

$$\mathfrak{w} = \text{curl}\ \mathfrak{v} \tag{37}$$

as an abbreviation for eq. (35) or (36), and when that equation is valid, we call $\mathfrak{w}$ the *vorticity* of the function $\mathfrak{v}$.

In order to avoid some misunderstandings that the beginner can easily fall prey to, I shall expressly point out the fact that even in, e.g., a pure vortex field, the motion at individual places in the field can be vortex-free, even though the velocity $\mathfrak{v}$ does not by any means vanish there. Later on, we will see that the seat of the vortex is often found only at isolated vortex filaments, while all remaining parts of the field are vortex-free. In contrast to that, e.g., the fluid that is found in a rotating vessel and is in a state of rest relative to that vessel will be vorticial at all locations.

CHAPTER TWO

LINEAR VECTOR FUNCTIONS.

§ 10. – Defining linear vector functions.

In ordinary analysis, an expression of the form:

$$y = ax + b \tag{38}$$

is referred to as a *linear function in one independent variable*, and in fact that is because the geometric representation of it will be a straight line. If we wished to adapt the concept of linear dependency to functions of directed quantities then we could not simply keep the Ansatz (38), since we would not possess a sufficiently broad knowledge of the couplings that exist between quantities of that kind from the outset. Rather, we would merely have to make the term "linear function" mean the simplest possible law for changing $\mathfrak{r}$ into a variable $\mathfrak{v}$ that corresponds to the one that is expressed by eq. (38) in that sense.

In order to clarify the meaning of that remark, I recall that every continuous function of x can be represented by eq. (38) for a sufficiently small interval around x. If one lets that equation be true for all values of x then the equation will represent the tangent to the curve that depicts the original function. The coefficient a in eq. (38) gives the law of the change in y for increasing x at the location considered up to infinitely-small quantities of higher order.

In the same sense, one can also speak of the desire to give the law by which an arbitrary continuous vector function $\mathfrak{v} = f(\mathfrak{r})$ changes at a well-defined location $\mathfrak{r}$ for an infinitely-small neighborhood that needs to be precise only to higher-order infinitesimals. If one temporarily lets that Ansatz be true for arbitrary values of $\mathfrak{r}$ then one will obtain a function that contacts the given function $\mathfrak{v} = f(\mathfrak{r})$ at that location, and indeed the contacting field that is found in that way will correspond precisely to the tangent to a curve in the case of functions of one scalar quantity. Such a field is called a *linear field*, and the associated function is called a *linear function*. Above all, we say that any two vector functions *contact* at a location $\mathfrak{r}$ when they coincide with each other up to second-order infinitesimals in a neighborhood of that location. It is not necessary to go further and say that two vector functions can also have second-order contact with each other, etc., and that contact does not need to be restricted to individual points. Rather, two vector functions can also contact each other along certain lines or surfaces. In that way, the geometry of fields proves to be much more diverse than the ordinary geometry of triply-extended space and in many regards suggests multidimensional geometry, while it takes on an entirely realistic meaning, in contrast to the latter.

Furthermore, for our purposes, linear vector functions will come under consideration, first and foremost, only to the extent that they contact a given function to higher order, since we will see, with no further discussion, that a linear field always extends to infinity, which is a case that has no interest for us. For that reason, I have also chosen the definition of a linear function to be the simplest of all other laws of change that follow from contacting functions. In the second place, one must admit that linear vector functions also occur for hyper-vector fields, and their treatment must not be omitted here, on just those grounds.

It still remains for us to decide which law of change should be regarded as the simplest one, in which we must always establish that it must be sufficiently general to allow contact with any arbitrarily-given continuous function at any location. There can be no doubt as to which Ansatz we must choose: The change in the same direction must be proportional to the path that we have laid out, and for a third direction of advance, the change must be given by the geometric sum of the changes for two directions of advance that lie in the same plane. In the form of an equation, that says that:

$$f(\mathfrak{r} + \mathfrak{r}_1 + \mathfrak{r}_2) - f(\mathfrak{r}) = [f(\mathfrak{r} + \mathfrak{r}_1) - f(\mathfrak{r})] + [f(\mathfrak{r} + \mathfrak{r}_2) - f(\mathfrak{r})]$$

or more briefly, that we have:

$$f(\mathfrak{r} + \mathfrak{r}_1 + \mathfrak{r}_2) = f(\mathfrak{r} + \mathfrak{r}_1) + f(\mathfrak{r} + \mathfrak{r}_2) - f(\mathfrak{r}). \tag{39}$$

The $\mathfrak{r}_1$ and $\mathfrak{r}_2$ in this are arbitrarily-chosen increments in the variable $\mathfrak{r}$. One can also regard eq. (39) itself as the defining equation for the linear function $f(\mathfrak{r})$. Of course, that has the drawback that one cannot directly glimpse the sense of that Ansatz and the basis for choosing it in exactly that form. In any event, eq. (39) must be chosen to be the starting point for the derivation of all remaining properties of the linear functions.

If one chooses $\mathfrak{r}_1$ and $\mathfrak{r}_2$ to be infinitely small, and one denotes, e.g., the change that $f(\mathfrak{r})$ experiences when one proceeds by $\mathfrak{r}_1$ by $d_1 f(\mathfrak{r})$ then it will also follow from eq. (39) that:

$$d_{1+2} f(\mathfrak{r}) = d_1 f(\mathfrak{r}) + d_2 f(\mathfrak{r}). \tag{40}$$

However, that equation is also fulfilled by any arbitrary continuous function up to second-order infinitesimals, since we can always calculate the change that $f(\mathfrak{r})$ experiences when we advance by $\mathfrak{r}_1 + \mathfrak{r}_2$ in such a way that we augment the change that belongs to $\mathfrak{r}_1$ by the one that arises when one proceeds from the endpoint of $\mathfrak{r}_1$ to the endpoint of the line segment $\mathfrak{r}_1 + \mathfrak{r}_2$. We replace the latter change with $d_2 f(\mathfrak{r})$ in eq. (40), and therefore the change that is drawn from starting point to the line segment that is equal and parallel to $\mathfrak{r}_2$. However, the concept of continuity is associated with the idea that an infinitely-small shift $\mathfrak{r}_1$ of the line segment $\mathfrak{r}_2$ to the change that belongs to $\mathfrak{r}_2$ can vary only infinitely little in comparison to the starting amount. In other words: Whereas we might also choose an infinitely-small line segment of given magnitude and direction inside of a infinitely-small neighborhood of the starting point, the change $d_2 f(\mathfrak{r})$ that belongs to $\mathfrak{r}_2$ must be the same up to second-order infinitesimals for continuous functions. If a special definition of continuity of a function were required then it would be included in that statement.

With that, one has proved that the Ansatz eq. (39) is, in fact, sufficiently general that a linear function can always be given that satisfies that Ansatz and which likewise contacts any given vector function at a prescribed point. The difference between linear functions and all other continuous functions consists of only the fact that eq. (40) is also fulfilled for finite values of $\mathfrak{r}_1$ and $\mathfrak{r}_2$ for linear functions, but for arbitrary functions, it is fulfilled for only infinitely-small values.

§ 11. – Coordinate representation.

If the increments $\mathfrak{r}_1$ and $\mathfrak{r}_2$ in $\mathfrak{r}$ have the same directions then eq. (40) will say that, as was desired all along, the increase in the function for a given direction of increase in the independent variable $\mathfrak{r}$ will be proportional to the absolute value of the increase in $\mathfrak{r}$.

One can further infer from equations (39) and (40) that a linear function is given completely as soon as one knows its value at an arbitrary reference point and the magnitudes of the increases for three directions that are not contained in a plane. One can then arrive at any other point of the field when one proceeds along paths that correspond to the three reference directions one after the other. When one applies eq. (40) (which must also be valid in a similar form for three increments $\mathfrak{r}_1$, $\mathfrak{r}_2$, $\mathfrak{r}_3$, moreover, as one can easily verify), one will then get the value of the function for any arbitrary $\mathfrak{r}$ as soon as the relevant defining data are given.

That situation intrinsically refers to the use of a coordinate system, and we therefore choose a rectangular one. Let the value of the linear function $\mathfrak{v}$ at the origin be $\mathfrak{v}_0$, and let the increases per unit length in the directions of the three *x*, *y*, *z* axes be $\mathfrak{r}_1$, $\mathfrak{r}_2$, $\mathfrak{r}_3$, resp. The linear function will then be represented by the equation:

$$\mathfrak{v} = \mathfrak{v}_0 + \mathfrak{r}_1 x + \mathfrak{r}_2 y + \mathfrak{r}_3 z, \tag{41}$$

and we have then found the desired generalization of eq. (38). In total, the complete description of a linear field then requires that we must be given four directed quantities, or in other words, twelve numbers.

I shall say that two given fields $\mathfrak{v}_1$ and $\mathfrak{v}_2$ *intersect* at a well-defined location when one has $\mathfrak{v}_1 = \mathfrak{v}_2$ at that location. Two linear fields cannot contact, since when they coincide up to higher-order infinitesimals in an infinitely-small neighborhood, that will be true everywhere; i.e., the two fields will coincide. If one writes out eq. (41) for two linear fields and observes that each of those two equations can be decomposed into component equations then one will see that two linear fields will generally intersect at a point. However, for special choices of defining data, they can also intersect in a line or a plane. If the point, line, or plane of intersection is shifted to infinity then those fields will be called *parallel*. If one would like to distinguish between those three cases more closely then one could say that they are *parallel* of *first*, *second, or third order*, resp.

I have cited those theorems precisely in order to better justify the term "linear" for those fields. One can make further use of the field concept for purely-geometric investigations, which should be obvious, anyway. However, I shall concern myself with such questions here.

For the examination of the further properties of linear fields, we can omit the constant term $\mathfrak{v}_0$ in eq. (41). The equation that thus arises:

$$\mathfrak{v} = \mathfrak{r}_1 x + \mathfrak{r}_2 y + \mathfrak{r}_3 z \tag{42}$$

when one thinks of assigning all possible values to the directed coefficient $\mathfrak{r}$ represents a sheaf of linear fields that all intersect at the coordinate origin. Any other field that does not belong to that sheaf will correspond to a field of sheaves that is parallel of third order to it in the aforementioned sense and differs from it only by the constant quantity $\mathfrak{v}_0$. The two fields will agree completely in regard to all other properties that will come under consideration, moreover, and we can then restrict ourselves to the study of fields of the sheaf. If one so desires, one can contract the scope of the fields that differ from each other essentially somewhat by regarding as equal all fields that differ from each other by only the unit of measurement or which can be made to overlap each other. However, I will refrain from doing that here in order to go further into those relations.

Since reference to a coordinate system was already made in eq. (42), it is recommended that one should decompose the directed quantities that enter into it along the coordinate directions. The equation will then decompose into three component equations, namely, with the notation c_{11} for the X-component of $\mathfrak{r}_1$, etc., one will have:

$$\left.\begin{aligned} v_1 &= c_{11}x + c_{21}y + c_{31}z, \\ v_2 &= c_{12}x + c_{22}y + c_{32}z, \\ v_3 &= c_{13}x + c_{23}y + c_{33}z. \end{aligned}\right\} \tag{43}$$

However, a more concise notation for that is desirable, in order for one to not need to write down the entire equation (43) whenever one would like to represent a linear vector function. Eq. (43) will then be equivalent to using the notation:

$$\mathfrak{v} = C(\mathfrak{r}). \tag{44}$$

We let C denote the linear operator by which $\mathfrak{v}$ is derived from $\mathfrak{r}$. We have only to observe that the operator C is always composed of nine components that are linked with the components of $\mathfrak{r}$ by the prescription in equations (43). The most general linear vector function can always be represented with the use of the operator symbol C in the form:

$$\mathfrak{v} = \mathfrak{v}_0 + C(\mathfrak{r}) . \tag{45}$$

§ 12. – Linear function of a unit vector.

One thinks of the center of the sheaf of fields that is represented by eq. (42) as the center of a sphere whose radius is equal to a unit length, which might be denoted by e. Any field will be known in all of space when the values of the field are given on the sphere, because at any other distance from the origin, the field quantity will point in the same direction that it has at the point on the sphere that lies along the ray that is likewise drawn through the origin, and the ratio of the absolute values will be the ratio of distances from the origin.

We would like to think of a line segment being drawn from each point of the unit sphere whose magnitude, direction, and sense depicts the field quantity at that location when one establishes an arbitrarily-chosen unit of measurement. One easily sees that the endpoints of that line are contained in an ellipsoid about the center. Namely, if one lets ξ, η, ζ denote the coordinates of the endpoints of that line segment then one will have:

$$\left.\begin{aligned} \xi &= x + c_{11}\,x + c_{21}\,y + c_{31}\,z, \\ \eta &= y + c_{12}\,x + c_{22}\,y + c_{32}\,z, \\ \zeta &= z + c_{13}\,x + c_{23}\,y + c_{33}\,z, \end{aligned}\right\} \tag{46}$$

and one will have the equation of a sphere:

$$x^2 + y^2 + z^2 = e^2, \tag{47}$$

moreover.

If one solves equations (46) for x, y, z and substitutes the values that are found in (47) then one will get an equation of degree two in ξ, η, ζ. The associated surface must be an ellipsoid, since its points are all finite, and the center of the ellipsoid must coincide with the origin, since the equation will always be fulfilled when one simultaneously changes the signs of ξ, η, ζ.

One will see from this that any linear field will generally have three mutually-perpendicular principal directions that are distinguished by their symmetry properties. An exception to that is when the determinant of the coefficients in equations (46) vanishes. In that case, the given field will intersect the field $\mathfrak{v} = 0$ along a line or a plane.

For a suitable choice of unit of measurement, with a reversal of direction for the whole field, if necessary, one can always arrange that the ellipsoid of the prior representation goes to an ellipse, so one principal axis of the ellipsoid will become zero. Mohr has given a very elegant representation of the stress state in a body with the help of that device. I shall cite the reference here (*), since the work is not very generally known amongst the readers that this book is intended for, while I would otherwise basically refrain from citing it on the same grounds as in my book on Maxwell's theory of electricity.

Above all, the interest in functions of a unit vector lies mainly in the realm of hyper-vector fields, so it will not be necessary to go into the topic here in more detail.

(*) Mohr, *Civil-Ingenieur*, 1882, pp. 126.

§ 13. – Sources and vortices for a linear vector function.

The linear field defines the simplest case of a vector field when one ignores the special cases that are already included in the concept of a linear field. It is therefore important to establish the distribution of sources and vortices in them, and all the more so since one will likewise find what type that distribution can have in an infinitely-small neighborhood of an arbitrary field. From equations (5) and (43), we find that:

$$q = \operatorname{div} \mathfrak{v} = c_{11} + c_{22} + c_{33} \tag{48}$$

for the linear field, which is therefore a uniform distribution of sources over the entire field. The vortices are also distributed uniformly over the entire field, since it will follow from eq. (35) that:

$$\mathfrak{w} = \mathfrak{i}\,(c_{23} - c_{32}) + \mathfrak{j}\,(c_{31} - c_{13}) + \mathfrak{k}\,(c_{12} - c_{21})\,. \tag{49}$$

The source intensity therefore depends upon only the components of C that have equal indices, while the vortex intensity depends upon the ones with unequal indices. At the same time, we see the conditions that must be fulfilled in order for a linear vector function to be either source-free or vortex-free. It can also be both source-free and vortex-free in all of space, moreover, since the field that is associated with it extends to infinity; it can even be infinitely large at infinity. The theorem that was proved in § **7** that a field cannot be simultaneously source-free and vortex-free in all of space will be true now under the assumption that was made there that the field cannot extend to infinity.

From equations (43), the geometric sum of two linear vector functions is again a linear vector function. Conversely, every linear function can be decomposed into two components, one of which is source-free and the other of which is vortex-free. Such a decomposition is of great interest in regard to the applications of the theory. Later on, we shall see that it is always possible uniquely for fields that do not extend to infinity. By contrast, the decomposition can be performed in infinitely many ways here. However, one of them is especially significant.

Namely, one can map the given function to a second one by transposing all of the indices in equations (43), such that, e.g., the value that was previously denoted by c_{12} will enter in place of c_{21} for the new function. The function that is thus found is said to be *conjugate* to the first one. The first one will then be conjugate to the second one, in its own right, so the relationship is reciprocal. If $c_{12} = c_{21}$, etc., from the outset then the function will called *self-conjugate*. From eq. (49), such a function will always be vortex-free. We would like to denote the operator that takes $\mathfrak{v} = C\,(\mathfrak{r})$ to its conjugate function by C_k, such that $\mathfrak{v}_k = C_k\,(\mathfrak{r})$ will denote the conjugate function.

By definition, the geometric sum $C\,(\mathfrak{r}) + C_k\,(\mathfrak{r})$ is always a self-conjugate function, and is therefore vortex-free. Its source-intensity is twice as large as it is for $C\,(\mathfrak{r})$ or $C_k(\mathfrak{r})$, since one generally has:

$$\operatorname{div} C\,(\mathfrak{r}) = \operatorname{div} C_k\,(\mathfrak{r}) \tag{50}$$

from eq. (48). By contrast, the geometric difference $C(\mathfrak{r}) - C_k(\mathfrak{r})$ is source-free, and its vortex intensity amounts to twice the vortex intensity of $C(\mathfrak{r})$, as one could infer directly from eq. (49). If we set:

$$\mathfrak{v} = C(\mathfrak{r}) = \tfrac{1}{2}\{[C(\mathfrak{r}) + C_k(\mathfrak{r})] + [C(\mathfrak{r}) - C_k(\mathfrak{r})]\} \tag{51}$$

then the decomposition of $\mathfrak{v}$ into a vortex-free part and a source-free one can be performed immediately, and indeed in such a way that both parts possess an especially simple structure.

§ 14. – Another representation of a linear vector function.

The two components into which $\mathfrak{v}$ is decomposed by eq. (51) can also be expressed in such a way that any reference to a coordinate system will be avoided. We next consider the first component $\tfrac{1}{2}[C(\mathfrak{r}) + C_k(\mathfrak{r})]$, about which we know that it defines a self-conjugate function, so it will be vortex-free. We know that any vortex-free field can be derived from a potential. Of course, the integration constant cannot be determined here in such a way that the potential vanishes at infinity, because the field extends to infinity and will itself be infinite there. It is simplest for us to choose that constant in such a way that the potential is zero at the origin, since its value still does not matter.

If we denote the first component of $\mathfrak{v}$ by $\mathfrak{v}_q$ then we will get the associated potential from eq. (13):

$$V = -\int_0^{\mathfrak{r}} \mathfrak{v}_q \, d\mathfrak{s} . \tag{52}$$

We can think of the path of integration as being chosen to be rectilinear from 0 to $\mathfrak{r}$. The integration can then be performed immediately, since $\mathfrak{v}_q$ increases uniformly from 0 to the final value, which will also be denoted by $\mathfrak{v}_q$, along the path of integration, while keeping the same direction. One then needs only to multiply the mean value $\tfrac{1}{2}\mathfrak{v}_q$ by the length $\mathfrak{r}$ of the entire path of integration to get:

$$V = -\tfrac{1}{2}\mathfrak{v}_q \, \mathfrak{r} . \tag{53}$$

If we once more go from the potential to the field $\mathfrak{v}_q$ then we will also have:

$$\mathfrak{v}_q = \tfrac{1}{2}\nabla \mathfrak{v}_q \, \mathfrak{r} . \tag{54}$$

One can also easily eventually convince oneself of the validity of this Ansatz by performing the operations that are suggested here according to the rules that were established before.

We now go on to the second component of $\mathfrak{v}$, which shall be denoted by $\mathfrak{v}_w$, so:

$$\mathfrak{v}_w = \tfrac{1}{2}[C(\mathfrak{r}) - C_k(\mathfrak{r})] .$$

A function with that structure has the peculiarity that the components of its operator will change sign when one transposes the two indices in c_{12}, etc., while the components that belong to equal indices will be zero. The conjugate of $\mathfrak{v}_w$ is the negative of $\mathfrak{v}_w$. Namely, when one writes out the components of $\mathfrak{v}_w$ as they are given in eq. (32):

$$\left.\begin{aligned} v_{w,1} &= \frac{c_{21}-c_{12}}{2}\,y + \frac{c_{31}-c_{13}}{2}\,z, \\ v_{w,2} &= \frac{c_{12}-c_{21}}{2}\,x + \frac{c_{32}-c_{23}}{2}\,z, \\ v_{w,3} &= \frac{c_{13}-c_{31}}{2}\,x + \frac{c_{23}-c_{32}}{2}\,y. \end{aligned}\right\} \tag{55}$$

If one compares the operator components in this equation with eq. (49) then one will see that their absolute values are one-half as large as the vorticity components in eq. (49), while the signs alternate between equal and opposite. If the components of $\mathfrak{w}$ are denoted by w_1 , w_2 , w_3 , as usual, then one can replace equations (55) with the following vector equation:

$$2\mathfrak{v}_w = \mathfrak{i}\,(w_2\,z - w_3\,y) + \mathfrak{j}\,(w_3\,x - w_1\,z) + \mathfrak{k}\,(w_1\,y - w_2\,x) . \tag{56}$$

In this expression, one recognizes the static moment of $\mathfrak{w}$ relative to the origin. The values in parentheses represent the static moment of $\mathfrak{w}$ relative to the three coordinate axes. However, a static moment relative to a point can be written down most simply as the vector product, or – to use Grassmann's terminology – the *exterior* product of the vector in question and the lever arm. We therefore replace eq. (56) with the notation:

$$\mathfrak{v}_w = \tfrac{1}{2}\mathbf{V}\,\mathfrak{w}\;\mathfrak{r} . \tag{57}$$

For the reader who is not yet familiar with vector calculus, that equation defines only an easily-understood abbreviation for the combination of components in eq. (56), which occurs frequently in mechanics, and the symbol gets its definition from that.

A static moment relative to a point is always a directed quantity that is perpendicular to the lever arm, so $\mathfrak{v}_w$ will also be perpendicular to $\mathfrak{r}$. For that reason, we can alter equations (53) and (54) in such a way that they will be simplified greatly. Instead of projecting the component $\mathfrak{v}_q$ of $\mathfrak{v}$ onto $\mathfrak{r}$, as was prescribed there, we can, in fact, also project all of $\mathfrak{v}$, since the other component $\mathfrak{v}_w$, as we just saw, is perpendicular to $\mathfrak{r}$, so the projection will contribute nothing. In that way, those equations will go to:

$$V = -\tfrac{1}{2}\,\mathfrak{v}\;\mathfrak{r} \qquad \text{and} \qquad \mathfrak{v}_q = \tfrac{1}{2}\nabla\mathfrak{v}\;\mathfrak{r}, \tag{58}$$

resp.

A linear vector function that intersects the field $\mathfrak{v} = 0$ at the origin can always be represented in the form:

$$\mathfrak{v} = \tfrac{1}{2}(\nabla\mathfrak{v}\ \mathfrak{r} + \mathrm{V}\mathfrak{w}\ \mathfrak{r}) \tag{59}$$

then.

In order to adapt that representation to the general case in which a constant term $\mathfrak{v}_0$ also appears, one must observe that the vorticity $\mathfrak{w}$ will not be affected by that at all, but only possibly the potential *V*, and therefore the first term in the parentheses in eq. (59), as well. One easily finds that:

$$\nabla\mathfrak{v}_0\ \mathfrak{r} = \mathfrak{v}_0\,, \tag{60}$$

and when one observes that, one will get the remarkable development:

$$\mathfrak{v} = \tfrac{1}{2}(\mathfrak{v}_0 + \nabla\mathfrak{v}\ \mathfrak{r} + \mathrm{V}\mathfrak{w}\ \mathfrak{r}) \tag{61}$$

for an arbitrary linear vector function.

The first term is constant, the second one originates in a potential, so it will be vortex-free, and the third one is source-free.

§ 15. – The inversion of linear functions.

When $\mathfrak{v} = C(\mathfrak{r})$, $\mathfrak{r}$ can also be regarded conversely as a linear function of $\mathfrak{v}$. Namely, that equation can be solved for $\mathfrak{r}$, in general, and one will then obtain an equation of the form:

$$\mathfrak{r} = K(\mathfrak{v})\,. \tag{62}$$

One can write down the nine components of the linear operator *K* directly when one recalls equations (43), which define the operator *C*. If one sets the determinant of the nine components of *C* in equations (43) equal to:

$$\begin{vmatrix} c_{11} & c_{21} & c_{31} \\ c_{12} & c_{22} & c_{32} \\ c_{13} & c_{23} & c_{33} \end{vmatrix} = \Delta, \tag{63}$$

to abbreviate, then one will get the components *x*, *y*, *z* of r by solving those equations:

$$x = \begin{vmatrix} v_1 & c_{21} & c_{31} \\ v_2 & c_{22} & c_{32} \\ v_3 & c_{23} & c_{33} \end{vmatrix} : \Delta, \qquad y = \begin{vmatrix} c_{11} & v_1 & c_{31} \\ c_{12} & v_2 & c_{32} \\ c_{13} & v_3 & c_{33} \end{vmatrix} : \Delta, \qquad \text{etc.} \tag{64}$$

Therefore, one has, e.g.:

$$k_{11} = (c_{22}\,c_{33} - c_{23}\,c_{32}) : \Delta \qquad \text{and} \qquad k_{21} = (c_{31}\,c_{23} - c_{21}\,c_{33}) : \Delta\,; \tag{65}$$

i.e., except for the constant divisor, each component of K is equal to the subdeterminant that belongs to oppositely-denoted components of C in Δ.

A deeper discussion of that situation is interesting in many aspects. Namely, one can derive some benefit from it for the investigation of rotations that a rigid body can perform about a point. Here, I will only point out the fact that the function $\mathfrak{v} = C(\mathfrak{r})$ will admit no inverse when the determinant Δ vanishes. The field $\mathfrak{v}$ will then cut the field $\mathfrak{v} = 0$ along a line that goes through the origin (and also a plane, in some exceptional cases), and it is precisely that case that is significant in the theory of rotations. The line of intersection will then correspond to the axis around which the rotation takes place.

Many interesting theorems can also be proved in regard to the composition of several linear operators with each other, so e.g., linear functions of the form:

$$\mathfrak{v} = A\,(B\,(\mathfrak{r})) = C\,(\mathfrak{r}),$$

in which C is now derived from the composition of the linear operators A and B in succession. On the one hand, I have, however, addressed such questions only slightly, and on the other hand, they are also not sufficiently significant for the purposes of the present book that I should go into them in more detail.

§ 16. – Stokes's theorem for a linear field.

Let an arbitrary closed curve be drawn in a linear field. We determine the value of the line integral $\int \mathfrak{v}\,d\mathfrak{s}$ for that path of integration that plays such an important role in the geometry of the field. In order to calculate that integral, we base it upon the representation of an arbitrary vector function that was given in eq. (61). The line integral will then be decomposed into three parts in that way, and we will get:

$$\int_A^A \mathfrak{v}\,d\mathfrak{s} = \tfrac{1}{2}\int_A^A \mathfrak{v}_0\,d\mathfrak{s} + \tfrac{1}{2}\int_A^A d\mathfrak{s}\,\nabla\mathfrak{v}\,\mathfrak{r} + \tfrac{1}{2}\int_A^A d\mathfrak{s}\,\mathbf{V}\mathfrak{w}\,\mathfrak{r}\,. \tag{66}$$

However, the first term on the right-hand side vanishes, since the factor $\mathfrak{v}_0$ is constant, and the geometric sum of all $d\mathfrak{s}$ will be zero for a closed curve. The second term will also vanish, since from eq. (17), $d\mathfrak{s}\;\nabla\mathfrak{v}\;\mathfrak{r}$ will given the change that $\mathfrak{v}\;\mathfrak{r}$ experiences under a displacement through $d\mathfrak{s}$, and the sum of those changes will be zero when one once more returns to the starting point, since $\mathfrak{v}\;\mathfrak{r}$ is necessarily a single-valued quantity. Only the third term will remains then, and that can be easily converted somewhat. Namely, one generally has $\mathfrak{A}\mathbf{V}\mathfrak{B}\mathfrak{C} = \mathfrak{B}\mathbf{V}\mathfrak{C}\mathfrak{A}$. In order to convince oneself of that fact, one needs only to examine that equation in coordinates, e.g., by writing down the components of $\mathrm{V}\mathfrak{B}\mathfrak{C}$ by comparing eq. (57) with eq. (56), and multiplying the $\mathfrak{i}$-component with A_1, the $\mathfrak{j}$-component with A_2, etc. I mention that the theorem about the commutability of factors in a product of that kind (the proof of which one will achieve along the way) is one of the most important in vector calculus. (All comments of that kind are unnecessary for the readers of my book on Maxwell's theory of electricity.) Furthermore, since the vorticity $\mathfrak{w}$ is a constant quantity for a linear field, from § **13**, we can move $\mathfrak{w}$ outside the integral sign and ultimately obtain:

$$\int_A^A \mathfrak{v}\, d\mathfrak{s} = \tfrac{1}{2}\mathfrak{w}\int_A^A \mathrm{V}\mathfrak{r}\, d\mathfrak{s}\,, \tag{67}$$

in place of eq. (66).

However, the expression that now appears on the right-hand side has a simple geometric meaning: namely, $\frac{1}{2}\mathrm{V}\ \mathfrak{r}\ d\mathfrak{s}$ is a directed quantity whose absolute value is equal to the area of a triangle whose vertex lies at the origin, while the opposite side is the line element $d\mathfrak{s}$ of the path of integration. Finally, the direction is perpendicular to that triangular surface.

If one also imagines $\mathfrak{w}$ as being constructed hydrodynamically in such a way that it gives a velocity (which is indeed constant here) to a second illustrative fluid then the scalar product $\frac{1}{2}\mathfrak{w}\mathrm{V}\ \mathfrak{r}\ d\mathfrak{s}$ will be the amount of fluid that flows through the previously-described triangle per unit time in this picture, since the current through a cross-section will always be found when one multiplies the cross-sectional area by the velocity component that is perpendicular to it.

The sign of the expression depends upon how the direction of traversal of $d\mathfrak{s}$ was chosen: It will be positive when the sequence $d\mathfrak{s}$, $\mathfrak{w}$, $\mathfrak{r}$ defines a right-handed system in space.

In performing the integration, we have to take the fluid current through all triangles that correspond to the line element $d\mathfrak{s}$ that belongs to the path of integration. However, in total, that will give the current through the surface of the cone whose vertex is the origin and whose base is the path of integration.

The line integral of $\mathfrak{v}$ is therefore equal to the amount of fluid that flows through the indicated surface of the cone for constant velocity $\mathfrak{w}$. In place of the surface of the cone, one can also use the base of the cone or any other surface segment that is bounded by the integration curve. One can draw very many surfaces of that kind, to which the conical surface that was just considered will itself belong. However, the same amount of fluid must flow through each of those surfaces in the field $\mathfrak{w}$. Namely, if one considers any two of those surfaces then they will collectively bound a volume that contains no sources, since $\mathfrak{w}$ is constant, and therefore div $\mathfrak{w} = 0$. Hence, from the incompressibility condition, just as much must flow out of the one surface as flows into the other. It is also unnecessary for us to refer to just the conical surface whose vertex lies at the origin in the statement of the theorem that we found in eq. (67): We can choose any arbitrary connected surface whose boundary curve is the path of integration. The amount of fluid that goes through that surface in the field $\mathfrak{w}$ is then just as large as the line integral over the boundary in the field $\mathfrak{v}$. If we recall the representation that we used before in eq. (6) for the flux through a surface then eq. (67) can also be replaced with:

$$\int_A^A \mathfrak{v}\, d\mathfrak{s} = \int \mathfrak{w}\,\mathfrak{N}\, df = \int \operatorname{curl}\mathfrak{v}\cdot\mathfrak{N}\, df\,. \tag{68}$$

The value of $\mathfrak{w}$ in eq. (37) has been substituted in the last form. In that form, the theorem also shows us clearly how preferable the choice of the quantity $\mathfrak{w}$ or curl $\mathfrak{v}$ is as a measure of vorticity. Whereas we originally (in § **9**) only found that $\mathfrak{w}$ must vanish if the current is to have no component that traverses a closed path at each location inside of an infinitely-small region, we are now in a position to give the precise connection between the field $\mathfrak{w}$ and the value of the line integral that was originally crucial for the definition of the vorticial motion.

Finally, it should be remarked that if both sides of the equations must also agree in sign then the direction of the unit normal $\mathfrak{N}$ in eq. (68) must be chosen in such a way that $d\mathfrak{s}$, $\mathfrak{N}$, $\mathfrak{r}$ defines a right-hand system in space, since we already saw that the line integral $\int \mathfrak{v}\, d\mathfrak{s}$ will be positive when $d\mathfrak{s}$, $\mathfrak{N}$, $\mathfrak{r}$ follow in the manner of a right-handed system, and that the surface integral $\int \mathfrak{w}\,\mathfrak{N}\, df$ will be likewise positive when one replaces $\mathfrak{w}$ with $\mathfrak{N}$ in that sequence.

§ 17. – Adapting Stokes's theorem to arbitrary fields.

Eq. (68) can be applied with no further discussion to an infinitely-small region in any arbitrary continuous field when one thinks of the field at that location as being replaced with the linear field that contacts it. However, eq. (68) will also remain valid unaltered for any arbitrary field for an integration path of finite measure. In order to see that, one lays an arbitrary surface through the path of integration such that path of integration defines the boundary curve of the surface segment. One then draws two systems of lines on the surface that subdivide the surface into infinitely many sections in such a way that each section lies inside of an infinitely-small region. One can the apply eq. (68) to each section, and once that has been done, one would like to add together all equations that one can write out according to the model of eq. (68). The sense of traversal of $d\mathfrak{s}$ must then be taken to be equal for the boundaries of all sections. The normals $\mathfrak{N}$ will then also go through the same side of the surface automatically.

One now sees immediately that the sum of the line integrals $\int \mathfrak{v}\, d\mathfrak{s}$ over the boundaries of all sections is just as large as the line integral over the boundary curve of the entire surface segment, and thus, over the original path of integration, since the boundary lines between neighboring sections will yield two contributions with equal values and opposite signs, which will mutually cancel then. All that remains then will be the contributions that originate on the boundary curve, which remain untouched by that mutual cancellation.

On the other side of the equation, in order to extend the surface integral over the entire surface, it is only necessary to sum. One then sees that one can, in fact, set:

$$\int_A^A \mathfrak{v}\, d\mathfrak{s} = \int \mathfrak{w}\, \mathfrak{N}\, df = \int \text{curl}\, \mathfrak{v} \cdot \mathfrak{N}\, df \tag{69}$$

for an arbitrary field and an arbitrary path of integration.

It already emerges from the proof that it is also entirely irrelevant here what surface spans the path of integration that will serve as the boundary curve. That also follows easily in a different way, and we will then point out an important property of the field $\mathfrak{w}$.

Namely, if one performs the operation div that is written down in eq. (5) on the value of $\mathfrak{w}$ that is defined by eq. (35) then one will get:

$$\text{div}\, \mathfrak{w} = \text{div curl}\, \mathfrak{v} = 0. \tag{70}$$

The field $\mathfrak{w}$, which emerges from an arbitrary field $\mathfrak{v}$ when one takes its vorticity, is always considered to be intrinsically source-free then. In fact, the same amount of fluid must therefore flow through any surface that spans the same boundary curve in the field $\mathfrak{w}$.

The theorem that is expressed by eq. (70) has great significance in the study of electricity. It says nothing less than the fact that electric currents must necessarily (and indeed on purely geometric grounds) traverse closed paths, assuming that one defines the electric current by its magnetic effects, as has always actually been done, namely – speaking more precisely –the vorticity of the magnetic field that it generates.

By contrast, Stokes's theorem, as it is expressed by eq. (69), is the most important tool for the further research into the properties of vector functions and the associated fields; it corresponds to a known theorem from the usual theory of functions that goes back to Cauchy. One first sees that this is the case most clearly with the notation of vector calculus, while the connection can easily remain unobserved when one knows Stokes's theorem only in its coordinate representation.

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CHAPTER THREE

THE SOURCE-FREE FIELD WITH ONE VORTEX FILAMENT.

§ 18. – Statement of the problem.

In Chapter One, we restricted ourselves to the investigation of fields in which no vortices were present, while an arbitrary distribution of sources could exist. Now we would like to examine precisely those vortex fields more thoroughly, so we can conversely bring about a simplification by assuming that the field is source-free. Later, we will see that we can consider an arbitrary field to be the sum of a vortex-free field that is created by a source and a source-free field that is created by a vortex. In fact, our task then comes down to that of discussing those two special cases in more detail.

In this chapter, we will then consider only those functions that satisfy the condition equation:

$$\operatorname{div} \mathfrak{v} = 0 \tag{71}$$

in all of space. All streamlines must then define closed lines. The line integral $\int \mathfrak{v}\, d\mathfrak{s}$ assumes a non-zero value for any streamline, and indeed a positive one, when we measure $d\mathfrak{s}$ in the direction of $\mathfrak{v}$.

However, in order to be able to pursue the investigation to its conclusion by the simplest means, we must temporarily introduce yet another assumption that we can drop in the next chapter. Namely, here, we would like to assume that the field is of a type that is simple enough for $\int \mathfrak{v}\, d\mathfrak{s}$ to always vanish for any closed line as long as that path of integration is a curve that is not chosen once and for all, but entirely arbitrarily. We shall refer to that curve as a *vortex line*, on grounds that will emerge directly.

If we bound a simply-connected region in that field that is not pierced by the assumed vortex lines then, by assumption, $\int \mathfrak{v}\, d\mathfrak{s}$ will be equal to zero for every closed integration path that lies inside of that region. $\mathfrak{v}$ can therefore be derived from a potential that is determined completely up to an arbitrary constant in that region. In that way, we will make it possible to carry out our examination with the tools of Chapter One; i.e., with ordinary potential theory. At the same time, it follows that $\mathfrak{w} = \operatorname{curl} \mathfrak{v}$ is equal to zero inside of the region. The fluid is not by any means vorticial then at all locations that lie outside the vortex line; i.e., the vorticity is restricted to only the line itself, which justifies the terminology that was chosen for the line.

Let $\int \mathfrak{v}\, d\mathfrak{s} = W$ for an integration path that encircles the vortex line once. We will soon see that W must necessarily have the same value for all such integration paths. We will likewise show that there are, in fact, functions of an increasingly diverse kind that fulfill all of the conditions that were imposed here.

We lay a surface through the path of integration, which is then its boundary curve, and that surface meets the vortex line at one point in the case where the two curves are of singly linked. From Stokes's theorem, we will then have:

$$W = \int \mathfrak{w}\,\mathfrak{N}\,df \;. \tag{72}$$

Everywhere outside of the vortex line, we will have $\mathfrak{w} = 0$. In order for $\mathfrak{w}$ to not itself be infinitely large on the vortex line (which is a case that physically meaningful only as a limiting case), we must then replace the vortex line with a vortex filament whose cross-section is assumed to be infinitely-small, while the vortex line itself will serve as the axis. Of course, W will still be infinitely small (and therefore $\mathfrak{v}$, as well) when we allow only finite values for $\mathfrak{w}$ in the vortex filament. That will be true for $\mathfrak{v}$ itself when we choose the path of integration to be small enough that it traverses the surface of the filament directly, since the cross-section of the vortex filament [and with it, from eq. (72), the value of W, as well] is small of second order for finite $\mathfrak{w}$, whereas the initial cross-section was small of order only one. For a finite $\mathfrak{w}$, an isolated vortex filament can therefore generate only a field that is infinitely-small of order one in the immediate neighborhood of the filament and infinitely-small of order two at finite distances from it.

The fact that we cannot assume that W has finite values for an isolated vortex filament also follows, moreover, from the fact that otherwise the velocity $\mathfrak{v}$ would be infinitely large in the immediate neighborhood of the vortex line. It therefore behaves in a manner that is entirely similar to the way that a point-like source behaves. For locations in the field that lie sufficiently far from the source or the vortex filament, one can always think of the source as being concentrated at a point or the vortex filament as being concentrated into a line. However, as long as we move into the neighborhood of the source or the vortex, we must think of both of them as being spatially-distributed. Hence, the field that is created by the source $q\;d\tau$ in the volume element $d\tau$ in their immediate neighborhood will be infinitely-small of order one, just as it is in the neighborhood of an isolated vortex filament with finite $\mathfrak{w}$.

Above all, one must regard it as a fact of greatest importance to the pure theory of functions, as well as the physical applications that are based upon the latter, that sources and vortices have such a close relationship to each other. In this book, I will refer to that fact quite often and at this point, I shall be content to point out the connection between the field $\mathfrak{v}$ and the vortex or system of vortices that belongs to a system of sources. In the theory of electrical action-at-a-distance, the sources of the electrostatic fields are regarded as the causes of those fields. In our purely-geometric theory of fields, there is no basis for such a viewpoint; we can just as well consider the field to be the cause of its source. In fact, both the vortex-free vector field and its system of sources mutually imply each other without allowing we to decide which of the two is more fundamental or important. Things are exactly the same for vortex filaments. We can probably consider the electrical current that flows in a linear conductor to be the cause of the magnetic field. For the geometric theory of fields, however, both of them come under consideration only in the form of vortex filaments and vortex fields, and thus, as things that are associated with each other. In fact, we have, in many aspects, already begun to invert the causal relationship by regarding the magnetic field as the cause of what we call the electrical current, which is then a consequence of it. That is due to the fact that the vortex field is regarded as the cause of its system of vortices. Here, we must stay clear of all such conjectures. However, for the sake of a more concise nomenclature, it will occasionally be permissible to say that the field is generated by a given vortex filament or (with the same significance) that the vortex filament is brought about by the field.

If one lays several surfaces through the aforementioned path of integration that link the vortex filament once then from Stokes's theorem – i.e., eq. (72), $-\int \mathfrak{w}\ \mathfrak{N}\ df$ will possess the same value for all of those surfaces that meet the cross-section of the vortex filament. Furthermore, that will also follow from the theorem that was expressed in eq. (70) that the field $\mathfrak{w}$ is always source-free.

At the same time, it emerges from the fact that when a field that is generated by a single vortex filament, the line integral $\int \mathfrak{w}\ \mathfrak{N}\ df$ must assume the same value for any integration path that encircles the vortex filament once, since that integral is, in any case, equal to the flux of the field $\mathfrak{w}$ through the cross-section of the vortex filament. The flux or quantity W in eq. (72) can then, in turn, be referred to as the *total intensity* or *strength* of the entire filament. In order to determine a vortex filament completely, it will suffice to give its centerline and its strength W, except that when one would like to assign it, say, the value that is determined more precisely by the field in the immediate neighborhood of the vortex filament, one would then have to introduce yet another given that says how the total intensity W is distributed over the cross-section of the filament. For the physical applications, one always deals with vortices of finite cross-sectional area and finite strength. At greater distances, one can frequently treat them like isolated vortex filaments with sufficient accuracy. When one is close to it, or in the space that the vortex itself occupies, one can decompose the vortex into a bundle of vortex filaments, such that the intensity of every vortex filament is infinitely small. The contribution that an isolated vortex filament itself makes to the total field at any location in its immediate neighborhood or on its surface or in its interior will always vanish then in comparison to the total effect. In fact, it will then suffice to calculate the field $\mathfrak{v}$ at a finite distance from the conducting line of an isolated vortex filament.

We must now solve the same problem for a vortex filament that was already solved in the first chapter for an elementary source $q\ d\tau$, *namely, the calculation of the field that is generated by the vortex filament at a finite distance from the filament* when the conducting line and the strength of the vortex are given. Of course, the solution of the problem will be more complicated here that it was there; however, we think it is important to emphasize the close relationship between the two problems.

§ 19. – Reducing the field to a vortex-free one.

One must address the study of vortex fields very early on in the theory of electricity. Naturally, one seeks to resolve problems of that kind in a manner that is similar to the way that one calculates with the vortex-free force fields that appear in laws of attraction, for which very useful methods have been found already. We already saw how closely such a process applies to a field that is otherwise everywhere vortex-free and possesses a singularity, in a sense, at an isolated vortex filament. In fact, it is only necessary to replace the vorticity field with a vortex-free one that is generated by sources and coincides with it up to a slight difference, or equals it at least to the extent that one will have the right to regard it as a substitute for the vortex-free field. It is self-explanatory that this wide open route can be used directly without one having to be careful, for the moment, to develop other methods that might be better suited to the spirit of the new problem. The celebrated theory of double layers arose in that way, which is perhaps the most shining example of how much one can attain by using basically unsuitable and inadequate tools.

In order to prepare our problem for solution in that way, I shall next show that the vorticity field is determined uniquely by its vortex filament and the conditions that it is source-free and must vanish at infinity. In order to avoid repetition, I would like to assume that the vortex system of the field is completely arbitrary, such that one does not need to restrict oneself to a single vortex filament.

Assume that it is possible for two source-free fields $\mathfrak{v}_1$ and $\mathfrak{v}_2$ to belong to the same system of vortices $\mathfrak{w}$. One will then form the difference field $\mathfrak{v} = \mathfrak{v}_1 - \mathfrak{v}_2$ in exactly the same way as was done in § **7** by a similar argument. One concludes from the defining equation (35) by which the concept of vorticity $\mathfrak{w}$ was introduced that:

$$\text{curl } \mathfrak{v} = \text{curl } \mathfrak{v}_1 - \text{curl } \mathfrak{v}_2$$

is equal to zero in the present case. One will likewise also have div $\mathfrak{v} = 0$, as was shown before on that previous occasion. Since the field should not extend to infinity, it follows, as before, that $\mathfrak{v}$ must be zero everywhere, and therefore $\mathfrak{v}_1 = \mathfrak{v}_2$.

We shall now return to the field with a single vortex filament and imagine drawing two infinitely-close surfaces in that field that have a form that is otherwise arbitrary, but with finite curvature, such that the contour of each of them reaches the vortex filament. Since the vortex filament must have an infinitely-small cross-section, we can think of the distance between two surfaces as being perhaps large enough that the vortex filament just completes the space between the two surfaces. We then assign a positive source to the one surface and a negative one to the other and consider the field that belongs to that source distribution to be vortex-free. Indeed, we can object to that construction on the grounds that a finite source productivity that is concentrated on a surface will contradict the basic rule that the spatial density of a source must remain finite. Meanwhile, we are not by any means dealing with an actual physical field here for which we would have to abide by that requirement, but with an artifice for deriving the correct results, such that we can always consider a distribution of this kind to be an acceptable limiting case. Later on, we might establish the fact that the result that is found in that way is actually correct in a way that is entirely independent of the way that it was derived.

For the time being, there is nothing more to be said about the vorticity itself. Rather, it shall be replaced with a double-layer field. One already sees with no calculation that the main part of the flux that flows from the positive source along the closed path (through the infinitely-thin intermediate space between the two surfaces) goes over to the neighboring sink. Meanwhile, a small fraction of it (which is infinitely small compared to the first part, as we will see) must also necessarily flow through external space, since the field now arises from only sources. In other words, it is vortex-free, and the line integral $\int \mathfrak{v}\, d\mathfrak{s}$ must then yield zero for any closed curve. We imagine a curve being drawn such that it crosses a line element in the internal space between two surfaces, while the curve otherwise remains in external space everywhere and closes beyond the boundary of the surface. The vortex filament that has temporarily forfeited its actual meaning will be linked by that curve once. The line element in the interior will make a contribution to the integral $\int \mathfrak{v}\, d\mathfrak{s}$ for that path of integration that will be negative in the event that we choose the sense of traversal of $d\mathfrak{s}$ in such a way that it opposes the current that exists in the internal space when it is in that space. In order for the entire integral to vanish, the rest of the path of integration must yield a positive contribution of equal absolute value. In that way, it is proved that a part of the total current must also, in fact, flow through the external space around the boundary of the surface from the positive source to the negative one.

At the same time, we recognize that the velocity in the external space must be infinitely small in comparison to the velocity in the internal space, since the integration over a finite path in the external space will yield only the same absolute value as the integration over the infinitely-small path in the internal space. We can also calculate the flow velocity in the internal space then. In order to do that, I shall bound both surfaces with two juxtaposed elements whose dimensions might be chosen to be small compared to the radii of curvature of the surface, but large in comparison to the distance h between the two surfaces. Let the areas of those two surface elements be denoted by df, while the source productivity per unit area at those two places is denoted by $+ q$ and $- q$, resp. Up to the vanishing fraction that goes through the external space, the amount $q\, df$ will flow through the internal space from the positive to the negative surface element per unit time. We can then set the flow velocity in the internal space equal to q and the contribution that the internal space makes to the line integral $\int \mathfrak{v}\, d\mathfrak{s}$ equal to $- qh$, which will be precise up to infinitely-small quantities.

We further infer from this that the part of the integration path that lies in the external space will yield the contribution $+ qh$ to the line integral. If we further agree that the source distribution on both sides is chosen such that the product qh has the same value at each location then the line integral will always have the same magnitude $+ qh$ for any arbitrary integration path that is not closed that we can draw around the boundary of the positive surface to the point on the negative surface.

Of course, once we have constructed a vortex-free field in this somewhat artificial way, we compare it to the vorticity field that originally interested us. It differs quite considerably from the field that the vortex filament generates without the aid of sources in the internal space, but is equal to it in all of the external space.

In both cases, the streamlines go around the boundary or around the vortex filament. For the double-layer field, the integration $\int \mathfrak{v}\, d\mathfrak{s}$ will give the constant value qh, with the exception of the infinitely-small piece in the internal space. For the vorticity field, one generally first obtains such a constant value when one includes the line element in the internal space in the path of integration. Here however, the contribution of that line element will be infinitely small in comparison to the value of the total integral, such that both fields will coincide in the external space up to infinitely-small quantities in that regard as well.

Once one has exhibited the desired field in the external space with the help of the double layer, one will then need only to once more extend the double layer, preserve the field in external space, and close the streamlines in the internal space by continuous extension in order to immediately obtain a field that coincides with the field of the vortex filament in all of its essential aspects.

In order for the strengths of both fields to be equally large, one must set:

$$W = qh. \tag{73}$$

One easily concludes that the field that is found in that way is identical to that of the vortex filament from the theorem that was proved in the introduction to this paragraph that only a single current distribution is possible that fulfills all conditions of the problem that might be concerned with the solution that is found here.

At the same time, that remark resolves a reservation that one might voice against the foregoing derivation. Namely, it might seem doubtful whether a streamline that starts from the positive surface of the double layer can reach the surface element on the negative surface that lies precisely opposite to the starting point by going through external space. One cannot, from the outset, rule out the possibility that the starting point and endpoint of an external streamline might be separated from each other by a finite distance. However, even when it is given that this is possible, once the streamline has been closed in the internal space, one must obtain a field that is source-free, and for which $\int \mathfrak{v}\, d\mathfrak{s} = W$ for the integration path that links the vortex filament, while $\int \mathfrak{v}\, d\mathfrak{s} = 0$ for any other closed integration path. That field must then be the desired vorticity field, and one concludes conversely from this that even in the field of the double layer, the starting point and endpoint of an external streamline must lie opposite to each other.

Perhaps it would be useful for me to add a comment. I have spoken repeatedly about an external space and an internal space that are separated from each other by two boundary surfaces. One should not understand that to mean that those boundary surfaces can have any influence on the current such as acting like impermeable walls. For us, they serve as only geometric loci through which the double layers would be assigned their locations in space and not as separation surfaces. Above all, in the general geometry of fields or in the theory of vector functions, one can never speak of actual separation surfaces by which, say, the flux of a field could be enclosed, even though such a picture might probably be quite permissible and useful for other investigations. Here, all of infinite space is available for our purpose of depiction, without limit, just as, e.g., the boundary of the graph paper is never meaningful in the depiction of the function of a scalar variable by a curve.

§ 20. – Solving the problem.

With those preliminaries, the field quantity $\mathfrak{v}$ easily can be calculated when the associated vortex filament is given. In order to do that, one needs only to apply the method that was presented in the first chapter to the source system of the double layer.

From eq. (28), a surface element df of the positive layer will yield a contribution of $+ q\, df / 4\pi a$ to the potential V. Apart from its sign, the associated element of the negative layer will differ by only the distance a to the reference point at which we would like to calculate V, as opposed to the previously-given value. The change in a can be denoted by $h \cdot da / dh$, if h is the distance between both surfaces, as before. If one sums the contributions of the associated elements of both layers then one will get:

$$dV = \frac{q\, df}{4\pi a^2} h \frac{da}{dh} \tag{74}$$

for the total contribution of df to the potential V. In order to get V, one needs only to integrate that expression over the entire surface. Meanwhile, first let a minor conversion be made by which one will arrive at the most elegant solution that Gauss contributed to the problem.

Let $d\varpi$ denote the spatial angle that the surface df (it is irrelevant whether one thinks of the positive or negative layer in this) subtends at the reference point. Hence, when multiplied by the square of the radius of a ball, $d\varpi$ is understood to mean an absolute number that will give the area of the section of its spherical surface that will define the solid angle when then vertex coincides with the center of the ball. All of space will then correspond to the solid angle $\varpi = 4\pi$; in short, the solid angle shall be employed to measure surface areas in precisely the same way that planar angles are used to measure arc length.

If df happens to be perpendicular to the distance a from the reference point to df then, by definition, the associated solid angle will be determined simply from the equation $a^2 d\varpi = df$. In other cases, one must replace df itself with the projection of df onto the spherical surface that is drawn from the reference point. However, dh is perpendicular to df, and da is perpendicular to the spherical surface, since da was the projection of dh onto the radius a. The angle between df and the spherical surface can be set equal to the angle between the two normals. One will then get $df \cdot da / dh$ for the projection of df onto the sphere, and therefore:

$$d\varpi = \frac{df}{a^2} \cdot \frac{da}{dh}. \tag{75}$$

Eq. (74) can then be written:

$$dV = \frac{1}{4\pi} qh\, d\varpi \tag{76}$$

with the use of the angle $d\varpi$, and that equation can be integrated over the entire sphere with no further discussion. If one then observes eq. (73) and one understands ϖ to mean the solid angle that the entire vortex filament subtends at the reference point then one will get simply:

$$V = \frac{1}{4\pi} W\, \varpi. \tag{77}$$

That is the solution that Gauss gave. One will see a special advantage that it possesses when one thinks of the reference point as being moved along a curve that links the vortex filament. In order to simplify the picture, one might think of the vortex filament as perhaps circular. One then starts from a reference point that lies in the plane of the circle, but outside of the interior of the circle. For that reference point, one must then set $\varpi = 0$, and one will therefore also have to set $V = 0$. One then displaces the reference point to the side of the plane of the circle that one might refer to as its "front" side. The vortex filament will now be projected through a conical surface from the reference point, and the conical angle will be ϖ. The conical angle will increase gradually when one goes further, and when the reference point has traversed the half of the path that links the vortex filament that lies on the front side, such that it will once more lie in the plane of the circle, and in fact in the interior of the circular surface, the solid angle will grow to 2π, while the potential will grow to $W / 2$. One moves the reference point further along its path until it enters the back side of the plane of the circle. If one now looks back then one will see the vortex filament subtend a solid angle that gradually decreases, just as it previously increased. However, one cannot change the direction that one is looking suddenly; as before, one must look in the direction of the reference point. Physiologically speaking, one will not see anything at all now, since the vortex filament will lie behind one. Geometrically speaking, however, one can always speak of the viewing angle that the vortex filament subtends along that line of sight. That viewing angle encompasses all of space with the exception of the angle that the vortex filament subtends when one reverses the line of sight. In that sense, the angle ϖ will grow ever broader when one proceeds forwards, and after we have traversed the entire path back to its starting point, it will grow to 4π – i.e., all of space – while it was originally equal to zero at the same place.

One sees that by this construction, the calculation already takes into account the fact that $\int \mathfrak{v}\, d\mathfrak{s}$ will be non-zero for a vortex filament that links the path of integration, and in fact, it will be equal to W, so V will also change by just as much. The potential that is represented by eq. (77) will be characterized as multi-valued in that way, and we would like to stress that fact more strongly by setting:

$$V = W\left(\frac{\varpi}{4\pi} \pm n\right), \tag{78}$$

instead of eq. (77), if n means an arbitrary *whole number*.

We started with the double layer and calculated V for external space. However, later on, the double layer vanished from our formulas completely. We have then found more than we could have originally expected for such a starting point, namely, a representation of the potential of the field that is generated by the vortex filament that is valid for all of space, but with the exclusion of the vortex filament itself and including the internal space of the double layer that was considered before. In fact, the introduction of the double layer has only a heuristic value. It came about by our ambition to reduce the vorticity field to a vortex-free one, but it is entirely dispensable for the proof that the solution that was found in eq. (78) is valid. In fact, one can be content to exhibit eq. (78) without introducing the double layer in any way, and one will then show, with little effort, that the vector field $\mathfrak{v}$ that emerges from that potential V possesses a vortex filament that coincides in form and strength with the one that was given originally. Due to the singlevaluedness of our solution, it will then follow immediately that it is given correctly by eq. (78).

Of course, an infinitely multi-valued quantity like V in eq. (78) cannot lay claim to any obvious physical meaning. It defines only an aid to calculation, from which the velocity $\mathfrak{v}$ of the field will be obtained by performing the operation $-\nabla$. One has:

$$\mathfrak{v} = -\frac{1}{4\pi} W \nabla \varpi \tag{79}$$

for $\mathfrak{v}$, and the problem that was posed has been solved with that.

§ 21. – Another form of the solution that was found.

The last equation shall now be converted further in such a way that we will actually calculate the change in the viewing angle ϖ for a displacement of the reference point. Let an infinitely-small displacement of the reference point in an arbitrary direction be denoted by $d\mathfrak{x}$, while the associated change in ϖ will be denoted by $d\varpi$. In order to find $d\varpi$, we can also leave the reference point in place and assign a translational displacement $-d\mathfrak{x}$ to the vortex filament. That is due to the fact that the vertices of the two solid angles that are being compared to each other can be made to overlap. In the main, both conical angles will also overlap, such that we need only to observe the small deviations that the circumferences experience.

When the line element $d\mathfrak{s}$ is given the displacement $-d\mathfrak{x}$, the vortex filament will describe a parallelogram surface whose area can be set equal to the absolute value of the exterior product $\mathrm{V}d\mathfrak{s}\, d\mathfrak{x}$. It belongs to a pyramid whose vertex lies at the reference point, and whose volume is equal to $\frac{1}{3}\mathfrak{a}\mathrm{V}d\mathfrak{s}\, d\mathfrak{x}$, if $\mathfrak{a}$ denotes the radius vector from the reference point to $d\mathfrak{s}$. The sign of that expression depends upon the direction in which one has chosen the sense of traversal of $d\mathfrak{s}$. However, in any event, once a sense of traversal has been fixed once and for all, the sign will alternate according to whether the pyramid represents an increase of the original conical volume or a decrease.

One finds the change in the solid angle that corresponds to the pyramid from the volume of the pyramid upon dividing by $a^3 / 3$. Therefore, except for the sign that has yet to be established, one will have, in all:

$$d\varpi = \int \frac{1}{a^3}\mathfrak{a}\mathrm{V}d\mathfrak{s}\, d\mathfrak{x} = d\mathfrak{x}\int \mathrm{V}\frac{\mathfrak{a}}{a^3}d\mathfrak{s} \;. \tag{80}$$

In the last conversion, use was made of a theorem that used before in § **16** in regard to calculating with the geometric product, and the constant quantity $d\mathfrak{x}$ was moved in front of the integral sign. Eq. (80) is true any arbitrary displacement $d\mathfrak{x}$, as long as it is only infinitely small. The factor that multiplies $d\mathfrak{x}$ is then the quantity that defined the negative gradient in ϖ in our previous terminology, and was found by means of the operator ∇ [cf., eq. (17)]. We then have:

$$\nabla\varpi = \int \mathrm{V}\frac{\mathfrak{a}}{a^3}\,d\mathfrak{s}\ , \tag{81}$$

in which the integration is naturally extended over the entire length of the vortex filament, as before. If we substitute that value in eq. (79) then we will get:

$$\mathfrak{v} = \frac{W}{4\pi}\int \mathrm{V}\frac{\mathfrak{a}}{a^3}\,d\mathfrak{s}\ , \tag{82}$$

except for the sign, whose ultimate definition was already omitted in the foregoing formulas.

The sign determination that still remains to be done is easily obtained from the remarks in § **16** afterwards. With the assumption that *W* is always reckoned to be positive and the sense of traversal of the vorticity is expressed by the direction of $d\mathfrak{s}$, the sequence $d\mathfrak{s}$, $\mathfrak{v}$, $\mathfrak{a}$ must lead to a right-handed system in space in order for eq. (82) to also have the correct sign. The arrow for $\mathfrak{v}$ will be determined from a given arrow for $d\mathfrak{s}$, and conversely. One can also express that by saying that the direction around which the vortex line flows, in conjunction with the direction $d\mathfrak{s}$ of the vortex filament itself, will lead to a right-handed screw.

Eq. (82) represents $\mathfrak{v}$ in the form of an integral over the closed conducting line of the vortex filament. It is tempting to resolve this integral into its individual elements, and therefore to establish that each element of length of the vortex filament will yield a contribution of:

$$d\mathfrak{v} = \frac{W}{4\pi}\mathrm{V}\frac{\mathfrak{a}}{a^3}\,d\mathfrak{s} \tag{83}$$

to the total field. The $d\mathfrak{v}$ that originates from an element of an isolated vortex filament is then perpendicular to the plane that goes through the reference point and $d\mathfrak{s}$, which points in the direction that is indicated by the outstretched left hand of the Ampèrian swimmer, is proportional to the vortex strength *W*, the length of $d\mathfrak{s}$, and the sine of the angle between $\mathfrak{a}$ and $d\mathfrak{s}$, and is inversely proportional to the square of the distance *a* between the reference point and the element of the vortex filament. We then have the *Biot-Savart* law, which plays such a significant role in the study of electromagnetism. However, at the same time, we see that its validity is not restricted to that special case, but that it is based upon a general property of vector functions.

That raises the question of whether we have actually chosen the simplest-possible case, as we did when we examined the field that was generated by a single vortex filament. It is tempting to think that the simplest case would be defined by the field of a single element of the vortex filament. However, that would be a mistake. If we wished to consider the field that is defined by eq. (83) then we would soon find that the vortex in it cannot be shrunk to the element $d\mathfrak{s}$ at all. Rather, as soon as one performs the operation curl on the value of $d\mathfrak{v}$ that was given by eq. (83), one will convince oneself that all of space will be filled with a system of vortices in that way. One should not have expected things to be otherwise from the outset, since we have already found previously that div $\mathfrak{w}$ is always zero, so an isolated element of a vortex filament with a beginning and an end would be a geometric impossibility.

The use of eq. (83) is then allowable only to the extent that one agrees to later integrate the expression over the entire extent of the vortex filament. Eq. (83) should always be considered to be only a preliminary form of eq. (82), whose validity is all that must be proved. There is absolutely no reason for one to speak of, e.g., the magnetic field that is generated by a current element (or of an elementary potential between two current elements, as in Helmholtz's older theory of electrodynamics). One will then fall victim to a fallacy that is rife in the history of the study of electricity.

§ 22. – Directed sources and Ampère's vortices.

The ambition to reduce the vortex filament to even simpler elements from which the latter can be thought of as being composed is completely justified in its own right. However, from the foregoing discussion, that process can happen only when each element again defines a closed vortex filament when taken by itself. The only simplification that our problem can admit will then consist of shrinking the vortex line to an infinitely-small region.

In fact, a vortex filament of finite extent will always be composed of a doubly-infinite number of vortex filaments, each of which extend over only an infinitely-small region, in a different way. A decomposition of that kind was already carried out in § **17**, and here I will be content to refer back to that discussion.

The decomposition into elementary vortices likewise shows a new aspect of the relationship that exists between sources and vortices and has also been emphasized repeatedly here. An elementary vortex – and thus, a closed vortex filament with everywhere infinitesimal dimensions – requires that a certain flux must flow through its opening, while a source requires that a certain flux must emanate from it in all directions. As the cause of a field, the source does not possess a well-defined direction, but the vortex probably does. One can regard the elementary vortex as a directed source, where one naturally has in mind that the elementary vortex does not allow newly-created fluid to stream into the field, as the source would, but only provokes a current in the fluid that is present already.

The complete exposition of the comparison encounters the difficulty that one cannot say how large the flux through the elementary vortex is (or in other words, the productivity of the directed source) without first making a more detailed analysis of the cross-section of the vortex filament and the distribution of vorticity over that cross-section.

We cannot therefore give too much significance to that comparison. I have mentioned it mainly because what one refers to as an applied force in the study of electricity has completely the character of a directed source. We can conclude from our discussion that we would do better to replace applied forces of that kind with the vortices that correspond to them.

Here, I would like to recall one of the most celebrated examples of an application of what I am trying to teach here. One is first compelled to study vortex fields in the study of magnetic phenomena. One initially assumes that the magnetic field of a steel magnet is vortex-free, but then finds that an electric current creates a magnetic vorticity field, which is how Ampère showed that one can best explain the phenomena in steel magnets when one regards their fields as vorticity fields. The molecular currents that that flow around an iron molecule, according to Ampère, are in fact basically only elementary vortices that explain the field.

In my book on Maxwell's theory, I have listed a series of reasons that contradict Ampère's hypothesis of molecular currents, and I am completely convinced that this hypothesis is false and untenable in its original form. The fact that the hypothesis has been preserved for so long and the fact that it might give a quite satisfactory account of many phenomena stems from the fact that, on the other hand, it also comes quite close to reality. It was a fortunate inspiration to liberate the study of magnetism from the idea that physical fields must always be regarded as vortex-free and derived from sources, which has been customary since the time of Newton. Today, one might be more certain that the magnetic fields in the interiors of hard magnetic bodies are not vortex-free and that any concept that avoids the use of magnetic masses and allows vortices must pave the way for a giant step forward in the understanding of magnetic fields.

None of the objections that have been raised against Ampère's theory since the time of its creation have been directed against the notion of elementary vortices, but only against the identification of the vortices with electric currents, since in order to explain the phenomena, one needs a vortex for the field $\mathfrak{B}$, and not for the field $\mathfrak{H}$, as in Ampère's theory.

I have briefly touched upon that, in part, still not completely explained physical problem in order to point out the close relationship that exists between the questions at issue and the general theory of vector functions. The theory of magnetism will first attain its definitive form once the entire scope has been examined thoroughly and consistently with the tools that are given by the geometry of vortex fields. Experimental arrangements of the kind that are familiar to the best known researchers in the theory of vortex-free fields cannot lead to any definitive results here, since experiments only allow one to resolve the admissibility of intuitions that were already considered to be possible up to now. They do not teach one about any new concepts that are probably still lacking (at least, for the moment). Those concepts must be defined in a different way.

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CHAPTER FOUR

THE VORTEX INTEGRATION OF SOURCE-FREE VECTOR FUNCTIONS.

§ 23. – The vector potential.

The problem of determining the field that is created by a given vortex filament was, in fact, solved already in the foregoing chapter. However, the solution was not entirely satisfactory, due to the detour that was taken in it. The reduction of the field to a vortex-free one that one introduced for the sake of examining the theorems that would be valid in that case by means of a double layer represented a gimmick that would ultimately drop out of the final result and that indeed showed us how to find the correct result, but by its application, we overlooked a method that would be better suited to the type of problem.

Instead of ordinary potential theory, which has a simple meaning only for vortex-free fields that is so rightly established that it can also lead to a solution here, in the case of vortex fields, one would, in fact, do better to abandon the auxiliary quantity V completely and look for a replacement for it. Due to the already oft-emphasized kinship between the two classes of fields, one might expect from the outset that one can also give a function for the vortex fields that relates to them in the same way that the potential V relates to the vortex-free fields that are due to sources.

We then seek a quantity from which the source-free field $\mathfrak{v}$ can be derived by means of a differential operator in just the same way that the vortex-field field was previously derived by the operator $-\nabla$. In other words, that also says that we would like to look for a different kind of geometric integration for a given source-free vector function $\mathfrak{v} = f(\mathfrak{r})$. The source quantity that we seek cannot be a scalar, since we know of only one kind of spatial differentiation for scalar fields (namely, ∇), and, as we saw before, it will not lead to the desired result. We must then seek to determine a vector function $\mathfrak{A} = F(\mathfrak{r})$ such that it can be regarded as an integral of the given function $\mathfrak{v} = f(\mathfrak{r})$; i.e., such that $\mathfrak{v}$ will emerge from $\mathfrak{A}$ by spatial differentiation. I therefore place some value upon emphasizing this relationship between the quantities $\mathfrak{A}$ and $\mathfrak{v}$ and the connection with the integration problem of the ordinary theory of functions.

We know of two kinds of spatial differentiation for vector functions: viz., the div and curl operators. Only the second one can come under consideration here, since the first one would lead to a scalar quantity. Only one possibility of deriving the desired integral remains open then, namely, determining $\mathfrak{A}$ in such a way that it satisfies the equation:

$$\mathfrak{v} = \text{curl}\,\mathfrak{A}. \tag{84}$$

Naturally, $\mathfrak{A}$ is still not determined completely by that equation: As with any integration, an arbitrary quantity can appear that plays the role of an integration constant. Namely, a term $\mathfrak{A}_0$ can be added to any solution of eq. (84), which might be an otherwise totally arbitrary function of $\mathfrak{r}$ that only has to satisfy the condition:

$$\text{curl}\,\mathfrak{A}_0 = 0. \tag{85}$$

The function $\mathfrak{A}_0$ is thus characterized as vortex-free, but it can be assigned an arbitrary system of sources div $\mathfrak{A}_0$.

Among all of the solutions of eq. (84) that are possible in that way, we would like to choose the simplest of them, namely, the value of $\mathfrak{A}$ that is likewise source-free in all of space. In order to determine the integration problem uniquely, we would then like to arbitrarily establish that the source quantity $\mathfrak{A}$ should fulfill the auxiliary condition:

$$\text{div}\,\mathfrak{A} = 0. \tag{86}$$

One obtains the fact that $\mathfrak{A}$ is determined uniquely by equations (84) and (86) and the obvious condition that it should vanish at infinity immediately from the theorem that was proved in § **19** that a source-free vector function that does not extend to infinity is defined uniquely by a system of vortices.

In the theory of electricity, the term *vector potential* is introduced for the integral of the function $\mathfrak{v}$ that is determined in that way. It does not emerge clearly enough from that terminology that one is dealing with a simple integration. It might also be permissible then to refer to the vector potential as the *vortex integral* of the function $\mathfrak{v}$. In fact, a more precise terminology that shows directly that one is dealing with the inversion of the differential operator curl is also desirable because one requires a symbol for the operator that cannot be abbreviated by pot, since that was used already with a different meaning. I shall therefore employ the abbreviation WJ (*Wirbelintegral*) for the vortex integral and set:

$$\mathfrak{A} = \text{WJ}\,\mathfrak{v} \tag{87}$$

as the inversion or solution of eq. (84) with the aforementioned auxiliary conditions.

§ 24. – Obtaining the integral.

We shall next perform a spatial differentiation of eq. (84). When we recall eq. (70), the operator div will yield div $\mathfrak{v} = 0$. We see from this that the vector potential is suited only to the investigation of source-free fields, just as the scalar potential V was suited to only the investigation of vortex-free ones. By contrast, the operator curl yields:

$$\text{curl}^2\,\mathfrak{A} = \text{curl}\,\mathfrak{v} = \mathfrak{w}. \tag{88}$$

The vorticity $\mathfrak{w}$ of the field $\mathfrak{v}$ is thus introduced here once more. Another conversion of this formula can be made that only comes down to a different arrangement of the individual terms on the left-hand side when one thinks of them as being developed into coordinate expressions. Namely, if $\mathfrak{F}$ denotes an entirely-arbitrary continuous field then one will always have:

$$\text{curl}^2\, \mathfrak{F} = \nabla\, \text{div}\, \mathfrak{F} - \nabla^2\, \mathfrak{F} . \tag{89}$$

One will find a proof of that theorem in my previous book [eq. (72), pp. 59], but it can also be easily proved by itself when one expresses the symbols that appear in it in coordinates using the prescription that was given before. When applied to the vector potential, and when one recalls eq. (86), eq. (89) will give:

$$\text{curl}^2\, \mathfrak{A} = -\, \nabla^2\, \mathfrak{A} . \tag{90}$$

With that conversion, which already plays an important role in the older theory of electricity, although it does not appear as clearly there as it does in eq. (90), due to the details of the calculations that are linked with it there, eq. (88) will go to:

$$\nabla^2\, \mathfrak{A} = -\, \mathfrak{w}. \tag{91}$$

That equation has just the form of the Laplace-Poisson equation for the scalar potential. The only difference between the two cases consists of the fact that eq. (32) refers to scalar quantities, while directed quantities enter into eq. (91). However, that difference does not prevent one from adapting the solution to the Laplace equation that was found before to the case that is now under scrutiny.

Namely, if one decomposes eq. (91) into its components along the directions of the three coordinate axes then one will get:

$$\nabla^2 A_1 = -\, w_1 , \qquad \nabla^2 A_2 = -\, w_2 , \qquad \nabla^2 A_3 = -\, w_3 \tag{92}$$

when one denotes the components in the usual way.

The component A_1 of the vector potential therefore likewise defines the scalar potential for a source-distribution of intensity w_1, and naturally the same remark is true for not only the other two coordinate axes, but also for any arbitrary direction in space onto which we project the vectors $\mathfrak{A}$ and $\mathfrak{w}$.

The unique solution to the Laplace equation with the condition that is also valid here that the potential should vanish at infinity was given by eq. (29), and we will then also have:

$$A_1 = \frac{1}{4\pi} \int^{\infty} \frac{w_1\, d\tau}{a} . \tag{93}$$

A_2 and A_3 can also be calculated in the same way, and the total $\mathfrak{A}$ can be composed by geometrically summing its components. One will once more find all of the components of $\mathfrak{w}$ in that way and obtain:

$$\mathfrak{A} = \frac{1}{4\pi}\int^{\infty} \frac{\mathfrak{w}\, d\tau}{a}. \tag{94}$$

It does not emerge from this derivation whether the function $\mathfrak{A}$ that was found likewise satisfies the condition that div $\mathfrak{A} = 0$. We shall convince ourselves that this is the case later. If we differentiate eq. (93) with respect to x and observe that only a will change under a shift of reference point then once we add $\partial A_2 / \partial y$ and $\partial A_3 / \partial z$ to the other two terms, we will get:

$$\operatorname{div} \mathfrak{A} = \frac{1}{4\pi}\int^{\infty} \mathfrak{w} \cdot \nabla \frac{1}{a}\, d\tau. \tag{95}$$

In order to calculate that integral, I start from the following equation:

$$\operatorname{div}\left(\frac{1}{a} \cdot \mathfrak{w}\right) = \frac{1}{a} \operatorname{div} \mathfrak{w} + \mathfrak{w} \cdot \nabla \frac{1}{a}, \tag{96}$$

which is merely an identity, as one can convince oneself by developing the individual terms [see *Maxw. Theorie*, eq. (78), pp. 61]. However, the first term on the right-hand side will vanish from eq. (70) and from the meaning that $\mathfrak{w}$ has in the present case. The value that was found for div $\mathfrak{A}$ before can also be replaced with:

$$\operatorname{div} \mathfrak{A} = \frac{1}{4\pi}\int^{\infty} \operatorname{div}\left(\frac{1}{a} \cdot \mathfrak{w}\right) d\tau. \tag{97}$$

However, the function $\frac{1}{a}\mathfrak{w}$ does not extend to infinity, since $\mathfrak{w}$ does not extend to infinity to begin with. When the integral is extended over all of infinite space, it will yield the value zero, as was found already in eq. (7). The required proof is complete with that, and eq. (94) will then, in fact, define the unique solution of eq. (84) with the auxiliary conditions that were imposed upon that solution.

Finally, I would like to point out that the agreement between the Laplace equations for the scalar and vector potentials once more confirms the close connection between sources and vortices. One has the reciprocal correspondence:

In vortex-free fields	In source-free fields
Source q	Vortex $\mathfrak{w}$
Scalar potential V	Vector potential $\mathfrak{A}$
Operator ∇ or div	Operator curl
Operator $-\nabla^2$	Operator curl^2
eq. (32)	eq. (91)
eq. (29)	eq. (94)

§ 25. – Connections between the functions $\mathfrak{A}$, $\mathfrak{v}$, $\mathfrak{w}$.

It is useful to once more clarify the reciprocal connection that exists between the three functions $\mathfrak{A}$, $\mathfrak{v}$, $\mathfrak{w}$. To that end, I shall give the following summary:

Any quantity in the sequence:

$$\mathfrak{A},\ \mathfrak{v},\ \mathfrak{w}$$

will be the vortex integral of the one that follows it, and conversely, it will be derived from the one that precedes it by the spatial differentiation curl, so:

$$\left.\begin{array}{ll} \mathfrak{v} = \text{curl}\,\mathfrak{A}, & \mathfrak{w} = \text{curl}\,\mathfrak{v} = \text{curl}^2\mathfrak{A}, \\ \mathfrak{v} = \text{WJ}\,\mathfrak{w}, & \mathfrak{A} = \text{WJ}\,\mathfrak{v} = \text{WJ}^2\mathfrak{w}. \end{array}\right\} \tag{98}$$

Once might also write the last formula as:

$$\mathfrak{A} = \text{pot}\,\mathfrak{w}, \tag{99}$$

such that operator symbol pot will be identical to WJ^2.

Naturally, one can also think of the sequence of functions $\mathfrak{A}$, $\mathfrak{v}$, $\mathfrak{w}$ as being continued arbitrarily in the same way to the right or left. The relations that are expressed by equations (98) will then exist between each of group of three successive terms in that entire sequence. That is a result of eq. (94), by which one will be in a position to actually perform the operator WJ^2 or pot.

In order to the find, e.g., the term that immediately precedes the term $\mathfrak{A}$ in the total sequence, which might be denoted by $\mathfrak{X}$, one sets:

$$\mathfrak{X} = \text{pot}\,\mathfrak{v} = \frac{1}{4\pi}\int^{\infty}\frac{\mathfrak{v}\,d\tau}{a}, \tag{100}$$

and one will then have:

$$\mathfrak{A} = \text{curl}\,\mathfrak{X}, \qquad \mathfrak{v} = \text{curl}\,\mathfrak{A} = \text{curl}^2\,\mathfrak{X}, \quad \mathfrak{w} = \text{curl}\,\mathfrak{v} = \text{curl}^3\,\mathfrak{X}, \tag{101}$$

and likewise:

$$\begin{gathered} \mathfrak{v} = \text{WJ}\,\mathfrak{w}, \qquad \mathfrak{A} = \text{WJ}\,\mathfrak{v} = \text{WJ}^2\,\mathfrak{w} = \text{pot}\,\mathfrak{w}, \\ \mathfrak{X} = \text{WJ}\,\mathfrak{A} = \text{WJ}^2\,\mathfrak{v} = \text{pot}\,\mathfrak{v} = \text{WJ}^3\,\mathfrak{w} = \text{WJ pot}\,\mathfrak{w} = \text{pot WJ}\,\mathfrak{w}. \end{gathered} \tag{102}$$

An important relationship between the terms in the sequence emerges from this summary. Namely, when $\mathfrak{v}$ is given, one can derive the associated vortex integral $\mathfrak{A}$ in two different ways: One might first take the curl of $\mathfrak{v}$ and look for the potential $\mathfrak{w}$ that is found from it using eq. (94). However, in place of that process (which is the only one that was discussed up to now), one can conversely also first define the potential $\mathfrak{X}$ of $\mathfrak{v}$

and then derive the desired function $\mathfrak{A}$ with the help of the curl operator. One will then have:

$$\mathfrak{A} = \text{WJ } \mathfrak{v} = \text{pot curl } \mathfrak{v} = \text{curl pot } \mathfrak{v} \,. \tag{103}$$

The operator symbols curl and pot can then be transposed in the sequence, as if one were dealing with only source-free fields, and when both of them are composed with each other, that will give the vortex integral WJ.

All of those considerations can also be adapted to the scalar potential V, the vector field $\mathfrak{v}$ that belongs to it, and the source system q. We can also extend the sequence V, $\mathfrak{v}$, q by arbitrarily many terms to the left and right so that any term can be obtained from the foregoing one by a spatial differentiation, and indeed it can be derived by alternating div and $-\nabla$. That kind of differentiation brings with it the fact that the terms in the sequence will be alternately directed and undirected quantities. For example, if one assumes that the terms before and after any term, which must both be vectors then, as well as being assumed to be vortex-free, then the sequence will possibly read like:

$$\mathfrak{M},\ V,\ \mathfrak{v},\ q,\ \mathfrak{p}.$$

One will then have:

$$V = \text{div } \mathfrak{M}, \qquad \mathfrak{v} = -\nabla V = -\nabla \text{ div } \mathfrak{M} = -\nabla^2 \mathfrak{M}. \tag{104}$$

In the last conversion, one must observe the identity eq. (89) and the condition that one should have curl $\mathfrak{M} = 0$. However, it follows from the last equation that the concept of the vector potential (or more precisely, the concept of a directed potential, in general) can also be adapted to the treatment of vortex-free fields. Namely, one gets from eq. (104) that:

$$\mathfrak{M} = \frac{1}{4\pi}\int^{\infty} \frac{\mathfrak{v}\, d\tau}{a}\,. \tag{105}$$

We already know that we can set:

$$q = \text{div } \mathfrak{v} = -\text{ div } \nabla V = -\nabla^2 V.$$

By contrast, we will arrive at a new relation when we focus on the term $\mathfrak{p}$ in our sequence. We will then have:

$$\mathfrak{p} = -\nabla q = -\nabla \text{ div } \mathfrak{v} = -\nabla^2 \mathfrak{v}, \tag{105}$$

in which the condition that $\mathfrak{v}$ should be vortex-free was essential. We get the solution:

$$\mathfrak{v} = \frac{1}{4\pi}\int^{\infty} \frac{\mathfrak{p}\, d\tau}{a} \tag{107}$$

from the last equation, and with that, we have, in fact, a second method for calculating the field quantity $\mathfrak{v}$ when the source q is given, along with the one that is usually given. In order to do that, we need only to construct the gradient field $\mathfrak{p} = - \nabla q$ of the source intensity q and define the potential from that using the usual rules of calculation.

These discussions can also be summarized in one equation that defines the counterpart to eq. (103), namely:

$$\text{pot } \nabla q = \nabla \text{ pot } q, \tag{108}$$

which is fulfilled for any scalar function q of a radius vector.

§ 26. – Solving the main problem with the help of the vector potential.

Previously, we proposed that the fundamental problem of the geometry of vortex fields was to find the field $\mathfrak{v}$ that belongs to a given vorticity distribution $\mathfrak{w}$. However, the function $\mathfrak{w}$ is necessarily source-free, so the solution to the main problem will already be included in the foregoing discussion. In fact, it only comes down to performing the vortex integration:

$$\mathfrak{v} = \text{WJ } \mathfrak{w} \tag{109}$$

on the given source-free function $\mathfrak{w}$. From eq. (103), that can always be done in two different ways that both lead to the same objective.

Ordinarily, one will first define the vector potential $\mathfrak{A}$ by the rules of calculation that were given in eq. (94) and one will then find that $\mathfrak{v} = \text{curl } \mathfrak{A}$, However, one can also conversely first derive an auxiliary quantity $\mathfrak{z}$ by setting:

$$\mathfrak{z} = \text{curl } \mathfrak{w} \tag{110}$$

and then get $\mathfrak{v}$ from that in the form of a vector potential:

$$\mathfrak{v} = \frac{1}{4\pi} \int^{\infty} \frac{\mathfrak{z} \, d\tau}{a} . \tag{111}$$

Naturally, that not only solves the problem for the case of a single vortex filament, but also for any system of vortices, with full generality, and that will show directly how much this method overlaps with the one that was discussed in the precious chapter.

Meanwhile, the equations also simplify considerably for a single vortex filament. When the vortex strength is denoted by W and an element of length of the centerline of the vortex filament is denoted by $d\mathfrak{s}$, as before, one will get:

$$\mathfrak{A} = \frac{W}{4\pi}\int\frac{d\mathfrak{z}}{a} \tag{112}$$

for the vector potential $\mathfrak{A}$.

The formulas of the previous chapter also emerge from this with no further discussion. The operation curl, which must be applied to the latter expression in order to obtain $\mathfrak{v}$, relates to only the variability of the position of the reference point. However, only the distance *a* depends upon the position of the reference point in eq. (112), and any element of the integral contributes to curl $\mathfrak{A}$ independently of any other one. Now since one has:

$$\text{curl}\left(\frac{1}{a}d\mathfrak{s}\right) = \frac{1}{a}\text{curl}\,d\mathfrak{s} + \mathrm{V}\left(\nabla\frac{1}{a}\right)d\mathfrak{s} \tag{113}$$

identically and in general, as one easily assures oneself by calculation [see *Maxw. Theorie*, eq. (80)], while $d\mathfrak{s}$ is constant here (so curl $d\mathfrak{s}$ will be zero), one will have:

$$\mathfrak{v} = \text{curl}\,\mathfrak{A} = \frac{W}{4\pi}\int\mathrm{V}\left(\nabla\frac{1}{a}\right)d\mathfrak{s}, \tag{114}$$

which will once more imply eq. (82) directly upon applying the operator ∇ to 1 / *a*.

That formula can also be derived by the second of the two methods that were just discussed. Of course, the second process is less suited to being applied to isolated vortex filaments, since it would require somewhat cumbersome calculations here. I shall thus be content to give a general outline of the solution.

One first focuses on the field $\mathfrak{z}$ – i.e., the vorticity that is produced by the given vortex filament. The field $\mathfrak{z}$ is contained entirely in the space that the vortex filament occupies. The streamlines of $\mathfrak{z}$ encircle the centerline of that filament; $\mathfrak{z}$ is zero everywhere outside of the filament, like $\mathfrak{w}$ itself. It would be simplest for one to imagine that the vorticity *W* is distributed uniformly over the cross-section of the filament up to the vicinity of the boundary, since one can distribute that vorticity arbitrarily. $\mathfrak{z}$ will then be concentrated on the surface of the filament, and it will be perpendicular to the longitudinal direction there. I draw a half-plane through the element $d\mathfrak{s}$ of the centerline of the vortex filament in an arbitrary direction and calculate how large the flux of $\mathfrak{z}$ that flows through the half-plane that belongs to $d\mathfrak{s}$ will be. With those preliminaries, I perform the integration that is prescribed by eq. (111) over the space of the element of the vortex filament that belongs to $d\mathfrak{s}$. If an element (that is a second-order infinitesimal) of the boundary of the cross-section is denoted by $d\mathfrak{s}'$ then one will get:

$$\frac{\mathfrak{w}\,d\mathfrak{s}}{4\pi}\int\frac{d\mathfrak{s}'}{a}$$

for that integral, where the integration is extended over the entire boundary of the cross-section.

However, with an extension of Stokes's theorem (for which I must refer to my previous book, § **31**), the line integral of a scalar over a closed curve can always be replaced with a surface integral over the surface that the curve encloses. From that theorem, one will always have:

$$\int \frac{1}{a} d\mathfrak{s}' = \int \mathrm{V}\mathfrak{N}\nabla\frac{1}{a} df . \tag{115}$$

The unit normal $\mathfrak{N}$ to the cross-section of the vortex filament that enters into this, whose element is denoted by *df*, points in the direction of the centerline so, except for its sign (the discussion of which can be skipped here), it points in the direction of $d\mathfrak{s}$. Furthermore, $\mathfrak{N}\nabla\frac{1}{a}$ can be regarded as constant over the entire cross-section, since the surface *f* is itself infinitely small and $\mathfrak{N}$ is constant. If one considers that the contribution to the integral (111) that is due to the element of the vortex filament can also be set equal to:

$$\frac{\mathfrak{w}\mathfrak{N}}{4\pi} f \,\mathrm{V} d\mathfrak{s}\nabla\frac{1}{a} \qquad \text{or} \qquad \frac{W}{4\pi}\mathrm{V} d\mathfrak{s}\nabla\frac{1}{a}$$

then that will once more lead one to eq. (114) when one performs the integration over $d\mathfrak{s}$ (except for the sign that was left undetermined).

§ 27. – Flux between two vortex filaments.

For the time being, let only one vortex filament be given in a field, and I will denote that filament by the index 1. In addition, a closed line shall be given that will be denoted by the index 2. One must then calculate the flux F_{12} that the vortex filament 1 communicates to the line 2, or as one can also say, the flux that is linked by the line 2. That flux will be represented by a surface integral over an otherwise-arbitrary surface whose boundary curve is the line 2, and the terminology originates in the fact that for the hydrodynamical construction of the vector functions, the surface integral will measure the fluid volume that flows through the surface per unit time. One then has:

$$F_{12} = \int \mathfrak{v}\mathfrak{N}\, df \tag{116}$$

for the defining equation for the flux F_{12} , when one expressly agrees that $\mathfrak{v}$ means the field that is created by the vortex filament 1 and that the integration is extended over a surface that is bounded by 2.

If one observes that one can set $\mathfrak{v} = \text{curl}\,\mathfrak{A}$ then one will also get from eq. (116) that:

$$F_{12} = \int \mathfrak{A}\, d\mathfrak{s}_2 \tag{117}$$

by an application of Stokes's theorem, in which $d\mathfrak{s}_2$ means a line element of curve 2 over which the integration is extended.

Finally, the value of $\mathfrak{A}$ that was established in eq. (112) can be substituted above, from which it will follow that:

$$F_{12} = \frac{W}{4\pi} \iint \frac{d\mathfrak{s}_1\, d\mathfrak{s}_2}{a}\,. \tag{118}$$

The double integral that appears in this equation depends upon only the form and mutual positions of the two lines 1 and 2. It already played an important role in the older theory of electricity, and in that context, it was referred to as the *coefficient of mutual induction* of the two lines 1 and 2. If we set:

$$L_{12} = \iint \frac{d\mathfrak{s}_1\, d\mathfrak{s}_2}{a}\,, \tag{119}$$

to abbreviate, then since the sequence of both integrations can be inverted, it will follow that:

$$L_{21} = L_{12}\,; \tag{120}$$

i.e., when the line 2 carries a vortex filament of the same strength that line 1 did before, line 1 will now be linked with a flux that is just as large as the flux that linked line 2 in the previous case.

One can also let the two lines 1 and 2 coalesce into one. The double integral (119) will then become logarithmically infinite, like the flux F that links the vortex filament itself, in the event that W has an infinite value. The basis for that is easy to see: For a finite value of the vorticity W that is concentrated on a line, $\mathfrak{v}$ will be infinitely large in the immediate neighborhood of the line. One can then only calculate the *coefficient of self-induction* of a vortex filament when the distribution of $\mathfrak{w}$ over the cross-section of the filament is given. The flux that is generated by a vortex filament, and is at the same time, linked by it, is carried at most by the close neighborhood of the filament, and therefore it is not permissible in such cases to think of the entire vorticity as being concentrated on the centerline.

§ 28. – Coefficient of induction between two coaxial circles.

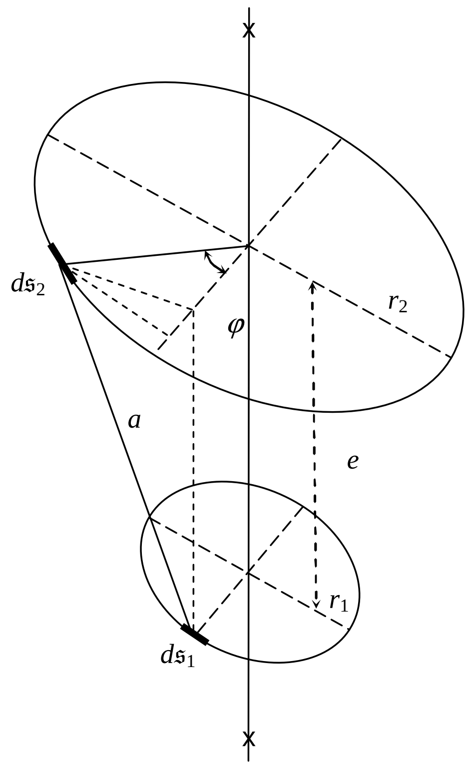
Up to now in this book, which is mostly devoted to the discussion of fundamental questions, I have avoided performing the peripheral calculations or going into the details very thoroughly. However, due to the considerable practical significance of the coefficient of induction between two coaxial circles, I shall make an exception for that topic.

When one refers to the figure below for the notations, one will have:

$$d\mathfrak{s}_1\, d\mathfrak{s}_2 = ds_1\, ds_2 \cos\varphi\,.$$

I next extend the integral (119) over the upper circle. One has:

$$a = \sqrt{e^2 + (r_2 \sin\varphi)^2 + (r_2 \cos\varphi - r_1)^2} = \sqrt{c^2 - 2\, r_1\, r_2 \cos\varphi}\,,$$



when one sets $c^2 = e^2 + r_1^2 + r_2^2$, to abbreviate. Since one further has $ds_2 = r_2\, d\varphi$, one will get:

$$\int \frac{\cos\varphi\, ds_2}{a} = 2r_2 \int_0^\pi \frac{\cos\varphi\, d\varphi}{\sqrt{c^2 - 2\, r_1\, r_2 \cos\varphi}}\,.$$

That integral is an elliptic one. In order to reduce it to its Legendre normal form, one sets:

$$\varepsilon^2 = \frac{2r_1\, r_2}{c^2}\,.$$

The quantity ε that is determined by that is always a proper fraction, since when one subtracts the numerator from the denominator, one will always get a positive result. The total L_{12} can now be written on the form:

$$L_{12} = \frac{4\pi\, r_1\, r_2}{c} \int_0^\pi \frac{\cos\varphi\, d\varphi}{\sqrt{1 - \varepsilon^2 \cos\varphi}}\,, \tag{121}$$

since the integration over ds_1 can be performed by simply multiplying by the circumference of the circle 1. From the symmetry about the common axis, one sees that each element ds_1 will make the same contribution to L_{12} .

With the substitution:

$$\varphi = \pi - 2\psi , \tag{122}$$

eq. (121) will go to:

$$L_{12} = \frac{4\pi r_1 r_2}{c\sqrt{1+\varepsilon^2}} \int_0^{\pi/2} \frac{4\sin^2\psi - 2}{\sqrt{1 - \frac{2\varepsilon^2}{1+\varepsilon^2}\sin^2\varphi}} d\psi , \tag{123}$$

and that expression can be further decomposed into the sum of a complete elliptic integral of the first kind and one of the second kind, such that one will ultimately get:

$$L_{12} = \frac{8\pi r_1 r_2}{c\,\varepsilon^2\sqrt{1+\varepsilon^2}} \left[F\left(\frac{\pi}{2}, k\right) - (1+\varepsilon^2) E\left(\frac{\pi}{2}, k\right) \right], \tag{124}$$

in which one sets:

$$k = \sqrt{\frac{2\varepsilon^2}{1+\varepsilon^2}} = 2\sqrt{\frac{r_1 r_2}{e^2 + (r_1 + r_2)^2}} , \tag{125}$$

to abbreviate. k is also always a proper fraction then, and L_{12} can then be found from eq. (124) with no further analysis with the help of Legendre's tables.

That development also allows one to calculate the coefficient of self-induction of a circular vortex filament to a suitable degree of approximation. Let the cross-section of the vortex filament over which the vorticity W is assumed to be distributed uniformly be a circle of radius ρ, which might be regarded as being very small compared to the radius r of the centerline of the vortex filament. One will then wish to calculate the total flux F that is created by the vortex filament and links the centerline of the filament.

To that end, I draw a circle that is concentric to the centerline whose radius is smaller than r by δ. Therefore δ shall be small compared to ρ and large compared to r, but otherwise chosen arbitrarily. We first calculate the flux that goes through the surface of that auxiliary circle. To that end, we set:

$$e = 0, \qquad r_1 = r, \qquad r_2 = r - \delta, \qquad \varepsilon = 1 - \frac{\delta^2}{2r^2}, \quad k = 1 - \frac{\delta^2}{4r^2}$$

in the foregoing development, in which small quantities of higher order are neglected. The modulus k in the elliptic integral that occurs in eq. (124) differs from unity by only a second-order infinitesimal. From known series developments (*), one can set:

$$F\left(\frac{\pi}{2}, k\right) = \ln\frac{8r}{\delta} + \left(\frac{1}{2}\right)^2 \left[\ln\frac{8r}{\delta} - 1\right]\frac{\delta^2}{4r^2} + \ldots,$$

(*) Cf., Schlömilch, *Compendium der höheren Analysis*, v. 2, 3rd ed., pp. 322, Braunschweig, 1879.

$$E\left(\frac{\pi}{2},k\right)=1+\frac{1}{2}\left[\ln\frac{8r}{\delta}-\frac{1}{1\cdot 2}\right]\frac{\delta^2}{4r^2}+\dots$$

Meanwhile, due to assumption that was made about the magnitude of the ratio δ / r, the second term in this development can be dropped. One will then find that:

$$L_{12}=4\pi r\left(\ln\frac{8r}{\delta}-2\right),$$

and the flux that goes through the surface of the auxiliary circle will then be:

$$F_{12}=W\,r\left(\ln\frac{8r}{\delta}-2\right).$$

Another flux flows through the annular surface of width $\delta-\rho$ that lies between the auxiliary circle and the boundary of the vortex filament that is easy to determine. The absolute value v of the velocity at a distance x from the centerline can then be set to:

$$v=\frac{W}{2\pi x}$$

in the strip that is currently of interest to us, and from that, the flux through the strip is found to be:

$$2\pi r\int_{\rho}^{\delta}v\,dx=Wr\ln\frac{\delta}{\rho}.$$

All that remains is the flux that flows through the interior of the vortex filament. We get:

$$v=\frac{W\,x}{2\pi\rho^2}$$

here in a similar way, and the flux will then be equal to:

$$2\pi r\int_{0}^{\rho}v\,dx=\frac{W\,r}{2}.$$

If we now combine all three terms together then we will get:

$$F=W\,r\left(\ln\frac{8r}{\rho}-\frac{3}{2}\right) \tag{126}$$

for the total flux F that links the centerline.

The quantity δ that was previously chosen arbitrarily will vanish from the ultimate formula. The coefficient of self-induction will follow from that equation when one divides by $W / 4\pi$, so:

$$L = 4\pi r \left(\ln \frac{8r}{\rho} - \frac{3}{2} \right). \tag{127}$$

§ 29. – Different interpretations for the vector potential.

The scalar potential V was introduced in § **5** as a line integral of the field quantity $\mathfrak{v}$, and the deep significance that the concept of potential enjoys in physics is based, in particular, upon the fact that this line integral will represent an amount of work done when the field $\mathfrak{v}$ is a force field. Meanwhile, once we have recognized the close relationship between the vector potential $\mathfrak{A}$ and the scalar potential V, we cannot avoid the question of whether we can also find a similar interpretation for $\mathfrak{A}$. However, since not much seems to have come to light from that investigation, I will touch upon it only quite briefly.

The origin of the entire splitting of the theory of vector functions into two closely-parallel parts, so the juxtaposition of sources and vortices, of scalar and vector potentials, lies in the two types of geometric products that one can form. Since V is obtained from the field quantity and the element of the integration path with the help of the inner product $\mathfrak{v}\, d\mathfrak{s}$, we must suspect from the outset that a corresponding representation that might be possible for $\mathfrak{A}$ can be obtained with only the help of the exterior product.

The next thing to do is to form a vector line integral of the form:

$$\mathfrak{K} = \int \mathrm{V}\, \mathfrak{v}\, d\mathfrak{s} \tag{128}$$

and examine whether $\mathfrak{A}$, or the difference between the $\mathfrak{A}$'s at the endpoints of the path of integration, can be represented in that way. However, in order for that to be true, the integral must be independent of the path of integration; i.e., it must vanish for every closed curve. I already proved in § **32** of my *Maxw. Theorie* that this can never happen (except in the trivial case of a constant field). One can therefore arrive at a unique value of $\mathfrak{A}$ with the help of an integral of the form (128) only when one makes a particular choice of the path of integration.

In fact, one can also come rather close to one's objective in that way. Namely, one selects an arbitrary constant direction in a field and draws a straight line to infinity in that direction from each point in the field. The vector integral $\mathfrak{K}$ for that path of integration (which extends from the given point to infinity) will then have the entirely-remarkable property that the field quantity $\mathfrak{v}$ can be derived by means of the operator curl. The proof of that is easy to carry out, but it shall be omitted here.

Nevertheless, the value of $\mathfrak{K}$ that is defined in that way will not be identical to the vector potential. Indeed, both of them belong to the same system of vortices, but $\mathfrak{K}$ is not source-free. For another choice of integration path, one might succeed in making div $\mathfrak{K}$ equal to zero, such that $\mathfrak{K}$ will then coincide with $\mathfrak{A}$ completely.

The concept of work that is represented by an inner product stands in opposition to the concept of static moment. In fact, the vector potential of a force field has the meaning and dimension of a static moment. If one thinks of assigning a well-defined unit of force to the field quantity $\mathfrak{v}$ at each point of the aforementioned integration path then all of those segments will fill up a certain surface that will give the value of the integral $\mathfrak{K}$ as the area of that moment surface. If one succeeds in arranging that div $\mathfrak{K}$ vanishes every point of a path of integration then the magnitude and direction of $\mathfrak{A}$ can also be represented quite intuitively by the area of that moment surface.

Naturally, I do not give very much weight to those considerations. They serve only to make more intuitive the concept of vector potential, which one cannot properly represent as long as it is introduced only as the source of the field, since one knows that it represents a static moment, in contradiction to the scalar potential, which represents an amount of work done. I do not remember having ever read that simple, but entirely relevant, remark anywhere.

§ 30. – Another derivation of Gauss's expression for the scalar potential of a vortex filament.

There might possibly be some interest in a small remark that I would like add here. In eq. (120), we found that $L_{12} = L_{21}$. The field $\mathfrak{v}$ that is created by an isolated vortex filament can then be calculated from that as follows with the use of the analysis in § **19**:

We place the reference point at which $\mathfrak{v}$ is to be determined at an arbitrary location in an infinitely-small planar surface of area f. Obviously, we will know $\mathfrak{v}$ when we are given the amount of flux that flows through f for each location on the surface. However, from the theorem that was just stated, the flux is just as large as the flux that a vortex filament of the same strength W communicates to the given vortex filament itself when the centerline of the new filament coincides with the contour of f. In order to calculate the flux, we replace the new vortex filament with a double layer $\pm Wf / h$. The positive part of that layer communicates the flux:

$$\frac{W\,f}{h}\frac{\varpi}{4\pi}$$

through the given vortex filament, when ϖ once more denotes the spatial angle that the given vortex filament subtends at the reference point. We also have the flux that originates from the negative layer. If we combine the two then we will get:

$$-\frac{W\,f}{h}\frac{1}{4\pi}\frac{d\varpi}{dh}h \qquad \text{or} \qquad -\,f\,\frac{d}{dh}\left(\frac{W\varpi}{4\pi}\right).$$

The component of $\mathfrak{v}$ that falls along the direction of h is therefore equal to the minus the differential quotient of $W\varpi / 4\pi$ over h ; i.e., $\mathfrak{v}$ can be derived from the potential:

$$V = \frac{W\varpi}{4\pi}$$

everywhere outside of the given vortex filament, which was to be proved. One might perhaps consider it to be an advantage of this derivation that the introduction of the solid angle ϖ into it is entirely natural.

CHAPTER FIVE

ARBITRARY FUNCTIONS. SPATIAL SUMS.

§ 31. – Arbitrary vector functions.

Up to now, we have always spoken of vector functions that were either only vortex-free or only source-free. It now remains for us to show that any arbitrary vector function that is assumed to be only continuous and to not extend to infinity can be reduced to those two forms.

When $\mathfrak{v} = f(\mathfrak{r})$ is given, one next defines the functions $q = \text{div}\,\mathfrak{v}$ and $\mathfrak{w} = \text{curl}\,\mathfrak{v}$, the first of which gives the associated system of sources, and the second of which gives the system of vortices. If q and $\mathfrak{w}$ are given then conversely $\mathfrak{v}$ will also be determined uniquely in that way, because the geometric difference of two solutions that can perhaps be given in all of space will be both vortex-free and source-free and must therefore necessarily vanish, from an argument that has already been used frequently.

All that is necessary then is for one to define the vortex-free field $\mathfrak{v}_1$ that belongs to q and the source-free field $\mathfrak{v}_2$ that belongs to $\mathfrak{w}$, using the prescription of the previous chapter. One will then hve that:

$$\mathfrak{v} = \mathfrak{v}_1 + \mathfrak{v}_2$$

is necessarily equal to the function that was given originally, and in that way, it is proved that any function can be decomposed into a vortex-free component and a source-free in a unique way. One now has the two components in hand, and one can then apply the rules that were developed before to them. One does not need to look for a link between the two, since both of them can generally be completely independent of each other. If a relationship exists between the two in some special case then the functions will be determined more precisely in that way; they will cease to be completely arbitrary.

One can, e.g., investigate the properties of functions for which both components are either equally-directed or perpendicular to each other everywhere or for which one component is a linear vector function of the other, etc. It might be that one can arrive at many interesting results in that way. However, the physical applications of the theory would hardly take on a special sense in that way.

Of greater interest is the study of functions that contain a scalar independent variable in addition to the vector. One then sets, say:

$$\mathfrak{v} = f(\mathfrak{r}, t)\,. \tag{129}$$

If the variable t means, e.g., time then that equation will represent a continuously-varying field, while the simpler equation $\mathfrak{v} = f(\mathfrak{r})$ will refer to a stationary field. The total differential $\delta\mathfrak{v}$ that one obtains when one increases $\mathfrak{r}$ and t by $\delta\mathfrak{r}$ and δt can then be set to [cf., eq. (59)]:

$$\delta\mathfrak{v} = \tfrac{1}{2}\nabla\mathfrak{v}\ \delta\mathfrak{r} + \tfrac{1}{2}\mathrm{V}\mathfrak{w}\ \delta\mathfrak{r} + \frac{\partial\mathfrak{v}}{\partial t}\cdot\delta t\,. \tag{130}$$

In general, the increases $\delta\mathfrak{r}$ and δt in that equation are completely independent of each other. In special cases (namely, in hydrodynamical investigations), it is often necessary to determine $\delta\mathfrak{v}$ more closely, such that one sets:

$$\delta\mathfrak{r} = \mathfrak{v}\ \delta t. \tag{131}$$

That is due to the fact that one follows the evolution of a particular material particle that has been singled out. One then refers to the quotient of $\delta\mathfrak{v}$ and δt as the *total differential quotient* of $\mathfrak{v}$ with respect to *t*, and eq. (30) will go to:

$$\frac{d\mathfrak{v}}{dt} = \tfrac{1}{2}\nabla\mathfrak{v}^2 + \tfrac{1}{2}\mathrm{V}\mathfrak{w}\ \mathfrak{v} + \frac{\partial\mathfrak{v}}{\partial t}. \tag{132}$$

In hydrodynamics, that equation is employed for the purpose of expressing the dependency of the field quantity $\mathfrak{v}$ on *t*, so expressing the partial differential quotient $\partial\mathfrak{v} / \partial t$ (which is basically the only thing that it comes down to) in terms of the forces that act upon the fluid. If *p* denotes the pressure, $\mathfrak{P}$ denotes the external force per unit volume (usually the weight), which we would like to assume can be derived from a potential, and μ denotes the specific mass then from the basic laws of dynamics (so from an experimental law that gets mixed with the function-theoretic investigation here), one will have:

$$\mu\frac{d\mathfrak{v}}{dt} = -\nabla p + \mathfrak{P} = -\nabla(p + P)$$

for a frictionless fluid, and one gets from eq. (132) that:

$$\frac{\partial\mathfrak{v}}{\partial t} = -\nabla\frac{p+P+L}{\mu} - \tfrac{1}{2}\mathrm{V}\mathfrak{w}\ \mathfrak{v}, \tag{133}$$

in which *L* denotes the *vis viva* per unit volume.

The process of investigating the evolution of a physical process is similar in all cases. One always seeks to present a differential equation of the type of eq. (133) that is based upon experimental facts or some hypothesis by which the partial differential quotient of the field quantity $\mathfrak{v}$ with respect to time is made to depend upon the instantaneous values of the field.

Often (and also in hydrodynamics, in particular), one arrives at some simple and quite remarkable results by such considerations when one examines, not the variation of $\mathfrak{v}$, but that of the vorticity $\mathfrak{w}$, as Helmholtz did. It follows from eq. (133):

$$\frac{\partial \mathfrak{w}}{\partial t} = -\tfrac{1}{2}\,\text{curl}\,\mathrm{V}\,\mathfrak{w}\;\mathfrak{v}, \tag{134}$$

and that expression can be converted even further using the formulas of vector analysis. However, in order to derive Helmholtz's celebrated theorem, it is simpler to calculate the change that the surface integral:

$$\int \mathfrak{w}\,\mathfrak{N}\,df\ ,$$

which is extended over a variable surface that always goes through the same material particles in the fluid, experiences in time. From Stokes's theorem, one will have:

$$\frac{d}{dt}\int \mathfrak{v}\,d\mathfrak{s} = \int \frac{d\mathfrak{v}}{dt}\,d\mathfrak{s} + \int \mathfrak{v}\cdot(d\mathfrak{s}\nabla)\,\mathfrak{v} = \int \frac{d\mathfrak{v}}{dt}\,d\mathfrak{s} + \tfrac{1}{2}\int (d\mathfrak{s}\nabla)\,\mathfrak{v}^2\ ,$$

since the variation that $d\mathfrak{s}$ experiences during the time element dt is equal to the path difference at the two endpoints of $d\mathfrak{s}$, which is therefore equal to $(d\mathfrak{s}\ \nabla)\ \mathfrak{v}\ dt$. However, the last integral above will vanish when we extend it over a closed curve, and one will have:

$$\frac{d}{dt}\int \mathfrak{w}\,\mathfrak{N}\,df = \int \frac{d\mathfrak{v}}{dt}\,d\mathfrak{s}\ , \tag{135}$$

which is, for the time being, not even based upon any physical hypothesis.

When one appeals to the basic equation of dynamics and the assumption that the external force $\mathfrak{P}$ is vortex-free, it will then follow that the vortex strength of a filament that is composed of the same fluid particles is constant in time.

Of course, those discussions can lead quite far afield from the realm of actual field geometry. In fact, the geometry of fields always extends into field kinematics or field mechanics as soon as one makes a definite assumption in regard to the connection between the scalar differentiation of $\mathfrak{v}$ with respect to t and the spatial differentiations that can be performed with respect to $\mathfrak{r}$. However, in the absence of such a connection, the rules by which the function depends upon $\mathfrak{r}$ and t are already included in the previous ones. It then seem preferable to me to exhibit a consistent example of the type of experimental facts or physical hypotheses that one might care to link with the rigorously-valid theorems on the general properties of vector functions in order to derive physical theories from them.

§ 32. – The field as a system of segments.

We shall now go on to the definition of spatial sums of various kinds. We understand a "spatial sum" to mean the result of any summation over all of infinite space. In § **4**, I already discussed the simplest kind of spatial sum that one can define from a given function $\mathfrak{v}$, namely, the integral $\int \mathfrak{v}\ d\tau$, which is extended over all of infinite space, and which I referred to there as the "field sum," in particular. There, we found that the field sum is equal to zero for a source-free field, and that for other fields, it can be easily calculated from the distribution of sources. It will then follow that, among other things, one always has:

$$\int^{\infty} \mathfrak{w}\, d\tau = 0, \tag{136}$$

since the function $\mathfrak{w}$ is always source-free. That equation defines a remarkable counterpart to the one $\int^{\infty} q\, d\tau = 0$ that was found before, and we see once more how sources and vortices agree in one important property.

Instead of simply summing the vectors $\mathfrak{v}\; d\tau$, one can also pose the problem of combining the $\mathfrak{v}\; d\tau$ in that same way that one combines forces that are applied to a rigid body. If we, in turn, think of $\mathfrak{v}\; d\tau$ as a force that acts upon a volume element then all of the forces that were obtained can be replaced with a resultant that goes through an arbitrarily-chosen point of application and a force-couple. The resultant corresponds to the field sum that was calculated before, while the moment of the force-couple is yet-to-be-determined. If we let $\mathfrak{r}$ denote the radius vector that points from the reference point of the resultant to $d\tau$, which is likewise the lever arm of the force $\mathfrak{v}\; d\tau$, then the moment of the result force-couple will be:

$$\mathfrak{M} = \int^{\infty} \mathrm{V}\,\mathfrak{v}\,\mathfrak{r}\, d\tau\,. \tag{137}$$

That expression will be independent of the choice of moment point from which the radius vector $\mathfrak{r}$ is drawn for the case of source-free fields. Namely, if one next extends the summation over a closed current tube then one will get:

$$\int f\, v \mathrm{V} d\mathfrak{s}\, \mathfrak{r} \qquad \text{or} \qquad F \int \mathrm{V} d\mathfrak{s}\, \mathfrak{r}\,,$$

when f means the cross-section, $d\mathfrak{s}$ means an element that points in the direction $\mathfrak{v}$ of the centerline, and $f\, v = F$ means the flux that goes through the current tube.

The integral extends over the centerline of the current tube, and it has a simple geometric meaning: Namely, $\mathrm{V} d\mathfrak{s}\; \mathfrak{r}$ will be given by twice the area of the triangle that has $d\mathfrak{s}$ for its base and the moment point for its opposite vertex. All of those triangles will collectively define the surface of a cone by which the centerline of the current tube will be projected from the moment point. However, all triangular surfaces must be summed geometrically in the integration; i.e., one must give consideration to the directions of their normals. If we imagine that a surface has been laid through the centerline in such a way that centerline defines the boundary of that surface and the surface itself, together with the aforementioned conical surface, bounds a conical space then $\int \mathrm{V} d\mathfrak{s}\; \mathfrak{r}$ can also be set equal to twice the area of that surface, up to sign, since one knows that the geometric sum of all surfaces that collectively define the surface of a body of arbitrary form is equal to zero. However, the aforementioned surface is completely independent of the position of the moment point, and the same thing will also be true for the moment $\mathfrak{M}$.

The result that one gets for a current tube can also be adapted to the total field with no further assumptions by summing over all current tubes in the total field.

The field sum and field moment $\mathfrak{M}$ are suitable quantities for measuring the strength of the excitations of an entire physical field. One appeals to the field sum for source-free fields and the field moment for vortex-free fields. A system of vortices $\mathfrak{w}$ is always source-free, and one can then regard the moment:

$$\mathfrak{W} = \int^{\infty} \mathrm{V}\,\mathfrak{w}\,\mathfrak{r}\,d\tau\,, \tag{138}$$

which is independent of the choice of moment point, as a quantity that characterizes the total strength of the system of vortices. How one determines it for an individual vortex filament will emerge from the foregoing discussion. The moment $\mathfrak{W}$ is equal to twice the product of the vortex strength and the area of the surface that encloses the filament in the event that it is planar. In the other case, the geometric sum of those surfaces must first be derived.

Naturally, all of those considerations are basically only special applications of the theory of the system of segments to the system of segments $\mathfrak{v}\;d\tau$ that are present here. Therefore, I would not like to stop to prove some theorems that the reader has already known for some time. Obviously, in vortex-free fields, one can, in fact, always give a degree to which the moment $\mathfrak{M}$ will vanish at all points from eq. (137). The source-free field corresponds to the case of a force-system that can be replaced with a single resultant. For a field that simultaneously includes both vortices and sources, one can always give a *central axis* that points in the same direction as the field-sum and also coincides with the direction of the moment $\mathfrak{M}$ for all points that lie on it.

§ 33. – The sum of the squares.

In many cases, another spatial sum proves to be much better suited to the task of characterizing the total content of a field by a single value or comparing the intensities of different fields on the whole, namely, the sum of the squares:

$$Q = \tfrac{1}{2}\int^{\infty} \mathfrak{v}^2\,d\tau\,. \tag{139}$$

That kind of appraisal has the advantage over the field sum $\mathfrak{F} = \int^{\infty} \mathfrak{v}\,d\tau$ that every volume element will yield a positive contribution and that Q will vanish only when the field itself vanishes everywhere. One will then have, in that expression, a well-defined measure of the total excitation that is present in a physical field $\mathfrak{v}$ under all circumstances.

However, the value of Q also has yet another advantage, namely, that it defines a measure of the energy in the field in many applications that result from experiments. Of course, that situation will not come under further consideration here, since we would not like to concern ourselves any further with applications. However, it does justify the fact that there is an otherwise-meaningless factor of $\frac{1}{2}$ in front of the integral that we would naturally do better to drop under other circumstances. Nonetheless, we are free to introduce it arbitrarily into the defining equation for Q in any event.

One can give some very remarkable theorems for the sum of the squares A that lend great significance to that expression, even when it is considered in a purely-analytical context. In order to derive the first theorem, I imagine that the field quantity $\mathfrak{v}$ has been decomposed into its vortex-free component $\mathfrak{v}_1$ and its source-free component $\mathfrak{v}_2$, as was discussed in § **31**, since Q will go to:

$$Q = \tfrac{1}{2}\int^{\infty} \mathfrak{v}_1^2\, d\tau + \tfrac{1}{2}\int^{\infty} \mathfrak{v}_2^2\, d\tau + \int^{\infty} \mathfrak{v}_1\, \mathfrak{v}_2\, d\tau\,.$$

However, the last integral, which is extended over all of infinite space, must always vanish. In order to prove that, I set $\mathfrak{v}_1 = -\nabla V$, where V means the scalar potential that belongs to $\mathfrak{v}_1$. One will then have:

$$\text{div}\,(\mathfrak{v}_2\, V) = V\,\text{div}\,\mathfrak{v}_2 - \mathfrak{v}_1\,\mathfrak{v}_2\,,$$

of which one convinces oneself immediately upon performing the operator div [cf., eq. (78) in *Maxw. Theorie*]. Since $\mathfrak{v}_2$ was source-free, the first term on the right-hand side will drop out, and one will have:

$$\int^{\infty} \mathfrak{v}_1\, \mathfrak{v}_2\, d\tau = -\int^{\infty} \text{div}\,(\mathfrak{v}_2\, V)\, d\tau\,.$$

The function $\mathfrak{v}_2 V$ does not extend to infinity, since that would follow already from the assumption that is always made about $\mathfrak{v}_2$ here, and since multiplication by V, which likewise vanishes at infinity, will diminish the order of magnitude of the product at great distances even further. Therefore, the spatial sum of all sources of $\mathfrak{v}_2 V$ will vanish, from § **4**, and we will, in fact, get:

$$\int^{\infty} \mathfrak{v}_1\, \mathfrak{v}_2\, d\tau = 0. \tag{140}$$

We then find:

$$Q = \tfrac{1}{2}\int^{\infty} \mathfrak{v}_1^2\, d\tau + \tfrac{1}{2}\int^{\infty} \mathfrak{v}_2^2\, d\tau = Q_1 + Q_2 \tag{141}$$

for the sum of the squares A, where the sum of the squares of the vortex-free and the source-free components are denoted by Q_1 and Q_2 in their own right. If Q then means the energy of a physical field $\mathfrak{v}$ then it will be equal to the sum of the total energies that are assigned to the two field components when one considers each of them by themselves. That shows us once more how useful the distribution of a vector function over the sources and vortices that it includes is.

Under the assumption that Q is, in fact, the correct expression for the energy of a certain physical field and that the energy distribution is further determined by physical phenomena – i.e., $\partial\mathfrak{v} / \partial t$ – eq. (121) will make it highly probable in such a case that each of the two components, independently of the other, will lead to physical phenomena that play out in parallel to each other without influencing each other. Naturally, that remark is true only approximately here. The phenomena also lie so close to each other that one cannot draw any definite conclusions from them, since the processes in nature are independent of the way that we temporarily regard them, and the only thing that we can do in order to study their laws consists of continuing to reshape the pictures that have been previously justified, while groping carefully and always being ready to replace them with others when their consequences do not overlap with experiments. Only the laws of field geometry will then define the fixed foundation from which we can depart with no reservations in that way.

§ 34. – Green's theorem and its extensions.

The sum of the squares Q_1 for the vortex-free components $\mathfrak{v}_1$ can also be set equal to another spatial sum, which I shall now derive. To that end, I shall start from the identity:

$$\text{div}\,(\mathfrak{v}_1\, V) = V\,\text{div}\,\mathfrak{v}_1 - \mathfrak{v}_1^2,$$

which has already been used before in a similar form. I can set q for $\mathfrak{v}_1$. After multiplying by $d\tau$ and integrating over all of space, it will then follow that:

$$Q_1 = \tfrac{1}{2}\int^{\infty} \mathfrak{v}_1^2\, d\tau = \tfrac{1}{2}\int^{\infty} V\, q\, d\tau, \tag{142}$$

since the spatial sum of div $(\mathfrak{v}_1\, V)$ will vanish, on grounds that were already discussed before. Eq. (142) (or really a somewhat more general conception of that equation that is of no interest to us here) corresponds to Green's theorem. The sum of the squares for a vortex-free field or for the vortex-free part of an arbitrary field can therefore be found already when one knows only the associated system of sources and the potential. At the same time, we see that when Q_1 gives the contribution to the field energy, two entirely different distributions of that energy over the individual volume elements will be geometrically possible in a legitimate way. Therein lies the root of the conflict between the two opposing views of the theories of action-at-a-distance and local action in physics, and especially in the theory of electricity.

The sum of the squares Q_2 for the source-free field components $\mathfrak{v}_2$ also admits a similar conversion. If $\mathfrak{A}$ denotes the vector potential from which $\mathfrak{v}_2$ can be derived then one will have:

$$\operatorname{div} V \mathfrak{A}\, \mathfrak{v}_2 = \mathfrak{v}_2^2 - \mathfrak{A}\, \mathfrak{w} \tag{143}$$

identically, from the best-known theorem of vector analysis [eq. (81) of *Maxw. Theorie*]. Meanwhile, one also easily convinces oneself of the identity of both sides of that equation with no further discussion by decomposing it into components and performing the prescribed operations.

The function $V\mathfrak{A}\,\mathfrak{v}_2$ cannot extend to infinity, on the same grounds that applied to the corresponding $V\mathfrak{v}_1$ in the vortex-free case, and therefore, the spatial sum that is defined by it will be equal to zero. We then get from eq. (143) that:

$$Q_2 = \tfrac{1}{2}\int^{\infty} \mathfrak{v}_2^2\, d\tau = \tfrac{1}{2}\int^{\infty} \mathfrak{A}\,\mathfrak{w}\, d\tau . \tag{144}$$

Remarks are also true for the meaning of that conversion that are completely analogous to the ones that were linked with Green's theorem.

The sum of the squares of an arbitrary function $\mathfrak{v}$ can always be represented in the form:

$$Q = \tfrac{1}{2}\int^{\infty} (Vq + \mathfrak{A}\,\mathfrak{w})\, d\tau \tag{145}$$

then.

It might be remarked belatedly at this point that one can use the last equation to prove that a function is defined uniquely by its source and its vortex in a different way from the proof that was described before in this book on the basis of merely imagining the streamlines. Namely, if $q = 0$ and $\mathfrak{w} = 0$ in all of space then it will follow from eq. (145) that $Q = 0$, and therefore one will also have $\mathfrak{v} = 0$, from a remark that was made before in the beginning of the previous paragraph. If one now considers two functions $\mathfrak{v}$ that belong to the same sources and vortices then their difference must, at the same time, be source-free and vortex-free, and therefore equal to zero everywhere. Both functions will then agree completely; i.e., there is only one function that simultaneously belongs to a given distribution of sources and vortices.

Finally, let me point out that the conversion that led to eq. (145) can also be generalized somewhat when one also directs one's attention to the terms that one can place before and after the two sequences of functions V, $\mathfrak{v}_1$, q and $\mathfrak{A}$, $\mathfrak{v}_2$, $\mathfrak{w}$, as was discussed in § **25**. If one considers, e.g., the sequence:

$$\mathfrak{M},\ V,\ \mathfrak{v}_1,\ q,\ \mathfrak{p}$$

in the sense that was discussed in § **25**, then one will have:

$$\operatorname{div}(\mathfrak{M}\, q) = V\, q - \mathfrak{M}\, \mathfrak{p} ,$$

and therefore also:

$$Q_1 = \tfrac{1}{2}\int^{\infty} \mathfrak{M}\,\mathfrak{p}\, d\tau . \tag{146}$$

Likewise, let the sequence of functions for the source-free field components be extended to:

$$\mathfrak{X},\ \mathfrak{A},\ \mathfrak{v}_2,\ \mathfrak{w},\ \mathfrak{y}\ .$$

From the theorem that was proved before in eq. (143), and when one recalls the connection that exists between the successive terms in that sequence, one will then have:

$$\operatorname{div} \mathrm{V}\ \mathfrak{X}\,\mathfrak{w} = \mathfrak{A}\,\mathfrak{w} - \mathfrak{X}\,\mathfrak{y}\ ,$$

and when one defines the spatial sum, one will also have:

$$Q_2 = \tfrac{1}{2}\int^{\infty} \mathfrak{X}\,\mathfrak{y}\,d\tau \tag{147}$$

then.

Therefore, in total, the sum of the squares of an arbitrary function will also be given by the expression:

$$Q = \tfrac{1}{2}\int^{\infty} (\mathfrak{M}\,\mathfrak{p} + \mathfrak{X}\,\mathfrak{y})\,d\tau\,, \tag{148}$$

which admits some conversions in its own right. Namely, one can (cf., § **25**) replace $\mathfrak{M} + \mathfrak{X}$ with pot $\mathfrak{v}$ and $\mathfrak{p} + \mathfrak{y}$ with $-\nabla^2\mathfrak{v}$. In addition, the spatial sums of $\mathfrak{M}\ \mathfrak{y}$ and $\mathfrak{X}\ \mathfrak{p}$ are equal to zero. Namely, one factor in each product is vortex-free and the other is source-free, and from eq. (140), the spatial sum of such a product will always be zero. That is because in the derivation of eq. (140), it was entirely essential for $\mathfrak{v}_1$ and $\mathfrak{v}_2$ to be introduced as components of an originally-given function $\mathfrak{v}$, but $\mathfrak{v}_1$ and $\mathfrak{v}_2$ can be otherwise-arbitrary functions when only the one is vortex-free and the other one is source-free. When one considers those remarks, it will also follow from eq. (148) that one has:

$$Q = -\tfrac{1}{2}\int^{\infty} \operatorname{pot}\mathfrak{v}\cdot\nabla^2\mathfrak{v}\,d\tau \tag{149}$$

for the sum of the squares of an arbitrary function $\mathfrak{v}$.

§ 35. – Spatial sum of a potential function.

Once more, we understand V to mean a scalar potential that belongs to a vortex-free field that does not extend to infinity. We will then always have:

$$\int^{\infty} V\,d\tau = 0. \tag{150}$$

In order to prove that, one can start from a field that is defined by two point-like sources $\pm q$. If the distances from the spatial element $d\tau$ to the two points are r_1 and r_2 then it will follow from § **8** that:

$$V = \frac{q}{4\pi}\left(\frac{1}{r_1} - \frac{1}{r_2}\right).$$

If one now imagines a plane that bisects the distance between the two source points perpendicularly then for any two spatial elements that are mirror images to that plane the contributions $V\,d\tau$ to the spatial sum will be just as large and of opposite sign. Eq. (150) is proved for the source-pair $\pm q$ with that remark. However, as was shown already in § **4**, any other system of sources can be constructed from a superposition of such source-pairs, and since V can be found for the entire system of sources by summing over the individual constituents, it will follow that eq. (150) will also remain valid for that case.

One can further conclude from this that the function $\mathfrak{M}$ = pot $\mathfrak{v}_1$ will not extend to infinity either when $\mathfrak{v}_1$ already satisfies that assumption, since V gives the system of sources for $\mathfrak{M}$, and eq. (150) will then embody the required condition.

The fact that one also has that:

$$\int^{\infty} \mathfrak{A}\, d\tau = 0 \tag{151}$$

already follows from the simple remark that the function $\mathfrak{A}$ is source-free, by its very definition. The field sum is always equal to zero for a source-free field.

§ 36. – The potential function as a spatial sum.

The spatial sums that were discussed before were constant values that characterized the total field. Nothing prevents us from increasing their number, and thus investigating, e.g., the properties of the sums $\int^{\infty} V^2\, d\tau$, $\int^{\infty} \mathfrak{v}^3\, d\tau$, etc. However, I shall skip over that, since such an investigation does not seem to promise to return very much.

However, we can also include a variable quantity in the element of a spatial sum that makes the sum itself become a function of that variable quantity. If we also still know nothing about the potential function then we will be led to take that step naturally. In fact, the simple path for generating a new function from a given one in that way obviously consists of defining a spatial sum that refers to only a well-defined point in the field – viz., the reference point – and which is therefore a function of the radius vector of that reference point. To that end, we will have to include the distance from the reference point to the volume element $d\tau$ in the element of summation in some way. That was, in fact, also done before at one point, namely, when the field moment for the reference point was derived in § **32**. However, it was shown there that this moment was independent of the position of the reference point for a source-free field, so it would define a constant spatial sum. Indeed, it will be a function of the position of the reference point in a source field. However, it extends to infinity, and therefore will not come under consideration here.

We must then look around for other couplings, and indeed, we will have to look for the simplest function that might lead to functions that do not reach infinity that comes into question in that way. Now, we indeed always have many possibilities to choose between. Among all of them, however, we must always focus upon the one that is closest to the spatial sums:

$$\int^{\infty}\frac{\mathfrak{v}\,d\tau}{a} \qquad \text{or} \qquad \int^{\infty}\frac{q\,d\tau}{a},$$

and with that, we will, in fact, be once more led to the potential. Perhaps there also other spatial sums of a similar kind whose closer examination would represent a fortuitous accoutrement (*einen glücklichen Griff*) for us.

§ 37. – Measuring the curvature of a field.

As we saw in the Chapter Two, inside of an infinitely-small neighborhood, we can represent an arbitrary continuous field by a linear vector function in the first approximation or replace it with the linear field that contacts it.

The fact that we referred to the type of agreement between both fields in that neighborhood as "contact" overlaps with the use that is made of that word elsewhere in geometry in that regard.

Just as one does not stop with looking for the tangents or tangent planes in the investigation of curves and surfaces, but must also consider the deviation of one tangent object from another inside of an infinitely-small region, one can also pose the same problem in the geometry of fields. What one calls "curvature" for curves and surfaces depends upon the type and magnitude of that deviation. Thus, I am justified in also adapting the concept of curvature to the case that occurs here, and in full generality I understand that to mean that property of the field that gives rise to the second-order infinitesimal deviation between the given field and the linear field that contacts it inside of an infinitely-small neighborhood.

Once the theory of the curvature of fields has been developed completely, it will naturally take on a much more multifaceted aspect than that of surfaces. Here, one can only make a first attempt at that. Namely, it would seem desirable to look for a value that could be employed as a measure of the total curvature of the field at a given location that would be similar to, say, the Gaussian curvature of a surface. Of course, I do not believe that one can lean upon that to any advantage here, and I would like to say that it seems entirely doubtful that one can say what quantity would be best suited to serve as a measure of the curvature of a field. In that regard, I ask that one should consider the following discussion to be only tentative.

One lays the origin of a rectangular coordinate system at the point of the field for which one would like to study the curvature behavior. From Taylor's development, one will then have that the X-component dv_1 of $d\mathfrak{v}$ is, precise to second-order infinitesimals:

$$\left.\begin{aligned} dv_1 &= \frac{\partial v_1}{\partial x}dx + \frac{\partial v_1}{\partial y}dy + \frac{\partial v_1}{\partial z}dz \\ &+ \frac{1}{2}\frac{\partial^2 v_1}{\partial x^2}dx^2 + \frac{1}{2}\frac{\partial^2 v_1}{\partial y^2}dy^2 + \frac{1}{2}\frac{\partial^2 v_1}{\partial z^2}dz^2 \\ &+ \frac{1}{2}\frac{\partial^2 v_1}{\partial x\,\partial y}dx\,dy + \frac{1}{2}\frac{\partial^2 v_1}{\partial x\,\partial z}dx\,dz + \frac{1}{2}\frac{\partial^2 v_1}{\partial y\,\partial x}dy\,dz \end{aligned}\right\} \qquad (152)$$

in the interior of a ball whose radius *r* is infinitely small of order one, and similarly for the other two components. The first-order terms in that development are of no further interest to us here; they overlap with the components of the contacting linear field. By contrast, the curvature of the field depends upon the second-order terms. If we denote the deviation between the field $\mathfrak{v}$ and the contacting linear field by $\delta\mathfrak{v}$ then we will get from eq. (152) upon combining the three components that:

$$\left.\begin{aligned} \delta\mathfrak{v} &= \frac{\partial^2\mathfrak{v}}{\partial x^2}\cdot\frac{dx^2}{2} + \frac{\partial^2\mathfrak{v}}{\partial y^2}\cdot\frac{dy^2}{2} + \frac{\partial^2\mathfrak{v}}{\partial z^2}\cdot\frac{dz^2}{2} \\ &+ \frac{\partial^2\mathfrak{v}}{\partial x\,\partial y}dx\,dy + \frac{\partial^2\mathfrak{v}}{\partial x\,\partial z}dx\,dz + \frac{\partial^2\mathfrak{v}}{\partial y\,\partial z}dy\,dz. \end{aligned}\right\} \qquad (153)$$

The second differential quotients of $\mathfrak{v}$ can be regarded as constant quantities inside of the infinitely-small ball for which the development is valid. If we imagine drawing any diameter through it and associating it with the deviation $\delta\mathfrak{v}$ that exists at that point then we will see that all of those segments are parallel to each other and that their magnitudes relate to each other like the squares of the distances from the center. The endpoints of those segments then lie along a parabolic arc that contacts the diameter. The curvature of the field along the diameter that was drawn is described completely by the magnitude of the radius of curvature of that arc and the direction of $\delta\mathfrak{v}$. In order to know the curvature of the field completely, one must be able to give those two data for every diameter of the ball.

Now, it seems to me that one will best summarize the total curvature when one takes the field sum of $\delta\mathfrak{v}$ over the volume of the infinitely-small ball. In order to arrive at a finite value, I then set:

$$\mathfrak{k} = \frac{4}{\Theta}\int^{r} \delta\mathfrak{v}\, d\tau\,, \qquad (154)$$

in which Θ means the moment of inertia of the infinitely-small ball of radius *r* with respect to a diameter. The factor 4 in the numerator was introduced at will in order to simplify the following formulas. I consider the vector $\mathfrak{k}$ to be a measure of the curvature of the field. I substitute the value $\delta\mathfrak{v}$ in eq. (153) and observe that the integral over the spherical volume of the form $\int^r dx\,dy\,dz$ will drop out because the ball can be split into two halves – e.g., by the *YZ*-plane – such that any two volume elements $d\tau$ of them that are mirror images of each other will contribute equal values of opposite sign to the integral. By contrast, e.g.:

$$\int^r dx^2\, d\tau = \tfrac{1}{2}\Theta$$

is, by definition, the moment of inertia. One will then get:

$$\mathfrak{k} = \frac{\partial^2 \mathfrak{v}}{\partial x^2} + \frac{\partial^2 \mathfrak{v}}{\partial y^2} + \frac{\partial^2 \mathfrak{v}}{\partial z^2} = \nabla^2\, \mathfrak{v} \tag{155}$$

for $\mathfrak{k}$.

We have thus, in fact, arrived at a very simple and, it seems to me, remarkable expression for the field curvature. If one expresses $\nabla^2\, \mathfrak{v}$ in terms of the sources q and the vorticity $\mathfrak{w}$ then one will also have:

$$\mathfrak{k} = \nabla q - \text{curl}\, \mathfrak{w}. \tag{156}$$

The total curvature $\mathfrak{k}$ will equal zero at every location in the field that contains neither a source nor a vortex (or where they are constant). Of course, that does not say that the field is not curved at all and is therefore linear. The curvatures along different diameters will then result only along directions that are partially opposed to each other such that the geometric sum of all curvatures would vanish.

If one would like to have a measure of curvature that vanishes only when the field is linear, so it would no longer be curved at all, then one must form the sum of the squares for $\delta\mathfrak{v}$ for the interior of the ball. That can be done easily with the help of the expression for $\delta\mathfrak{v}$ in eq. (153). However, one will come to a rather long-winded expression in that way, into which I meanwhile do not know how to go any further.

Finally, I shall remark that one can also make eq. (153) true for finite values of the coordinates *dx dy dz*. In that way, one will come to a second-order field that osculates the given one. Before one can discuss the theory of the curvature of fields in detail, one must naturally discuss the properties of second-order fields thoroughly, perhaps in a manner that is similar to what was done with linear fields in Chapter Two of this book.

Meanwhile, I shall omit such an investigation. I consider the grand prize in this overview of the curvature properties to be the intuitive geometric interpretation that the operator ∇^2, which occurs so frequently in potential theory, acquires from eq. (155).

Any vector function can also be represented by a spatial transformation. One needs only to assign the vector $\mathfrak{v} = f(\mathfrak{r})$ to the point $\mathfrak{r}$ with a suitable unit of measurement and to associate the endpoint of the segment in the transformed space to the starting point of the segment in the original space. The study of transformation groups has now become so much better developed and in so much depth that presumably all of the properties of vector functions that were discussed here have already been worked out for some time in a much more general form. However, in order to make that treasure trove useful for physical theories, one must call upon an extended mathematical knowledge that a theoretical physicist would probably possess only quite rarely. On the other hand, the mathematician for whom that assumption is applicable will ordinarily be less inclined to emphasize the topics from among the extensive array that he has mastered that are suitable for practical applications in a correspondingly simpler form. He is almost exclusively interested in the applications of the advanced mathematical topics to other purely-mathematical problems. The practitioner (as he will be called here, for the sake of comparison) will then need to rely upon himself. Such a step in the direction of self-reliance was what led to the writing of this volume. I make no claim at all to being a mathematician nor am I trying to introduce myself as such a thing with this book, but I wish only that by my work, I have been of service to others that might find themselves in the same position.

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