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The fundamental concepts of analytical mechanics as the foundations of quantum and wave mechanics ([1])

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([1]) The following report is the extended version of a talk that was given at the VI Congress of Russian physicists in Nizhni-Novgorod on 10 August 1928.

Introduction.

Today, we know that classical mechanics does not include natural laws of general validity, but only a mathematical paradigm with the help of which certain groups of phenomena can be described to an approximation that is more or less adequate. For phenomena for which the paradigm does not suffice, one always seeks to find a new one by "generalizing" the Ansätze of classical mechanics. However, "generalization" is an infinitely-multivalued process. It comes down to knowing which components of the old paradigm should be preserved as essential and which ones are not, and one will get different "generalizations" according to the choice of those components. We would perhaps like to enumerate three types of them.

First, one regards the paradigm of the force law: force = mass × acceleration as essential to mechanics and seeks the generalization in the form of ever more complicated functions for the dependency of the force on position and velocity and ever more complicated assumptions about the distribution of mass. That standpoint corresponds to what was previously referred to as the "mechanistic picture of nature," and which one held to be the only satisfactory theory in the quest for knowledge out of custom and increasingly under the conscious and unconscious influence of more archaic metaphysical theories and motivations. Those attempts have actually already failed due to the impossibility of including the electromagnetic field equations in that paradigm in a natural way. The assumptions about forces and mass distributions soon became so complicated that the quest was abandoned. It is often overlooked that this complexity is required, not only on the grounds of the "convenience" of abandoning that path, but also due to the fact that it is only under simple assumptions about the forces that **Newton**ian mechanics will say anything at all about nature. When one interprets it as a mere mathematical formalism that everything is ultimately subject to, the **Newton**ian Ansatz reads: "Motions can be represented according to the paradigm 'force = mass × acceleration' under simple assumptions about the forces."

The second type of generalization is relativistic mechanics. There, one always abandons the representation of the electromagnetic field by generalizing **Newton**'s laws, but still wishes to understand at least the motions of masses in the field by analogy with classical mechanics. One gives up the **Newton**ian paradigm and no longer tries to represent accelerations of the masses in a simple way as functions of the relative positions of the bodies, but rather one keeps **Newton**ian mechanics as the essential part of **Hamilton**'s variational principle, and puts a more general quantity that is invariant under the group of **Lorentz** transformations under the integral sign in place of the classical **Hamilton**ian function since the known **Maxwell-Lorentz** field equations will also be invariant under that group. In that way, the space and time coordinates will play a similar role in the integrand, and one will obtain new equations of motion by setting its first variation equal to zero by the same formalism as in classical mechanics. I shall pass over the entirely-similar generalization that exists in the general theory of relativity and start from the fact that relativistic mechanics is also not in a position to the represent the processes in the atom that lead to the appearance of line spectra.

After several groping attempts, one arrives at a third generalization of the **Newton**ian paradigm, namely quantum mechanics, as it was developed in the work of **Heisenberg**, **Born**, **Jordan**, **Schrödinger**, Dirac, *et al*., in recent years. One now regards something in classical mechanics to be essential that most physicists had actually considered to be only a mathematical

gimmick, namely, the representation of mechanics by the **Poisson** bracket expressions and the theory of contact transformations. The concepts of mass and force have lost their meanings, and only the abstract form of classical mechanics in which everything is expressed in terms of abstract group-theoretic concepts is suitable for generalization. When we then consider classical mechanics from standpoint of quantum mechanics looking backwards, that will give an entirely different emphasis in regard to what the important and unimportant parts of physics are. Just as we suddenly notice new insights into classical mechanics from the standpoint of relativistic mechanics, such as the fact that space and time coordinates can be considered to have the same status in it and the fact that by defining the equations of motion by using **Hamilton**'s principle, one will get the laws of impulse and energy as components of one and the same vector equation, just as is true in relativistic mechanics, so also will one see that when one looks back from the standpoint of quantum mechanics, many of its theorems had already been formally modeled in classical mechanics without anyone having ascribed any importance to that form.

In what follows, we would now like to give a presentation of classical mechanics from that standpoint such that the conceptual structures and theorems that have proved to be essential for the purpose of making the transition to quantum mechanics will emerge in it most rigorously. To that end, we will make much use of a mathematical theory that has still been employed explicitly in the presentations very little up to now, namely, the theory of function groups that was developed by **S. Lie** ([1]).

2. – Introduction of the bracket expressions.

According to classical mechanics, every mechanical system is characterized by the fact that one is given the dependency of its **Hamiltonian** function H on the state of the system. When we have a system with n degrees of freedom before us, that state is determined by giving the n generalized coordinates $q_1, q_2, \ldots, q_n$ and the n generalized impulse components $p_1, p_2, \ldots, p_n$. Any function of the $2n$ quantities $q_1, q_2, \ldots, q_n, p_1, p_2, \ldots, p_n$ is called a *state function* of the system. The temporal change in the state quantities under the motion is the system is defined by **Hamilton**'s canonical equations of motion, so by:

$$\frac{dq_j}{dt} = \frac{\partial H}{\partial p_j}, \quad \frac{dp_j}{dt} = -\frac{\partial H}{\partial q_j} \qquad (j = 1, 2, \ldots, n). \tag{1}$$

It will then follow that the temporal change in an arbitrary state function f is:

$$\frac{df}{dt} = \sum_{j=1}^{n}\left(\frac{\partial f}{\partial q_j}\frac{dq_j}{dt} + \frac{\partial f}{\partial p_j}\frac{dp_j}{dt} \right). \tag{2}$$

([1]) **Sophus Lie**, *Gesammelte Abhandlungen*, Bd. III, Christiania and Leipzig, 1922. pp. 32, *et seq.* (*illegible*)

When we substitute (1) in that and define the **Poisson** bracket expression of two state functions f and g by:

$$(f, g) = \sum_{j=1}^{n} \left(\frac{\partial f}{\partial p_j} \frac{\partial g}{\partial q_j} - \frac{\partial f}{\partial q_j} \frac{\partial g}{\partial p_j} \right) , \tag{3}$$

that will give the simple formula:

$$\frac{df}{dt} = (H, f) . \tag{4}$$

When we choose the state function f to be the state quantities q_j, p_j themselves, it will include the equations of motion (1) as special cases, which can then be written in the form:

$$\frac{dq_j}{dt} = (H, q_j) , \quad \frac{dp_j}{dt} = (H, p_j) \qquad (j = 1, 2, \ldots, n). \tag{5}$$

When the bracket expression of two state functions f and g vanish, one says that they are "involutory" or "in involution with each other." If their bracket expression is equal to 1 then one says that they are "canonical conjugate" to each other, so:

$$\begin{aligned} &(f, g) = 0: \quad f \text{ and } g \text{ are in involution,} \\ &(f, g) = 1: \quad f \text{ and } g \text{ are canonical conjugate.} \end{aligned} \tag{6}$$

3. – Integrals and clock readings.

When a state function f does not change its value in time during the entire motion of the system, we call it an *integral* ([1]) of the system. It follows from (4) that f is an integral of the system if and only if:

$$(H, f) = 0 , \tag{7}$$

so f is in involution with the **Hamilton**ian function. Since H is involutory with itself, from the definition (3), H will itself be an integral that one calls the *energy integral* in mechanics. When a state function changes in time in such a way that its change gives us the elapsed time itself, knowing f can be employed in the definition of a clock. We then call f a "clock reading" (*Uhrenausdruck*). We must then have $df / dt = 1$, and therefore, from (4):

$$(H, f) = 1 . \tag{8}$$

The clock readings are then canonically conjugate to H.

([1]) One often employs the term "integral" in the more general sense and then refers to the state functions that are called integrals here, more precisely, as time-independent integrals.

One can easily gain an overview of the number of differential integrals of the system in the usual way. The general integral of the $2n$ first-order differential equations in (1) will give the $2n$ state quantities q_j, p_j as functions of t, and the $2n$ arbitrary constants c_1, c_2, …, c_{2n} . However, in so doing, one should notice that one of those constants can only be added to t since t does not enter into the equations explicitly, but only dt. We can then calculate the general solutions in such a way that we regard the state quantities as functions of $t - t_0$ and the $2n - 1$ arbitrary constants c_1, c_2, …, c_{2n-1}, in which t_0 is also an arbitrary constant. If we solve the $2n$ equations that were thus obtained for the $2n$ quantities constants c_1, c_2, …, c_{2n-1}, $t - t_0$ then we will get relations of the form:

$$\begin{aligned} &f_j(q_2,\ldots,q_n,p_2,\ldots,p_n) = c_j \qquad (j = 1,2,\ldots,2n-1), \\ &f_{2n}(q_2,\ldots,q_n,p_2,\ldots,p_n) = t - t_0 . \end{aligned} \tag{9}$$

In the f_2, …, f_{2n-1}, we will then have $2n - 1$ independent integrals before us, and indeed all such things. Each of the other ones can be expressed in terms of them. At the same time, we have arrived at a clock reading in the form of f_{2n} .

From (7), the question of how many independent integrals there are can also be posed as: How many state functions are there that are in involution with a given one? We would like to answer that question by regarding it as a special case of a more general question that **S. Lie** had answered.

4. – Complete systems of linear first-order partial differential equations.

One can first generalize the last question that was posed by saying that in place of the one state function H, one has r of them, and then posing the problem: How many state functions f are there that are in involution with r given ones g_1, g_2, …, g_r ? Obviously, the desired f must satisfy the following equations:

$$(g_1, f) = 0 , \qquad (g_2, f) = 0 , \qquad \ldots, \qquad (g_r, f) = 0 . \tag{10}$$

They are r linear homogeneous first-order partial differential equations for f as a function of the $2n$ state quantities q_1, …, q_n , p_1, …, p_n . We next consider an arbitrary system of that sort. When we define the operator A_j by:

$$A_j (f) = \sum_{k=1}^{n} \left(a_{kj} \frac{\partial f}{\partial q_k} + a'_{kj} \frac{\partial f}{\partial p_k} \right) , \tag{11}$$

in which a_{kj}, a'_{kj} are arbitrary state functions, the aforementioned general system will have the form:

$$A_j (f) = 0 \qquad\qquad (j = 1, 2, \ldots, r). \tag{12}$$

In so doing, we assume that the r equations (12) are linearly independent, i.e., that not all of them can be obtained from the other ones as a linear combination, and even when we allow coefficients that are themselves state functions. It is clear that equations (12) will have just as many fewer

common solutions as their number gets larger. When $r = 2n$, one easily sees that they have no common solution at all besides the trivial one f = const. That is because in that case, we have $2n$ ordinary linear homogeneous equations with non-vanishing determinant in the $2n$ unknown quantities $\frac{\partial f}{\partial q_k}, \frac{\partial f}{\partial p_k}$ before us, from which, the vanishing of those quantities will follow, and therefore the constancy of f. However, if $r < 2n$ then it can be shown that the number of common solutions can never be larger than $2n - r$.

It is initially clear that applying the operators A_k to one of eq. (12) will produce expressions of the form $A_k (A_j (f))$ that also contain the second partial derivatives of f with respect to the state quantities. However, if one forms combinations of the form $A_k (A_j (f)) - A_j (A_k (f))$ then one will easily see from the definition (11) that the second derivatives will cancel, and expressions will once more arise that are linear and homogeneous in the $\frac{\partial f}{\partial q_k}, \frac{\partial f}{\partial p_k}$, so they will have precisely the same form as $A_k (f)$ itself. It is also clear that the common solutions of the r equations (12) will also be solutions of the equations:

$$A_k (A_j (f)) - A_j (A_k (f)) = 0 \qquad (k = 1, 2, \ldots, r) \qquad (13)$$

which have entirely the same form as them. When the combinations of the form (13) are not linearly independent of the r expressions $A_k (f)$, one can simply add the r equations in (12) to eq. (13) and obtain a system with precisely the same common solutions f, but which consists of more than r equations. Therefore, we shall assume that no relations of the form:

$$A_k (A_j (f)) - A_j (A_k (f)) = \sum_{s=1}^{r} C_s A_s (f) \qquad (14)$$

exist. We can continue that process by defining more combinations of the form (13) with the left-hand sides of the now-augmented equations (12). Ultimately, we will be led to a system of equations of the form:

$$A_j (f) = 0 \qquad (j = 1, 2, \ldots, r, r + 1, \ldots, r + l) , \qquad (15)$$

in which each combination (13) the left-hand sides can be represented as a linear combination of the $r + l$ quantities $A_j (f)$ themselves in the manner of (14). If that is first achieved for $r + l = 2n$ then, from what was said before, there will be no common solutions f to eq. (15) at all, so none for eq. (12), either. We would then like to assume that this "saturation state" of the system (12) can already be achieved for $r + l < 2n$. We call the "saturated" system (15) a *complete* system of first-order linear partial differential equations. We can then reduce the integration of each system (12) to that of a complete system since both of them will possess the same common solutions f. We would then like to assume, for simplicity, that (12) itself is already a complete system.

It can now be shown ([1]) that every complete system possesses precisely $2n - r$ different solutions. That is, there are $2n - r$ state functions between which no identity exists, and which are all common solutions to (12). Furthermore, each common solution to (12) can be expressed identically in terms of the aforementioned $2n - r$ special solutions. In that way, one now understands identities to mean not only linear homogeneous ones, but also general ones. Since every incomplete system of r equations can be extended to a complete one with more than r equations, it will obviously have less than $2n$ common solutions.

5. – Complete systems and function groups.

If we return once more to the question of the common solutions of (1) then we can formulate it as follows: When do the r equations (10) define a complete system? **Sophus Lie** has shown ([2]) that this comes down to the **Poisson** bracket expressions of the $g_1, \ldots, g_r$. He proved the fundamental theorem: Eq. (10) will be a complete system when all of the bracket expressions of the g_j can be expressed identically in terms of the function g_j themselves, so it will have $2n - r$ different solutions. The assumptions about the g_j can then be expressed as: They must be mutually independent (and not just linearly independent), and their bracket expressions should be identically representable in the following way:

$$(g_j, g_k) = f_{jk}(g_1, g_2, \ldots, g_r) \qquad (j, k = 1, 2, \ldots, r) . \tag{16}$$

If we would like to prove **Lie**'s theorem then we would have to start from the identity:

$$A_j(f) = (g_j, f) , \tag{17}$$

which is true with the notation of the previous section.

By calculation and recalling (16), one will arrive at the identity:

$$A_k(A_j(f)) - A_j(A_k(f)) = ((g_j, g_k), f) = (f_{jk}, f) . \tag{18}$$

If we then employ the property of the bracket expression eq. (21) that will be proved in the next section then that, along with (17), will yield:

$$A_k(A_j(f)) - A_j(A_k(f)) = \frac{\partial f_{jk}}{\partial g_1} A_1(f) + \frac{\partial f_{jk}}{\partial g_2} A_2(f) + \cdots + \frac{\partial f_{jk}}{\partial g_r} A_r(f) . \tag{19}$$

Therefore, from the criterion that was summarized in eq, (14) and eq. (17), the system (10) will prove to be a complete system. According to **S. Lie**, one says that a system of r state functions g_1,

([1]) One can find the proof, which is not discussed here, in, e.g., **E. Goursat**, *Vorlesungen über die Integration der partiellen Differentialgleichungen erster Ordnung*, translated by **H. Maser**, Leipzig, 1893. That is by far the most recommendable textbook on this topic, which is now becoming important to physicists.

([2]) **S. Lie**, *Gesammelte Abhandlungen*, Bd. III, 1922.

…, g_r that are different from each other, but the bracket expressions can be expressed in terms of the g_1, …, g_r themselves, define a *function group*. With that group, one can also calculate all state functions that can be expressed in terms of the g_1, …, g_r that define the group. One then has the theorem:

If r state functions define a function group then there will be $2n - r$ *state functions f that are in involution with all of them.*

Before we go on, some rules of calculation for **Poisson** brackets shall be reviewed that will also be important in what follows.

6. Rules of calculation for operating with bracket expressions.

It will follow immediately from the definition (3) of the bracket expression that for the two state functions *f* and *g*, one has:

$$(f, g) = -(g, f) , \qquad (f, f) = 0 . \tag{20}$$

When *F* is a function of the state functions f_1, f_2, …, and one replaces *f* in (3) with $F(f_1, f_2, \ldots)$ and performs the differentiations, that will give:

$$(F, g) = \sum_k \frac{\partial F}{\partial f_k} (f_k, g) . \tag{21}$$

The process of forming the bracket then has the character of an ordinary differential process. We will see that when we also understand *G* to mean a function of the state functions g_1, g_2, … and form (*F*, *G*). Analogous to (21), that will then give:

$$(F, G) = \sum_{j,k} \frac{\partial (F,G)}{\partial (f_j, g_k)} (f_j, g_k) . \tag{22}$$

Each numerical combination *j*, *k* occurs only once in the sum, up to the order of the two. The coefficients in (22) are the functional determinants:

$$\frac{\partial (F,G)}{\partial (f_j, g_k)} \qquad \text{(illegible).} \tag{23}$$

We have a further analogy to differentiation in the bracket expression of a product *f g* with a third state function *h*. Namely, we have:

$$(f g, h) = f (g, h) + g (f, h) . \tag{24}$$

Finally, when we form the bracket expression of a bracket expression (f, g) and a state function h and calculate according to (3), we will get the ***Jacobi*** *identity:*

$$((f, g), h) + ((g, h), f) + ((h, f), g) = 0 \, . \tag{25}$$

7. – Polar groups and distinguished functions.

According to **S. Lie**, one can further show that the functions that are in involution with the r functions of the group according to no. (?), which we would like to call $f_1, f_2, \ldots, f_{2n-r}$, will themselves define a function group. One calls it the *polar group* to the original group. In that sense, since the $2n - 1$ integrals of our mechanical system are all in involution with H, they will define a function group, namely, the polar group of H. The theorem that the bracket expression (f_j, f_k) of two integrals is itself an integral of the equations of motion (1) can now be regarded as a special case of that theorem. Namely, since the integrals define a group, the bracket expression (f_j, f_k) can be expressed in terms of the integrals $f_1, f_2, \ldots, f_{2n-1}$ themselves. so it must be itself an integral.

One can also raise the question now of when $2n - 1$ state functions $f_1, f_2, \ldots, f_{2n-1}$ can be regarded as integrals of a mechanical system. From (7), one must then be able to find a state function H that is in involution with all f_j, so it will satisfy the equations:

$$(H, f_j) = 0 \qquad\qquad (j = 1, 2, \ldots, 2n - 1). \tag{26}$$

In order for that to be true, eq. (26) must define a complete system. That will certainly be the case if and only if the $f_1, f_2, \ldots, f_{2n-1}$ define a function group. For $2n - 1$ state functions, one can the find a **Hamilton**ian function that they belong to as integrals of the equations of motion when the $2n - 1$ state functions define a group.

When a function belongs to the polar group, as well as the original one, one calls it a *distinguished function* of the latter. It must then be in involution with all functions of the latter. The **Hamilton**ian function H is obviously the only distinguished function in the group of integrals of a mechanical system. In that way, the energy integral plays an essentially-different role mathematically than all other integrals.

8. – Canonical form for a group and canonical transformations.

The totality of all state functions obviously defines a group as well. One can choose $2n$ mutually independent state functions to be its "basis." One then says that every function that can be expressed in terms of the functions of the basis belongs to the group. An obvious choice for the basis for the group of all state functions would be the $2n$ state quantities $q_1, \ldots, q_n, p_1, \ldots, p_n$. From the definition (3) of the bracket expression, every q_j is canonically conjugate to the corresponding p_j and any two other state quantities are in involution with each other. One then has:

$$(p_j, p_k) = 0\,, \quad (q_j, q_k) = 0\,, \quad (p_j, q_k) = \begin{cases} 0 & \text{for} \quad j \neq k\,, \\ 1 & \text{for} \quad j = k\,. \end{cases} \tag{27}$$

If we can arrange the basis functions of a group into two rows in such a way that the functions in the same row are in involution with each other, but the ones in different rows, but the same column, are canonically conjugate then we will say that the basis in question is a *canonical form* for the group. The group of all (or what amounts to the same thing, 2*n*) state functions can then admit the following canonical representation:

$$\begin{matrix} p_1\,, & p_2\,, & \cdots & p_n \\ q_1\,, & q_2\,, & \cdots & q_n\,. \end{matrix} \tag{28}$$

If we introduce 2*n* new variables $Q_1, \ldots, Q_n, P_1, \ldots, P_n$ by means of the 2*n* independent equations:

$$\begin{aligned} Q_j &= Q_j\,(q_1, \ldots, q_n, p_1, \ldots, p_n)\,, \\ P_j &= P_j\,(q_1, \ldots, q_n, p_1, \ldots, p_n) \end{aligned} \qquad (j = 1, 2, \ldots, n) \tag{29}$$

then in order for the new functions to also define a canonical representation of the group of all state functions of the form:

$$\begin{matrix} P_1\,, & P_2\,, & \cdots & P_n \\ Q_1\,, & Q_2\,, & \cdots & Q_n \end{matrix} \tag{30}$$

obviously the relations must exist:

$$(P_j, P_k) = 0\,, \quad (Q_j, Q_k) = 0\,, \quad (P_j, Q_k) = \begin{cases} 0 & \text{for} \quad j \neq k\,, \\ 1 & \text{for} \quad j = k\,. \end{cases} \tag{31}$$

One calls the transformation that is represented by (29) a *canonical transformation* or a *contact transformation*. The canonical form of the group (28) will be preserved by it. That raises the question now of the extent to which one can put arbitrary function groups into a canonical form. **S. Lie** has also given an exhaustive answer to that question ([1]).

If, say, the *r* state functions $f_1, f_2, \ldots, f_r$ define the basis of an *r*-term function group then, according to **Lie**, one can find *r* mutually-independent functions of the $f_1, \ldots, f_r$, so a new basis for the same group, such that those new functions, which are expressed in terms of the state functions themselves that one would like to denote by $P_1, P_2, \ldots, P_\alpha$; $Q_1, Q_2, \ldots, Q_\beta$ ($\alpha + \beta = r$), will satisfy the relations (31). If, say $\beta = \alpha + s$, then one can also write them with the outward appearance of a canonical group, namely:

([1]) **S. Lie**, *loc. cit.*, pp. 42, *et seq.*

$$\begin{aligned} &P_1, P_2, \ldots, P_n \\ &Q_1, Q_2, \ldots, Q_\alpha, Q_{\alpha+1}, \ldots, Q_{\alpha+s} \quad (\alpha + s = \beta). \end{aligned} \tag{32}$$

According to **Lie**, one can further extend the functions $P_1 \ldots, P_\alpha$, $Q_1, \ldots, Q_{\alpha+s}$ to a $2n$-parameter group by adding $2n - r = 2n - \alpha - \beta$ more functions such that the P_j, Q_j satisfy the relations (31). That is: One can convert any group $f_1, f_2, \ldots, f_r$ into its canonical form (32), and indeed by way of a contact transformation (29), (31) that takes the $q_1, \ldots, q_n, p_1, \ldots, p_n$ to $f_1, \ldots, f_r$. It is clear with no further analysis from the representation (32) that our group includes precisely $s = \beta - \alpha$ distinguished functions $Q_{n+1}, \ldots, Q_{n+s}$, i.e., the ones that are in involution with all of the group parameters. **Lie** could show that this number remains the same for any form of the group that emerges from the canonical one by a contact transformation. He inferred the important theorem (which we will only briefly mention here) that two function groups with the same number r of parameters and the same number s of distinguished functions can also be taken to each other by a contact transformation, since the two will assume the same canonical form (32).

9. – Systems in involution.

When any two of r state functions $f_1, \ldots, f_r$ are in involution with each other, we call the f_j an r-parameter *system in involution*. If we define r independent function of the f_j, say $F_1, F_2, \ldots, F_r$, then we can easily show that they will also be a system in involution. We merely need to express the bracket expressions (F_j, F_k) in terms of the (f_j, f_k) using (22). If we wish to know the number of functions in a group that are in involution with each other then we will need only to consider the group in its canonical form (32). Since only functions in the same row or ones in different rows that are not in the same column are in involution here, a system in involution can consist of at most $\alpha + s = \beta$ parameters. A group with r parameters and s distinguished functions will then include systems in involution with up to $\alpha + s = (r + s) / 2$ parameters. However, since obviously we always have $\alpha + s \leq n$, the only systems in involution will have at most n parameters. More than n state functions can never be in involution with each other. A function group that consists of an n-parameter system in involution will then have n distinguished functions. It will then be identical to its polar group, because there can be no further $(n+1)^{\text{th}}$ function that is in involution with all n of them.

The n state functions $f_1, f_2, \ldots, f_r$ might perhaps define as "complete" system in involution, as we would now like to call such an n-parameter system. By setting the f_j equal to zero, we will get n equations between the $2n$ state quantities $q_1, \ldots, q_n, p_1, \ldots, p_n$. If we assume that they are soluble then we can solve the relations $f_j = 0$ for the n quantities $p_1, \ldots, p_n$ and n relations of the form:

$$p_j = \pi_j (q_1, q_2, \ldots, q_n) = 0 \qquad (j = 1, 2, \ldots, n). \tag{33}$$

One can easily show that their left-hand sides also define a system in involution, just as in the original form. By calculating the bracket expression using the definition (3), we will get:

$$(p_j - \pi_j\,,\, p_k - \pi_k) = \frac{\partial \pi_j}{\partial q_k} - \frac{\partial \pi_k}{\partial q_j}\ . \tag{34}$$

Since the expressions in (33) define a system in involution, it will follow from (34) that the linear differential expression $\pi_1\, dq_1 + \pi_2\, dq_2 + \ldots + \pi_n\, dq_n$ defines a complete differential, such that a function S of the $q_1, \ldots, q_n$ will exist, such that:

$$p_j = \pi_j\,(q_1, q_2, \ldots, q_n) = \frac{\partial S}{\partial q_j} \qquad (j = 1, 2, \ldots, n). \tag{35}$$

10. – Canonical form of the group of integrals of a mechanical system.

Since this group consists of $2n - 1$ parameters, according to (32) its canonical form can take a form in which a single function has no counterpart in the other row. Since the energy H is the single distinguished function of the group, it must be the single function that stands along. (At most a function of H alone can replace it.) The table for the group of integrals that corresponds to (32) must then take the following form:

$$\begin{array}{cccc} H & P_2 & \ldots & P_n \\ & Q_2 & \cdots & Q_n \end{array} \tag{36}$$

If we would like to extend the group (36) to the group of all state functions then only one more function must be added on the left, but it must be in involution with all of the other integrals. From no. **3**, the missing quantity Q_1 is, however, obviously a clock reading, or if one prefers, time itself. One can then obtain the group of integrals of a mechanical system in a certain sense by leaving all functions of a clock reading out of the group.

11. – The integration problem from the standpoint of the group picture.

Integrating the equations of motion amounts to nothing but finding the $2n - 1$ integrals in (36). However, the problem can be expressed as follows: Take the group of all state functions in the simple form (28) and put it into a form (30) such that the canonical form will remain preserved, so eq. (31) will be true, and one will have:

$$H\,(q_1, \ldots, q_n, p_1, \ldots, p_n) = P_1\,, \tag{37}$$

which is why the group will assume the form (36). Otherwise stated: Find a contact transformation (29), (31) such that when it is solved for the state quantities $q_1, \ldots, q_n, p_1, \ldots, p_n$ and replaced in $H\,(q_1, \ldots, q_n)$, the identity (37) will be fulfilled. Namely, we will then have collectively $2n - 1$

integrals in the $P_2, \ldots, P_n$, $Q_2, \ldots, Q_2$, which are all in involution with H, together with H, and a clock reading in Q_1. If we set:

$$\begin{gathered} H = c_1, \quad P_j = c_j, \quad Q_j = c_{n+j-1}, \quad (j = 2,3,\ldots,n), \\ Q_1 = t - t_0 \end{gathered} \tag{38}$$

then when we solve those equations for the $2n$ state quantities $q_1, \ldots, p_n$, we will get the latter as functions of time t and the $2n$ arbitrary constants $t_0, c_1, c_2, \ldots, c_{2n-1}$.

However, one easily sees that the integration can also be performed already when the contact transformation (29), (31) satisfies, not the identity (37), but the much less demanding identity:

$$H(q_1, \ldots, q_n, p_1, \ldots, p_n) = E(P_1, P_2, \ldots, P_n), \tag{39}$$

in which E is an arbitrary function of the $P_1, \ldots, P_n$. We shall next show that under the assumption (39), the $P_1, \ldots, P_n$ in the group with the form (38) will become integrals of our mechanical system.

Namely, from (21), (39), and (31), one will have:

$$(H, P_j) = (E, P_j) = \sum_{k=1}^{n} \frac{\partial E}{\partial P_k}(P_k, P_j) = 0. \tag{40}$$

From (7), the P_j are integrals then, so they will be constant during the motion. Therefore, that will also be true for the:

$$\omega_j = \frac{\partial E}{\partial P_j}. \tag{41}$$

By contrast, if we define (H, Q_j) then (21), (37), (31) will likewise imply that:

$$(H, Q_j) = (E, Q_j) = \sum_{k=1}^{n} \frac{\partial E}{\partial P_k}(P_k, Q_j) = \frac{\partial E}{\partial P_j} = \omega_j. \tag{42}$$

From (4), we will then have:

$$\frac{dQ_j}{dt} = \omega_j, \qquad Q_j = \omega_j t + \beta_j, \tag{43}$$

in which the β_j, together with the P_j, are $2n$ arbitrary integration constants. By making the replacements:

$$P_j = \alpha_j, \qquad Q_j = \omega_j t + \beta_j \qquad (j = 1, 2, \ldots, n) \tag{44}$$

in (29) and solving for the $2n$ state quantities $q_1, \ldots, q_n, p_1, \ldots, p_n$, we will get them as functions of t and the $2n$ constants $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n$. We shall call a system of canonical variables Q_j, P_j that behave as in (39) in such a way that the H can be expressed in terms of the P_j alone the canonical variables that are "adapted" to the mechanical problem.

12. – Representing the equations of motion in terms of only bracket expressions.

We now think of the equations of transformation by which the Q_j, P_j were introduced as having the solved form:

$$\begin{aligned} q_j &= q_j(Q_1,\ldots,Q_n,P_1,\ldots,P_n), \\ p_j &= p_j(Q_1,\ldots,Q_n,P_1,\ldots,P_n) \end{aligned} \qquad (j = 1, 2, \ldots, n). \tag{45}$$

Since we have a contact transformation before us, the relations:

$$(q_j, q_k) = (p_j, p_k) = 0\ , \qquad (p_j, q_k) = \begin{cases} 1 & \text{for} \quad j = k \\ 0 & \text{for} \quad j \neq k \end{cases} \tag{46}$$

will be satisfied. It is self-explanatory that one understands the bracket expressions (q_j, q_k), etc., in that to mean:

$$(q_j, q_k) = \sum_{l=1}^{n} \left(\frac{\partial q_j}{\partial P_l}\frac{\partial q_k}{\partial Q_l} - \frac{\partial q_j}{\partial Q_l}\frac{\partial q_k}{\partial P_l} \right), \quad \text{etc.} \tag{47}$$

In order to carry out the integration as in no. **11**, the Q_j, P_j that were introduced in eq. (45) must be adapted variables, so the identity would have to follow from (45). However, if we were to define the bracket expressions of both sides of that with q_j (p_j, resp.), in which those expressions are understood to be as in (47), then it would follow that:

$$(H, q_j) = (E, q_j), \qquad (H, p_j) = (E, p_j) \qquad (j = 1, 2, \ldots, n). \tag{48}$$

Those equations, together with (46), define a system of partial differential equations for the $2n$ functions q_j, p_j that were introduced in (45) in terms of the $2n$ variables Q_j, P_j . We easily see that any solution of that system will already lead us to the general solution of the system of equations. In order to do that, one must only show that the identity (30) will again follow from (48) and (46). That is because in this case, from no. **11**, we have in eqs. (45) and (44) a solution to the equations of motion with $2n$ arbitrary constants before us.

Therefore, we must only show that the vanishing of the partial derivatives of H with respect to all Q_j follows from eqs. (48) and (46), because that is the gist of the identity (39). However, we have:

$$\frac{\partial H}{\partial Q_l} = \sum_{j=1}^{n} \left(\frac{\partial H}{\partial q_j}\frac{\partial q_j}{\partial Q_l} + \frac{\partial H}{\partial p_j}\frac{\partial p_j}{\partial Q_l} \right) \qquad (j = 1, 2, \ldots, n)\ . \tag{49}$$

Eq. (21) gives the following formulas for the bracket expressions in (48):

$$(H, q_j) = \sum_{k=1}^{n} \left[\frac{\partial H}{\partial q_k}(q_k, q_j) + \frac{\partial H}{\partial p_j}(p_k, q_j) \right],$$

(50)

$$(H, p_j) = \sum_{k=1}^{n} \left[\frac{\partial H}{\partial q_k}(q_k, p_j) + \frac{\partial H}{\partial p_j}(p_k, p_j) \right],$$

so from (46):

$$(H, q_j) = \frac{\partial H}{\partial p_j}, \qquad (H, p_j) = -\frac{\partial H}{\partial q_j} \qquad (j = 1, 2, \ldots, n). \tag{51}$$

By contrast, one will get the following expressions for the left-hand sides of (48) and (47):

$$(E, q_j) = \sum_{k=1}^{n} \frac{\partial E}{\partial P_k}\frac{\partial q_j}{\partial Q_k}, \qquad (E, p_j) = \sum_{k=1}^{n} \frac{\partial E}{\partial P_k}\frac{\partial p_j}{\partial Q_k} \qquad (j = 1, 2, \ldots, n). \tag{52}$$

When one substitutes from (48), (51), (52) in (49), that will give:

$$\frac{\partial H}{\partial Q_l} = \sum_{k,j} \frac{\partial E}{\partial P_k}\left(\frac{\partial p_j}{\partial Q_l}\frac{\partial q_j}{\partial Q_k} - \frac{\partial p_j}{\partial Q_k}\frac{\partial q_j}{\partial Q_l} \right) \qquad (l = 1, 2, \ldots, n). \tag{53}$$

If we introduce the symbols:

$$[Q_l, Q_k] = \sum_{j=1}^{n} \left(\frac{\partial p_j}{\partial Q_l}\frac{\partial q_j}{\partial Q_k} - \frac{\partial p_j}{\partial Q_k}\frac{\partial q_j}{\partial Q_l} \right),$$

$$[P_l, Q_k] = \sum_{j=1}^{n} \left(\frac{\partial p_j}{\partial P_l}\frac{\partial q_j}{\partial Q_k} - \frac{\partial p_j}{\partial Q_k}\frac{\partial q_j}{\partial P_l} \right), \qquad \text{etc.,} \tag{54}$$

then it will follow from (53) that:

$$\frac{\partial H}{\partial Q_l} = \sum_{k,j} \frac{\partial E}{\partial P_k}[Q_l, Q_k] \qquad (l = 1, 2, \ldots, n). \tag{55}$$

Whereas the **Poisson** bracket expressions are defined from every pair of functions of the 2n state quantities $Q_1, Q_2, \ldots, P_1, P_2, \ldots, P_n$ according to (47), the bracket expressions (54) arise from the 2n state functions $q_1, q_2, \ldots, p_1, p_2, \ldots, p_n$, and a pair of state quantities. One often calls them the **Lagrange** bracket expressions. Certain relations exist between the two types of bracket expressions that we will state here without proof, which we can, however, easily verify by calculation using the rules in no. **6**. Namely, when we appeal to eq. (45) in the solved form (29) for the **Poisson** expressions, so we define the parentheses by (3), we will have:

$$\sum_{j=1}^{n} (Q_j, Q_k)[Q_j, Q_l] = \begin{cases} 1 & \text{for} \quad k = l \\ 0 & \text{for} \quad k \neq l. \end{cases} \tag{56}$$

Therefore, any of the quantities $Q_1, \ldots, Q_n, P_1, \ldots, P_n$ can replace the Q_j, Q_k in that. If the functions $Q_1, \ldots, P_n$ of the $q_1, \ldots, p_n$ define a contact transformation, so eq. (31) will be true for the **Poisson**

bracket expressions, then due to (56), the same relations will be true for the **Lagrange** brackets, so:

$$\left.\begin{aligned} [Q_j, Q_k] &= [P_j, P_k] = 0, \\ [P_j, Q_k] &= \begin{cases} 1 & \text{for} \quad j = k \\ 0 & \text{for} \quad j \neq k. \end{cases} \end{aligned}\right\} \tag{57}$$

However, it will then follow from (55) that:

$$\frac{\partial H}{\partial Q_l} = 0 \qquad (l = 1, 2, \ldots, n). \tag{58}$$

However, the identity (39) will follow from that, and it is therefore shown that the integration of the equations of motion of classical mechanics can be reduced to the integration of the partial differential equations (48), (46).

13. – Analogy to quantum mechanics.

Let us write eqs. (46), (48) for a mechanical system with one degree of freedom that can be established by the canonical coordinates q, p. We then give a function H (q, p) of the state q, p, and we must look for three unknown functions of the two variables Q, P, namely, q (Q, P), p (Q, P), E (Q, P), of which the last one can depend upon only P (as we knew from the outset). In order to calculate those three functions, three partial differential equations:

$$(H, q) = (E, q), \qquad (H, p) = (E, p), \qquad (p, q) = 1 \tag{59}$$

are available.

Here, the complete analogy to the basic equations of quantum mechanics, as they were posed by **Heisenberg**, is clarified. We also have three differential equations for the system of one degree of freedom before us, not two as in the usual form of the equations of motion. The last one corresponds to the quantum condition. It is not the two coordinates q and p as functions of time that are regarded as the unknowns, but q, p, and W as functions of the two variables Q and P. That comes down to the fact that it is not an individual trajectory that appears as the solution, but the entire system of trajectories. According to (44), the dependency of Q corresponds to the various points of a trajectory that differs by only P when one goes from one trajectory to another. The functions q, p, and E correspond to matrices in quantum mechanics. The fact that E depends upon only P is again found in quantum mechanics in the fact that the matrix that corresponds to E is a diagonal matrix.

14. – Complete systems in involution and wave functions.

We have shown that we can find a system of n integrals $P_1, P_2, \ldots, P_n$, so a “complete” system in involution with the help of a contact transformation (45), (46), from which the identity (39) will

follow, which introduces the "adapted" canonical variables. Now, we can also easily show that conversely, if we know a complete system in involution then that will suffice for us to introduce adapted canonical variables by a contact transformation, and thus to be able to solve the integration problem. Let, say:

$$f_j\,(q_1, \ldots, q_n, p_1, \ldots, p_n) = P_j \qquad (j = 1, 2, \ldots, n) \qquad (60)$$

be n integrals of the equations of motion that are in involution. The P_j in that are integration constants. If we solve eq. (60) for the $p_1, \ldots, p_n$ then we will get the p_j in eq. (33) as functions of the q_j and the integration constants P_j . According to eq. (35), there will then be a function S of the q_j and P_j such that:

$$\begin{aligned} p_j &= \pi_j(q_1,\ldots,q_n,P_1,\ldots,P_n) \\ &= \frac{\partial S\,(q_1,\ldots,q_n,P_1,\ldots,P_n)}{\partial q_j} \quad (j=1,2,\ldots,n). \end{aligned} \qquad (61)$$

If we introduce the n quantities $Q_1, \ldots, Q_n$ by way of the equations:

$$Q_j = \frac{\partial S\,(q_1,\ldots,q_n,P_1,\ldots,P_n)}{\partial P_j} \qquad (j = 1, 2, \ldots, n) \qquad (62)$$

then the $2n$ eqs. (61), (62) will define a contact transformation between q_j, p_j and the Q_j , P_j . Namely, one can show (which will be only stated without proof here) that the relations (31) [(46), resp.] will follow from eq. (61), (62) for an S of arbitrary form. One further easily sees that the Q_j, p_j that were introduced by eqs. (61), (62) are adapted variables. Namely, since (60) says that the P_j will be integrals of the equations of motion when they are considered to be functions of the q_j, p_j , from (7), one will have the n equations:

$$(H, P_j) = 0 \qquad (j = 1, 2, \ldots, n), \qquad (63)$$

in which the bracket expressions are understood to mean what they did in (3). Due to (21), one then has:

$$(H, P_j) = \sum_{k=1}^{n} \left[\frac{\partial H}{\partial Q_k}(Q_k, P_j) + \frac{\partial H}{\partial P_k}(P_k, P_j) \right] \qquad (j = 1, 2, \ldots, n), \qquad (64)$$

from which it will follow from (31) that:

$$(H, P_j) = -\frac{\partial H}{\partial Q_j} \qquad (j = 1, 2, \ldots, n). \qquad (65)$$

However, from (63), H cannot include the Q_j then, so the identity (39) will be true, and the Q_j, P_j will be adapted variables. One can then reduce the integration problem to the search for the function $S\,(q_1, \ldots, q_n, P_1, \ldots, P_n)$. One calls it the action function, or with an interpretation that we shall soon speak of, the "wave function."

15. – The Hamilton-Jacobi integration method.

From what has been done up to now, one can immediately give a partial differential equation that the wave function S must satisfy as a function of the $q_1, \ldots, q_n$. Namely, by substituting (61) in the identity (39), one can obtain the condition that S must satisfy in order for the Q_j, P_j that were introduced by the contact transformation (61), (62) to be adapted variables:

$$H\left(q_1, q_2, \ldots, q_n, \frac{\partial S}{\partial q_1}, \frac{\partial S}{\partial q_2}, \ldots, \frac{\partial S}{\partial q_n} \right) = E\,(P_1, P_2, \ldots, P_n)\,. \tag{66}$$

E is an unknown function in that, of which we know only that it depends upon only the P_j . Eq. (66) is called the ***Hamilton-Jacobi*** *partial differential equation*. If $S\,(q_1, \ldots, q_n, P_1, \ldots, P_n)$ is any solution that includes the n independent integration constants $P_1, \ldots, P_n$ (one calls such a solution complete) then one will obtain the general of solution of the equations from (61), (61), in conjunction with (44) and (41), namely, the $2n$ quantities $q_1, \ldots, q_n, p_1, \ldots, p_n$ as functions of time t and the $2n$ arbitrary constants $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n$. One can easily obtain the $n - 1$ separate equations that determine the geometric form of the "trajectories" in the n-dimensional space of the $q_1, \ldots, q_n$ in that solution. To that end, one chooses one of the n constants $P_1, \ldots, P_n$ to be energy itself, so one sets, perhaps, $P_n = E$. The **Hamilton-Jacobi** eq. (66) will then read simply:

$$H\left(q_1, \ldots, q_n, \frac{\partial S}{\partial q_1}, \ldots, \frac{\partial S}{\partial q_n} \right) = P_n = E\,, \tag{67}$$

and the wave function S will depend upon the n constants $P_1, \ldots, P_{n-1}, E$. Eqs. (42), (44) will then read:

$$\begin{aligned} &\omega_j = 0 \quad (j = 1, 2, \ldots, n-1), \qquad \omega_n = 1\,, \\ &Q_j = \beta_j \qquad\qquad\qquad\qquad (j = 1, 2, \ldots, n-1), \\ &Q_n = t + \beta_n = t - t_0\,, \end{aligned} \tag{68}$$

and as a result of eq. (62):

$$\frac{\partial S\,(q_1, \ldots, q_n, P_1, \ldots, P_{n-1}, E)}{\partial P_j} = \beta_j \qquad (j = 1, 2, \ldots, n-1), \tag{69}$$

$$\frac{\partial S}{\partial E} = t - t_0\,. \tag{70}$$

In eq. (69), we then have the desired $n - 1$ relations between the n quantities $q_1, \ldots, q_n$ before us that defines a curve in n-dimensional q-space. Eq. (70) then gives us a clock reading, so the time evolution of the motion. In the solution to the **Hamilton-Jacobi** equation (66), we then have a means before us of finding n integrals of the equations of motion that are in involution, so a complete system in involution of the type (60). We will get it from (61) by calculating the P_j and functions of the q_j, p_j .

16. – Separation of variables.

The simplest case of a system in involution of the form (60) is the one in which the f_j include only one of the q_j and the p_j that belongs to it, so one of the form:

$$f_j\,(q_j, p_j) = P_j \qquad (j = 1, 2, \ldots, n)\,. \tag{71}$$

One easily sees that, from (3), the bracket expressions (f_j, f_k) always vanish for functions of that form. If eq. (61) is supposed to represent that system in involution then p_j can depend upon only q_j, so S must be a sum of terms only one of which includes q_j, so:

$$S = \sum_{j=1}^{n} S_j(q_j)\,. \tag{72}$$

Initially, one can always try to see if it would be possible to find a solution of eq. (67) that has the special form (72) then. In that case, it is possible to find a complete system in involution of the simple form (71). One then says that: The problem is integrable by "separation of the variables." The significance of that method consists of the fact that in the cases where it is applicable, there will be integrals of the simple form (71) such that one can see the property that they define a system in involution immediately from them.

17. – Waves and trajectories.

If we consider the action function S of a mechanical system that has a given energy value E then that S, which occurs in eqs, (69), (70), depends upon $n - 1$ arbitrary constants $P_1, \ldots, P_{n-1}$. We then have ∞^{n-1} functions of the coordinates $q_1, \ldots, q_n$ before us, or if we set:

$$S\,(q_1, \ldots, q_n, P_1, \ldots, P_{n-1}) = \tau - \tau_0\,, \tag{73}$$

when $\tau - \tau_0$ is constant, we will have a family of ∞^{n-1} surfaces in n-dimensional q-space. We call them the "wave surfaces" that belong to our mechanical problem.

We next imagine that the constant τ_0 has been chosen such that for $t = 0$, the surface of the family that corresponds to the parameter values $P_1, \ldots, P_{n-1}$ will go through the point $q_1^{(0)}, \ldots, q_n^{(0)}$. It will then follow from (73) that $\tau_0 = -\,S(q_1^{(0)}, \ldots, q_n^{(0)}, P_1, \ldots, P_{n-1})$, and the family (73) can be written in the form:

$$S\,(q_1, \ldots, q_n, P_1, \ldots, P_{n-1}) - S(q_1^{(0)}, \ldots, q_n^{(0)}, P_1, \ldots, P_{n-1}) = \tau\,. \tag{74}$$

We can now demand that τ should change continuously, while the $P_1, \ldots, P_{n-1}$ remain constants. The individual surfaces of the family (74) will then change continuously, as well. When we regard,

say τ, as a fictitious time, we can see a sort of wave propagation in the continuous change in the wave surfaces with τ. If we then consider a wave surface that goes through a certain spatial point $q_1^{(0)}, \ldots, q_n^{(0)}$ at a certain time τ, e.g., it goes through $q_1^{(0)}, \ldots, q_n^{(0)}$ at time $\tau = 0$, then we can choose the parameters $P_1, \ldots, P_{n-1}$ such that the wave normal has a well-defined direction at that point in time. A direction in n-dimensional space is, in fact, given by the ratios of n quantities, say, $\lambda_1 : \lambda_2 : \ldots : \lambda_n$. We must only choose the $P_1, \ldots, P_{n-1}$ such that the proportion:

$$\frac{\partial S}{\partial q_1} : \frac{\partial S}{\partial q_2} : \cdots : \frac{\partial S}{\partial q_n} = \lambda_1 : \lambda_2 : \ldots : \lambda_n \tag{75}$$

will exist for $q_1 = q_1^{(0)}$, $q_1 = q_1^{(0)}, \ldots, q_1 = q_1^{(0)}$. The desired direction for the surface normal is then fixed. However, the proportion (75) corresponds to precisely $n - 1$ equations for the $n - 1$ quantities $P_1, \ldots, P_{n-1}$. When the direction of the wave normal at the point $q_1^{(0)}, \ldots, q_n^{(0)}$ is also prescribed, the family (74) will then read:

$$S(q_1,\ldots,q_n,P_1^{(0)},\ldots,P_{n-1}^{(0)}) - S(q_1^{(0)},\ldots,q_n^{(0)},P_1^{(0)},\ldots,P_{n-1}^{(0)}) = \tau, \tag{76}$$

when $P_1^{(0)}, \ldots, P_{n-1}^{(0)}$ are the solutions of eq. (73). Therefore, when τ varies continuously through all real values, there will be only one surface that exists at each τ, viz., a "wave train." We shall also consider the wave trains now that simultaneously go through the point $q_1^{(0)}, \ldots, q_n^{(0)}$ that is described in (76), but whose wave normals do not coincide precisely with the one at that point. We will then obtain a family of waves whose directions of propagation at the point will fill up a thin spatial cone with its vertex at that point. One will get its analytical representation by simply replacing $P_1^{(0)}, \ldots, P_{n-1}^{(0)}$ in (76) with somewhat-different values $P_1^{(0)} + \Delta P_1, \ldots, P_{n-1}^{(0)} + \Delta P_{n-1}$:

$$S(q_1,\ldots,q_n,P_1^{(0)}+\Delta P_1,\ldots,P_{n-1}^{(0)}+\Delta P_{n=1}) - S(q_1^{(0)},\ldots,q_n^{(0)},P_1^{(0)}+\Delta P_1,\ldots,P_{n-1}^{(0)}+\Delta P_{n-1}) = \tau. \tag{77}$$

We can now pose the questions of whether all of those waves that go through the point $q_1^{(0)}, \ldots, q_n^{(0)}$ at $\tau = 0$ will once more all come together at a point in q-space at any other time point τ, and where all of those points will be found at each time point τ. To that end, we develop the left-hand side of (77) in a **Taylor** series and truncate it after the first term. We will then get:

$$S(q_1,\ldots,q_n,P_1^{(0)},\ldots,P_{n-1}^{(0)}) - S(q_1^{(0)},\ldots,q_n^{(0)},P_1^{(0)},\ldots,P_{n-1}^{(0)}) + \sum_{j=1}^{n-1}\left[\frac{\partial S}{\partial P_j^{(0)}} - \left(\frac{\partial S}{\partial P_j^{(0)}}\right)_0\right]\Delta P_j = \tau, \tag{78}$$

in which:

$$\frac{\partial S}{\partial P_j^{(0)}} = \frac{\partial S(q_1,\ldots,q_n,P_1^{(0)},\ldots,P_{n-1}^{(0)})}{\partial P_j^{(0)}},$$

$$\left(\frac{\partial S}{\partial P_j^{(0)}}\right)_0 = \frac{\partial S(q_1^{(0)},\ldots,q_n^{(0)},P_1^{(0)},\ldots,P_{n-1}^{(0)})}{\partial P_j^{(0)}}. \qquad (j = 1, 2, \ldots, n-1) \qquad (79)$$

Should eq. (78) be fulfilled for the entire family of waves (77), so for all values of $\Delta P_1, \ldots, \Delta P_{n-1}$, and certain common values of the $q_1, \ldots, q_n$, and τ, then those $q_1, q_2, \ldots, q_n$, and τ would have to satisfy the equations:

$$\frac{\partial S}{\partial P_j^{(0)}} = \left(\frac{\partial S}{\partial P_j^{(0)}}\right)_0 \qquad (j = 1, 2, \ldots, n-1), \qquad (80)$$

$$S(q_1,\ldots,q_n,P_1^{(0)},\ldots,P_{n-1}^{(0)}) - S(q_1^{(0)},\ldots,q_n^{(0)},P_1^{(0)},\ldots,P_{n-1}^{(0)}) = \tau. \qquad (81)$$

If we compare eq. (80) with eq. (69) then we will see that they are the $n - 1$ equations of the trajectories of our mechanical problem. [We must only denote the constants on the right-hand side of eq. (80) by β_j and the parameters of the action function by P_j, instead of $P_j^{(0)}$, again.] Since (81) further coincides with (76), we can say: If we assign rays to the wave propagation in such a way that we pursue the motion of the points at which neighboring wave trains intersect at equal times then the propagation of the wave along the ray will be simply the motion along the trajectory of the mechanical problem, but the velocity will be given by the law (76) of wave propagation, not by the velocity of the mechanical system, which is indeed also defined only for the individual mass-points. However, that definition of ray corresponds entirely to the way that one finds it in the optics of rays, when based upon the undulatory theory. That is because we have not, in fact, done anything but pursue the points at which neighboring waves with equal phase (for us that means: equal values of τ) will intersect.

With that analogy between mechanical systems and the study of waves, there are two things to be observed: First of all, the wave function S that appears in all of our equations depends upon not only the parameters P_j, but also upon the energy E, as is clear from eqs. (67), (69), (70). We can then actually write $S(q_1, q_2, \ldots, q_n, E, P_1, \ldots, P_n)$ everywhere. That is, every value of energy for the mechanical system is assigned a different family of waves (74). Secondly (and this is much more important), the time τ that we have introduced into (73) is identical to the time t by which we follow the motion of our mechanical system, but we have introduced τ only to give a simplest-possible wave-theoretic interpretation of the action function S.

If we now look for a different wave-theoretic interpretation in which an action function appears that will be independent of energy E and in which the wave propagation is described in the time t of the mechanical system then we will arrive at the wave that **Schrödinger** associated with each mechanical system in his wave mechanics.

18. – Schrödinger waves.

If we define a function W of the arguments $q_1, \ldots, q_n, P_1, \ldots P_{n-1}$, E, t by:

$$W = S\,(q_1, \ldots, q_n, P_1, \ldots P_{n-1}) - E\,t\,, \tag{82}$$

then it will follow immediate with the help of (70) that the derivative of W with respect to E will vanish, so E will not appear in W as an argument. When W_0 means a constant, the equation:

$$W = S - E\,t = W_0 \tag{83}$$

will define a surface in n-dimensional q-space for every value of t, so a wave train, together with its law of propagation in the time t that describes out mechanical system. The waves that are represented by eq. (83) are the waves that are associated with our mechanical system according to **Schrödinger**'s wave mechanics. Their speed of propagation u at any spatial point is obviously:

$$u = \frac{dn}{dt}\,, \tag{84}$$

if n is the normal to the wave surface at the point in question. However:

$$\frac{dn}{dt} = \frac{dn}{dW}\frac{dW}{dt}\,. \tag{85}$$

Nonetheless, since:

$$\frac{dW}{dn} = |\,\mathrm{grad}\,W\,|\,, \tag{86}$$

in which the two vertical lines mean the absolute value of the vector that they enclose, (84), (85), (86), (83) will give:

$$u = \frac{E}{|\,\mathrm{grad}\,W\,|} = \frac{E}{|\,\mathrm{grad}\,S\,|}\,. \tag{87}$$

By the way, it should be remarked that from (74), the wave picture in no. **17** for the speed of propagation at the time value τ will imply that:

$$\frac{dn}{d\tau} = \frac{dn}{dS}\frac{dS}{d\tau} = \frac{1}{|\,\mathrm{grad}\,S\,|}\,. \tag{88}$$

However, the absolute value of grad S at a point in q-space can be calculated from the **Hamilton-Jacobi** differential equation (67). Namely, when the kinetic energy of our system is expressed in terms of the generalized coordinates q_j and impulse components p_j in the form:

$$K = \tfrac{1}{2}\sum_{j,k} a_{jk}\, p_j\, p_k \ , \tag{89}$$

in which the a_{jk} depend upon only the $q_1, \ldots, q_n$, but the potential energy reads $\Phi\ (q_1, \ldots, q_n)$, then $H = K + \Phi$, and eq. (67) will assume the form:

$$\tfrac{1}{2}\sum_{j,k} a_{jk}\frac{\partial S}{\partial q_j}\frac{\partial S}{\partial q_k} + \Phi\ (q_1, \ldots, q_n) = E \ , \tag{90}$$

when we once more replace the p_j with the derivatives of S with respect to q_j . On the other hand, the gradient of S in our generalized coordinate system has the components $\frac{\partial S}{\partial q_1}, \ldots, \frac{\partial S}{\partial q_n}$. From the rules of calculation for vector-analytic covariants in curvilinear coordinates, its absolute value will then read:

$$|\ \text{grad}\ S\ |^2 = \sum_{j,k} a_{jk}\frac{\partial S}{\partial q_j}\frac{\partial S}{\partial q_k}, \tag{90.a}$$

analogous to (89). However, it will then follow from eq. (90) that:

$$|\ \text{grad}\ S\ | = \sqrt{2(E-\Phi)}\ , \tag{91}$$

and for the speed of propagation u, eq. (87) will yield:

$$u = \frac{E}{\sqrt{2(E-\Phi)}}\ . \tag{92}$$

If we now consider those of the wave that are represent by (83) that go through the point $q_1^{(0)}, \ldots, q_n^{(0)}$ in q-space at time $t = 0$ then we must set the constant $W_0 = S(q_1^{(0)},\ldots,q_n^{(0)},P_1,\ldots,P_{n-1},E)$ for any system of values of the $P_1, \ldots P_{n-1}, E$, and eq. (83) will read:

$$S\ (q_1, \ldots, q_n, P_1, \ldots, P_{n-1}, E) - S(q_1^{(0)},\ldots,q_n^{(0)},P_1,\ldots,P_{n-1},E) = E\ t\ , \tag{93}$$

analogous to (74). If we now also prescribe the direction of the wave normal at that point then when the energy value E is prescribed, in analogy to (75), we can determine the $P_1, \ldots P_{n-1}$, and we will get a completely-determined wave that is given by:

$$S(q_1,\ldots,q_n,P_1^{(0)},\ldots,P_{n-1}^{(0)},E) - S(q_1^{(0)},\ldots,q_n^{(0)},P_1,\ldots,P_{n-1},E) = E\ t\ , \tag{94}$$

analogous to (76), when the $P_1^{(0)}$, …, $P_{n-1}^{(0)}$ are just the values that correspond to the chosen direction for the wave normal. If we once more consider a family of waves whose directions of propagation fill up a small cone at $q_1^{(0)}$, …, $q_n^{(0)}$ then we will have to replace the $P_j^{(0)}$ in (94) with the values $P_j^{(0)} + \Delta P_j$, and we will get:

$$S(q_1,\ldots,q_n,P_1^{(0)}+\Delta P_1^{(0)},\ldots,P_{n-1}^{(0)}+P_{n-1}^{(0)},E)-S(q_1^{(0)},\ldots,q_n^{(0)},P_1^{(0)}+\Delta P_1^{(0)},\ldots,P_{n-1}^{(0)}+\Delta P_{n-1}^{(0)},E) = E\,t\,, \tag{95}$$

analogous to (77). It will then follow, exactly as with (77), that eqs. (78) to (81) will yield:

$$\frac{\partial S}{\partial P_j^{(0)}} = \left(\frac{\partial S}{\partial P_j^{(0)}}\right)_0 = \beta_j \qquad (j = 1, 2, \ldots, n-1), \tag{96}$$

$$S(q_1,\ldots,q_n,P_1^{(0)},\ldots,P_{n-1}^{(0)})-S(q_1^{(0)},\ldots,q_n^{(0)},P_1,\ldots,P_{n-1}) = E\,t\,. \tag{97}$$

The points at which neighboring waves of the same phase meet, so whether the waves reinforce each other by interference, in the sense of the undulatory theory, will move along the trajectories of the mechanical problem that are given by eq. (69). However, due to (97), the speed at which those points move forward is given by the speed of propagation of the **Schrödinger** waves (83), so by eq. (92). The result will be somewhat different when we consider a family of waves in which the individual waves differ, not only by their directions of propagation, but by their energy values E for the corresponding mechanical problem. We will also have to replace E with the value $E_0 + \Delta E$ then, in which E_0 is a fixed value and ΔE is small compared to E_0 . The "wave group" that is thus described will then be represented by:

$$S(q_1,\ldots,q_n,P_1^{(0)}+\Delta P_1^{(0)},\ldots,P_{n-1}^{(0)}+P_{n-1}^{(0)},E_0+\Delta E) - S(q_1^{(0)},\ldots,q_n^{(0)},P_1^{(0)}+\Delta P_1^{(0)},\ldots,P_{n-1}^{(0)}+\Delta P_{n-1}^{(0)},E_0+\Delta E) = (E_0+\Delta E)\,t\,. \tag{98}$$

We can also write eq. (98) as a **Taylor** development in the form:

$$S(q_1,\ldots,q_n,P_1^{(0)},\ldots,P_{n-1}^{(0)},E_0)-S(q_1^{(0)},\ldots,q_n^{(0)},P_1^{(0)},\ldots,P_{n-1}^{(0)},E_0) + \sum_{j=1}^{n-1}\left[\frac{\partial S}{\partial P_j^{(0)}}-\left(\frac{\partial S}{\partial P_j^{(0)}}\right)_0\right]\Delta P_j + \left[\frac{\partial S}{\partial E_0}-\left(\frac{\partial S}{\partial E_0}\right)_0\right]\Delta E = E_0\,t + t\,\Delta E\,. \tag{99}$$

If we once more ask what the time point might be at which all waves in the group (99) that meet at the point $q_1^{(0)}$, …, $q_n^{(0)}$ at time $t = 0$ will meet again at any point then, analogous to (80), upon comparing the coefficients of ΔP_j, ΔE, we will get the equation:

$$\frac{\partial S}{\partial P_j^{(0)}} = \left(\frac{\partial S}{\partial P_j^{(0)}} \right)_0 = \beta_j \qquad (j = 1, 2, \ldots, n-1), \tag{100}$$

$$\frac{\partial S}{\partial E_0} = t + \left(\frac{\partial S}{\partial E_0} \right)_0 = t - t_0 \, . \tag{101}$$

We will then get not only eq. (69), but also eq. (70), and the see that: The points at which the waves of the group (99) reinforce by interference due to the agreement in phase wander not only along the trajectories of the mechanical problem, but also with the velocity with which the point with the coordinates $q_1, \ldots, q_n$ moves, which is the one that can represent the entire motion of the mechanical system.

However, one easily sees that the validity of (99) still does not follow from eqs. (100), (101). Rather, in order for that to be true, it is also necessary that (94) must be fulfilled with $E = E_0$. However, since $q_1, \ldots, q_n$ can already be calculated as functions of t from (100) and (101) alone, that equation will not generally be fulfilled by substituting those values in (94). One will not have:

$$S(q_1, \ldots, q_n, P_1^{(0)}, \ldots, P_{n-1}^{(0)}, E_0) - E_0 \, t = S(q_1^{(0)}, \ldots, q_n^{(0)}, P_1^{(0)}, \ldots, P_{n-1}^{(0)}, E_0) \, ,$$

but something that deviates from it somewhat:

$$S(q_1, \ldots, q_n, P_1^{(0)}, \ldots, P_{n-1}^{(0)}, E_0) - E_0 \, t = S(q_1^{(0)}, \ldots, q_n^{(0)}, P_1^{(0)}, \ldots, P_{n-1}^{(0)}, E_0) + \Delta S_0 \, . \tag{102}$$

Instead of (98), we will then consider the wave group:

$$\begin{aligned} &S(q_1, \ldots, q_n, P_1^{(0)} + \Delta P_1^{(0)}, \ldots, P_{n-1}^{(0)} + P_{n-1}^{(0)}, E_0 + \Delta E) \\ &\quad - S(q_1^{(0)}, \ldots, q_n^{(0)}, P_1^{(0)} + \Delta P_1^{(0)}, \ldots, P_{n-1}^{(0)} + \Delta P_{n-1}^{(0)}, E_0 + \Delta E) - \Delta S_0 = (E_0 + \Delta E) \, t \, , \end{aligned} \tag{103}$$

in which ΔS_0 is a quantity that does not depend upon the parameters of the wave group $\Delta P_1^{(0)}, \ldots, P_{n-1}^{(0)}, \Delta E$ ([1]). The wave group (103) differs from the original one (98) by only the fact that the waves no longer go through the point $q_1^{(0)}, \ldots, q_n^{(0)}$ at the time $t = 0$, but at a different time. However, since that is the same moment for all intermediate terms of the wave group, the phase difference at any time-point t will not vary, and all of the calculations that led from (98) to (100), (101) can be repeated precisely with eq. (103), but with the difference that the quantity ΔS_0 will be added to the constant $S(q_1^{(0)}, \ldots, q_n^{(0)}, P_1^{(0)} + \Delta P_1^{(0)}, \ldots, E_0 + \Delta E)$ everywhere, but E_0 will not affect the differentiations with respect to the parameters $\Delta P_j^{(0)}$ that appear. The representative point of

([1]) It should be noted that ΔS_0 will probably have a different value for every time-point t in the motion of the representative point. However, that means only that every such position will arise by the interference of another wave group. In the consideration of the propagation of the waves (102), (103) themselves, e.g., the calculation of the speed of propagation, ΔS_0 can be regarded as a constant.

the mechanical system whose motion is given by (100), (101) can no longer be regarded as a result of the combined effect of the wave group (98) whose intermediate terms all go through a point $q_1^{(0)}, \ldots, q_n^{(0)}$ at time $t = 0$, but the position of the representative point will arise at each time-point t by the combined effect (interference) of a wave group (103) that goes through the point $q_1^{(0)}, \ldots, q_n^{(0)}$ at a moment that depends upon t.

19. – The complete figure of wave surfaces and trajectories.

When we are given a complete system in involution of the form (60), according to no. **14**, a wave function $S(q_1, \ldots, q_n, P_1, \ldots, P_n)$ will be determined by it. When we assign constant values $P_j = \alpha_j$ to the P_j according to (44), S will be a function of the $q_1, \ldots, q_n$ that is defined in all of q-space and depends upon n parameters $\alpha_1, \ldots, \alpha_n$. A family of surfaces in q-space will then be determined by:

$$S(q_1, \ldots, q_n, P_1, \ldots, P_n) = \tau \tag{104}$$

when τ varies continuously, and that will then be a family of wave surfaces for our problem according to our interpretation in no. **17**. If we once more choose the energy integral to be the integral that we start from, and set, say $\alpha_n = E$, then a family of ∞^{n-1} trajectories can be derived from the wave function S according to (69) that can be represented in the form:

$$\frac{\partial S}{\partial \alpha_j} = \beta_j \qquad (j = 1, 2, \ldots, n-1) \tag{105}$$

with our current notation. When we have an arbitrary complete system in involution, the ∞^{n-1} trajectories that can be derived from S can no longer be written in the form (105), but from (62), (44), in the somewhat-more-complicated form:

$$\frac{\dfrac{\partial S}{\partial \alpha_1} - \beta_1}{\omega_1} = \frac{\dfrac{\partial S}{\partial \alpha_2} - \beta_2}{\omega_2} = \ldots = \frac{\dfrac{\partial S}{\partial \alpha_n} - \beta_n}{\omega_n} \tag{106}$$

when the energy can be expressed in terms of the $\alpha_1, \ldots, \alpha_n$ in the form $E(\alpha_1, \ldots, \alpha_n)$, and from (41):

$$\omega_j = \frac{\partial S}{\partial \alpha_j} \qquad (j = 1, 2, \ldots, n). \tag{107}$$

In any event, the wave function S can also be associated with ∞^{n-1} trajectories here by using eq. (106) ([1]). One now calls the ∞^1 wave surfaces (104), together with the ∞^{n-1} trajectories that are associated with them by eq. (105) or (106), a "complete figure" of wave surfaces and trajectories. That concept was coined by **C. Carathéodory** for a general variational problem. Since S depends upon n parameters $\alpha_1, \ldots, \alpha_n$, there will be ∞^n complete figures, and as a result, $\infty^{n-1} \cdot \infty^n = \infty^{2n-1}$ trajectories in total. The set of all trajectories can then be arranged into complete figures that each have ∞^{n-1} trajectories with the help of a complete system in involution, or what amounts to the same thing, the solution to the **Hamilton-Jacobi** partial differential equation (66). We now ask what the relationship might be that exists between the direction of the trajectory and the wave surface. The direction of the trajectory is given by the velocity vector with the components $\dot{q}_1, \ldots, \dot{q}_n$. However, it can also be derived when we are given the impulse vector with the components $p_1, \ldots, p_n$, because it would follow from (1), (89), and $H = K + \Phi$ that the $\dot{q}_j$ can be calculated from the p_j according to the rule:

$$\dot{q}_j = \frac{\partial K}{\partial p_j} = \sum_{l=1}^{n} a_{jl}\, p_l \qquad (j = 1, 2, \ldots, n). \tag{108}$$

One often calls the p_j the "canonical" direction coordinates. Now, the direction of a trajectory at the point $q_1, \ldots, q_n$ is given by eq. (61). When we denote a line element on the wave surface by $dq_1, \ldots, dq_n$, we will have:

$$\sum_{j=1}^{n} \frac{\partial S}{\partial q_j} dq_j = 0\,. \tag{109}$$

since S = const. Therefore, according to (61), the relation:

$$\sum_{j=1}^{n} p_j\, dq_j = 0 \tag{110}$$

will exist between a line element on the wave surface that is represented by the ordinary direction components $dq_1, \ldots, dq_n$ and an element of the trajectory that is defined by the canonical direction quantities $p_1, \ldots, p_n$. In the terminology of the calculus of variations, one then says: The trajectory is "transverse" to the wave surface. If the $q_1, \ldots, q_n$ are coordinates such that the *vis viva* can be written in the form:

$$K = \frac{1}{2m} \sum_{j=1}^{n} p_j^2\,. \tag{111}$$

([1]) That is because the curves that are represented by (106) depend upon only the $n - 1$ differences of the n parameters β_j / ω_j.

then from (108), the $\dot{q}_j = (1/m)\, p_j$, so the ordinary direction quantities will then be proportional to the canonical p_j. The "transversality condition" (110) then goes over to the "orthogonality condition," which says that the trajectory is perpendicular to the wave surface. In the simplest case where the q_j are Cartesian coordinates of the three-dimensional space, the "complete figure" will then consist of ∞^1 wave surfaces that cover space simply and whose ∞^2 orthogonal trajectories are the trajectories.

In general, a certain system of values for the $\alpha_1, \ldots, \alpha_n$ will define a certain complete figure. Every system of values for the $\beta_1, \ldots, \beta_{n-1}$ will define a trajectory that belongs to that figure.

20. – Conception of the complete figure as a "point-cloud."

A velocity with the components $\dot{q}_1, \ldots, \dot{q}_n$ is defined at every point in q-space by eqs. (108), (61) when one has a complete figure, so analytically speaking, when one is given the wave function $S(q_1, \ldots, q_n, \alpha_1, \ldots, \alpha_n)$. Namely, one has:

$$\dot{q}_j = \sum_{l=1}^{n} a_{jl} \frac{\partial S}{\partial q_l} \qquad (j = 1, 2, \ldots, n)\,. \tag{112}$$

We now suppose that all of q-space is filled with representative points of mechanical systems. When we have ordinary three-dimensional space before us, that will mean simply that it is filled with material points, or as **L. de Broglie** said it, we have a "cloud of material points." A motion of that point-cloud is then defined by eq. (112). If we regard it as the motion of a space-filling continuous mass then eq. (112) will represent a stationary flow since the velocity depends upon only the position $q_1, \ldots, q_n$. One also speaks of the flow of an "incoherent" medium since the individual points of the "cloud" do not interact with each other.

In order for us to be dealing with an actual flow, the number of particles in a volume element cannot change under the motion that (112) represents, i.e., the continuity equation of hydrodynamics must be satisfied. When one denotes the number of particles per unit volume, so the spatial density, by $\sigma(q_1, \ldots, q_n)$, the continuity condition will read:

$$\sum_{l=1}^{n} \frac{\partial (\sigma \dot{q}_j)}{\partial q_l} = 0\,, \tag{113}$$

so from (112):

$$\sum_{l=1}^{n} \sum_{l=1}^{n} \frac{\partial}{\partial q_l} \left(\sigma\, a_{jl} \frac{\partial S}{\partial q_l} \right) = 0\,. \tag{114}$$

If the complete figure, so S, is given then the particle density σ must satisfy the first-order partial differential equation (114). Only entirely special densities $\sigma(q_1, \ldots, q_n)$ will belong to each complete figure if it is to be interpreted as a flow. In that sense, a complete figure (a point-cloud

flow, resp.) is determined by the two functions S and σ that satisfy the two partial differential equations (90), (114).

21. – Point-clouds and the undulatory theory.

Up to now, we have used the word "wave" only in the sense of geometrical optics, i.e., only as a geometric representation for the set of light rays. The wave surface determines the light rays as its orthogonal ("transverse," resp.) trajectories. In that way, nothing was said about any periodicity, so of frequency. However, in **Schrödinger**'s wave mechanics, those waves are regarded as actual waves in the sense of the theory of undulatory theory. Based upon what **de Broglie** said, the phase of the wave is defined by the quantity W in eq. (83). The quantity ψ that propagates in the form of a wave will then be represented by the formula:

$$\psi = Ae^{iHt/\hbar} = A(q_1,\ldots,q_n)e^{iS(q_1,\ldots,q_n)/\hbar}e^{-iEt/\hbar}. \tag{115}$$

The $\hbar$ is a constant that must be introduced in order to make the exponent of the exponential function into a dimensionless quantity. It will be identified with **Planck**'s constant of action. A is the amplitude, and the frequency ω is coupled with E by the relation:

$$E = \omega\hbar. \tag{116}$$

Since the speed of propagation u is given by eqs. (87), (92), we will get:

$$\lambda = \frac{2\pi u}{\omega} = \frac{2\pi\hbar}{|\operatorname{grad} S|} = \frac{2\pi\hbar}{\sqrt{2(E-\Phi)}} \tag{117}$$

for the wavelength λ. If a wave train, in the sense of geometrical optics, is actually represented by (115) then it must go to a plane wave in the small. Stated more precisely: A and grad S will vary only slightly in a spatial region with the same order of magnitude as the wavelength. We shall now assume that ψ satisfies the ordinary wave equation with the speed of propagation u, so:

$$\operatorname{div}\operatorname{grad}\psi = \frac{1}{u^2}\frac{\partial^2\psi}{\partial t^2}. \tag{118}$$

We then ask what relations slowly-varying A and S must satisfy in order for A and S to be a solution of (118): **De Broglie**, and more generally **A. Sommerfeld** ([1]), have shown that S must then be the wave function, and $\sigma = A^2$ is the density of a point-cloud, in the sense of no. **20**. We next write (115) in the form:

([1]) In his contribution to **Ph. Frank** and **R. Mises**, *Die Differential und Integralgleichungen der Mechanik und Physik*, Bd. II, pp. 484, *et seq.*

$$\psi = \varphi(q_1,\ldots,q_n)e^{-i\omega t}, \tag{119}$$

such that:

$$\varphi = Ae^{iS/\hbar}. \tag{120}$$

Upon substituting (119) in (118), one will then get the following differential equation for φ:

$$\text{div grad}\,\varphi + \frac{\omega^2}{u^2}\varphi = 0\,, \tag{121}$$

which one often calls the "abbreviated" wave equation or the differential equation for the "temporal amplitude" φ. One understands the "spatial" amplitude to then mean the function of position A, which can obviously be interpreted as the wave amplitude for slowly-varying A and grad S. Since the vector-analytic covariant of grad φ is expressed in terms of the generalized coordinates $q_1, \ldots, q_n$ by:

$$\text{div grad}\,\varphi = \sum_{j,k}\frac{\partial}{\partial q_j}\left(a_{jk}\frac{\partial\varphi}{\partial q_k}\right), \tag{122}$$

when one substitutes (120) in (121), one will get the following differential equation for φ:

$$\sum_{j,k}\frac{\partial}{\partial q_j}\left(a_{jk}\frac{\partial\varphi}{\partial q_k}\right)+\frac{\omega^2}{u^2}\varphi = 0\,. \tag{123}$$

If we replace φ in (123) with its value in (120) and employ the expressions (90.a) and (122), as well as the analogous one:

$$(\text{grad}\,A,\,\text{grad}\,S) = \sum_{j,k}a_{jk}\frac{\partial S}{\partial q_j}\frac{\partial S}{\partial q_k} \tag{124}$$

for the vector-analytic covariant in general coordinates, then we will get the following differential equation for A and S:

$$\left[\frac{i^2}{\hbar^2}A\,|\,\text{grad}\,S\,|^2 + A\frac{\omega^2}{u^2}\right]+\frac{i}{\hbar}\Big|\,2\,(\text{grad}\,A,\text{grad}\,S)+A\,\text{div grad}\,S\,\Big|+\text{div grad}\,A = 0\,. \tag{125}$$

The terms in the first square bracket are large compared to all other terms for short wavelengths (so for large frequencies ω), due to the slow variation of A and grad S. They must then nearly vanish in their own right. Due to (116) and (117), it will then follow that one has, approximately:

$$\text{grad}\,S^2 = \frac{E^2}{u^2} = 2\,(E-\Phi)\,, \tag{126}$$

so the **Hamilton-Jacobi** partial differential equation (90). Due to the slow variation of A and $\operatorname{grad} S$ over the span of a wavelength, $\lambda \operatorname{grad} A$ and $\lambda \operatorname{div} \operatorname{grad} S$ must be small compared to A and $\operatorname{grad} S$, respectively. However, due to (117), it will follow that the terms in the second square bracket in eq. (123) are small compared to the first one, but large compared to the last term $\operatorname{div} \operatorname{grad} A$. Therefore, one must also have:

$$2A(\operatorname{grad} A, \operatorname{grad} S) + A^2 \operatorname{div} \operatorname{grad} S = 0 \tag{127}$$

then. However, when we set:

$$\sigma = A^2 , \tag{128}$$

that equation will be identical to eq. (114), due to (122) and (124). Nonetheless, for wavelengths that are short enough, many wavelengths will fit inside of a domain in which A and grad S can be regarded as constant, so the wave train (115) will be nothing but the representation of a bundle of rays (or more precisely, a congruence of rays) in the sense of geometrical optics, so it is also an approximate solution to the wave equation (118) of undulatory optics. A "point-cloud," as it was represented by (90), (114), can also be represented by (115) then when A is the amplitude and S is the wave phase of an actual solution to the basic equation (118) of the undulatory theory in the limiting case of waves that are short compared to the inhomogeneity of the medium, which will be the case in which wave optics can be replaced by geometrical optics. In spirit of **de Broglie** and **Schrödinger**, classical mechanics, as "geometrical mechanics," then seems to be a limiting case of "wave mechanics." In the limiting case of short wavelengths, the waves go to a point-cloud whose density is proportional to the square of the amplitude, according to (128).

22. – When does an individual trajectory define a complete figure?

For a certain choice of $\alpha_1, \ldots, \alpha_n$, a wave-function $S(q_1, \ldots, q_n, \alpha_1, \ldots, \alpha_n)$ will define a certain complete figure in the sense of no. **19**, and when one varies the $\alpha_1, \ldots, \alpha_n$, that will define the totality of all complete figures that subsume the totality of all trajectories. For that reason, one can find a complete figure that belongs to any trajectory, and indeed it will generally be infinite. Namely, for a given choice of the wave function S, the constants $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n$ in eqs. (61), (105) can be determined such that those equations will represent a certain trajectory. However, there are infinitely-many solutions S to the **Hamilton-Jacobi** differential equation (67), one can regard every trajectory as belonging to a complete figure in infinitely-many ways. That figure will be established when one fixes the function S and the values of the α_j . The values of β_j will determine only the individual trajectories inside of that figure then. Knowing the complete figure alone will already make it possible to determine the impulse components $p_1, \ldots, p_n$ for the trajectory that goes through any point $q_1, \ldots, q_n$ of a figure without having to know the β_j . Namely, from eq. (61), they are simply:

$$p_j = \frac{\partial S(q_1, \ldots, q_n, \alpha_1, \ldots, \alpha_n)}{\partial q_j} \qquad (j = 1, 2, \ldots, n) . \tag{129}$$

Conversely, when the p_j are given as functions of the $q_1, \ldots, q_n$, i.e., the impulses of the trajectories of our figure that go through each point in q-space, S (so the complete figure) can be established only when the p_j are functions of position such that $p_1\, dq_1 + \ldots + p_n\, dq_n$ is a complete differential. That is because in that case, there will always be a function S that satisfies eq. (129).

Now, it is obvious that the case can occur in which the p_j are determined as functions of $q_1, \ldots, q_n$ in all of q-space. Namely, let n integrals of the system be given, so n relations of the form:

$$f_j\,(q_1, \ldots, q_n, p_1, \ldots, p_n) = \alpha_j \qquad (j = 1, 2, \ldots, n) \tag{130}$$

that all states q_j , p_j must satisfy along trajectories. Those integrals should be arranged such that all trajectories that satisfy the relations (130) for given values of the α_j must run inside of a finite region in q-space for fixed values of the integration constants $\alpha_1, \ldots, \alpha_n$. Moreover, every trajectory shall have the property that it comes arbitrarily close to each point in q-space that is inside of the aforementioned region in the course of time. That means: Any further integral that gets added to the n integrals (130) shall be only a "quantitative" one, not a "qualitative" one, as **Levi-Civita** phrased it. Namely, one understands the latter type to mean that even with a small indeterminacy in the integration constants [so when the quantities α_{n+1} in $f_{n+1}\,(q_1, \ldots, p_n) = \alpha_{n+1}$ fluctuate between two close limits $\alpha_{n+1} - \Delta\alpha_{n+1}$ and $\alpha_{n+1} + \Delta\alpha_{n+1}$], one selects a smaller region from the n-dimensional region in q-space that is determined by the n integrals (130) in which the trajectories must run if they are to also satisfy the relations $f_{n+1}\,(q_1, \ldots, p_n) = \alpha_{n+1}$ for an ill-defined α_{n+1} . It is clear that in order for such a "qualitative" integral to exist, the assumption that the trajectory will come arbitrarily close to the point of the region that is bounded by eq. (130) cannot be fulfilled.

By contrast, should the n integrals (130) be "qualitative" ones, i.e., when one solves them for the $p_1, \ldots, p_n$, the latter must be functions of the $q_1, \ldots, q_n$ that are single-valued or finitely-multivalued at every point in q-space. Every trajectory that comes close to a point in q-space will always have impulse components there that assume values that differ only slightly from the ones that they had already assumed at a previous visit to that neighborhood. We shall now follow a trajectory that the integrals (130) satisfy along its entire evolution in the subset of q-space that it fills up. At each point $q_1, \ldots, q_n$ along the curve, a certain system of values for the impulse components $p_1, \ldots, p_n$ will be determined by it, and indeed the one that is given by solving eq. (130) for the $p_1, \ldots, p_n$. Let it be given by:

$$p_j = \pi_j\,(q_1, \ldots, q_n)\,. \tag{131}$$

On the other hand, we can consider the trajectory in question to be like any other one that belongs to a complete figure. A wave function S will then belong to it from which the impulse components p_j can be calculated by eq. (129). Those p_j are then, on the one hand, given in the entire subregion of q-space by the solutions (131) to eq. (130), and on the other hand, by eq. (129) at the places where out trajectory runs through. The relations:

$$p_j = \pi_j\,(q_1, \ldots, q_n) = \frac{\partial S}{\partial q_j} \tag{132}$$

must then be true at all of those places. However, since our assumption that (130) are qualitative (or as one often says, single-valued) integrals implies that the p_j can be only finitely-multivalued functions of the $q_1, \ldots, q_n$ (so they must also assume neighboring values at neighboring points), if eq. (132) is true for all points along the trajectory then it cannot break down at any other point of our n-dimensional region, either, since indeed the trajectory comes arbitrarily close to all of its points. However, it will then follow that $\pi_1\, dq_1 + \ldots + \pi_n\, dq_n$ is a complete differential. Therefore, the function S, and therefore a complete figure, will be established by the $\pi_1, \ldots, \pi_n$, so also along our individual trajectory. At the same time, it will follow from no. **9** that the integrals (130) must be a complete system in involution.

We can then summarize: If the n integrals (130) are the only qualitative integrals, and every trajectory that satisfies them fills up a finite region of the n-dimensional q-space that is determined by (130) in the sense that it comes arbitrarily close to every point then a complete figure will be determined by every trajectory, and indeed in a finitely-multivalued way, since the p_j are given in a finitely-multivalued way by solving eq. (130). Since every such figure also corresponds to a wave-train, every trajectory of the type that was described is also associated with a wave-train in the sense of no. **17** in a finitely-multivalued way. The n integrals (130) themselves will always define a complete system involution then. It is only such a thing that can define an n-dimensional region that can be filled up with trajectories in the given way ([1]).

23. – The multidimensional harmonic oscillator as an example of no. 22.

As a simple example of the case that was treated in no. **22**, one ordinarily considers the multidimensional linear oscillator whose periods in the different directions have irrational ratios. We restrict shall ourselves to the oscillations of a mass-point of unit mass in three-dimensional space about the coordinate origin as the equilibrium point. Let q_1, q_2, q_3 Cartesian coordinates, and let p_1, p_2, p_3 be the corresponding impulse components. The **Hamiltonian** function then reads:

$$H = \tfrac{1}{2}(p_1^2 + p_2^2 + p_3^2 + \omega_1^2 q_1^2 + \omega_2^2 q_2^2 + \omega_3^2 q_3^2)\ , \tag{133}$$

in which $\omega_1, \omega_2, \omega_3$ are the frequencies of oscillation in each of the three axis directions. If we denote the energy by α_3 then, according to (67), the **Hamilton-Jacobi** partial differential equation will read:

$$\tfrac{1}{2}\sum_{j-1}^{3}(p_j^2 + \omega_j^2 q_j^2) = \alpha_3 \tag{134}$$

here, and three integrals of our mechanical system will read:

([1]) **A. Einstein**, Verh. d. D. Phys. Ges.**10** (1917), pp. 82-92.

$$p_j = \sqrt{2\alpha_j - \omega_j^2 q_j^2} \qquad (j = 1, 2),$$
$$p_3 = \sqrt{2(\alpha_3 - \alpha_1 - \alpha_2) - \alpha_3^2 q_3^2}\,, \tag{135}$$

which is already solved for p_j in the form of (131). Due to the double-valuedness of the square root, every point in q-space will belong to eight systems of value for the p_1, p_2, p_3. Since every p_j depends upon only one q_j, we are dealing with the case of separation of variables in no. **16**, so the expression $p_1\, dq_1 + \ldots + p_n\, dq_n$ will obviously be a complete differential, and the wavefunction will be given by a mere quadrature as:

$$S(q_1, q_2, q_3, \alpha_1, \alpha_2, \alpha_3) = \sum_{j-1}^{3} \int \sqrt{2\alpha_j - \omega_j^2 q_j^2}\, dq_j + \int \sqrt{2(\alpha_3 - \alpha_1 - \alpha_2) - \omega_3^2 q_3^2}\, dq_3\,. \tag{136}$$

Due to the eight possible systems of values for the p_j, there are also eight expressions the wave function. Nonetheless, since the quantity S itself will only change sign upon inverting the signs of all p_j, which means that there can be no other complete figure, only four complete figures will be defined by the three integrals (135), which obviously define a complete system in involution. However, our system possesses no other "qualitative" integral in addition to the integrals (135). Namely, since the expression under the square root in (135) must be positive for every possible state of the system, the q_j must lie between fixed limits, namely:

$$q_1 \text{ must lie between } -\frac{2\alpha_1}{\omega_1^2} \text{ and } +\frac{2\alpha_1}{\omega_1^2}\text{, etc.}$$

However, all trajectories run inside of a cuboid subregion of q-space that is defined by the three constants $\alpha_1, \alpha_2, \alpha_3$. When we assume that the oscillation frequencies have irrational ratios, every trajectory will come arbitrarily close to each point of the aforementioned cube, so they will fill it up in a certain sense. From what was said up to now, every individual trajectory will define such a cube, and therefore well-defined systems of values for the p_j at each point in that region, so the integrals (135), as well. Due to the multivaluedness that was spoken of before, every trajectory will belong to precisely four figures, or as one can also say, four wave-trains that it can be associated with in the spirit of wave mechanics. Such systems, whose trajectory can then be associated with only a *finite* number of wave-trains, have played a significant role in the development of quantum theory as a nondegenerate multiply-periodic system. We see that they also assume an especially-distinguished position from the standpoint the association of trajectories and wave-trains that is required by wave mechanics ([1]).

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([1]) One can find a thorough presentation of the parts of classical mechanics that are most important in quantum mechanics in *Handbuch der Physik* (ed. by **Geiger** and **Scheel**) in Bd. V (contribution by **L. Nordheim** and **E. Fues**), and Bd. XX (contribution by **A. Landé** on optics, mechanics, and wave mechanics), and furthermore, in **M. Born**, *Lehrbuch der Atommechanik*, and in the new edition of **Riemann-Weber**, *Differentialgleichungenen der Physik* (ed. by **Ph. Frank** and **R. von Mises**) in Bd. II. (Contribution by **Ph. Frank** on *Analytical Mechanics*)