DIFFERENTIAL GEOMETRY

AND

ANALYTICAL MECHANICS

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To Christiane,

Who was not sparing in his pains

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FOREWORD

"There are already several treatises on mechanics, but the plan of this one is entirely new. I propose to reduce the theory of that science and the art of solving the problems that pertain to it to some general formulas whose simple development will give all of the equations that are necessary for the solution of each problem."

"This book will be useful in another way, moreover: It will unite the various principles that were found up to now in order to facilitate the solution of the questions of mechanics and present them from a common viewpoint, while showing the links between them and their mutual dependency, and to make it more possible to judge their exactitude and their scope."

"I have divided it into two parts: Statics, or the theory of equilibrium, and dynamics, or the theory of motion, and in each of those two parts, I have treated solid bodies and fluids separately."

"One will find no figures at all in this book. The methods that are presented here demand neither constructions nor geometric or mechanical arguments, but only algebraic operations that are subject to a regular and uniform progression. Those who love analysis will behold with pleasure the fact that mechanics will become a new branch of it, and it would be gratifying to me to know that I have thus extended that domain."

J. L. LAGRANGE
Mécanique Analytique, 1811

INTRODUCTION

The lectures of Élie Cartan on integral invariants, which continue to be strikingly topical today, marked the beginning of what one can call "modern analytical mechanics": Indeed, the intrinsic formulation of the equations of dynamics, and not the variational one, appeared in them for the first time. More recently, the work of A. Lichnerowicz, F. Gallisot, and J. Klein have clearly exhibited the fact that differential geometry is the natural context in which to base the foundations of analytical mechanics.

The first contribution that this geometric formalism makes is very neat distinction between the Hamiltonian aspect of mechanics and its Lagrangian aspect. Certainly, it has been known for some time that Hamilton's equations are "covariant," while the Lagrange equations are "contravariant." Today, one interprets the former as a dynamical system on the cotangent space to the configuration manifold and the latter as a dynamical system on the tangent bundle to that manifold.

The Hamiltonian aspect is linked with the existence of a canonical symplectic structure on any cotangent bundle that is determined by the Liouville form. The techniques of differential calculus on manifolds then permit one to pursue the ideas of Élie Cartan in order to obtain an intrinsic formulation of Hamilton's equations. As F. Gallisot has shown, one can then interpret the classical results on first integrals and the cases of integrability geometrically.

The Lagrangian aspect is more complex. According to J. Klein, it is linked with the existence of a differential calculus on a tangent bundle that is much richer than the one on an arbitrary differentiable manifold. Upon utilizing the geometric structure of that space, one can indeed define some differential operators that will lead to the Lagrange equations of a mechanical system, and always by means of the techniques of symplectic geometry.

The link between those two aspects is finally assured by the Legendre transformations, which exhibits a duality between them, in some sense.

The first part of this book is a presentation of differential geometry that covers one part of the Certificate Program C.3: exterior calculus, vector bundles, differentiable manifolds, differential and integral calculus on manifolds. One is assumed to know only the elements of linear algebra, general topology, and local differential calculus (such as, for example, what is taught in the first year of proficiency).

The second part is dedicated to analytical mechanics. Furthermore, it includes a study of the classes of differential forms, as well as a presentation of the geometry of tangent spaces and their differential calculus.

This book has its origin in a series of presentations that were made in 1967 in Strasbourg in the context of a seminar on trajectories. The interest that was shown by P. Cartier then proved to be decisive in their publication. Moreover, the author had numerous conversations with G. Reeb and J. Martinet that were quite useful in the preparation of the manuscript.

Strasbourg, February 1968.

CHAPTER I

THE ALGEBRA OF EXTERIOR FORMS

In sections 1, 2, and 3, A will denote a unitary commutative ring. In sections 4, 5, and 6, one supposes, moreover, that A is a unitary algebra over the field $\mathbb{Q}$ of rationals. Finally, in sections 7 and 8, A will denote a commutative field with characteristic zero.

All modules will be unitary modules over A.

§ 1. – Duality and orthogonality.

1.1. Definition. – *Let $(E_i)_{1 \le i \le p}$ and F be $p + 1$ modules. A map $\alpha : E_1 \times \ldots \times E_p \to F$ is a **multilinear** map if the map:*

$$x \mapsto \alpha(e_1, \ldots, e_{i-1}, x, e_{i+1}, \ldots, e_p)$$

is a linear map of E_i into F for every index i and every element $e_j \in E_j$, $j \neq i$.

One also says a *bilinear map* when $p = 2$ and *a multilinear form* when $F = A$.

If $E_1 = \ldots = E_p = E$ and $F = A$ then one says that α is a *multilinear form of degree p on E*.

The set $\mathbf{L}^p(E)$ of multilinear forms of degree p on E is canonically endowed with the structure of a module over A.

1.2. – Let h be a linear map from a module E into a module F and let α be an element of $\mathbf{L}^p(F)$:

$$h^*\alpha : (e_1, \ldots, e_p) \mapsto \alpha(h\, e_1, \ldots, h\, e_p)$$

is a multilinear form of degree p on E : $h^*\alpha$ is the *reciprocal image form to α* under h.

The map h^* is a linear map of $\mathbf{L}^p(F)$ into $\mathbf{L}^p(E)$. If k is a linear map of F into a module G then one will have $(k \circ h)^* = h^* \circ k^*$. If h is the identity map of E then h^* will be the identity map of $\mathbf{L}^p(E)$. Consequently, if h is an isomorphism of E to F then h^* will be an isomorphism of $\mathbf{L}^p(E)$ to $\mathbf{L}^p(F)$, and one will have $(h^*)^{-1} = (h^{-1})^*$.

1.3. Definition. – *Let E be a module. The module $E^* = \mathbf{L}^1(E)$ is called the **dual** of E.*

If e is an element of E and α is an element of E^* then one lets $< e, \alpha >$ denote the value $\alpha(e)$ of α on e. $(e, a) \mapsto\ < e, \alpha >$ is the canonical bilinear form on $E \times E^*$.

1.4. – If $(e_i)_{1 \leq i \leq n}$ is a basis for E then one can define elements ε_j of E^* by $< e_i , \varepsilon_j > = \delta_{ij}$. Those elements form a basis for E^*. $(\varepsilon_i)_{1 \leq i \leq n}$ is the *dual basis* to the basis (e_i).

In particular, if A is a field and E is a finite-dimensional vector space over A then E and E^* will have the same dimension.

1.5. Proposition. – *Let G be a free module. If the sequence* $0 \to E \xrightarrow{h} F \xrightarrow{k} G \to 0$ *is exact then the same thing will be true for the sequence* $0 \to G^* \xrightarrow{k^*} F^* \xrightarrow{h^*} E^* \to 0$.

Proof: Recall, first of all, that the exactness of the sequence $0 \to E \xrightarrow{h} F \xrightarrow{k} G \to 0$ corresponds to the following hypotheses:

– h is an injective linear map.
– k is a surjective linear map.
– The image of h equal to the kernel of k (Im h = Ker k).

One easily shows (with no other hypothesis on G) that k^* is injective, h^* is surjective, and $\text{Im } k^* = \text{Ker } h^*$.

Let $(g)_{i\in I}$ be a basis for G and let $(f)_{i\in I}$ be a basis for F such that $k(f_i) = g_i$ for any i. The module F is the direct sum of the image of h and the sub-module G' that is generated by the family $(f)_{i\in I}$.

An element α of E^* determines a linear form on $h(E)$, and that form prolongs (for example, by giving the value of 0 to G') to a linear form β on F. One will then have $h^*\beta = \alpha$, which shows that h^* is surjective.

Q. E. D.

1.6. Definition. – *Let E be a module. The dual* E^{**} *of* E^* *is called the* ***bidual*** *of E.*

For any element e of E, the map $\alpha \mapsto < e, \alpha >$ is a linear form $\tilde{e}$ on E^* and $e \mapsto \tilde{e}$ is a linear map of E into E^{**}.

1.7. Proposition. – *If E possesses a finite basis then the map* $e \mapsto \tilde{e}$ *is an isomorphism of E onto* E^{**}.

Proof: Let $(e_i)_{1 \leq i \leq n}$ be a basis for E and let $(\varepsilon_i)_{1 \leq i \leq n}$ be the dual basis for E^*.

If $e = \sum a_i e_i$ is an element of E such that $\tilde{e} = 0$ then one will have $< e, \varepsilon_i > = a_i = 0$ for every i, and consequently, $\tilde{e} = 0$.

If ω is a linear form on E^* then the element $e = \sum \omega(\varepsilon_i) e_i$ of E will verify $< e, \varepsilon_i > = \omega(e_i)$ for any i. One will then have $\omega = \tilde{e}$.

Q. E. D.

Under the hypotheses of Proposition 1.7., one can identify E and E^{**} *by means of that isomorphism.*

1.8. Definition. – *Let F be a sub-module of a module E. The* ***orthogonal complement*** $F^{\perp}$ *of F is the sub-module of forms on E that are zero on F.*

1.9. Proposition. – *Let F be a sub-module of a module E, and let q be the projection of E onto E*/*F. The map* q^* *is an isomorphism of* $(E/F)^*$ *onto* $F^{\perp}$.

1.10. Corollary. – *If A is a field and E is a finite-dimensional vector field onto A then one will have that* $\dim F^{\perp} = \operatorname{codim} F$.

1.11. Corollary. – *With the hypotheses of corollary 1.10 (and the convention of 1.7), one will have* $F^{\perp\perp} = F$.

Indeed, $F^{\perp\perp}$ contains F, and $\dim F^{\perp\perp} = \operatorname{codim} F^{\perp} = \dim F$, moreover.
One can generalize Proposition 1.9 in the following way:

1.12. Proposition. – *Let F be a sub-module of a module E, and let q be the projection of E onto E*/*F. The map* q^* *is an isomorphism of* $\mathbf{L}^p(E/F)$ *onto the sub-module L of forms* α *in* $\mathbf{L}^p(E)$ *such that* $\alpha(e_1, \ldots, e_p) = 0$ *if one of the* e_i *is in F.*

In what follows, $\mathbf{L}^p(E/F)$ *will be identified with the sub-module L of* $\mathbf{L}^p(E)$.

1.13. Proposition. – *Let* F_1 *and* F_2 *be two sub-modules of a module E. One has:*

$$\mathbf{L}^p(E / (F_1 + F_2)) = \mathbf{L}^p(E/F_1) \cap \mathbf{L}^p(E/F_2) .$$

§ 2. – Exterior forms.

Let $\mathfrak{S}_p$ be the group of permutations of the set $\{1, \ldots, p\}$. Let denote ι the identity permutation, and let $\varepsilon(s)$ be the signature of a permutation s of $\mathfrak{S}_p$.

2.1. – Let α be a multilinear form of degree p on a module E, and let s be an element of $\mathfrak{S}_p$:

$$s\,\alpha : (e_1, \ldots, e_p) \mapsto \alpha(e_{s^{-1}(1)}, \ldots, e_{s^{-1}(p)})$$

is a multilinear form of degree p on E.

One has $\iota\,\alpha = \alpha$ and $(st)\,\alpha = s\,(t\alpha)$ for any s and t in $\mathfrak{S}_p$. $\alpha \mapsto s\,\alpha$ is therefore an automorphism of $\mathbf{L}^p(E)$ for any $s \in \mathfrak{S}_p$.

2.2. Definition. – *Let α be a multilinear form of degree p over a module E. α is an* ***antisymmetric multilinear form*** *if $s\alpha = e(s)\,\alpha$ for any $s \in \mathfrak{S}_p$.*

One also says that α is an *exterior form of degree p* over E, or even that a is an *exterior p-form* over E.

The set $\mathbf{A}^p(E)$ of exterior p-forms over E is a sub-module of $\mathbf{L}^p(E)$, and one will have $\mathbf{A}^1(E) = \mathbf{L}^1(E) = E^*$.

If h is a linear map of E into a module F then $h^*(\mathbf{A}^p(F))$ will be contained in $\mathbf{A}^p(E)$. Consequently, if F is a sub-module of a module E then one can identify (Prop. 1.12) $\mathbf{A}^p(E/F)$ with a sub-module of forms α in $\mathbf{A}^p(E)$ such that $\alpha(e_1, \ldots, e_p) = 0$ if one of the e_i is in F.

Exercise. – If 2 is invertible in A then a form $\alpha \in \mathbf{A}^p(E)$ will be antisymmetric if and only if $\alpha(e_1, \ldots, e_p) = 0$ when two of the e_i are equal.

2.3. Proposition. – *Let F_1 and F_2 be two sub-modules of a module E. One has:*

$$\mathbf{A}^p(E/(F_1 + F_2)) = \mathbf{A}^p(E/F_1) \cap \mathbf{A}^p(E/F_2) .$$

2.4. Corollary. – *If F_1 is contained in F_2 then $\mathbf{A}^p(E/F_2)$ will be contained in $\mathbf{A}^p(E/F_1)$.*

2.5. Theorem. – *If the module E has a basis of n elements then $\mathbf{A}^p(E)$ will have a basis of $\binom{n}{p}$ elements.*

Proof: Let $(e_i)_{1 \leq i \leq n}$ be a basis for E. An element of $\mathbf{A}^p(E)$ is determined by its values on the sequences $(e_{i_1}, \ldots, e_{i_p})$ such that:

$$1 \leq i_1 < \ldots < i_p \leq n .$$

One associates every increasing sequence $(i_1, \ldots, i_p)$, $1 \leq i_1 < \ldots < i_p \leq n$, with the element $\varepsilon_{i_1 \cdots i_p}$ of $\mathbf{A}^p(E)$ that is defined by:

$$\varepsilon_{i_1 \cdots i_p}(e_{i_1}, \ldots, e_{i_p}) = 1 ,$$
$$\varepsilon_{i_1 \cdots i_p}(e_{j_1}, \ldots, e_{j_p}) = 0$$

if the increasing sequence $(j_1, \ldots, j_p)$ is different from the sequence $(i_1, \ldots, i_p)$. Those elements, which are $\binom{n}{p}$ in number, are independent in $\mathbf{A}^p(E)$, and any exterior p-form α can be written:

$$\alpha = \sum_{1\le i_1<\cdots<i_p\le n} \alpha(e_{i_1},\ldots,e_{i_p})\,\varepsilon_{i_1\cdots i_p}$$

Q. E. D.

2.6. Corollary. – *If the module E has a basis of n elements then* $\mathbf{A}^n(E)$ *will have basis of* 1 *element, and* $\mathbf{A}^p(E) = 0$ *for* $p > n$.

Exercise. – Let E be a free module that has an infinite basis. For any $p > 0$, $\mathbf{A}^p(E)$ will be a non-zero free module.

2.7. Corollary. – *If the module E has a basis of n elements then all of its bases will have n elements.*

2.8. Definition. – *Let E be a module that has a basis of n elements. A* ***volume*** *form on E is an element* $v \in \mathbf{A}^n(E)$ *that defines a basis for* $\mathbf{A}^n(E)$.

Exercises:

i) Any volume form on E can be written $w = a\, v$, where a is an invertible element of A.

ii) If h is an endomorphism of E then one will have $h^*v = (\det h)\, v$.

2.9. Proposition. – *Let a be a multilinear form of degree p over a module E. The form* $\mathfrak{a}(\alpha) = \sum_{s\in\mathfrak{S}_p} \varepsilon(s)\, s\,\alpha$ *is antisymmetric.*

Proof: Let t be a permutation of $\mathfrak{S}_p$. One has:

$$\begin{aligned} t\,\mathfrak{a}(\alpha) &= \sum_{s\in\mathfrak{S}_p} \varepsilon(s)\, t(s\,\alpha) \\ &= \varepsilon(t)\sum_{s\in\mathfrak{S}_p} \varepsilon(ts)(ts)\,\alpha \\ &= \varepsilon(t)\sum_{r\in\mathfrak{S}_p} \varepsilon(r)\, r\,\alpha = \varepsilon(t)\,\mathfrak{a}(\alpha)\,. \end{aligned}$$

Q. E. D.

The map $\mathfrak{a}$ is therefore a linear map of $\mathbf{L}^p(E)$ into $\mathbf{A}^p(E)$. $\mathfrak{a}$ is the *antisymmetrization operator*, and $\mathfrak{a}(\alpha)$ is the *antisymmetrization* of the form α. One has:

$$[\mathfrak{a}\,(\alpha)]\,(e_1, \ldots, e_p) = \sum_{s\in\mathfrak{S}_p} \varepsilon(s)\,\alpha(e_{s^{-1}(1)}, \ldots, e_{s^{-1}(p)})$$
$$= \sum_{s\in\mathfrak{S}_p} \varepsilon(s)\,\alpha(e_{s(1)}, \ldots, e_{s(p)})\,.$$

2.10. Proposition. – *If α is an exterior p-form on a module E then* $\mathfrak{a}\,(\alpha) = p!\;\alpha$.

2.11. Proposition. – *Let h be a linear map of a module E into a module F and let α be a form in* $\mathbf{L}^p\,(E)$. *One will have* $h^*(\mathfrak{a}\,(\alpha)) = \mathfrak{a}(h^*\alpha)$.

§ 3. – Tensor Product.

Let α be a multilinear form of degree p and let β be a multilinear form of degree q over a module E.

$$(e_1, \ldots, e_{p+q}) \mapsto \alpha\,(e_1, \ldots, e_p)\;\beta\,(e_{p+1}, \ldots, e_{p+q})$$

is a multilinear form of degree $p + q$ on E.

3.1. Definition. – *The **tensor product** of the forms* $\alpha \in \mathbf{L}^p\,(E)$ *and* $\beta \in \mathbf{L}^q\,(E)$ *is the form* $\alpha\beta \in \mathbf{L}^{p+q}\,(E)$ *that is defined by:*

$$\alpha\beta\,(e_1, \ldots, e_{p+q}) = \alpha\,(e_1, \ldots, e_p)\;\beta\,(e_{p+1}, \ldots, e_{p+q})\,.$$

3.2. Proposition. – *The tensor product is a bilinear map of* $\mathbf{L}^p\,(E) \times \mathbf{L}^q\,(E)$ *into* $\mathbf{L}^{p+q}\,(E)$. *Moreover, one has* $\alpha\,(\beta\,\gamma) = (\alpha\beta)\,\gamma$ *for* $\alpha \in \mathbf{L}^r\,(E)$, $\beta \in \mathbf{L}^s\,(E)$, *and* $\gamma \in \mathbf{L}^t\,(E)$. (Associativity of the tensor product)

3.3. Proposition. – *Let h be a linear map of a module E into a module F and let* $\alpha \in \mathbf{L}^p\,(F)$ *and* $\beta \in \mathbf{L}^q\,(F)$. *One has* $h^*(\alpha\beta) = h^*(\alpha)\,h^*(\beta)$.

3.4. Proposition. – *Let α be a multilinear form of degree p and let β be multilinear form of degree q on a module E. One has* $\mathfrak{a}\,(\mathfrak{a}\,(\alpha)\,\beta) = p!\;\mathfrak{a}\,(\alpha\beta)$ *and* $\mathfrak{a}\,(\mathfrak{a}\;\alpha\,(\beta)) = p!\;\mathfrak{a}\,(\alpha\beta)$.

Proof: Indeed:

$$\mathfrak{a}\,(\mathfrak{a}\,(\alpha)\,\beta) = \sum_{s\in\mathfrak{S}_{p+q}} \sum_{t\in\mathfrak{S}_p} \varepsilon(s)\,\varepsilon(t)\,s((t\alpha)\beta)$$
$$= \sum_{s\in\mathfrak{S}_{p+q}} \sum_{t\in\mathfrak{S}_p} \varepsilon(st)\,(st)(\alpha\beta)\,,$$

(after identifying $t \in \mathfrak{S}_p$ with the permutation of $\{1, \ldots, p + q\}$ such that $t(i) = i$ for $i > p$), and that:

$$= \sum_{r\in\mathfrak{S}_{p+q}} \varepsilon(r)\, r(\alpha\beta) = p!\ \mathfrak{a}(\alpha\beta)\ .$$

One likewise proves the second equality.

Q. E. D.

3.5. Proposition. – *Let a be a multilinear form of degree p and let b be a multilinear form of degree q over a module E. One has* $\mathfrak{a}(\alpha\beta) = (-1)^{pq}\ \mathfrak{a}(\beta\alpha)$.

Proof: Let t be the permutation of $\mathfrak{S}_{p+q}$ that is defined by:

$$\begin{aligned} t(i) &= q + i && \text{for} && 1 \le i \le p\ , \\ t(p + i) &= i && \text{for} && 1 \le i \le q\ . \end{aligned}$$

The signature of t is $(-1)^{pq}$, and one has:

$$\begin{aligned} (\mathfrak{a}(\alpha\beta))(e_1, \ldots, e_{p+q})\ \alpha &= \sum_{s\in\mathfrak{S}_{p+q}} \varepsilon(st)\,\alpha(e_{st(1)},\ldots,e_{st(p)})\,\beta(e_{st(p+1)},\ldots,e_{st(p+q)}) \\ &= (-1)^{pq} \sum_{s\in\mathfrak{S}_{p+q}} \varepsilon(s)\,\alpha(e_{s(q+1)},\ldots,e_{s(p+q)})\,\beta(e_{s(1)},\ldots,e_{s(q)}) \\ &= (-1)^{pq}\ (\mathfrak{a}(\beta\alpha))(e_1, \ldots, e_{p+q})\ . \end{aligned}$$

Q. E. D.

§ 4. – Exterior product.

From now on, suppose that A is a commutative unitary algebra over the field $\mathbb{Q}$ *of rationals. One identifies* $\mathbb{Q}$ *with the sub-algebra of A that is generated by unity.*

4.1. Definition. – *Let* α *be an exterior p-form and let* β *be an exterior q-form on a module E. The* ***exterior product*** *of* α *and* β *is the exterior* $(p + q)$*-form:*

$$\alpha \wedge \beta = \frac{1}{p!q!}\,\mathfrak{a}(\alpha\beta)\ .$$

One then has:

$$(\alpha \wedge \beta)(e_1, \ldots, e_{p+q}) = \frac{1}{p!q!} \sum_{s\in\mathfrak{S}_{p+q}} \varepsilon(s)\,\alpha(e_{s(1)},\ldots,e_{s(p)})\,\beta(e_{s(p+1)},\ldots,e_{s(p+q)})\ .$$

Exercise. – When $p = 1$, one can write:

$$(\alpha \wedge \beta)\,(e_1, \ldots, e_{p+q}) = \sum_{s \in \mathfrak{S}_{q+1}} (-1)^{i-1}\,\alpha(e_i)\,\beta(e_1, \ldots, e_{i-1}, e_{i+1}, \ldots, e_{q+1})\,.$$

4.2. Proposition. – *The exterior product is a bilinear map of* $\mathbf{A}^p(E) \times \mathbf{A}^q(E)$ *into* $\mathbf{A}^{p+q}(E)$.

4.3. Proposition (anti-commutativity of the exterior product). – *Let α be an exterior p-form and let β be an exterior q-form over a module E. One has:*

$$\alpha \wedge \beta = (-1)^{pq}\,\beta \wedge \alpha\,.$$

That result is an immediate consequence of Proposition 3.5

4. 4. Corollary. – *If α is an exterior form of odd degree then one will have:*

$$\alpha \wedge \alpha = 0.$$

4.5. Proposition (associativity of the exterior product). – *Let α be an exterior p-form, let β be an exterior q-form, and let γ be an exterior r-form on a module E. One will then have:*

$$\alpha \wedge (\beta \wedge \gamma) = (\alpha \wedge \beta) \wedge \gamma\,.$$

Proof: Indeed:

$$\begin{aligned} \alpha \wedge (\beta \wedge \gamma) &= \frac{1}{p!(q+r)!}\ \mathfrak{a}\,(\alpha\,(\beta \wedge \gamma)) \\ &= \frac{1}{p!(q+r)!}\frac{1}{q!r!}\ \mathfrak{a}\,(\mathfrak{a}\ \alpha\,(\beta\gamma)) \\ &= \frac{1}{p!\ q!\ r!}\ \mathfrak{a}\,(\alpha\,(\beta\gamma)) \qquad \text{(Prop. 3.4).} \end{aligned}$$

One likewise has:

$$(\alpha \wedge \beta) \wedge \gamma = \frac{1}{p!\ q!\ r!}\ \mathfrak{a}\,(\alpha\,(\beta\gamma))\,.$$

Q. E. D.

4.6. Proposition. – *Let $(\alpha_i)_{1 \le i \le p}$ be linear p-forms over a module E and let $(e_i)_{1 \le i \le p}$ be p elements in E. One has:*

$$(\alpha_1 \wedge \ldots \wedge \alpha_p)\,(e_1, \ldots, e_p) = \det(< e_j, \alpha_i >)\,.$$

Indeed, one deduces from the preceding proof that:

$$\alpha_1 \wedge \ldots \wedge \alpha_p = \mathfrak{a}\,(\alpha_1, \ldots, \alpha_p)\,.$$

4.7. Definition. – *An exterior form α of degree p on a module E is* ***decomposable*** *if there exist linear p-forms $(\alpha_i)_{1 \le i \le p}$ such that $\alpha = \alpha_1 \wedge \ldots \wedge \alpha_p$.*

4.8. Proposition. – *Let $(e_i)_{1 \le i \le n}$ be a basis for a module E and let $(\varepsilon_i)_{1 \le i \le n}$ be the dual basis for E^*. The decomposable forms:*

$$\varepsilon_{i_1} \wedge \ldots \wedge \varepsilon_{i_p} \qquad 1 \le i_1 < \ldots < i_p \le n$$

constitute a basis for $\mathbf{A}^p(E)$.

Proof: One has, in fact, that:

$$(\varepsilon_{i_1} \wedge \ldots \wedge \varepsilon_{i_p})(e_{j_1} \wedge \ldots \wedge e_{j_p}) = \begin{cases} 1 & \text{if } i_r = j_r \text{ for every } r \\ 0 & \text{otherwise} \end{cases}$$

for any pair $(i_1, \ldots, i_p)$ and $(j_1, \ldots, j_p)$ of increasing sequences in $\{1, \ldots, n\}$. Consequently, $\varepsilon_{i_1} \wedge \ldots \wedge \varepsilon_{i_p} = \varepsilon_{i_1 \cdots i_p}$ (with the same notations as in the proof of Theorem 2.5).

Q. E. D.

4.9. Corollary. – *The exterior n-form $\varepsilon_1 \wedge \ldots \wedge \varepsilon_n$ is a volume form on E, and any element of $\mathbf{A}^n(E)$ is decomposable.*

4.10. Proposition. – *Any exterior p-form is the sum of decomposable p-forms.*

However, it should be pointed out (see § **8**) that not every exterior p-form is decomposable.

4.11. Proposition. – *Let h be a linear map of a module E into a module F and let $\alpha \in \mathbf{A}^p(E)$ and $\beta \in \mathbf{A}^q(E)$. One has:*

$$h^*(\alpha \wedge \beta) = (h^*\alpha) \wedge (h^*\beta)\,.$$

That result is a direct consequence of Propositions 2.11 and 3.3.

§ 5. – The algebra of exterior forms.

5.1. – One agrees to set $\mathbf{A}^0(E) = A$ for any module E and to extend the exterior product to forms of degree 0 by:

$$\alpha \wedge \beta = \beta \wedge \alpha = \alpha\,\beta \qquad \text{if} \qquad \alpha \in A = \mathbf{A}^0(E) \qquad \text{and} \qquad \beta \in \mathbf{A}^q(E), \quad q \ge 0.$$

The exterior product, thus-extended, once more verifies Propositions 4.2, 4.3, and 4.5.

If h is a linear map of a module E into a module F then one also agrees to take h^* : $\mathbf{A}^0$ (E) → $\mathbf{A}^0$ (E) to be the identity map on A. Proposition 4.11 will then be once more verified.

5.2. – Let $\mathbf{A}$ (E) = $\sum_{F \geq 0} \mathbf{A}^p(E)$ be the direct sum of the modules $\mathbf{A}^p$ (E). The elements of $\mathbf{A}$ (E) are called *exterior forms on E*. One can extend the exterior product to the module $\mathbf{A}$ (E) by bilinearity in such a manner as to endow it with the structure of an algebra.

If h is a linear map of a module E into a module F then the maps h^* : $\mathbf{A}^p$ (F) → $\mathbf{A}^p$ (E), $p \geq 0$, will determine a linear map h^* : $\mathbf{A}$ (F) → $\mathbf{A}$ (E).

5.3. Definition. – *The* ***algebra of exterior forms*** *on a module E is the direct sum* $\mathbf{A}$ (E) = $\sum_{F \geq 0} \mathbf{A}^p(E)$, *endowed with the structure of an algebra that is defined by the exterior product.*

If E is the zero module (0) then one will have $\mathbf{A}$ (E) = $\mathbf{A}^0$ (E) = A.

The various propositions of section 4 then allow us to state:

5.4. Theorem. – *The algebra of exterior forms on a module E is an associative, unitary, graded, and anti-commutative algebra* (see Chap. IV, **§ 1**).

5.5. Theorem. – *If the module E possesses a finite basis then its algebra of exterior forms will be generated by its elements of degree* 0 *and* 1.

Exercise. – If E possesses a finite basis that has m elements then $\mathbf{A}$ (E) will possess a finite basis with 2^m elements.

5.6. Theorem. – *Let h be a linear map of a module E into a module F. The map* h^* : $\mathbf{A}$ (F) → $\mathbf{A}$ (E) *will then be a homomorphism of algebras.*

Consequently, if F is a sub-module E then $\mathbf{A}$ (E/F) will be identified with a sub-module of $\mathbf{A}$(E). (See 2.2.)

5.7. Proposition. – *Let* F_1 *and* F_2 *be two sub-modules of a module E. One has* $\mathbf{A}$ (E / ($F_1 + F_2$)) = $\mathbf{A}$ (E / F_1) ∩ $\mathbf{A}$ (E / F_2)) .

5.8. Corollary. – *If* F_1 *is contained in* F_2 *then* $\mathbf{A}$ (E / F_2)) *is a sub-algebra of* $\mathbf{A}$ (E / F_1)).

§ 6. – Interior product.

Let α be an exterior p-form, $p \geq 0$, on a module E, and let x be an element of E.

$$i(x)\,\alpha : (e_1, \ldots, e_{p-1}) \mapsto \alpha(x, e_1, \ldots, e_{p-1})$$

is an exterior $(p-1)$-form on E and the map $i(x) : \alpha \mapsto i(x)\,\alpha$ will be a linear map of $\mathbf{A}^p(E)$ into $\mathbf{A}^{p-1}(E)$.

One can extend that map to an endomorphism of $\mathbf{A}(E)$ by agreeing to set $i(x)\,\alpha = 0$ when $\alpha \in \mathbf{A}^0(E) = A$.

6.1. Definition. – *The endomorphism $i(x)$ of $\mathbf{A}(E)$ is called the **interior product** by an element x of E.*

6.2. Proposition. – *Let x and y be two elements of E. The following properties can be verified:*

i) $i(x + y) = i(x) + i(y)$.
ii) $i(a\,x) = a\,i(x)$, $a \in A$.
iii) $i(x)\,i(y) = -\,i(y)\,i(x)$.
iv) $i(x)\,i(x) = 0$.

6.3. Proposition. – *Let α be an exterior p-form on a module E, let β be an exterior q-form on E, and let x be an element of E. One has:*

$$i(x)\,(\alpha \wedge \beta) = (i(x)\alpha) \wedge \beta + (-1)^p\, \alpha \wedge i(x)\,(\beta)\ .$$

Proof: One can suppose that $p \geq 1$ and $q \geq 1$.

If one agrees to set $x = e_1$ then one will have:

$$(i(x)\,(\alpha \wedge \beta))\,(e_2, \ldots, e_{p+q}) = \alpha \wedge \beta\,(e_1, \ldots, e_{p+q})$$

$$= \frac{1}{p!\,q!} \sum_{s \in \mathfrak{S}_{p+q}} \varepsilon(s)\,\alpha(e_{s(1)}, \ldots, e_{s(p)})\,\beta(e_{s(p+1)}, \ldots, e_{s(p+q)})\ .$$

Let $\mathfrak{S}' = \{s \in \mathfrak{S}_{p+q} \mid s^{-1}(1) \leq p\}$ and $\mathfrak{S}'' = \{s \in \mathfrak{S}_{p+q} \mid s^{-1}(1) > p\}$. $\mathfrak{S}_{p+q}$ is the union of $\mathfrak{S}'$ and $\mathfrak{S}''$. One identifies $\mathfrak{S}_{p+q-1}$ with the set of permutations r of $\mathfrak{S}_{p+q}$ such that $r(1) = 1$.

For any $s \in \mathfrak{S}_{p+q}$, let t_s (t'_s, resp.) denote the transposition of $\mathfrak{S}_{p+q}$ that exchanges 1 and $s^{-1}(1)$ ($p + 1$ and $s^{-1}(1)$, resp.). One can write:

$$\frac{1}{p!q!}\sum_{s\in\mathfrak{S}'}\varepsilon(s)\alpha(e_{s(1)},\dots,e_{s(p)})\beta(e_{s(p+1)},\dots,e_{s(p+q)})$$
$$=\frac{1}{p!q!}\sum_{s\in\mathfrak{S}'}\varepsilon(t_s s)\alpha(x,e_{st_s(2)},\dots,e_{st_s(p)})\beta(e_{st_s(p+1)},\dots,e_{st_s(p+q)})$$
$$=\frac{p}{p!q!}\sum_{r\in\mathfrak{S}_{p+q-1}}\varepsilon(r)\alpha(x,e_{r(2)},\dots,e_{r(p)})\beta(e_{r(p+1)},\dots,e_{r(p+q)})$$

(because for each $r\in\mathfrak{S}_{p+q-1}$ there exist p permutations $s\in\mathfrak{S}'$ such that $st_s = r$):

$$= (i(x)\,\alpha)\wedge\beta)\,(e_2,\dots,e_{p+q})\,.$$

Let u be the permutation of $\mathfrak{S}_{p+q}$ that is defined by $u(1) = p + 1$, $u(i) = i - 1$ for $2 \leq i \leq p + 1$ and $u(i) = i$ for $i > p + 1$. One has $\varepsilon(u) = (-1)^p$.

One can write:

$$\frac{1}{p!q!}\sum_{s\in\mathfrak{S}''}\varepsilon(s)\alpha(e_{s(1)},\dots,e_{s(p)})\beta(e_{s(p+1)},\dots,e_{s(p+q)})$$
$$=\frac{1}{p!q!}\sum_{s\in\mathfrak{S}''}\varepsilon(t'_s s)\alpha(e_{st'_s(2)},\dots,e_{st'_s(p)})\beta(x,e_{st'_s(p+1)},\dots,e_{st'_s(p+q)})$$
$$=\frac{(-1)^p}{p!q!}\sum_{s\in\mathfrak{S}''}\varepsilon(ut'_s s)\alpha(e_{st'_s u(2)},\dots,e_{st'_s u(p)})\beta(x,e_{st'_s u(p+1)},\dots,e_{st'_s u(p+q)})$$
$$=\frac{(-1)^p}{p!q!}\,q\sum_{s\in\mathfrak{S}''_{p+q-1}}\varepsilon(r)\alpha(e_{r(2)},\dots,e_{r(p)})\beta(x,e_{r(p+1)},\dots,e_{r(p+q)})$$
$$=(-1)^p\,(\alpha\wedge(i(x)\,\beta))\,(e_2,\dots,e_{p+q})\,.$$

Q. E. D.

6.4. *Remark.* – If F is a sub-module of a module E then $\mathbf{A}(E/F)$ will be the set of exterior forms $\alpha\in\mathbf{A}(E)$ such that $i(x)\,\alpha = 0$ for all $x\in F$. Furthermore, if A is a field of characteristic zero and E is a finite-dimensional vector space over A then F will be equal to a subspace G of $x\in E$ such that $i(x)\,\alpha = 0$ for any $\alpha\in\mathbf{A}(E/F)$. Indeed, one has $F\subset G$ and $\mathbf{A}(E/G) = \mathbf{A}(E/F)$. Now, if H is a subspace of codimension m on E then $\mathbf{A}(E/H)$ will have dimension 2^m. Consequently, dim G = dim F and $F = G$.

§ 7. – Associated system and rank of an exterior form.

One now supposes that A is a commutative field with characteristic zero. Let E denote a vector space of finite dimension n over A.

7.1. Proposition. – *Let* $(\alpha_i)_{1 \le i \le p}$ *be linear p-forms on E. In order for the forms* (α_i) *to be independent in E, it is necessary and sufficient that:*

$$\alpha_1 \wedge \ldots \wedge \alpha_p \neq 0 \, .$$

Proof: If the forms (α_i) are dependent then one can write one of them as a function of the others, and as a result (Prop. 4.4), one will have $\alpha_1 \wedge \ldots \wedge \alpha_p = 0$.

If the forms (α_i) are independent then one can find a basis $(e_i)_{1 \le i \le n}$ for E such that if $(\varepsilon_i)_{1 \le i \le n}$ is the dual basis for E^* then one will have $\varepsilon_i = \alpha_i$ for $i \le p$. $\varepsilon_1 \wedge \ldots \wedge \varepsilon_n$ will then be a volume form on E, and consequently $\alpha_1 \wedge \ldots \wedge \alpha_p \neq 0$.

Q. E. D.

7.2. Proposition. – *Let F be a subspace of E. The sub-algebra* **A** (E/F) *of* **A** (E) *is generated by* $A + F^{\perp}$.

Indeed (Theorem 5.5), **A** (E/F) is generated by the set A of its elements of degree 0 and the set $F^{\perp}$ (Prop. 1.9.) of its elements of degree 1.

7.3. Proposition. – *Let* α *be an exterior form on E. There exists one and only one subspace F of E that has the following properties:*

i) $\alpha \in$ **A** (E/F) .

ii) *If G is a subspace of E such that* $\alpha \in$ **A** (E/F) *then G will be contained in F.*

Proof. – Let $\mathcal{F}$ be the family of subspaces H of E such that $\alpha \in$ **A** (E/F). $\mathcal{F}$ is not vacuous (viz., it contains (0)). Since the dimension of E is finite, the family $\mathcal{F}$, when ordered by inclusion, will contain maximal elements, i.e., subspaces H such that $G \in \mathcal{F}$ and $G \supset H$ will imply that $G = H$.

However, $\mathcal{F}$ can contain only one maximal element, because (Prop. 5.7) if F_1 and F_2 are two subspaces of $\mathcal{F}$ then $F_1 + F_2$ will also be in $\mathcal{F}$.

Q. E. D.

The remark 6.4. leads to the following characterization of the associated subspace:

7.5. Definition. – *The* ***associated subspace*** $A(\alpha)$ *to a form* $\alpha \in$ **A** (E) *is the set of all* $x \in E$ *such that* $i(x)\,\alpha = 0$.

7.6. Corollary. – *If* α *is a non-zero linear form on E then the associated subspace to* α *will be the hyperplane that is defined by* α.

7.7. Definition. – *Let α be an exterior form on E. The* ***associated system*** *to α is the subspace $A^*(\alpha) = (A(\alpha))^{\perp}$ of E.*

If α is an exterior form of degree 0 then one will have $A^*(\alpha) = (0)$. If α is a linear form then $A^*(\alpha)$ will be the subspace of E that is generated by α.

7.8. Proposition. – *The associated system to a form $\alpha \in \mathbf{A}(E)$ is the smallest of the subspaces F of E^* such that α belongs to the sub-algebra of* **A** (*E*) *that is generated by A + F^*.*

That proposition is an immediate consequence of Propositions 7.2. and 7.3.

7.9. Proposition. – *Let a be a non-zero exterior p-form on E, $p \geq 2$, and let h be the multilinear map of E^{p-1} into E^* that is defined by:*

$$h(x_1, \ldots, x_{p-1}) = i(x_1) \ldots i(x_{p-1})\ \alpha .$$

The associated system to α is the subspace of E^* that is generated by the image of h.

Proof: Since:

$$(i(x_1) \ldots i(x_{p-1})\ \alpha)(x) = (-1)^{p-1}\ i(x_1) \ldots i(x_{p-1})\ i(x)\ \alpha = 0$$

for any $x \in A(\alpha)$ and any $(x_1, \ldots, x_{p-1}) \in E^{p-1}$, the subspace I of E^* that is generated by the image of h is contained in $A^*(\alpha)$.

One can then find a basis $(e_i)_{1 \leq i \leq n}$ for E such that the dual basis $(\varepsilon_i)_{1 \leq i \leq n}$ for E^* will have the following properties:

i) $\varepsilon_1, \ldots, \varepsilon_r$ is a basis for I.

ii) $\varepsilon_1, \ldots, \varepsilon_s$, $s \geq r$, is as basis for $A^*(\alpha)$.

If ε_s does not belong to I then one can write $\alpha = \alpha' \wedge \varepsilon_s + \beta$, where α' is a non-zero $(p-1)$-form that belongs to the sub-algebra of **A** (*E*) that is generated by $\varepsilon_1, \ldots, \varepsilon_{s-1}$, along with β.

Let $(x_1, \ldots, x_{p-1})$ be an element of E^{p-1} such that $\alpha'(x_{p-1}, \ldots, x_1) = a \neq 0$. Since $i(x)\alpha' = i(x - \varepsilon_s(x)\ e_s)\ \alpha'$, one can suppose that x_i verifies $\varepsilon_s(x_i) = 0$, $i = 1, \ldots, p-1$. One will then have:

$$i(x_1) \ldots i(x_{p-1})\ \alpha = a\,\varepsilon_s + \sum_{i<s} a_i\,\varepsilon_i ,$$

which is impossible since ε_s is not in I.

Q. E. D.

7.10. Corollary. – *Let α be an exterior form of degree 2 on E, and let $(e_i)_{1 \leq i \leq n}$ be a basis for E. The associated system $A^*(\alpha)$ to α is generated by the forms $i(e_k)\,\alpha$, $k = 1, \ldots, n$.*

The equations of the associated subspace $A(\alpha)$ are then equal to $i(e_k)\,\alpha = 0$, $k = 1, \ldots, n$.

7.11. Definition. – *Let α be an exterior form on E. The* ***rank*** *of α is the dimension of the associated system $A^*(\alpha)$.*

The rank of α is the "the smallest number of independent linear forms that are necessary for one to express α."

The rank of the form α is also equal to the codimension of the associated subspace $A(\alpha)$.

7.12. *Examples:*

i) An exterior form of degree has rank 0.

ii) A non-zero exterior form of degree 1 has rank 1.

7.13. Proposition. – *Let α be a non-zero exterior p-form on E. The rank of α is greater than p (and less than n). It is equal to p if and only if α is decomposable.*

Proof: For any subspace F of E, $\mathbf{A}^p(E/F)$ will be zero as soon as p becomes greater than the codimension of F. The rank of α will then be greater than p. If α is decomposable then there will exist independent linear p-forms $\varepsilon_1, \ldots, \varepsilon_p$ on E such that $\alpha = \varepsilon_1 \wedge \ldots \wedge \varepsilon_p$. The associated system $A^*(\alpha)$ will then be the subspace of E^* that is generated by $\varepsilon_1, \ldots, \varepsilon_p$, and consequently, α will have rank p.

If α has rank p then $A^*(\alpha)$ will possess a basis $\varepsilon_1, \ldots, \varepsilon_p$ that has p elements. One will then have $\alpha = a\,\varepsilon_1 \wedge \ldots \wedge \varepsilon_p$, which shows that α is decomposable.

Q. E. D.

7.14. Corollary. – *A non-zero exterior form of degree n on E has rank n.*

7.15. Proposition. – *Any non-zero exterior form of degree $n - 1$ on E is decomposable.*

Proof: Let α be an exterior $(n - 1)$-form on E and let h be the linear map of E^* into $\mathbf{A}^n(E)$ that is defined by $h(\varepsilon) = \varepsilon \wedge \alpha$.

Since $\mathbf{A}^n(E)$ has dimension 1 over A, the kernel K of h will have dimension n or $n-1$, and one can find a basis $(\varepsilon_i)_{1 \le i \le n}$ for E^* such that $(\varepsilon_i)_{1 \le i \le n-1}$ will be a basis for K if h if non-zero.

One can write $\alpha = \sum_{1 \le i \le n} a_i\, \varepsilon_1 \wedge \ldots \wedge \varepsilon_{i-1} \wedge \varepsilon_{i+1} \wedge \ldots \wedge \varepsilon_n$. The map h is then determined by $h(\varepsilon_i) = (-1)^{i-1} a_i\, \varepsilon_1 \wedge \ldots \wedge \varepsilon_n$. Consequently:

– If h is zero then α will be zero.

– If h is non-zero then one will have $a_i = 0$ for $i < n$, $a_n \neq 0$, and α can be written in the form of $a_n\, \varepsilon_1 \wedge \ldots \wedge \varepsilon_n$.

Q. E. D.

7.16. Corollary. – *A non-zero exterior form of degree $n-1$ on E has rank $n-1$.*

7.17. Corollary. – *A non-zero exterior form of degree $n-2$ on E has rank $n-2$.*

Proof: A non-zero exterior form of degree $n-2$ on E can have rank $n-2$, $n-1$, or n.

A decomposable form has rank $n-2$. On the other hand, if E is a four-dimensional space and $(\varepsilon_i)_{1 \le i \le n}$ is a basis for E^* then $\alpha = \varepsilon_1 \wedge \varepsilon_2 + \varepsilon_3 \wedge \varepsilon_4$ will have rank 4 (Prop. 8.4). It will then remain to be shown that the rank of a form of degree $n-2$ cannot be $n-1$.

Let α be an exterior form of degree $n-2$ and rank $n-1$, and let $F = A(\alpha)$ be the associated subspace to α. E/F will be a space of dimension $n-1$. Since α is a form of degree $n-2$ in $\mathbf{A}(E/F) \subset \mathbf{A}(E)$, it will be decomposable, so its rank will be $n-2$, which is a contradiction.

Q. E. D.

§ 8. – Exterior forms of degree 2.

8.1. Theorem. – *Let α be an exterior form of degree 2 on E. There exists a basis $(e_i)_{1 \le i \le n}$ for E and an even integer $2s \le n$ such that:*

i) $\alpha(e_{2i-1}, e_{2i}) = -\alpha(e_{2i}, e_{2i-1}) = 1$ *for $i \le s$.*

ii) *All of the other values of $\alpha(e_i, e_j)$ are zero.*

Proof: One uses recurrence on the dimension n of E when the result is trivial for $n = 1$. One then supposes that $\alpha \neq 0$.

Let e_1 and e_2 be two vectors in E such that $\alpha(e_1, e_2) = 1$. e_1 and e_2 generate a subspace F of dimension 2 in E.

Let G be the set of all $x \in E$ such that $\alpha(e_1, x) = \alpha(e_2, x) = 0$. G is the intersection of the two hyperplanes H_1 and H_2 whose equations are $i(e_1)\,\alpha = 0$ and $i(e_2)\,\alpha = 0$, respectively. One has $H_1 \cap F = (e_1)$ and $H_2 \cap F = (e_2)$. Consequently, G will be a supplement to F.

One can find a basis $(e_i)_{1 \leq i \leq n}$ for G and an even number $2s \leq n$ such that:

i) $\alpha(e_{2i-1}, e_{2i}) = -\alpha(e_{2i}, e_{2i-1}) = 1$ for $2 \leq i \leq s$.

ii) All other values of $\alpha(e_i, e_j)$, $i, j > 2$ are zero.

The basis $(e_i)_{1 \leq i \leq n}$ then possesses the desired properties.

Q. E. D.

8.2. Corollary. – *Let α be an exterior form of degree 2 on E. There exists an even integer $2s \leq n$ and $2s$ independent linear forms $(\varepsilon_i)_{1 \leq i \leq 2s}$ on E such that:*

$$\alpha = \varepsilon_1 \wedge \varepsilon_2 + \ldots + \varepsilon_{2s-1} \wedge \varepsilon_{2s} .$$

One can choose ε_1 arbitrarily in $A^*(\alpha)$, moreover.

Proof: Let γ be a form on $A^*(\alpha)$. There exist two vectors e_1 and e_2 in E such that $i(e_2)\,\alpha = -\gamma$ and $\alpha(e_1, e_2) = 1$.

The preceding proof permits one to obtain a basis $(e_i)_{1 \leq i \leq n}$ for E that has the properties that were stated in 8.1. In particular, $< e_1, \gamma > = 1$ and $< e_i, \gamma > = 0$ for $i > 1$. The form γ will then be the element e_1 of the dual basis $(\varepsilon_i)_{1 \leq i \leq n}$ for E^*, and one will have:

$$\alpha = \varepsilon_1 \wedge \varepsilon_2 + \ldots + \varepsilon_{2s-1} \wedge \varepsilon_{2s}$$

in that basis.

Q. E. D.

8.3. Corollary. – *An exterior form of degree 2 on E has even rank.*

Indeed, with the preceding notations, if α is a non-zero form then the associated system to α will be the subspace of E^* that is generated by the forms $\varepsilon_1, \ldots, \varepsilon_{2s}$. α then has rank $2s$. (That shows, in particular, that the integer $2s$ that enters into 8.1. and 8.2 depends upon only α.)

8/4/ Proposition. – *Let α be an exterior form of degree 2 on E. In order for α to have rank $2s$, it is necessary and sufficient that one should have $\alpha^s \neq 0$ and $\alpha^{s+1} = 0$.*

Indeed (always with the preceding notations), if α has class $2s$ then one will have:

$$\alpha^s = s!\ \varepsilon_1 \wedge \varepsilon_2 + \ldots + \varepsilon_{2s-1} \wedge \varepsilon_{2s} \neq 0 ,$$

$$\alpha^{s+1} = 0.$$

8.5. Proposition. – *Let α be an exterior form of degree 2 on E. If α has rank 2s then the forms α, α^2, …, α^s will all have the same associated system.*

Indeed, for $r \le s$, one has:

$$\alpha^r = r! \sum_{1 \le i_1 < \cdots < i_r \le s} \varepsilon_{2i_1 - 1} \wedge \varepsilon_{2i_1} \wedge \ldots \wedge \varepsilon_{2i_r - 1} \wedge \varepsilon_{2i_r} .$$

Consequently (Prop. 7.9), the forms ε_1, …, ε_{2s} belong to the associated system to α^r.

8.6. Definition. – *A* ***symplectic structure*** *on E is defined when one is given an exterior form α of degree 2 and maximum rank n on E.*

One then says that (E, α) is a symplectic vector space. The dimension of E is necessarily even (Corollary 8.3).

8.7. Proposition. – *Let E be a vector space of even dimension $n = 2m$, and let α be an exterior form of degree 2 on E. The following properties are equivalent:*

i) *(E, α) is a symplectic vector space.*

ii) *α^m is a volume form on E.*

iii) *$x \mapsto i(x)\,\alpha$ is an isomorphism of E onto E^*.*

That equivalence is an immediate consequence of the Propositions 8.4 and 7.9.

One says that α is a *symplectic form* on E.

8.8. Lemma. – *Let (E, α) and (F, β) be two symplectic vector spaces with the same dimension. A linear map $h : E \to F$ such that $h^*\beta = \alpha$ is an isomorphism.*

In particular, if h is an endomorphism of E such that $h^*\alpha = \alpha$ then h will be an automorphism with determinant 1 on E.

8.9. Definition. – *Let (E, α) and (F, β) be two symplectic vector spaces. A* ***symplectic isomorphism*** *of E to F is a linear isomorphism $h : E \to F$ such that $h^*\beta = \alpha$.*

8.10. Proposition. – *Let* (E, α) *be a symplectic vector space. The set* $\mathrm{Sp}\,(E, \alpha)$ *of symplectic automorphisms of* (E, α) *is a subgroup of the group* $\mathrm{SGl}\,(E)$ *of automorphisms of determinant* 1 *on E.*

Exercise. – If E has dimension two then any automorphism of determinant 1 on E will be a symplectic automorphism. (This result is not true when E has dimension greater than two.)

8.11. *Remark.* – Let $(\varepsilon_i)_{1 \leq i \leq 2m}$ be a basis for E^* such that:

$$\alpha = \varepsilon_1 \wedge \varepsilon_2 + \ldots + \varepsilon_{2m-1} \wedge \varepsilon_{2m} ,$$

and let $(e_i)_{1 \leq i \leq 2m}$ be the dual basis on E. [One says that (e_i) is a *symplectic basis* for (E, α).]

The matrix $J = (\alpha\,(e_i, e_j))$ for α in the basis (e_i) will then have the form:

$$\begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & 1 \\ & & & -1 & 0 \end{pmatrix}.$$

Let h be an endomorphism of E and let M be the matrix of h in the basis (e_i). In order for h to be a symplectic automorphism of (E, α), it is necessary and sufficient that one should have:

$$^tM\,J\,M = J\,.$$

Appendix: Orientations on real vector spaces.

Let E be a real vector space of finite dimension n. The space $\mathbf{A}^n\,(E)$ of exterior n-forms on E is one-dimensional, and the relation $w = \lambda v$, $\lambda > 0$, is an equivalent relation on $\mathbf{A}^n\,(E) - \{0\}$ that two equivalence classes.

A.1. Definition. – *An* ***orientation*** *of E is an equivalence class of* $\mathbf{A}^n\,(E) - \{0\}$ *under the relation* $w = \lambda v$, $\lambda > 0$.

A space E then possesses two distinct orientations. When one has made a choice of one orientation, one says that E is an *oriented vector space*.

A volume form $v$on E determined an orientation of E by its equivalence class. One also says that v is an orientation on E.

A basis for E determines an orientation on E (Prop. 4.10). In particular, the vector space $\mathbb{R}^n$ will always be oriented by its canonical basis.

A.2. Definition. – *An automorphism h of E* ***preserves the orientation*** *if v and h^*v define the same orientation on E for any volume for $v \in \mathbf{A}^n(E)$.*

A.3. Proposition. – *Let E be an oriented vector space. In order for an automorphism h of E to preserve the orientation on E, it is necessary and sufficient that the determinant of h should be positive.*

A.4. Proposition. – *Let E be an oriented vector space. The set* $\mathrm{Gl}^+(E)$ *of automorphisms of E that preserve the orientation is a subgroup of index* 2 *of the group* $\mathrm{Gl}(E)$ *of all automorphisms of E.*

Let (E, α) be a symplectic vector space of dimension $2m$. The volume form α^m defines an orientation on E, namely, the *canonical* orientation of the symplectic vector space (E, α). One always endows E with that orientation.

A.5. Proposition. – *A symplectic automorphism preserves the orientation.*

CHAPTER II

VECTOR BUNDLES

All of the vector spaces considered are real. All of the vector bundles are real and finite-dimensional.

§ 1. – Locally-trivial fiber bundles.

1.1. Definition. *Let F be a topological space. A* ***locally-trivial fiber bundle*** *with fiber F is a triplet $\eta = (E, p, B)$, in which:*

– *E and B are topological spaces,*
– *$p : E \to B$ is a continuous, surjective map,*

and they satisfy the following condition:

(L.-T.) *for any point b in B, there exists an open neighborhood U of b and a homeomorphism $\Phi : p^{-1}(U) \to U \times F$ such that $p_1 \circ \Phi = p$* (in which p_1 denotes the projection of $U \times F$ onto U).

One says that:

E is the *total space* of η,
B is the *base*,
p is the *projection*,
$F_b = p^{-1}(b)$ is the *fiber over the point b* in B.

A pair (U, Φ) of the type that intervenes in the condition (L.-T.) is called a *chart* on η. Let $\mathcal{C}$ denote the set of charts of η, i.e., the set of pairs (U, Φ) that consist of an open set U of B and a homeomorphism $\Phi : p^{-1}(U) \to U \times F$ such that $p_1 \circ \Phi = p$.

If (U, Φ) and (V, Ψ) are two charts on η such that $U \cap V \neq \varnothing$ then one writes $\Psi\,\Phi^{-1}(b, f) = (b, g(b)(f))$, $(b, f) \in (U \cap V) \times F$, in which g is a map of $U \cap V$ into the group of homeomorphisms of F.

If A is a subset of B then $\eta|_A = (p^{-1}(A), p, A)$ will be a locally-trivial bundle with fiber F and base B. $\eta|_A$ is called the *restriction of η to A*.

If A' is a subset of A then one will have $\eta|_{A'} = (\eta|_A)|_{A'}$.

1.2. – One immediately deduces the following properties from Definition 1.1:

i) Each fiber of η is isomorphic to F.

ii) The projection p is an open map.

iii) The base B is the quotient topological space of E by the equivalence relation whose classes are the fibers of η.

Exercise. – If B and F are separable (locally compact, locally connected, locally path-connected, compact, paracompact, connected, path-connected, resp.) topological spaces then the same thing will be true for E.

1.3. Definition. – *Let $\eta = (E, p, B)$ be a locally-trivial fiber bundle and let A be a subset of B. A* ***section*** *of η over A is a continuous map $s : A \to E$ such that $p \circ s$ is the identity map of A.*

A section $s : A \to B$ is therefore a homeomorphism of A onto s (A).

1.4. Definition. – *Let $\eta = (E, p, B)$ and $\eta' = (E', p', B')$ be two locally-trivial bundles (with possibly distinct fibers). A* ***homomorphism*** *of η into η' is a pair (H, h) of continuous maps $H : E \to E'$ and $h : B \to B'$ such that $p' \circ H = h \circ p$.*

That is therefore equivalent to saying that H takes the fiber over b into the fiber over h (b), or even that the diagram:

$$\begin{tikzcd} E \arrow[r, "H"] \arrow[d, "p"'] & E' \arrow[d, "p'"] \\ B \arrow[r, "h"'] & B' \end{tikzcd}$$

commutes.

If the map $H : E \to E'$ takes fibers to fibers then it will determine the map h completely. That is why one also writes that H is a homomorphism of η into η' (and even of E into E') over h.

Let (U, Φ), $\Phi : p^{-1}(U) \to U \times F$ and (U', Φ'), $\Phi' : p^{-1}(U) \to U' \times F'$ be charts on η and η'. If $h(U) \cap U' \neq \varnothing$ then one can write:

$$\Phi' H \Phi^{-1}(b, f) = (h(b), l(b)(f)), \qquad (b, f) \in (U \cap h^{-1}(U')) \times F,$$

in which l is a map $U \cap h^{-1}(U')$ into the set of continuous maps from F into F'.

If H is the identity map on E and h is the identity map on B then (H, h) will be the identity homomorphism of η.

Let A be a subset of B and let i (I, resp.) be the canonical injection of A into B ($p^{-1}(A)$ into $p^{-1}(B)$, resp.). (I, i) is the canonical homomorphism of $\eta|_A$ into η.

Let (H, h) be a homomorphism of η into η' and let (K, k) be a homomorphism of η' into a bundle η''. $(K \circ H, k \circ h)$ is a homomorphism of η into η'', viz., the composite homomorphism.

An isomorphism of η onto η' is therefore a homomorphism (H, h) for which H and h are homeomorphisms. Moreover, in order for that to be true, it suffices that H should be a homeomorphism. In that case, η and η' are homeomorphic fibers.

1.5. Definition. – *Let $\eta = (E, p, B)$ and $\eta' = (E', p', B')$ be two locally-trivial bundles over the same base B (but with possibly distinct fibers). A **homomorphism of η into η' over** B is a homomorphism $(H, h) : \eta \to \eta'$ for which h is the identity map of B.*

If η and η' are isomorphic over B then one says that η and η' are *equivalent*. The composition of two homomorphisms over B is once more a homomorphism over B.

1.6. *Examples:*

i) *Trivial bundle.* – The bundle $\theta = (B \times F, p_1, B)$ is a locally-trivial bundle with fiber F, namely, θ is the trivial bundle with base B and fiber F.

For example, the cylinder $S^1 \times [-1, +1]$ is the total space of the trivial bundle that has the circle S^1 for its base and the line segment $[-1, +1]$ for its fiber.

More generally, $\eta = (E, p, B)$ is a trivial bundle with base B and fiber F then there exists a homomorphism H of η onto θ over B. One says that H is a *trivialization* of η.

A chart (U, Φ) on a locally-trivial bundle η is therefore a trivialization of $\eta|_U$.

ii) *Möbius band.* – Let D be the band $\mathbb{R} \times [0, +1]$ in the plane $\mathbb{R}^2$. The Möbius band is the quotient space E of D that is obtained by identifying the points (x, y) and $(x + 1, 1 - y)$. One denotes the projection of D onto E by ϖ.

The map $(x, y) \mapsto e^{2\pi ix}$ determines a continuous map p of E onto the circle S^1 and $\eta = (E, p, S^1)$ locally-trivial bundle with fiber $[0, +1]$.

Indeed, let $x = e^{2\pi ix}$ be a point on S^1. $U = S^1 - (-x)$ is an open subset of x, and the restriction of ϖ to $W =]\,\xi - 1/2, \xi + 1/2\,[\times [0, 1]$ is a homeomorphism of W onto the open subset $\varpi(W) = p^{-1}(U)$. The map $\Phi : \varpi(u, v) \mapsto (e^{2\pi ix}, v)$ then defines a chart (U, Φ) of η.

One might point out that the bundle η is not trivial. Indeed, the set ∂E of points of E that do not possess a neighborhood that is homeomorphic to an open subset of $\mathbb{R}^2$ (see Chap. III, **§ 1**) is connected, whereas that set has two components for the cylinder.

iii) *Tangent bundle to the sphere* S^2. – Let E be the pairs (u, v) in $\mathbb{R}^3 \times \mathbb{R}^3 = \{(x, y, z ; \xi, \eta, \zeta)\}$ such that $\| u \| = 1$ and $< u, v > = 0$, and let $p : (u, v) \mapsto u$ be projection of E onto the unit sphere S^2. $\eta = (E, p, S^2)$ is a locally-trivial bundle with fiber $\mathbb{R}^2$.

Indeed, let U_1 (U_2, U_3, resp.) be the open subset of S^2 that is defined by $| x | < 1$ ($| y | < 1$, $| z | < 1$). $S^2 = U_1 \cap U_2 \cap U_3$. The homeomorphisms:

$$\Phi_1 : p^{-1}(U_1) \to U_1 \times \mathbb{R}^2, (x, y, z ; \xi, \eta, \zeta) \mapsto (x, y, z ; \eta z - \zeta y, \xi) ,$$

$$\Phi_2 : p^{-1}(U_2) \to U_2 \times \mathbb{R}^2, (x, y, z ; \xi, \eta, \zeta) \mapsto (x, y, z ; \zeta x - \xi z, \eta) ,$$

$$\Phi_3 : p^{-1}(U_3) \to U_3 \times \mathbb{R}^2, (x, y, z ; \xi, \eta, \zeta) \mapsto (x, y, z ; \xi y - \eta x, \zeta)$$

permit one to define a chart (U_i, Φ_i) on a neighborhood of a point b of U_i.

1.17 Theorem. – *Let $\eta' = (E', p', B')$ be a locally-trivial bundle with fiber F, and let h be a continuous map from a space B into B'. There exists:*

– *a locally-trivial bundle with fiber F, $\eta = (E, p, B)$,*
– *a continuous map H of E into E'*

that has the following properties:

i) *H is a homomorphism of η into η' over h.*

ii) *If $z = (D, \pi, B)$ is a locally-trivial bundle with base B and K is a homomorphism of ε into η' over h then there will exist one and only one homomorphism L of ε into η over B such that $p \circ L = \pi$ and $H \circ L = K$.*

Moreover, η (and the homomorphism H) is determined up to an equivalence by the preceding properties.

One can summarize the situation in Theorem 1.7. by the following commutative diagram:

$$\begin{tikzcd}
D \arrow[dr, dashed, "L"] \arrow[drr, "K"] \arrow[ddr, "\pi"'] & & \\
& E \arrow[r, dashed, "H"] \arrow[d, "p"'] & E' \arrow[d, "p'"] \\
& B \arrow[r, "h"'] & B'
\end{tikzcd}$$

Proof: Let E be the set of pairs (b, e') in $B \times E'$ such that $h(b) = p'(e')$, and let p : $(b, e') \mapsto b$ be the projection of E onto B.

Let b be a point of B and let $\Phi : p'^{-1}(V) \to V \times F$ be a trivialization of the restriction of η' to an open neighborhood V of $h(b)$. Let U be the open subset $h^{-1}(V)$ of B. The map $\Phi : (b, e') \mapsto (b, p_2\,\Phi'(e'))$ is a homeomorphism of $p^{-1}(U)$ onto $U \times F$ [its inverse is $(b, f) \mapsto (b, \Phi'^{-1}(h(b), f))\}$. The pair (U, Φ) is therefore a chart of η over the open neighborhood U of b, which shows that η is a locally-trivial bundle with fiber F.

The continuous map $H : (b, e') \mapsto e'$ of E into E' will then be a homomorphism of η into η' over h.

Under the hypotheses of *ii*), the conditions that were imposed upon L will then lead one to take $L(d) = (\pi(d), K(d))$.

Finally, let $\hat{\eta} = (\hat{E}, \hat{p}, B)$ be a locally-trivial bundle with fiber F and let $(\hat{H}, h)$ be a homomorphism of $\hat{\eta}$ into η' that verifies the properties *i*) and *ii*). There will exist a unique homomorphism L ($\hat{L}$, resp.) of $\hat{\eta}$ into η (of η into $\hat{\eta}$, resp.) such that $p \circ L = \hat{p}$ and $H \circ L = \hat{H}$ ($\hat{p} \circ \hat{L} = p$ and $\hat{H} \circ \hat{L} = H$, resp.). One will then have $p \circ (L \circ \hat{L}) = p$ and $H \circ (L \circ \hat{L}) = H$. Consequently (from the uniqueness in *ii*), $L \circ \hat{L}$ will be the identity isomorphism of η. Similarly, $L \circ \hat{L}$ is the identity isomorphism of $\hat{\eta}$. That shows that L is an isomorphism of $\hat{\eta}$ onto η over B.

Q. E. D.

The bundle $\eta = (E, p, B)$, thus-constructed, is called the *reciprocal image bundle* of η' by h. One denotes it by $\eta = h^*(\eta')$.

For any point b in B, H will be a homeomorphism of F_b onto $F_{h(b)}$.

One deduces the following corollaries from the uniqueness of the reciprocal image bundle:

1.8. Corollary. – *If h is the identity map on B' then $h^*(\eta')$ and η' will be equivalent.*

1.9. Corollary. – *If* $\hat{h}$ *is a continuous map of a space* B' *into B then the reciprocal image bundles* $(h \circ \hat{h})^*(\eta')$ *and* $\hat{h}^*(h^*(\eta'))$ *will be equivalent.*

1.10. Corollary. – *If B is a subset of* B' *and if h is the canonical injection of B into W then* $h^*(\eta')$ *and the restriction* $\eta'|_B$ *of* η' *to B will be equivalent.*

§ 2. – Vector bundles.

One always endows real, finite-dimensional vector spaces with their (well-defined) topologies as normed vector spaces.

If F and F' are two real, finite-dimensional vector spaces then the set Hom (F, F') of linear maps from F into F' will also be a finite-dimensional vector space. In particular, End (F) = $\text{Hom}(F, F)$ is a real, finite-dimensional algebra, and the group Gl (F) of automorphisms of F is an open subset of End (F). The canonical map Hom $(F, F') \times F \to F'$ and Gl $(F) \times F \to F$ are continuous.

2.1. Definition. – *Let F be a real vector space of finite-dimension n, and let* η (E, p, B) *be a locally-trivial bundle with fiber F. A* ***vector bundle*** *structure on* η *is determined by the given of a family* $\hat{\mathcal{A}} = \{(U_\alpha, \Phi_\alpha)\} \subset \mathcal{C}$ *of charts on* η *that has the following properties:*

(V. B)$_\text{I}$ (U_α) *is an open covering of B,*

(V. B)$_\text{II}$ *For any pair* (α, β) *such that* $U_\alpha \cap U_\beta \neq \emptyset$, *one will have:*

$$\Phi_\beta \Phi_\alpha^{-1}(b, f) = (b, g_{\beta\alpha}(b) f) \qquad (b, f) \in (U_\alpha \cap U_\beta) \times F,$$

in which $g_{\beta\alpha}$ *is a continuous map of* $U_\alpha \cap U_\beta$ *into* Gl (F) (see 1.1).

(V. B)$_\text{III}$ *If* $\mathcal{B} \supset \hat{\mathcal{A}}$ *is a family of charts on h that has the properties* (V. B.)$_\text{I}$ *and* (V. B.)$_\text{II}$ *then* $\mathcal{B} = \hat{\mathcal{A}}$.

Denote one such bundle $\eta = (E, p, B; \hat{\mathcal{A}})$, or even better, one often denotes it by $\eta = (E, p, B)$. One says that η is a (real) n-dimensional vector bundle.

The set $\hat{\mathcal{A}}$ is the *atlas* of the vector bundle η, and the elements of $\hat{\mathcal{A}}$ are the *vector charts* of η. The continuous maps $g_{\beta\alpha} : U_\alpha \cap U_\beta \to$ Gl (F) are the *changes of charts* in the atlas $\hat{\mathcal{A}}$.

2.2. Lemma. – *The changes of charts have the following property:*

$$g_{\gamma\beta}(b)\, g_{\beta\alpha}(b) = g_{\gamma\alpha}(b) \qquad \textit{for all} \;\; b \in U_\gamma \cap U_\beta \cap U_\alpha .$$

In particular:

– $g_{\alpha\alpha}(b) = \iota$ for any $b \in U_\alpha$ [in which ι is the identity of Gl (F)],
– $g_{\alpha\beta}(b) = (g_{\beta\alpha}(b))^{-1}$ for any $b \in U_\alpha \cap U_\beta$.

More generally, an *atlas* on η is a subset $\mathcal{A}$ of $\hat{\mathcal{A}}$ that has the properties (V. B.)$_{\mathrm{I}}$ and (V. B.)$_{\mathrm{II}}$. One then says that $\hat{\mathcal{A}}$ is *the maximal atlas* of η. That notion is, in fact, justified by the following proposition:

2.3. Proposition. – *Let F be a finite-dimensional vector space and let* $\eta = (E, p, B)$ *be a locally-trivial bundle with fiber F. If* $\mathcal{A} = \{(U_\alpha, \Phi_\alpha)\}$ *is a family of local charts on* η *that has the properties* (V. B.)$_{\mathrm{I}}$ *and* (V. B.)$_{\mathrm{II}}$ *then there will exist one and only one subset* $\hat{\mathcal{A}}$ *of* $\mathcal{C}$ *that contains* $\mathcal{A}$ *and defines the structure of a vector bundle over* η.

Proof: Let $\hat{\mathcal{A}}$ be the set of charts (U, Φ) on η that have the following property: For any chart (U_α, Φ_α) of $\mathcal{A}$ such that $U_\alpha \cap U \neq \varnothing$, one has:

$$\Phi_\alpha\, \Phi^{-1}(x, f) = (x, g_\alpha(x) f), \qquad (x, f) \in (U_\alpha \cap U) \times F,$$

in which g_α is a continuous map of $U_\alpha \cap U$ into Gl (F). The set $\hat{\mathcal{A}}$ contains $\mathcal{A}$. It therefore verifies (F. V.)$_{\mathrm{I}}$.

Let (U, Φ) and (V, Ψ) be two charts on $\hat{\mathcal{A}}$ such that $U \cap V \neq \varnothing$. For any point x in $U \cap V$, there exists a chart (U_α, Φ_α) in $\mathcal{A}$ such that $x \in U_\alpha$. One can then write:

$$\Phi_\alpha\, \Phi^{-1}(y, f) = (y, g_\alpha(y) f), \qquad (y, f) \in (U_\alpha \cap U) \times F,$$
$$\Phi_\alpha\, \Psi^{-1}(y, f) = (y, \gamma_\alpha(y) f), \qquad (y, f) \in (U_\alpha \cap V) \times F,$$

in which g_α (γ_α, resp.) is a continuous map of ($U_\alpha \cap U$, resp.) into Gl (F).

If one writes:

$$\Psi\, \Phi^{-1}(y, f) = (y, g(y) f), \qquad (y, f) \in (U_\alpha \cap U) \times F$$

then one will have:

$$g(y) = (\gamma_\alpha(y))^{-1} g_\alpha(y) \qquad \text{for any} \qquad y \in U_\gamma \cap U_\beta \cap U_\alpha .$$

Consequently, g is a continuous map of $U \cap V$ into Gl (F), which shows that $\hat{\mathcal{A}}$ verifies (F. V.)$_{\mathrm{II}}$.

Finally, $\hat{\mathcal{A}}$ satisfies (F. V.)$_{\mathrm{III}}$, by the construction itself.

Q. E. D.

One also denotes the vector bundle $(E, p, B\,;\,\mathcal{A})$ by $\eta = (E, \mathrm{p}, B\,;\,\mathcal{A})$.

2.4. Corollary. – *Let F be a finite-dimensional vector space and let $\eta = (E, p, B)$ be a locally-trivial bundle with fiber F. In order for two atlases $\mathcal{A}$ and $\mathcal{A}'$ on η to define the same vector bundle structure, it is necessary and sufficient that the following property should be verified:*

If (U_α, Φ_α) is a chart on $\mathcal{A}$ and $(U'_\gamma, \Phi'_\gamma)$ is a chart on $\mathcal{A}'$ such that $U_\alpha \cap U'_\alpha \neq \emptyset$ then one will have:

$$\Phi'_\gamma\, \Phi_\alpha^{-1}(b, f) = (b, g_{\gamma\alpha}(b)\, f)\,, \qquad (b, f) \in (U'_\gamma \cap U_\alpha) \times F\,,$$

in which $g_{\gamma\alpha}$ is a continuous map of $U'_\gamma \cap U_\alpha$ into $\mathrm{Gl}\,(F)$.

2.5. *Examples:*

i) *Trivial vector bundle*. – Let I be the identity map of the product $B \times F$ to itself. The chart (B, I) forms an atlas $\mathcal{A}$ on the trivial bundle $\theta = (B \times F, p_1, B)$: $\theta = (B \times F, p_1, B\,;\,\mathcal{A})$ is the trivial vector bundle with base B and fiber F.

In particular, the trivial bundle with fiber (0) is the *null vector bundle* with base B.

ii) *Tangent bundle to the sphere* S^2. – With the notations of the example *iii*) in 1.6, let $\mathcal{A}$ be the set of charts (U_i, Φ_i), $i = 1, 2, 3$; $\eta = (E, p, B\,;\,\mathcal{A})$ is a vector bundle with fiber $\mathbb{R}^2$.

Indeed, the changes of charts are represented by the following matrices:

$$g_{2,1}(u) = \frac{-1}{y^2 + z^2}\begin{pmatrix} x\,y & z \\ -z & x\,y \end{pmatrix},$$

$$g_{3,2}(u) = \frac{-1}{z^2 + x^2}\begin{pmatrix} y\,z & x \\ -x & y\,z \end{pmatrix},$$

$$g_{1,3}(u) = \frac{-1}{x^2 + y^2}\begin{pmatrix} z\,x & y \\ -y & z\,x \end{pmatrix}.$$

Let D be the set of pairs $((u, v), (u', v'))$ in $E \times E$ such that $u = u'$ and $< u, v > = < u, v' > = 0$. The maps:

$$\Sigma : D \to E\,, \qquad ((u, v), (u', v')) \to (u, v + v')\,,$$

$$\mu : \mathbb{R} \times E \to E\,, \quad (\lambda, (u, v)) \to (u, \lambda\, v)$$

are continuous and induce the structure of a two-dimensional vector space on each fiber of η.

Indeed, that situation is a general property of vector bundles, as the following theorem shows.

2.6. Theorem. – *Let $\eta = (E, p, B)$ be a vector bundle with fiber F, and let $D = \bigcup_{b\in B} F_b \times F_b$ be the set of pairs (e, e') in $E \times E$ such that $p(e) = p(e')$. There exists:*

– *a section $s_0 : b \mapsto 0_b$ of η over B,*
– *a continuous map $\Sigma : (e, e') \mapsto e + e'$ of D into E,*
– *a continuous map $\mu : (\lambda, e) \mapsto \lambda e$ of $\mathbb{R} \times E$ in E*

that have the following properties for any point b of B:

i) $\Sigma(F_b \times F_b) \subset F_b$,

ii) $\mu(\mathbb{R} \times F_b) \subset F_b$,

iii) *Σ and μ define a vector space structure on F_b that is isomorphic to F and has 0_b for its zero.*

One says that:

– s_0 is the *zero section* of η (one generally writes 0 for 0_b),
– $e + e'$ is the *sum* of e and e' [when $p(e) = p(e')$],
– λe is the *product of e by the scalar λ.*

Proof: Let $\hat{A} = \{(U_\alpha, \Phi_\alpha)\}$ be the maximal atlas on η. For any chart (U_α, Φ_α) in $\hat{A}$, one defines:

– A section $(s_0)_\alpha : U_\alpha \to E$ by $(s_0)_\alpha(b) = \Phi_\alpha^{-1}(b, 0)$.
– A continuous map Σ_α of the open subset $D_\alpha = \bigcup_{b\in U_\alpha} F_b \times F_b$ of D in into E by:

$$\Sigma_\alpha(e, e') = \Phi_\alpha^{-1}(p(e), p_2\Phi_\alpha(e) + p_2\Phi_\alpha(e')).$$

– A continuous map μ_α of $\mathbb{R} \times p^{-1}(U_\alpha)$ into E by:

$$\mu_\alpha(\lambda, e) = \Phi_\alpha^{-1}(p(e), \lambda p_2\Phi_\alpha(e)).$$

One then has $p\,\Sigma_\alpha(e, e') = p(e) = p(e')$ and $p\,\mu_\alpha(\lambda, e) = p(e)$.

Let (U_α, Φ_α) be a second chart in $\hat{A}$ such that $U_\alpha \cap U_\beta \neq \varnothing$. One will then have:

$$\begin{aligned}
(s_0)_\alpha(b) &= \Phi_\alpha^{-1}(b,0) = \Phi_\beta^{-1}\Phi_\beta\,\Phi_\alpha^{-1}(b,0) \\
&= \Phi_\beta^{-1}(b, g_{\beta\alpha}(b)0) = \Phi_\beta^{-1}(b,0) \\
&= (s_0)_\beta(b) \qquad \text{for any} \qquad b \in U_\alpha \cap U_\beta .
\end{aligned}$$

$$\begin{aligned}
\Sigma_\alpha(e, e') &= \Phi_\alpha^{-1}(p(e), p_2\,\Phi_\alpha(e) + p_2\,\Phi_\alpha(e')) \\
&= \Phi_\beta^{-1}\Phi_\beta\,\Phi_\alpha^{-1}(p(e), p_2\,\Phi_\alpha(e) + p_2\,\Phi_\alpha(e')) \\
&= \Phi_\beta^{-1}(p(e), g_{\beta\alpha}(p(e))[p_2\,\Phi_\alpha(e) + p_2\,\Phi_\alpha(e')]) \\
&= \Phi_\beta^{-1}(p(e), p_2\,\Phi_\beta(e) + p_2\,\Phi_\beta(e')) \\
&= \Sigma_\beta(e, e') \qquad \text{for any pair} \qquad (e, e') \in D_\beta \cap D_\alpha
\end{aligned}$$

$$\begin{aligned}
\mu_\alpha(\lambda, e) &= \Phi_\alpha^{-1}(p(e), \lambda p_2\,\Phi_\alpha(e)) \\
&= \Phi_\beta^{-1}\Phi_\beta\,\Phi_\alpha^{-1}(p(e), g_{\beta\alpha}(p(e))[\lambda\, p_2\,\Phi_\alpha(e)]) \\
&= \Phi_\beta^{-1}(p(e), \lambda\, p_2\,\Phi_\beta(e)) \\
&= \mu_\beta(\lambda, e) \qquad \text{for any pair} \qquad (\lambda, e) \in \mathbb{R} \times p^{-1}(U_\alpha \cap U_\beta)
\end{aligned}$$

There will then exist some continuous applications $s_0 : B \to E$, $\Sigma : D \to E$, and $\mu : \mathrm{R} \times E \to E$ such that $s_0|_{U_\alpha} = (s_0)_\alpha$, $\Sigma|_{D_\alpha} = \Sigma_\alpha$ and $\mu|_{\mathbb{R}\times p^{-1}(U_\alpha)} = \mu_\alpha$. The verifications of the properties *i*), *ii*), and *iii*) will then become immediate.

Q. E. D.

One easily shows that if one wishes to define the maps s_0, Σ, and μ then one can restrict oneself to an arbitrary atlas on η. Consequently, the operations that were constructed in the example *ii*) will coincide with the ones in Theorem 2.6.

2.7. Corollary. – *Let $\eta = (E, p, B)$ be a vector bundle and let A be a subset of B. The maps s_0, Σ, and μ induce the structure of a vector space on the set of sections of η over A whose zero is the zero section $s_0|_A$.*

More generally, if $\lambda : A \to \mathbb{R}$ is a continuous function and if s is a section of η over A then s : $b \mapsto \lambda(b)\, s(b)$ will be a section of η over A.

Let s be a section of η over B and let $\mathcal{A} = \{(U_\alpha, \Phi_\beta)\}$ be an atlas on η. For any chart (U_α, Φ_α) on η, one will have $s(b) = \Phi_\alpha^{-1}(b, s_\alpha(b))$, $b \in U_\alpha$, in which s_α is a continuous map of U_α into F. One will then have $s_\beta(b) = g_{\alpha\beta}(b)\, s_\alpha(b)$, $b \in U_\beta \cap U_\alpha$.

Conversely, when one is given a family of continuous maps $s_\alpha : U_\alpha \to F$ that verify the preceding relations, that will determine a section s of η over B.

For example, the zero section s_0 is determined by the constant maps $s_\alpha : b \to 0$ of U_α into F.

2.8. Definition. – *Let* $\eta = (E, p, B)$ *and* $\eta' = (E', p', B')$ *be two vector bundles with fibers* F *and* F' *that are defined by the maximal atlases* $\hat{\mathcal{A}} = \{(U_\alpha, \Phi_\beta)\}$ *and* $\hat{\mathcal{A}}' = \{(U'_\alpha, \Phi'_\alpha)\}$. *A homomorphism* (H, h) *of* h *into* η' *is a* ***vector bundle homomorphism*** *if it satisfies the following condition:*

(H) *For any chart* (U_α, Φ_α) *in* $\hat{\mathcal{A}}$ *and any chart* $(U'_\gamma, \Phi'_\gamma)$ *in* $\hat{\mathcal{A}}'$ *such that* $h(U_\alpha) \cap U'_\gamma \in \varnothing$, *one will have:*

$$\Phi'_\gamma \, H \, \Phi_\alpha^{-1}(b, f) = (h(b), h_{\gamma\alpha}(b) f), \quad (b, f) \in (U_\alpha \cap h^{-1}(U'_\gamma)) \times F,$$

in which $h_{\gamma\alpha}$ *is a continuous map of* $h^{-1}(U'_\gamma)) \cap U_\alpha$ *into* $\mathrm{Hom}(F, F')$ (see 1.4).

Moreover, it would suffice that the condition (H) is verified by arbitrary atlases $\mathcal{A}$ and $\hat{\mathcal{A}}$ that define η and η'.

The identity homomorphism is a vector bundle homomorphism. The composition of two vector bundle homomorphisms is again a vector bundle homomorphism.

On what follows, one will write simply "homomorphism" for "vector bundle homomorphism," and one will say that a vector bundle is trivial if it is equivalent to the trivial bundle in Example 2.5.

Exercise. – Let $\eta = (E, p, B)$ and $\eta' = (E', p', B')$ be two vector bundles and let (H, h) be a homomorphism of η into η'. If H is a homeomorphism of E onto E' then (H^{-1}, h^{-1}) will be a homomorphism (of vector bundles).

2.9. Lemma. – *With the same notations as the ones in the definition* 2.8, *one has:*

$$h_{\gamma\beta}(b)\, g_{\beta\alpha}(b) = h_{\gamma\alpha}(b), \qquad b \in h^{-1}(U'_\gamma) \cap U_\beta \cap U_\alpha,$$

$$g'_{\delta\gamma}\, h(b)\, h_{\gamma\alpha}(b) = h_{\delta\alpha}(b), \qquad b \in h^{-1}(U'_\delta \cap U'_\gamma) \cap U_\alpha,$$

(*in which* $g_{\beta\alpha}$ *and* $g'_{\delta\gamma}$ *denote the changes of charts* η *and* η').

2.10. Theorem. – *Let* $\eta = (E, p, B)$ *and* $\eta' = (E', p', B')$ *be two vector bundles with fibers* F *and* F' *that are defined by the atlases* $\mathcal{A} = (U_\alpha, \Phi_\alpha)$ *and* $\hat{\mathcal{A}} = (U'_\gamma, \Phi'_\gamma)$. *Let* h *be a continuous map of* B *into* B' *and let* $h_{\gamma\alpha} : h^{-1}(U'_\gamma) \cap U_\alpha \to \mathrm{Hom}(F, F')$ *be a family of continuous maps that* *verify the relations of Lemma 2.9. There will then exist one and only one homomorphism H of η into η' over h such that the condition (H) in 2.8. is satisfied.*

Proof: For any chart (U_α, Φ_α) in $\mathcal{A}$ and any chart $(U'_\gamma, \Phi'_\gamma)$ in $\hat{\mathcal{A}}$ such that $h(U_\alpha) \cap U'_\gamma \neq \emptyset$, one must set:

$$H(e) = \Phi'^{-1}_\gamma(h(b), h_{\gamma\alpha}(b)\, p_2\, \Phi_\alpha(e)), \qquad e \in U_\alpha \text{ and } b = p(e).$$

That choice is possible because if e is in $h^{-1}(U'_\delta \cap U'_\gamma) \cap U_\beta \cap U_\alpha$ then one will have:

$$\begin{aligned} H(e) &= \Phi'^{-1}_\gamma(h(b), h_{\gamma\alpha}(b)\, p_2\, \Phi_\alpha(e)) \\ &= \Phi'^{-1}_\delta \Phi_\delta \Phi'^{-1}_\gamma(h(b), h_{\gamma\alpha}(b)\, p_2\, \Phi_\alpha \Phi^{-1}_\beta \Phi_\beta(e)) \\ &= \Phi'^{-1}_\delta(h(b), g'_{\delta\gamma}\, h_{\gamma\alpha}(b)\, g_{\alpha\beta}\, p_2\, \Phi_\beta(e)) \\ &= \Phi'^{-1}_\delta(h(b), h_{\delta\beta}(b)\, p_2\, \Phi_\beta(e)) \end{aligned}$$

Q. E. D.

For example, the constant maps $h_{\gamma\alpha} : b \to 0$ of $h^{-1}(U'_\gamma) \cap U_\alpha$ into Hom (F, F') determine a homomorphism $(0, h)$ of η into η'. One says that $(0, h)$ is a *zero homomorphism* (over h).

2.11. Proposition. – *Let $(H, h) : \eta \to \eta'$ be a homomorphism of vector bundles. One has:*

$$\begin{aligned} &H(0) = 0, \\ &H(e + e') = H(e) + H(e'), \\ &H(\lambda\, e) = \lambda\, H(e). \end{aligned}$$

Conversely, if $H : E \to E'$ is a continuous map that takes fibers to fibers linearly then H will be a vector bundle homomorphism.

2.12. Proposition. – *Let $\eta = (E, p, B)$ and $\eta' = (E', p', B')$ be two vector bundles, and let h be a continuous map of B into B'. The sum and scalar product maps induce a vector space structure on the set* Hom$_h$ (η, η') *of homomorphisms of η into η' over h that has the null homomorphism for its zero.*

The proofs of those two propositions present no difficulties.

2.13. Theorem. – *Let $\eta' = (E', p', B')$ be a vector bundle with fiber F and let h be a continuous map of a space B into B'. Let $\eta = (E, p, B)$ be the reciprocal image bundle $h^*(\eta')$ and let H be the canonical homomorphism of η into η'. There exists a vector space structure on h that has the following properties:*

i) (H, h) *is a vector bundle homomorphism.*

ii) *If* $\varepsilon = (D, p, B)$ *is a vector bundle and if* $K : e \mapsto \eta'$ *is a vector bundle homomorphism over* h *then the associated homomorphism* $L : e \mapsto \eta$ *is a vector bundle homomorphism.*

Moreover, that vector bundle structure on η *is determined by the conditions i*) *and ii*), *up to an equivalence.*

Proof: Let $\hat{\mathcal{A}}' = \{(U'_\alpha, \Phi'_\alpha)\}$ be the maximal atlas on η' and let $U_\alpha = h^{-1}(U'_\alpha)$. For any α, $\Phi_\alpha : (b, e') \mapsto (b, p_2 \Phi'_\alpha(e'))$ will be a homomorphism of $p^{-1}(U_\alpha)$ onto $U_\alpha \times F$ that defines a chart (U_α, Φ_α) on η. The set $\mathcal{A} = \{(U_\alpha, \Phi_\alpha)\}$ will then be an atlas for η. Indeed, if $U_\alpha \cap U_\beta \neq \varnothing$ then one will have:

$$\Phi_\beta \Phi_\alpha^{-1}(b, f) = (b, g'_{\beta\alpha}(h(b) f), \quad (b, f) \in (U_\alpha \cap U_\beta) \times F$$

(in which $g'_{\beta\alpha}$ denotes the changes of charts in the atlas $\hat{\mathcal{A}}'$).

The homomorphism (H, h) is then a vector bundle homomorphism because:

$$\begin{aligned} \Phi'_\beta H \Phi_\alpha^{-1}(b, f) &= \Phi'_\beta \Phi'^{-1}_\beta(h(b), f) \\ &= (h(b), g'_{\beta\alpha}(h(b) f) . \end{aligned}$$

Under the hypotheses on *ii*), one easily verifies that L is a vector bundle homomorphism.

Finally, the proof of uniqueness (up to equivalence) of the vector bundle structure on η is analogous to the one that was given in 1.7.

Q. E. D.

Under the hypotheses of 2.13, let $\eta = h^*(\eta')$ denote the vector bundle, thus-defined, in what follows. Moreover, one can define η by starting from an arbitrary atlas on η'.

2.14. Theorem. – *Let* $\mathcal{U} = (U_\alpha)$ *be an open covering on a space* B *and let* F *be a finite-dimensional vector space. If* $U_\alpha \cap U_\beta \neq \varnothing$ *then let* $g_{\beta\alpha} : U_\beta \cap U_\alpha \to \mathrm{Gl}(F)$ *be a family of continuous maps such that:*

$$g_{\gamma\beta}(x)\, g_{\beta\alpha}(x) = g_{\gamma\alpha}(x), \qquad \textit{for any} \qquad x \in U_\gamma \cap U_\beta \cap U_\alpha .$$

There will then exist one and only one (up to equivalence) vector bundle with fiber F *for which the maps* $g_{\beta\alpha}$ *are the changes of charts in the atlas* $\mathcal{A}$.

One says that $(U_\alpha, g_{\alpha\beta})$ is a *cocycle* over B with values in Gl (F) (subordinate to the open covering $\mathcal{U}$).

An atlas $\mathcal{A} = (U_\alpha, \Phi_\alpha)$ on a vector bundle η then determines a cocycle $(U_\alpha, g_{\alpha\beta})$ that characterizes η up to equivalence.

Proof: Let Σ be the topological sum of the products $U_\alpha \times F$ and let ρ be the equivalence relation on Σ that identifies the pairs $(x, e) \in U_\alpha \times F$ and $(y, f) \in U_\beta \times F$ when $x = y$ and $f = g_{\beta\alpha}(x)\, e$ (the conditions imposed on $g_{\beta\alpha}$ imply that ρ is indeed an equivalence relation). ρ is an open equivalence relation.

Let π be the projection of Σ onto the quotient space E / ρ. The continuous map of Σ onto B that is defined by the first projection $U_\alpha \times F$ onto U_α is compatible with ρ. It will then determine a continuous, surjective map $p : E \to B$.

The triplet $\eta = (E, p, B)$ is then a locally-trivial bundle with fiber F : The restriction π_α of π to $U_\alpha \times F$ is indeed a homeomorphism onto $p^{-1}(U_\alpha)$ such that $p \circ \pi_\alpha(x, f) = x$ and $(U_\alpha, \pi_\alpha^{-1})$ is a chart on η.

The set $\mathcal{A} = \{(U_\alpha, \pi_\alpha^{-1})\}$ is an atlas on η because when $U_\beta \cap U_\alpha \neq \varnothing$, one will have:

$$\pi_\beta^{-1}\, \pi_\alpha(b, f) = (b, g_{\beta\alpha}(b) f), \qquad (b, f) \in (U_\beta \cap U_\alpha) \times F .$$

Consequently, $\eta = (E, p, B ; \mathcal{A})$ is a vector bundle with fiber F for which the maps $g_{\beta\alpha}$ are changes of charts.

Now, let $\eta' = (E', p', B'; \mathcal{A}')$ be a vector bundle with fiber F that is defined by an atlas $\mathcal{A}' = \{(U_\alpha, \Phi_\alpha)\}$ that also has the applications $g_{\beta\alpha}$ for changes of charts.

The continuous map $\hat{H}$ of Σ into E' that is equal to Φ_α^{-1} on $U_\alpha \times F$ is compatible with ρ. It will then determine a continuous map H of E into E' such that $p' \circ H = p$. The homomorphism (H, h) of η into η' over B will then be a vector bundle isomorphism. Indeed, one has:

$$\begin{aligned} \Phi_\beta H\, \pi_\alpha(b, f) &= \Phi_\beta\, \Phi_\alpha^{-1}(b, f) \\ &= (b, g_{\beta\alpha}(b) f), \qquad (b, f) \in (U_\beta \cap U_\alpha) \times F. \end{aligned}$$

Q. E. D.

Under the hypotheses of Theorem 2.14, $\eta = (E, p, B ; \mathcal{A})$ will denote the vector bundle that is constructed in the preceding proof from now on.

One proves the following proposition in an analogous fashion.

2.15. Proposition. – *Let* $(U_\alpha, g_{\beta\alpha})$ *and* $(U'_\gamma, g'_{\delta\gamma})$ *be two cocycles on a space B with values in the same linear group* Gl (F). *In order for the bundles* η *and* η' *that are defined by those cocycles to be equivalent, it is necessary and sufficient that when* $U'_\gamma \cap U_\alpha \neq \varnothing$, *there should exist a family of continuous maps* $h_{\gamma\alpha} : U'_\gamma \cap U_\alpha \to$ Gl (F) *that verify the following relations:*

$$h_{\gamma\beta}(b)\, g_{\beta\alpha}(b) = h_{\gamma\alpha}(b)\,, \qquad b \in U'_\gamma \cap U_\beta \cap U_\alpha\,,$$
$$g_{\delta\gamma}(b)\, h_{\gamma\alpha}(b) = h_{\delta\alpha}(b)\,, \qquad b \in U'_\delta \cap U'_\gamma \cap U_\alpha\,.$$

2.16. Corollary. – *Let* $(U_\alpha, g_{\beta\alpha})$ *and* $(U_\alpha, g'_{\beta\alpha})$ *be two cocycles on a space B with values in the same linear group* Gl (F) *that are subordinate to the same open covering* (U_α) *of B. In order for the bundles* η *and* η' *that are defined by those cocycles to be equivalent, it is necessary and sufficient that there should exist a family of continuous maps* $h_\alpha : U_\alpha \to$ Gl (F) *such that:*

$$g'_{\beta\alpha}(b)\, h_\alpha(b) = h_\beta(b)\, g_{\beta\alpha}(b)\,, \qquad \textit{for any } b \in U_\alpha \cap U_\beta\,.$$

Indeed, with the same notations as in 2.15, it suffices to set $h_\alpha = h_{\alpha\alpha}$ [then $h_{\beta\alpha}(b) = h_\beta(b)\, g_{\beta\alpha}(b)$].

§ 3. – Associated bundles. Orientation.

3.1. – Let $\eta = (E, p, B)$ be a vector bundle with fiber F, and let $(U_\alpha, g_{\beta\alpha})$ be the cocycle over B that is associated with the maximal atlas on η.

Let l be a continuous homomorphism of the group Gl (F) into the group $\mathrm{Gl}(F')$ be automorphism of a finite-dimensional vector space F'. The continuous maps $g'_{\beta\alpha} = \lambda \circ g_{\beta\alpha}$ of $U_\alpha \cap U_\beta$ into $\mathrm{Gl}(F')$ verify the relations:

$$g'_{\gamma\beta}(b)\, g'_{\beta\alpha}(b) = g'_{\gamma\alpha}(b) \qquad \text{for any } b \in U_\gamma \cap U_\beta \cap U_\alpha\,.$$

$(U_\alpha, g'_{\beta\alpha})$ is then a cocycle over B with values in the group $\mathrm{Gl}(F')$.

3.2. Definition. – *Let F and* F' *be two finite-dimensional vector space and let l be a continuous homomorphism of* Gl (F) *into* $\mathrm{Gl}(F')$. *Let* η *be a vector bundle with base B and fiber F and for which one lets* $(U_\alpha, g_{\beta\alpha})$ *denote the cocycle that is associated with its maximal atlas. The bundle* η_λ *with base B and fiber* F' *that is determined by the cocycle* $(U_\alpha, \lambda \circ g_{\beta\alpha})$ *is called the* ***associated bundle*** *to* η *for the homomorphism* λ.

If $\mathcal{A}$ is an atlas on η that defines a cocycle $(V_\gamma, g'_{\delta\gamma})$ over B then the bundle with base B and fiber F' that is determined by the cocycle $(V_\gamma, \lambda \circ g'_{\delta\gamma})$ will be equivalent to $\eta\lambda$.

The associated bundles to a trivial vector bundle are also trivial then.

3.3. *Examples: bundle of exterior p-forms.*

Let $\mathbf{A}^p(F)$ be the vector space of exterior p-forms on F. The map $\alpha \mapsto (\alpha^*)^{-1}$ is a continuous homomorphism λ_p of Gl (F) into Gl $(\mathbf{A}^p(F))$. The associated bundle $\mathbf{A}^p(\eta) = \eta_{\lambda_p}$ is called the *bundle of exterior p-forms over* η.

The bundle $\eta^* = \mathbf{A}^1(\eta)$ is also called the *dual bundle* to η.

3.4. Proposition. – *Let $\eta = (E, p, B)$ be a vector bundle with fiber F and let $D^m = \bigcup_{b\in B}(F_b)^m \subset E^m$ be the set of sequences $(e_1, \ldots, e_m) \in E^m$ such that $p(e_1) = \ldots = p(e_m)$. The vector space of sections over B of the bundle $\mathbf{A}^m(\eta)$ is isomorphic to the set of continuous functions $\sigma : D^m \to \mathbb{R}$ whose restrictions to $(F_b)^m$ are exterior p-forms on F_b for every $b \in B$.*

Proof: Let $\hat{\mathcal{A}} = \{(U_\alpha, \Phi_\alpha)\}$ be the maximal atlas of η and let $(U_\alpha, g_{\alpha\beta})$ be the cocycle that is associated with $\hat{\mathcal{A}}$.

A section s of $\mathbf{A}^m(\eta)$ over B is determined by a family of continuous maps $s_\alpha : U_\alpha \to \mathbf{A}^m(F)$ such that:

$$\sigma_\beta(b) = (g^*_{\beta\alpha})^{-1}(b)\, s_\alpha(b), \qquad b \in U_\beta \cap U_\alpha .$$

As in the proof of Theorem 2.6, one then verifies that those maps σ_α determine a continuous function σ on such that for each point $b \in B$, the restriction of σ to $(F_b)^m$ will be an exterior p-form on F_b.

Conversely, by the preceding equality, such a continuous function will determine a section of $\mathbf{A}^m(\eta)$ over B, and those two correspondences will be the inverse isomorphisms to each other.

Q. E. D.

3.5. *Remark.* – The preceding isomorphism is compatible with restrictions, in the following sense: If A is a subset of B and s is a section of $\mathbf{A}^m(\eta)$ over B that corresponds to a numerical function s on D^m then the restriction of s to A corresponds to the restriction of σ to the subset $\bigcup_{b\in B}(F_b)^m$ of D^m.

In particular, take A to be a point x in B, which will show that the fiber of $\mathbf{A}^m(\eta)$ over x is isomorphic to the space of exterior p-forms on the fiber F_x of η over x.

In what follows, one will denote a section of $\mathbf{A}^m(\eta)$ *and the corresponding function on* D^m *by the same symbols.*

3.6. Corollary. – *Let* η *and* η' *be two vector spaces and let* (H, h) *be a homomorphism of* η *into* η'. *If s is a section of* $\mathbf{A}^m(\eta')$ *then the map* $(e_1, \ldots, e_m) \mapsto s\,(H\,e_1, \ldots, H\,e_m)$ *will determine a section* $(H, h)^*s$ *of* $\mathbf{A}^m(\eta)$.

The map $s \mapsto (H, h)^*s$ is then a linear map of the vector space of sections of $\mathbf{A}^m(\eta')$ into the vector space of sections of $\mathbf{A}^m(\eta)$. In particular, if (H, h) is the canonical homomorphism of the restriction $\eta\,|_A$ into h then $(H, h)^*$ will be the restriction homomorphism for sections (Remark 3.5).

3.7. Definition. – *An n-dimensional vector bundle is* ***orientable*** *if the bundle* $\mathbf{A}^n(\eta)$ *of exterior n-forms over* η *is trivial.*

3.8. Proposition. – *Let* η *be an n-dimensional vector bundle. In order for* η *to be orientable, it is necessary and sufficient that there should exist a non-zero section of the bundle* $\mathbf{A}^n(\eta)$.

Since $\mathbf{A}^n(\eta)$ is a one-dimensional vector bundle, Proposition 3.8. is a consequence of the following result:

3.9. Proposition. – *In order for a one-dimensional vector bundle to be trivial, it is necessary and sufficient that it should possess a non-zero section.*

Proof: The necessary condition is obvious. Therefore, suppose conversely that $\eta = (E, p, B)$ is a vector bundle whose fiber F is one-dimensional, and for which there exists a non-zero section $s : B \to E$.

Let λ be an isomorphism of F onto $\mathbb{R}$. The map $H : B \times F \to E$ defined by $H\,(b, f) = \lambda\,(f)\,s\,(b)$ is an isomorphism of the trivial bundle $\theta = (B \times F, p_1, B)$ onto η : Indeed, if $\{(U_\alpha, \Phi_\alpha)\}$ is an atlas on η, and if the section s is determined by some maps $s_\alpha : U_\alpha \to F$ then one will have:

$$\Phi_\alpha\,H\,(b, f) = \Phi_\alpha\,(\lambda\,(f)\,s\,(b)) = (b, l\,(f)\,s_\alpha\,(b)) .$$

Q. E. D.

3.10. Lemma. – *Let* $\eta = (E, p, B)$ *be a one-dimensional trivial bundle and let* s_1 *and* s_2 *be two non-zero sections of* η. *There exists a continuous function* $\lambda : B \to \mathbb{R} - (0)$ *such that* $s_2 = \lambda\,s_1$.

The proof of that lemma presents no difficulty.

Let $\eta = (E, p, B)$ be an orientable n-dimensional vector bundle and let Γ_0 be the set of non-zero sections of $\mathbf{A}^n(\eta)$. The relation $s_2 = \lambda\,s_1$, in which λ is a continuous, strictly-positive function on B, is an equivalence relation on Γ_0.

3.11. Definition. – *Let $\eta = (E, p, B)$ be an orientable, n-dimensional vector bundle and let Γ_0 be the set of non-zero sections of $\mathbf{A}^n(\eta)$. An **orientation** on η is an equivalence class of Γ_0 under the relation $s_2 = \lambda\, s_1$, in which λ is a continuous, strictly-positive function on B.*

An orientation of η determines an orientation of each of its fibers.

When one makes a choice of orientation, one says that η is an *oriented vector bundle*.

A non-zero section s of $\mathbf{A}^n(\eta)$ determines an orientation on η by its equivalence class. One also says that s is an orientation on η.

3.12. Proposition. – *Let $\eta = (E, p, B)$ be an orientable, n-dimensional vector bundle whose base B is connected. The fiber η possesses two and only two distinct orientations.*

Indeed, any continuous, non-zero function on B is either strictly positive or strictly negative.

If s is a section of $\mathbf{A}^n(\eta)$ that defines an orientation on η then the second orientation on η will be defined by the section $-s$.

3.13. Definition. – *Let η and η' be two n-dimensional vector bundles that are oriented by non-zero sections v and w of $\mathbf{A}^n(\eta)$ and $\mathbf{A}^n(\eta')$, resp. An isomorphism (H, h) of η and η' is **compatible with the orientations** if v and $(H, h)^* w$ define the same orientation on η.*

If $\eta = \eta'$ and $v = w$ then one also says that (H, h) *preserves the orientation* on η. (H, h) *reverses the orientation* of h if v and $-(H, h)^* v$ define the same orientation.

3.14. Theorem. – *Let B be a paracompact, locally-connected topological space, and let η be a vector bundle with base B whose fiber F is an oriented, n-dimensional vector space. In order for h to be orientable, it is necessary and sufficient that there should exist an atlas $\mathcal{A}$ on η that defines a cocycle $(U_\alpha, g_{\beta\alpha})$ on B such that for any pair (α, β) and any point $b \in U_\beta \cap U_\alpha$, the automorphisms $g_{\beta\alpha}(b)$ preserve the orientation.*

Proof: Let $\mathcal{A} = \{(U_\alpha, \Phi_\alpha)\}$ be an atlas of η that defines a cocycle $(U_\alpha, g_{\beta\alpha})$ on B. Since B is locally connected, each connected component of U_α will be an open set in B. One can then suppose that the open sets U_α are connected.

Let s be a non-zero section of $\mathbf{A}^n(\eta)$ over B that is determined by a family of continuous maps $s_\alpha : U_\alpha \to \mathbf{A}^n(F)$. For any chart (U_α, Φ_α) of $\mathcal{A}$, one can write $[(\Phi_\alpha^{-1})^*(s|_{U_\alpha})](b) = s_\alpha(b) = \lambda_\alpha(b)\, v$, in which v is a volume form that defines the orientation on F and $\lambda_\alpha : U_\alpha \to \mathbb{R}$ is a continuous non-zero function.

One can then suppose (after having possibly composed Φ_α with a symmetry with respect to a hyperplane in F) that each function λ_α is strictly positive. Under those conditions, one has:

$$s_\beta(b) = \lambda_\beta(b)\, v = \det(g_{\beta\alpha}(b))\, s_\alpha(b) = \det(g_{\beta\alpha}(b))\, \lambda_\alpha(b)\, v\,,$$

and consequently:

$$\det(g_{\beta\alpha}(b)) = \lambda_\beta(b) / \lambda_\alpha(b) > 0\,.$$

That shows that the condition is necessary.

Conversely, suppose that $\mathcal{A} = \{(U_\alpha, \Phi_\alpha)\}$ is an atlas on η for which the changes of charts $g_{\beta\alpha}$ have their values in the subgroup $\mathrm{Gl}^+(F)$. Since B is paracompact, one can suppose that the open covering (U_α) of B is locally finite. There will then exist a partition of unity (φ_α) that is subordinate to the covering (U_α). [So (φ_α) is a family of continuous maps $\varphi_\alpha : B \to [0, 1]$ such that $\overline{\varphi_\alpha^{-1}(]0,1[)} \subset U_\alpha$ for any α, and $\sum_\alpha \varphi_\alpha(b) = 1$ for any $b \in B$] ([1]).

For every chart (U_α, Φ_α) on η, $\sigma_\alpha = (\Phi_\alpha)^* v$ (in which v also denotes the section $b \mapsto (b, v)$ of the trivial bundle $(U_\alpha \times \mathbf{A}^n(F), p_1, U_\alpha)$ is a section of $\mathbf{A}^n(\eta)$ over U_α and the section $(\varphi_\alpha|_{U_\alpha})\sigma_\alpha$ prolongs to a section s_α of $\mathbf{A}^n(\eta)$ over B by way of the zero section over $B - U_\alpha$.

Since the open covering (U_α) is locally-finite, the sum $s = \sum_\alpha s_\alpha$ will be a section of $\mathbf{A}^n(\eta)$ over B, and that section will not go to zero over B. Indeed, let b be a point of B and let $U_{\alpha_1}, \ldots, U_{\alpha_r}$ be the open subsets of the covering (U_α) that include b. One will have:

$$\Phi_{\alpha_1}(s(b)) = \left(b, \left(\sum_{i=1}^{r} \varphi_{\alpha_1}(b) \det g_{\alpha_1\alpha_i}(b)\right) v\right),$$

and consequently $s(b)$ will be non-zero.

That shows that the condition is also sufficient.

Q. E. D.

§ 4. –Sub-bundles. Quotient bundles. Whitney sums.

4.1. Proposition. – *Let F be a finite-dimensional vector space and let F' be a subspace of F. Let $\eta = (E, p, B)$ be a vector bundle with fiber F and let $\eta' = (E', p', B)$ be a vector bundle with the same base B and fiber F'. Finally, let H be an injective homomorphism of η' into η over B. One can find an atlas $\mathcal{A} = \{(U_\alpha, \Phi_\alpha)\}$ for η that has the following properties for any α:*

i) There exists a homeomorphism $\Phi'_\alpha : p'^{-1}(U_\alpha) \to U_\alpha \times F'$ such that (U_α, Φ'_α) is a vector bundle chart on η'.

([1]) A construction of a partition of unity when X is a differentiable manifold (Prop. 2.1.2) is given in Chapter III.

ii) $\Phi_\alpha H \Phi'^{-1}_\alpha(b, f) = (b, f), (b, f) \in (U_\alpha, \Phi'_\alpha)$.

Moreover, under those conditions:

iii) *The set of charts* (U_α, Φ'_α) *is an atlas* $\mathcal{A}'$ *on* η'.

iv) *The changes of charts* $g_{\beta\alpha}$ *of in the atlas* $\mathcal{A}$ *leave* F' *invariant.*

v) *The changes of charts* $g'_{\beta\alpha}$ *in the atlas* $\mathcal{A}'$ *are the restrictions of* $g_{\beta\alpha}$ *to* F'.

One says that η' is a *sub-bundle* of η (subordinate to *H*).

Proof: Let $\mathcal{B} = \{(U_\alpha, \Psi_\alpha)\}$ be an atlas of η such that for any α, there exists a (vector) trivialization Φ'_α of $\eta'|_{U_\alpha}$. One will then have:

$$\Psi_\alpha H \Phi'^{-1}_\alpha(b, f) = (b, h_\alpha(b) f), \qquad (b, f) \in U_\alpha \times F',$$

in which h_α is a continuous map of U_α into Hom (F', F).

For each α, one can find a continuous map g_α of U_α into Gl (F) such that $g_\alpha(b)\, h_\alpha(b)$ is the canonical injection of F' into F for every point $b \in U_\alpha$, even if it means refining the open covering (U_α).

The map $\Phi_\alpha : e \mapsto (p(e), g_\alpha(p(e), p_2\, \Psi_\alpha(e))$ determines a vector chart (U_α, Φ_α) on η, and the set of charts (U_α, Φ_α) is an atlas $\mathcal{A}$ for η that verifies the conditions *i*), *ii*), and *iii*).

The properties *iv*) and *v*) are then immediate consequences of Lemma 2.9.

Q. E. D.

One easily proves the following converse:

4.2. Proposition. – *Let* $(U_\alpha, g_{\beta\alpha})$ *be a cocycle on B with values in* Gl (F) *such that the* $g_{\beta\alpha}$ *leave the subspace* F' *of F invariant. Let* $\eta = (E, p, B)$ *be a vector bundle with fiber F and let* $\eta' = (E', p', B)$ *be a vector bundle with the same base B and fiber* F'. *Finally, let K be a surjective homomorphism of* η *onto* η' *over B. One can find an atlas* $\mathcal{A} = \{(U_\alpha, \Phi_\alpha)\}$ *on* η *that has the following properties for any* α:

i) *There exists a homeomorphism* $\Phi'_\alpha : p'^{-1}(U_\alpha) \to U_\alpha \times F'$ *such that* (U_α, Φ'_α) *is a vector chart on* η'.

ii) $\Phi'_\alpha K \Phi_\alpha^{-1}(b, f) = (b, q(f))$, $(b, f) \in U_\alpha \times F$.

Moreover, under those conditions:

iii) *The set of charts* (U_α, Φ'_α) *will be an atlas* $\mathcal{A}'$ *for* η'.

iv) *The changes of charts* $g_{\beta\alpha}$ *of the atlas* $\mathcal{A}$ *are compatible with* q.

v) *The changes of charts* $g'_{\beta\alpha}$ *of the atlas* $\mathcal{A}'$ *are the quotients of the* $g_{\beta\alpha}$.

One says that η' is a *quotient bundle* of η (subordinate to K).

4.4. Proposition. – *Let* $(U_\alpha, g_{\beta\alpha})$ *be a cocycle on B with values in* Gl (F) *such that the* $g_{\beta\alpha}$ *are compatible with the projection* $q : F \to F'$. *Let* $\eta = (E, p, B)$ *and* $\eta' = (E', p', B)$ *be vector bundles over B with fibers F and* F', *resp., that are associated with that cocycle. There exists one and only one surjective homomorphism K of* η *into* η' *over B that has the following property:*

$$\text{for any index } \alpha, \qquad h_{\alpha\alpha}(b, f) = (b, q(f)), \quad b \in U_\alpha \text{ and } f \in F.$$

4.5. Definition. – *Let* $\eta_i = (E_i, p_i, B)$, $i = 1, 2, 3$, *be three vector bundles with base B and fibers* F_1, F_2, *and* F_3, resp., *and let H* (*K*, *resp.*) *be a homomorphism of* η_1 *into* η_2 (η_2 *into* η_3, *resp.*) *over B*. *The sequence* $0 \to \eta_1 \xrightarrow{H} \eta_2 \xrightarrow{K} \eta_3 \to 0$ *is an* ***exact sequence*** *of vector bundles if the sequence:*

$$0 \to (F_1)_b \xrightarrow{H} (F_2)_b \xrightarrow{K} (F_3)_b \to 0$$

is exact for every b in B.

Under those condition, the dimension of η_2 will be the sum of the dimensions of η_1 and η_3, and the composition $K \circ H$ will be the zero homomorphism of η_1 into η_3 over B.

4.6. Proposition. – *Under the hypotheses of Proposition 4.1, there exists a vector bundle* η'' $= (E'', p'', B)$ *with B and fiber* $F'' = F / F'$, *and a surjective homomorphism of* η *into* η' *over B such that the sequence* $0 \to \eta' \xrightarrow{H} \eta \xrightarrow{K} \eta'' \to 0$ *is exact.*

Moreover, η'' *(and the homomorphism K) is determined by that condition, up to equivalence.*

One says that η'' is the *quotient bundle of* η *by (the sub-bundle)* η'.

4.7. Proposition. – *Under the hypotheses of Proposition 4.3, there exists a vector bundle* η'' $= (E'', p'', B)$ *with B and fiber* $F'' = q^{-1}(0) \subset F$, *and an injective homomorphism H of* η'' *in h over B such that the sequence:*

$$0 \to \eta'' \xrightarrow{H} \eta \xrightarrow{K} \eta' \to 0$$

is exact.

Moreover, η'' *(an homomorphism H) is determined by that condition, up to equivalence.*

4.8. Proposition. – *Let* $\eta = (E, p, B)$ *and* $\eta' = (E', p', B)$ *be two vector bundles with the same B, and let H be a homomorphism of* η *into* η' *over B of constant rank (the rank of* $H : F_b \to F'_b$ *is independent of b). One will then have:*

i) $\ker H = (H^{-1}s'_0(B), p, B)$ *is a sub-bundle of* η'' *.*

ii) $\operatorname{Im} H = (H(E),\ p', B)$ *is a sub-bundle of* η' *.*

iii) $\ker H$ *and* $\operatorname{Im} H$ *are determined by the exact sequence* $0 \to \ker H \to \eta \xrightarrow{H} \operatorname{Im} H \to 0$, *up to equivalence.*

One says that $\ker H$ is the *kernel* of H, and $\operatorname{Im} H$ is the *image* of H.

The proofs of those three propositions present no difficulties.

4.9. – Let $\eta = (E, p, B)$ and $\eta' = (E', p', B')$ be two vector bundles with fibers F and F', resp., that are defined by their maximal atlases $\hat{\mathcal{A}} = \{(U_\alpha, \Phi_\alpha)\}$ and $\hat{\mathcal{A}}' = \{(V_\gamma, \Psi_\gamma)\}$, resp. The triplet $\eta \times \eta' = (E \times E', p \times p', B \times B')$ is a locally-finite bundle with fiber $F \times F'$, and the set $\hat{\mathcal{A}} \times \hat{\mathcal{A}}' = \{(U_\alpha \times V_\gamma, \Phi_\alpha \times \Psi_\gamma)\}$ defines a vector bundle structure on $\eta \times \eta'$.

4.10. Definition. – *Let* $\eta = (E, p, B\ ;\ \hat{\mathcal{A}})$ *and* $\eta' = (E', p', B'; \hat{\mathcal{A}}')$ *be two vector bundles. The vector bundle* $\eta \times \eta' = (E \times E', p \times p', B \times B'; \hat{\mathcal{A}} \times \hat{\mathcal{A}}')$ *is called the* ***product vector bundle*** *of* η *and* η'.

The dimension of $\eta \times \eta'$ is therefore the sum of the dimensions of η and η'.

Exercise. – Let $\mathcal{A} = \{(U_\alpha, \Phi_\alpha)\}$ be an atlas on η and $\mathcal{A}' = \{(V_\gamma, \Psi_\gamma)\}$ is an atlas on η' $\mathcal{A} \times \mathcal{A}' = \{(U_\alpha \times V_\gamma, \Phi_\alpha \times \Psi_\gamma)\}$ will be an atlas on $\eta \times \eta'$.

4.11. – Let p_1 (p_2, resp.) be the projection of $B\times B'$ onto B (B', resp.). The bundle $p_1^*(\eta)$ is a bundle with fiber F that whose base is the product $B\times B'$ and whose total space is the set D of triplets $(b, b', e) \in B\times B'\times E$ such that $p(e) = b$.

Let $I_1 : D \to E\times E'$ and $P_1 : E\times E' \to D$ be continuous maps that are defined by $I_1(b, b', e) = (e, 0_{b'})$ and $P_1(e, e') = (p(e), p'(e'), e)$, resp. One verifies that I_1 and P_1 are vector bundle homomorphisms over $B\times B'$ such that $P_1\circ I_1$ is the identity automorphism on $p_1^*(\eta)$.

One similarly defines vector bundle homomorphisms $I_2 : p_2^*(\eta') \to \eta\times\eta'$ and $P_2 : \eta\times\eta' \to p_2^*(\eta')$ such that $P_2\circ I_2$ will be the identity automorphism on $p_2^*(\eta')$.

One then has:

4.12. Proposition. *The sequences:*

$$0\to p_1^*(\eta)\xrightarrow{I_1}\eta\times\eta'\xrightarrow{P_2} p_2^*(\eta')\to 0,$$
$$0\to p_2^*(\eta')\xrightarrow{I_2}\eta\times\eta'\xrightarrow{P_1} p_1^*(\eta)\to 0$$

are exact.

4.13. Definition. – *Let $\eta = (E, p, B)$ and $\eta' = (E', p', B)$ be two vector bundles with the same base B. The* ***Whitney sum*** *$\eta\oplus\eta'$ of η and η' is the inverse image bundle of the product bundle $\eta\times\eta'$ under the diagonal map $d : b \mapsto (b, b)$ of B into $B\times B$.*

One denotes the Whitney sum bundle $\eta\oplus\eta'$ by $(E\oplus E', p\oplus p', B)$.

The dimension of $\eta\oplus\eta'$ is the sum of the dimensions of η and η'.

The total space $E\oplus E'$ of $\eta\oplus\eta'$ is the set of triplets:

$$(b, e, e') \in B\times E\times E' \qquad \text{such that} \qquad p(e) = p'(e') = b\,.$$

It is therefore homeomorphic to the set D of pairs $(e, e') \in E\times E'$ such that $p(e) = p'(e')$. The projection $p\oplus p'$ is transformed into the map $\pi : (e, e') \mapsto p(e) = p'(e')$ by that homeomorphism.

4.14. – As in 4.11, the maps:

$$I_1 : E \to E\oplus E',\ e \mapsto (e, 0), \qquad P_1 : E\oplus E' \to E,\ (e, e') \mapsto e,$$
$$I_2 : E' \to E\oplus E',\ e' \mapsto (0, e'), \qquad P_2 : E\oplus E' \to E',\ (e, e') \mapsto e'$$

are vector bundle homomorphisms that have the following properties:

i) $P_1\circ I_1$ is the identity automorphism of η.

ii) $P_2 \circ I_2$ is the identity automorphism of η'.

iii) The sequence $0 \to \eta \xrightarrow{I_1} \eta \oplus \eta' \xrightarrow{P_2} \eta' \to 0$ is exact.

iv) The sequence $0 \to \eta' \xrightarrow{I_2} \eta \oplus \eta' \xrightarrow{P_1} \eta \to 0$ is exact.

Exercises.

i) The sum map $\Sigma : (e,\ e') \mapsto e + e'$ (Theorem 2.6) is a homomorphism of $\eta \oplus \eta$ into η over B.

ii) If η and η' are two orientable vector bundles over B then $\eta \oplus \eta'$ will be orientable. Moreover, orientations on η and η' will determine an orientation on $\eta \oplus \eta'$.

iii) Let η and η' be two vector bundles. The fiber product is $\eta \times \eta'$ is equivalent to the Whitney sum $p_1^*(\eta) \oplus p_2^*(\eta')$.

4.15. Proposition. – *Let $\eta = (E, p, B)$ and $\eta' = (E', p', B)$ be two n-dimensional vector bundles over a space B. In order for η' to be equivalent to the dual η^* of η, it is necessary and sufficient that there should exist a continuous map h : $E \oplus E' \to \mathbb{R}$ such that the restriction of h to each fiber of $\eta \oplus \eta'$ will be a non-degenerate bilinear form.*

The proof of that proposition is left to the reader.

CHAPTER III

DIFFERENTIABLE MANIFOLDS

Let $\mathbb{R}^m$ denote the real numerical space of dimension m (and endow it with its Euclidian norm), and let $x_1, \ldots, x_m$ denote the canonical coordinates on $\mathbb{R}^m$. One identifies $\mathbb{R}^m \times \mathbb{R}^n$ with $\mathbb{R}^{m+n}$, and one identifies $\mathbb{R}^{m-1}$ with the hyperplane in $\mathbb{R}^m$ whose equation is $x_m = 0$.

One writes "differentiable" to mean "indefinitely differentiable."

Let H^m be the half-space in $\mathbb{R}^m$ that is defined by $x_m \geq 0$. A map h of an open subset of H^m into a space $\mathbb{R}^n$ is *differentiable* if there exists an open set V in $\mathbb{R}^n$ that contains U and a differentiable map $g : V \to \mathbb{R}^n$, such that $h = g|_U$. The restriction of h to $U \cap \mathbb{R}^{m-1}$ is then a differentiable map.

One assumes the following two fundamental results:

i) *Invariance of the dimension:* For $m \neq n$, an open subset in H^m will not be homeomorphic to an open subset of H^n .

ii) *Invariance of the boundary:* Let U and V be two open subsets of H^m and let h be a homeomorphism of U onto V. One will then have $h(U \cap \mathbb{R}^{m-1}) = V \cap \mathbb{R}^{m-1}$.

§ 1. – Differential structures.

1.1. Definition. – *A **manifold (topological, with boundary) of dimension** m is a non-vacuous topological space that is separable and has a denumerable basis of open subsets, and every point of it possesses an open neighborhood that is homeomorphic to an open subset of* H^m .

A 0-dimensional manifold is a denumerable discrete space.

A non-vacuous open subset of an m-dimensional manifold is again an m-dimensional manifold.

Let M^m be an m-dimensional manifold. A *chart* in M^m is a pair (U, φ) that consists of an open set U in M^m and a homeomorphism φ of U onto an open subset in $\mathbb{R}^m$ or H^m . One says that the numerical functions $y_1 = x_1 \circ \varphi, \ldots, y_m = x_m \circ \varphi$ form a *system of local coordinates* on the open set U.

1.2. Proposition. – *The dimension of a manifold is a topological invariant: Two homeomorphic manifolds will have the same dimension.*

That proposition is an immediate generalization of the theorem of the invariant of dimension.

1.3. – Let M^m be an *m*-dimensional manifold. Let Int M^m denote the set of points of M^m that possess an open neighborhood that is homeomorphic to an open set in $\mathbb{R}^m$, and let $\partial M^m = M^m$ – Int M^m be the complement to Int M^m (that distinction is justified by the theorem on the invariance of the boundary). One will then have:

i) Int M^m is a non-vacuous open subset of M^m.

ii) Int M^m is an *m*-dimensional manifold such that ∂ (Int M^m) = $\varnothing$.

iii) ∂M^m is a closed subset that is nowhere-dense in M^m.

iv) If ∂M^m is non-vacuous then it will be an $(m - 1)$-dimensional manifold such that $\partial(\partial M^m) = \varnothing$.

v) If $m = 0$ then $\partial M^m = \varnothing$.

One says that Int M^m is the *interior* of M^m and that ∂M^m is the *boundary* of M^m. If $\partial M^m = \varnothing$ then one also says that M^m is a *manifold without boundary.*

1.4. Proposition. – *Let M^m and N^n be two manifolds of dimensions m and n, respectively. The product space $M^m \times N^n$ is a manifold of dimension m + n whose boundary is $\partial(M^m \times N^n) = (\partial M^m) \times N^n \cup M^m \times \partial(N^n)$.*

Indeed, the product $H^m \times H^n$ is the homeomorphic (but not diffeomorphic) to H^{m+n}.

One says that $M^m \times N^n$ is the *product manifold* of the manifolds M^m *and* N^n.

1.5. *Examples:*

i) *Vector spaces.* – A real vector space of finite dimension *m* is an *m*-dimensional manifold without boundary.

ii) *Circle.* – The unit circle S^1 is a compact manifold without boundary of dimension 1. Indeed, for any point $x = e^{2\pi i\xi}$, $0 \leq \xi \leq 1$, of S^1, the map $t \mapsto e^{2\pi it}$ will determine a homeomorphism φ_x of the open neighborhood $U_x = S^1 - \{-x\}$ of x onto the segment $]\,\xi - 1/2, \xi + 1/2\,[$.

iii) *Möbius band.* – The Möbius band E is a compact, two-dimensional manifold with boundary: Indeed, with the same notations as in Example 1.6 of Chapter II, any point x of E can be written $x = \varpi(u, v)$, $v < 1$. The projection v will then determine a homeomorphism φ_u of an open neighborhood U_u of x onto an open subset $V_u =]\,u - 1/2, u + 1/2\,[\times [0, 1[$ of H^2.

The boundary $\partial E = \varpi(\mathbb{R} \times \{0\}) = \varpi(\mathbb{R} \times \{1\})$ of E is homeomorphic to the circle S^1.

iv) *Sphere.* – The unit sphere S^2 in $\mathbb{R}^3$ is a compact, two-dimensional manifold without boundary: Indeed, let $U_{i,\varepsilon}$, $i = 1, 2, 3$, and $\varepsilon = \pm 1$ be the open sets in S^2 that are defined by $\varepsilon x_i > 0$. The maps:

$$\varphi_{1,\varepsilon} : U_{1,\varepsilon} \to \mathbb{R}^3, (x_1, x_2, x_3) \mapsto (x_2, x_3) ,$$

$$\varphi_{2,\varepsilon} : U_{2,\varepsilon} \to \mathbb{R}^3, (x_1, x_2, x_3) \mapsto (x_1, x_3) ,$$

$$\varphi_{3,\varepsilon} : U_{3,\varepsilon} \to \mathbb{R}^3, (x_1, x_2, x_3) \mapsto (x_1, x_2)$$

define charts in the neighborhood of each point of S^2.

1.6. Definition. – *Let M^m be an m-dimensional topological manifold. A* ***differentiable manifold*** *structure on M^m is defined when one is given a family $\hat{\mathcal{A}} = \{(U_i, \varphi_i)\}$ of charts on M^m that have the following properties:*

(D. M.)$_{\mathrm{I}}$ *(U_i) is an open covering of M^m.*

(D. M.)$_{\mathrm{II}}$ *If $U_i \cap U_j \neq \varnothing$ then $\varphi_j\, \varphi_i^{-1}$ is a differentiable map of $\varphi_i(U_i \cap U_j)$ into $\varphi_j(U_i \cap U_j)$.*

(D. M.)$_{\mathrm{III}}$ *If $\mathcal{B} \supset \hat{\mathcal{A}}$ is a family of charts on M^m that has the properties* (D. M.)$_{\mathrm{I}}$ *and* (D.M.)$_{\mathrm{II}}$ *then $\mathcal{B} = \hat{\mathcal{A}}$.*

One denotes such a manifold by $(M^m, \hat{\mathcal{A}})$, or even more often by M^m, and one says that $(M^m, \hat{\mathcal{A}})$ is an *m-dimensional differentiable manifold*.

The set $\hat{\mathcal{A}}$ is the *atlas* of the differentiable manifold $(M^m, \hat{\mathcal{A}})$, and the elements of $\hat{\mathcal{A}}$ are the *differentiable charts* on M^m. The differentiable maps $\varphi_j\, \varphi_i^{-1}$: $\varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$ are *changes of charts* of the atlas $\hat{\mathcal{A}}$. More generally, an *atlas* on $(M^m, \hat{\mathcal{A}})$ is a subset $\mathcal{A}$ of $\hat{\mathcal{A}}$ that has the properties (D. M.)$_{\text{I}}$ and (D. M.)$_{\text{II}}$. One then says that $\hat{\mathcal{A}}$ is the *maximal* atlas of M^m. That notion is, in fact, justified by the following proposition:

1.7. Proposition. – *Let M^m be an m-dimensional topological manifold, and let $\mathcal{A} = \{(U_i, \varphi_i)\}$ be a family of charts on M^m that have the properties* (D. M.)$_{\text{I}}$ *and* (D. M.)$_{\text{II}}$. *There exists one and only one family $\hat{\mathcal{A}}$ of charts on M^m that contains $\mathcal{A}$ and defines the structure of a differentiable manifold on M^m.*

Proof. – Let $\hat{\mathcal{A}}$ be the set of charts (U, φ) on M^m that have the following property: For any chart (U_i, φ_i) of $\mathcal{A}$ such that $U_i \cap U_j \neq \varnothing$, $\varphi_i \varphi^{-1}$ and $\varphi\varphi_i^{-1}$ are differentiable maps. The set $\hat{\mathcal{A}}$ contains $\mathcal{A}$. It will then verify (D. M.)$_{\text{I}}$.

Let (U, φ) and (V, ψ) be two charts of $\hat{\mathcal{A}}$ such that $U \cap V \neq \varnothing$. For any point x of $U \cap V$, there exists a chart (U_i, φ_i) of $\mathcal{A}$ such that $x \in U_i$. One will then have $\varphi\psi^{-1} = (\varphi\varphi_i^{-1})(\varphi\psi^{-1})$, and the map $\varphi\psi^{-1}$ will be differentiable at the point $\psi(x)$. That shows that $\hat{\mathcal{A}}$ verifies (D. M.)$_{\text{II}}$.

Finally, $\hat{\mathcal{A}}$ will satisfy (D. M.)$_{\text{III}}$, by the construction itself.

Q. E. D.

One can also let $(M^m, \mathcal{A})$ denote the differentiable manifold $(M^m, \hat{\mathcal{A}})$ then.

1.8. Corollary. – *Let M^m be an m-dimensional topological manifold. In order for two differentiable manifold structures on M^m that are defined by atlases $\mathcal{A} = \{(U_i, \varphi_i)\}$ and $\mathcal{A}' = \{(V_k, \psi_k)\}$ to be identical, it is necessary and sufficient that the following property should be verified:*

For any chart (U_i, φ_i) of $\mathcal{A}$ and any chart (V_k, ψ_k) of $\mathcal{A}'$ such that $U_i \cap U_j \neq \varnothing$, the maps $\psi_k \varphi_i^{-1}$ and $\varphi_i \psi_k^{-1}$ are differentiable.

For example, the two charts $(\mathbb{R}, x \mapsto x)$ and $(\mathbb{R}, x \mapsto x^3)$ define two distinct differentiable manifold structures on the real line $\mathbb{R}$.

1.9. *Examples:*

i) *Vector spaces.* – Let E be a vector space of finite dimension m, and let h be an isomorphism of E onto $\mathbb{R}^m$. The chart (E, h) on E defines a differentiable manifold structure on E, and that structure is independent of the choice of isomorphism h.

The finite-dimensional real vector spaces will always be endowed with the differentiable manifold structure, thus-defined.

ii) Let I be an interval in $\mathbb{R}$ and let j be the injection of I into $\mathbb{R}$. The chart (I, j) defines a differentiable manifold structure on I.

iii) *Open submanifold.* – Let M^m be a differentiable manifold that is defined by its maximal atlas $\hat{\mathcal{A}} = \{(U_i, \varphi_i)\}$, and let V be an open subset of M^m. The set $\hat{\mathcal{A}}|_V$ of charts (U_i, φ_i) of $\hat{\mathcal{A}}$ such that $U_i \subset V$ determines a differentiable manifold structure on V: That structure is the differentiable manifold structure that is induced on V by M^m.

As a result, one can endow an open subset V in a differentiable manifold M^m with the induced differentiable manifold structure. One then says that V is an open subset of M^m.

In particular, if E is a real, finite-dimensional vector space then the group Gl (E) is an open submanifold of the vector space End (E).

iv) *Circle.* – The two charts (U_1, φ_1) and (U_{-1}, φ_{-1}) [see 1.5, *ii*)] define a differentiable manifold structure on the circle S^1. Indeed, one has:

$$\varphi_{-1}\,\varphi_1^{-1}(t) = \begin{cases} t+1 & \text{for} \quad -\tfrac{1}{2} < t < 0, \\ t & \text{for} \quad 0 < t < \tfrac{1}{2}. \end{cases}$$

v) *Möbius band.* – One similarly verifies that the four charts (U_u, φ_u), $u = 0, 1/2, 1, 3/2$ [see 1.5, *iii*)] define a differentiable manifold structure on the Möbius band.

vi) *Sphere.* – The six charts $(U_{i,\varepsilon}, \varphi_{i,\varepsilon})$ [see 1.5, *iv*)] define a differentiable manifold structure on the sphere S^2. Indeed:

$$\varphi_{2,\varepsilon}\,\varphi_{1,\varepsilon}^{-1}(x, y) = (\varepsilon\sqrt{1-x^2-y^2}, y),$$
$$\varphi_{3,\varepsilon}\,\varphi_{2,\varepsilon}^{-1}(x, y) = (x, \varepsilon\sqrt{1-x^2-y^2}),$$
$$\varphi_{1,\varepsilon}\,\varphi_{3,\varepsilon}^{-1}(x, y) = (y, \varepsilon\sqrt{1-x^2-y^2}).$$

1.10. Proposition. – *Let M^m be an m-dimensional differentiable manifold with a non-vacuous boundary that is defined by its maximal atlas $\hat{\mathcal{A}} = \{(U_i, \varphi_i)\}$. The set $\mathcal{B}$ of charts (V, ψ) on ∂M^m for which there exists a chart (U, φ) in $\hat{\mathcal{A}}$ such $V = U \cap \partial M^m$ and $\psi = \varphi|_V$ define a differentiable manifold structure of dimension $m - 1$ on the boundary ∂M^m of M^m.*

In what follows, one will always endow that boundary of a differentiable manifold with the differentiable manifold structure, thus-defined.

Proof: Let (V_j, ψ_j) and (V_k, ψ_k) be two charts in $\mathcal{B}$ such that $V_j \cap V_k \neq \varnothing$ and they are restrictions of differentiable charts (U_j, φ_j) and (U_k, φ_k) on M^m. If:

$$\varphi_k\, \varphi_j^{-1}(x_1, \ldots, x_m) = (h_1\,(x_1, \ldots, x_m), \ldots, h_m\,(x_1, \ldots, x_m))$$

then one will have $h_m\,(x_1, \ldots, x_{m-1}, 0) = 0$. Consequently:

$$\psi_k\, \psi_j^{-1}(x_1, \ldots, x_{m-1}) = (h_1\,(x_1, \ldots, x_{m-1}, 0), \ldots, h_{m-1}\,(x_1, \ldots, x_{m-1}, 0))$$

$\psi_k\, \psi_j^{-1}$ is therefore a differentiable map of $\psi_j\,(V_j \cap V_k)$ into $\psi_k\,(V_j \cap V_k)$.

Q. E. D.

Exercise. – If $\mathcal{A}$ is an arbitrary atlas of $(M^m, \hat{\mathcal{A}})$ then $\mathcal{A}$ will induce an atlas on ∂M^m that determines the structure of a differentiable manifold on ∂M^m.

1.11. *Example.* – One always endows the half-space H^m with the differentiable manifold structure that is defined by the chart (H^m, identity). The differentiable manifold structure that is induced on the boundary $\partial H^m = \mathbb{R}^{m-1}$ is the canonical structure on $\mathbb{R}^{m-1}$ then.

§ 2. – Differentiable maps.

2.1. Definition. – *Let M^m and N^n be two differentiable manifolds. A continuous map $h : M^m \to N^n$ is a* ***differentiable map*** *of M^m into N^n if the following condition is satisfied:*

(D. M.) *For any differentiable chart (U, φ) on M^m and any differentiable chart (V, ψ) on N^n such that $h\,(U) \cap V \neq \varnothing$, $\psi \circ h \circ \varphi^{-1}$ will be a differentiable map of $\varphi(U \cap h^{-1}(V))$ into $\psi(V)$.*

It would then suffice that the condition (D. M.) should be satisfied by the charts of arbitrary atlases on M^m and N^n.

The identity map of a differentiable manifold into itself is a differentiable map. The composition of two differentiable maps will again be a differentiable map.

However, if h is a differentiable homeomorphism then h^{-1} will not necessarily be differentiable. For example, $x \mapsto x^3$ is a differentiable homeomorphism of $\mathbb{R}$, but $x \mapsto \sqrt[3]{x}$ is not a differentiable map.

2.2. Definition. – *A* ***differentiable curve*** *in a differentiable manifold* M^m *is a differentiable map of an interval I into* M^m.

2.3. *Examples:*

i) If V is an open submanifold of a differentiable manifold M^m then the injection of V into M^m will be a differentiable map.

ii) If M^m is a differentiable manifold with a non-vacuous boundary then the injection of ∂M^m into M^m will be a differentiable map.

iii) Let M^m and N^n be two differentiable manifolds. A constant map of M^m into N^n is a differentiable map.

iv) The injection of S^2 into $\mathbb{R}^3$ is a differentiable map (see 1.5 and 1.9).

2.4. *Remarks:*

i) In the case of open subsets of $\mathbb{R}^m$, this new notion of a differentiable map coincides with the classical one.

ii) The notion of differentiability for a map is a local one: A continuous map $h : M^m \to N^n$ is differentiable if and only if any point x of M^m possesses an open subset V such that $h\,|_V$ is differentiable.

iii) Let $h : M^m \to N^n$ be a differentiable map and let (U, φ) be a differentiable chart on M^m, while (V, ψ) is a differentiable chart on N^n such that $h(U) \cap V \neq \varnothing$. One can write:

$$\psi h \varphi^{-1}(x_1, \ldots, x_m) = (h_1(x_1, \ldots, x_m), \ldots, h_m(x_1, \ldots, x_m)) .$$

If y_i, $i = 1, \ldots, m$ (z_j, $j = 1, \ldots, n$, resp.) denote the local coordinates on (U, φ) [(V, ψ), resp.] then one often says that $z_j = h_j(y_1, \ldots, y_m)$, $j = 1, \ldots, n$, is the *local expression for h* [in the charts (U, φ) and (V, ψ)].

A differentiable map is therefore a map that is expressed locally by differentiable functions.

2.5. Definition. – *Let* M^m *and* N^n *be two differentiable manifolds. A* ***diffeomorphism*** *of* M^m *onto* N^n *is a homeomorphism* $h : M^m \to N^n$ *such that h and* h^{-1} *are differentiable maps.*

The two manifolds M^m and N^n will then have the same dimension, and h^{-1} will be a diffeomorphism of N^n onto M^m. One says that M^m and N^n are two *diffeomorphic* manifolds.

2.6. *Examples:*

i) Let (U, φ) be a chart on a differentiable manifold M^m. In order for (U, φ) to be a differentiable chart on M^m, it is necessary and sufficient that φ should be a diffeomorphism of U onto $\varphi(U)$

ii) Let F be a finite-dimensional vector space. The map $h \mapsto h^{-1}$ is a diffeomorphism of $\mathrm{Gl}(F)$ onto itself.

iii) The map $x \mapsto \sqrt[3]{x}$ of $\mathbb{R}$ into itself is a diffeomorphism of the two differential structure that were defined in 1.8.

2.7. Proposition. – *Let (U_i) be an open covering of a topological manifold M^m, and let φ_i be a homeomorphism of U_i onto a differentiable manifold N_i for every i. If $\varphi_j\,\varphi_i^{-1}$ is a diffeomorphism of $\varphi_i\,(U_i \cap U_j) \subset N_i$ onto $\varphi_j\,(U_i \cap U_j) \subset N_j$ then there will exist one and only one differentiable manifold structure on M^m for which the homeomorphisms are diffeomorphisms.*

Indeed, for one such structure, if (V, ψ) is a differentiable chart on N_i then $(\varphi_i^{-1}(V), \psi \circ \varphi_i)$ must be a differentiable chart on M^m, and one easily verifies that the set of charts on M^m that are constructed in that way will define a differentiable manifold structure that has the desired property.

One then says that the differentiable manifold M^m is obtained by "gluing" the manifolds N_i together.

2.8. A differentiable function on a differentiable manifold M^m is a differentiable map of M^m into $\mathbb{R}$. The set $\mathcal{D}\,(M)$ of differentiable functions on M^m is a commutative algebra with unity over $\mathbb{R}$. One identifies $\mathbb{R}$ with the subalgebra of constant functions on M^m.

Let f be a differentiable function on M^m. The *support* of f is the adherence of the set of points $x \in M^m$ such that $f(x) \neq 0$. The support of f is therefore a closed subset of M^m.

A family (θ_i) of functions in $\mathcal{D}\,(M)$ in is a *locally-finite family* if any point x of M^m possesses an open neighborhood V_x such that all of the restrictions $\theta_i\,|_{V_x}$ will be zero except for a finite number of them.

Under those conditions, one can define a function $\theta: x \mapsto \sum_i \theta_i(x)$, and that function will be differentiable on M^m: $\theta = \sum_i \theta_i$ is the sum of the locally-finite family (θ_i). That sum will then possess all of the algebraic properties of finite sums.

If h is a differentiable map of M^m into a differentiable manifold N^n then $h^*: f \mapsto h \circ f$ will be a unitary homomorphism of the algebra $\mathcal{D}$ (N) into the algebra $\mathcal{D}$ (M) (that is compatible with locally-finite sums). In particular, if V is an open set in M^m then the injection of V into M^m will induce the restriction homomorphism of $\mathcal{D}$ (M) into $\mathcal{D}$ (N).

2.9. Proposition. – *Let M^m and N^n be two differentiable manifolds and let h be a continuous map of M^m into N^n. In order for h to be a differentiable map, it is necessary and sufficient that for every function f $\in \mathcal{D}$ (N), $h \circ f$ should belong to $\mathcal{D}$ (M).*

It remains for us to prove that the condition is sufficient. In order to do that, we shall utilize the following lemma:

2.10. Lemma. – *There exists a positive, differentiable function θ on $\mathbb{R}^n$ such that:*

$$\begin{aligned} \theta(x) &= 1 \qquad \textit{for} \qquad \|x\| \leq 1, \\ \theta(x) &= 0 \qquad \textit{for} \qquad \|x\| \geq 2. \end{aligned}$$

Proof of lemma: The function $h : \mathbb{R} \to \mathbb{R}$ that is defined by:

$$h(t) = \begin{cases} e^{-1/t} & \text{for} \quad t > 0, \\ 0 & \text{for} \quad t < 0 \end{cases}$$

is a differentiable function on $\mathbb{R}$.

The function:

$$\theta(x) = \frac{h(2 - \|x\|)}{h(2 - \|x\|) + h(\|x\| - 1)}$$

then possesses the desired properties.

Q. E. D.

Proof of Proposition 2.9: Let x be a point of M^m and let (U, φ) be a local chart on M^m that contains x. One can find a local chart (V, ψ) on N^n that contains h (x) and is such that the image of ψ is $\mathbb{R}^n$ or H^n, and $\psi(h(x)) = 0$.

Let z_i, $i = 1, \ldots, n$ be the system of local coordinates that is defined by y on the open set V. The functions $\theta(\psi)\, z_i \in \mathcal{D}(V)$ extend by zeroes to $N^n - V$ into functions $\zeta_i \in \mathcal{D}(V)$ such that $\zeta_i = z_i$ on an open neighborhood $W \subset V$ of $h(x)$.

The functions $\zeta_i \circ h$ then belong to $\mathcal{D}(M)$, and one will have:

$$h(u) = \psi^{-1}(z_1(h(u)), \ldots, z_n(h(u))) = \psi^{-1}(\zeta_1(h(u)), \ldots, \zeta_n(h(u))) \qquad \text{for every } u \in U \cap h^{-1}(W),$$

which shows that h is differentiable on a neighborhood of z.

Q. E. D.

One can remark that the first part of that proof is a justification for the following proposition:

2.11. Proposition. – *Let M^m be a differentiable manifold, let U be an open neighborhood of a point x of M^m, and let f be a differentiable function on U. There exists a differentiable function g on M^m such that $g = f$ on a neighborhood of x.*

2.12. Proposition. – *Let $\mathcal{U} = (U_i)$ be an open covering of a differentiable manifold M^m. There exists an open covering $\mathcal{V} = (V_k)$ of M^m that is locally-finite and finer than $\mathcal{U}$, and a partition of unity (θ_k) that is subordinate to the covering $\mathcal{V}$ such that each function θ_k will be differentiable on M^m.*

In other words, $\mathcal{V} = (V_k)$ is an open covering of M^m, and (θ_k) is a locally-finite family of differentiable functions on M^m that have the following properties:

i) Any point of M^m possesses a neighborhood that meets only a finite number of open sets in $\mathcal{V}$.

ii) Any open set of $\mathcal{V}$ is contained in an open set of $\mathcal{U}$.

iii) $\sum \theta_k = 1$.

iv) The support of θ_k is contained in V_k.

Observe that such an open covering is $\mathcal{V}$ denumerable.

One then says that (θ_k) is a *differentiable partition of unity.*

The proof of that proposition utilizes the following lemma:

2.13. Lemma. – *Let $\mathcal{U} = (U_i)$ be an open covering of an m-dimensional differentiable manifold M^m. There exists an atlas $\{(V_k, \psi_k)\}$ of M^m that has the following properties:*

i) *$\mathcal{V} = (V_k)$ is an open covering of M^m that is locally-finite and finer than $\mathcal{U}$.*

ii) *The image of ψ_k is $\mathbb{R}^m$ or H^m.*

iii) *The open sets $W_k = \{v \in V_k \mid \| \psi_k \| < 1\}$ define an open covering of M^m.*

Proof: Since M^m is a locally-compact space that has a denumerable basis of open sets, there exists a family $(K_r)_{r\in N}$ of compacta in M^m that has the following properties:

– K_r is contained in the interior of K_{r+1} .

– $M^m = \cup K_r$.

For any point x of $L_r = \overline{K_{r+1} - K_r}$, there exists an open subset $U_{i(x)}$ of $\mathcal{U}$ and a local chart (V_x, ψ_k) on such that:

– $x \in V_x$ and $\psi_k (x) = 0$.

– $V_x \subset (K_{r+2} - K_{r-1}) \cap U_{i(x)}$.

– $\psi_k (V_x)$ is equal to $\mathbb{R}^m$ or H^m .

Let W_x be the open set of V_x that is defined by $\| \psi_k (v) \| < 1$. Since L_r is compact, there exists a finite family $x_1, \ldots, x_{s(r)}$ such that $W_{x_1}, \ldots, W_{x_{s(r)}}$ will be an open covering of L_r .

The set of all local charts, thus-chosen, will then be an atlas for M^m that has the desired properties.

Q. E. D.

Proof of the proposition 2.12: With the same notations as in Lemmas 2.10 and 2.13., the functions $\theta = \theta \circ \psi_k \in \mathcal{D}(V_k)$ will extend by zeroes to $M^m - V_k$ into positive functions $\hat{\theta}_k \in \mathcal{D}(M)$ that are equal to 1 on W_k and have its support contained in V_k .

The family $(\hat{\theta}_k)$ is a locally-finite family of functions then, and the sum $\sum_k \hat{\theta}_k$ is not annulled on M^m .

The differentiable functions $\theta_k = \hat{\theta}_k / \sum \hat{\theta}_k$ will then determine a partition of unity subordinate to the open covering $\mathcal{V}$.

Q. E. D.

§ 3. – Product manifolds. Differentiable vector bundles.

3.1. – Let M^m be an m-dimensional differentiable manifold without boundary, and let N^n be an n-dimensional differentiable manifold, possibly with a boundary, that are defined by maximal atlases $\hat{\mathcal{A}} = \{(U_i, \varphi_i)\}$ and $\hat{\mathcal{B}} = \{(V_k, \psi_k)\}$. Since $\mathbb{R}^m \times H^m = H^{m+n}$, the pairs $(U_i \times V_k, \varphi_i \times \psi_k)$ are local charts on the product manifold $M^m \times N^n$, and the set $\hat{\mathcal{A}} \times \hat{\mathcal{B}} = \{(U_i \times V_k, \varphi_i \times \psi_k)\}$ of local charts will be an atlas of a differentiable structure on $M^m \times N^n$.

3.2. Definition. – *Let* $(M^m, \hat{\mathcal{A}})$ *and* $(N^n, \hat{\mathcal{B}})$ *be two differentiable manifolds such that* $\partial M^m = \varnothing$. *The differentiable manifold* $(M^m \times N^n, \hat{\mathcal{A}} \times \hat{\mathcal{B}})$ *is called the* ***(differentiable) product manifold*** *of the manifolds* $(M^m, \hat{\mathcal{A}})$ *and* $(N^n, \hat{\mathcal{B}})$.

3.3. *Remarks:*

i) If $\mathcal{A} = \{(U_i, \varphi_i)\}$ is an atlas on M^m *and* $\mathcal{B} = \{(V_k, \psi_k)\}$ is an atlas on N^n then $\mathcal{A} \times \mathcal{B} = \{(U_i \times V_k, \varphi_i \times \psi_k)\}$ will be an atlas for $M^m \times N^n$.

ii) The canonical isomorphism of $\mathbb{R}^m \times \mathbb{R}^n$ onto $\mathbb{R}^{m+n}$ is a diffeomorphism. (That diffeomorphism will then justify the identification of $\mathbb{R}^m \times \mathbb{R}^n$ with $\mathbb{R}^{m+n}$.)

The verifications of the following proposition, which are stated under the same hypotheses as in 3.1, are immediate.

3.4. Proposition. – *The boundary of* $M^m \times N^n$ *is the product manifold* $M^m \times \partial N^n$.

3.5. Proposition. – *The projections* $p_1 : M^m \times N^n \to M^m$ *and* $p_2 : M^m \times N^n \to N^n$ *are differentiable maps.*

3.6. Proposition. – *For any point u (v, resp.) of* M^m *(*N^n*, resp.), the map* $i_u : y \mapsto (u, y)$ *(*$j_v : x \mapsto (x, v)$*, resp.) is a differentiable map of* N^n *(*M^m*, resp.) into* $M^m \times N^n$.

3.7. Proposition. – *In order for a continuous map h of a differentiable manifold V^p into $M^m \times N^n$ to be differentiable, it is necessary and sufficient that the maps $p_1 \circ h$ and $p_2 \circ h$ should be differentiable.*

By contrast, it is well-known that if $k: M^m \times N^n \to V^p$ is a continuous map then the hypothesis of the differentiability of all maps $k \circ i_u: N^n \to V^p$ and $k \circ j_v: M^m \to V^p$ will not suffice for one to assert that k is a differentiable map.

3.8. Proposition. – *Let M_1, M_2, and M_3 be three differentiable maps such that $\partial M_1 = \partial M_2 = \varnothing$. The canonical homeomorphism of $(M_1 \times M_2) \times M_3$ onto $M_1 \times (M_2 \times M_3)$ is a diffeomorphism.*

That proposition therefore justifies the suppression of the parentheses in the products of differential manifolds.

3.9 Proposition. – *Let F, F', and F'' be three finite-dimensional vector spaces. The maps:*

$(f, f') \mapsto f + f'$	*of*	$F \times F$ *into* F,
$(\lambda, f) \mapsto \lambda f$	*of*	$\mathbb{R} \times F$ *into* F,
$(h, f) \mapsto h(f)$	*of*	$\mathrm{Hom}(F, F') \times F$ *into* F',
	or of	$\mathrm{Gl}(F) \times F$ *into* F,
$(h, k) \mapsto k\,h$	*of*	$\mathrm{Hom}(F, F') \times \mathrm{Hom}(F', F'')$ *into* $\mathrm{Hom}(F, F'')$
	or of	$\mathrm{Gl}(F) \times \mathrm{Gl}(F)$ *into* $\mathrm{GL}(F)$

are differentiable.

3.10. Definition. – *Let F be a real, n-dimensional vector space, and let B be an m-dimensional differentiable manifold. A vector bundle $\eta = (E, p, B)$ with fiber F and base B is a* ***differentiable vector bundle*** *if it possesses an atlas $\mathcal{A} = \{(U_i, \Phi_i)\}$ for which the changes of charts $g_{ij}: U_j \cap U_i \to \mathrm{Gl}(F)$ are differentiable maps.*

One says that $\mathcal{A}$ is a differentiable atlas on η.

3.11. Lemma. – *Let η is a differentiable vector bundle. There exists one and only one differentiable atlas $\mathcal{B}$ of η and that contains all of the differentiable atlas of η.*

The proof of that lemma is analogous to the proof of Proposition 2.3 of Chapter II.

The atlas $\mathcal{B}$ is the maximal (differentiable) atlas of the differentiable vector bundle η. A chart (U, Φ) on $\mathcal{B}$ is a differentiable vector chart on η.

3.12. Proposition. – *Let $\eta = (E, p, B)$ be an n-dimensional differentiable vector bundle whose base B is an m-dimensional differentiable manifold. The total space E of η is a topological manifold of dimension m + n, and there exists one and only one differentiable manifold structure on E such that for any differentiable chart (U, Φ) of h, Φ is a diffeomorphism of the open set $p^{-1}(U_i)$ onto the product manifold $U_i \times F$. For that differentiable manifold structure, the projection p is a differentiable map.*

Proof: Let $\mathcal{B} = \{(U_i, \Phi_i)\}$ be the maximal differentiable atlas of η. For any pair (i, j) such that $U_i \cap U_j \neq \emptyset$, the map $\Phi_j \Phi_i^{-1} : (x, f) \mapsto (x, g_{ji}(x) f)$ will be a diffeomorphism of $\Phi_i(U_i \cap U_j)$ onto $\Phi_j(U_i \cap U_j)$.

The existence and uniqueness of the differentiable manifold structure on E is therefore a consequence of Proposition 2.7.

Since $p = p_1 \circ \Phi_i$ on $p^{-1}(U_i)$, the projection p will be differentiable.

Q. E. D.

As a result, one will always endow the total space of a differentiable vector bundle with the structure of a differentiable manifold, thus-defined. (That structure is, in fact, independent of the choice of differentiable atlas on η.)

3.13. Corollary. – *Let $\eta = (E, p, B)$ and $\eta' = (E', p', B')$ be two differentiable vector bundles with fibers F and F', and let (H, h) be a homomorphism of η into η' such that η is a differentiable map. In order for H to be a differentiable map of E into E', it is necessary and sufficient that the following condition should be verified:*

For any chart (U, Φ) of a differentiable atlas for η and any chart (V, Ψ) of a differentiable atlas for η' such that $h(U) \cap V \neq \emptyset$, one will have $\Psi H \Phi^{-1}(x, f) = (h(x), g(x) f)$, $(x, f) \in (h^{-1}(V) \cap U) \times F$, in which g is a differentiable map of $h^{-1}(V) \cap U$ into $\mathrm{Hom}(F, F')$

One then says that (H, h) is a *differentiable homomorphism.*

As a result, one says that a differentiable vector bundle is trivial if it is differentiably isomorphic to the trivial bundle $\theta = (B \times F, p_1, B)$. Indeed, one can show that a (continuously) trivial bundle is differentiably trivial by differentiable approximations.

3.14. Proposition. – *If η and η' are two differentiable vector bundles then $\eta \times \eta'$ will be a differentiable vector bundle. Moreover, if η and η' have the same base then $\eta \oplus \eta'$ will also be a differentiable vector bundle.*

3.15. Proposition. – *Let $\eta = (E, p, B)$ be a differentiable vector bundle. The zero section $s_0 : B \to E$, the sum map $\Sigma : E \oplus E \to E$, and the scalar product $\mu : \mathbb{R} \times E \to E$ are differentiable maps.*

3.16. Proposition. – *Let $\eta' = (E', p', B')$ be a differentiable vector bundle, and let h be a differentiable map of a differentiable manifold B into B'. The inverse image bundle $\eta = h^*(\eta')$ is a differentiable vector bundle, and the canonical homomorphism (H, h) of η into η' will be a differentiable homomorphism.*

Furthermore, if (K, h) is a differentiable homomorphism of a differentiable vector bundle ε with base B into η' then the associated homomorphism of ε into η will be differentiable.

The verification of those results will present no difficulties.

§ 4. – Tangent bundle.

4.1. – Let M^m be an m-dimensional differentiable manifold that is defined by its maximal atlas $\hat{\mathcal{A}} = \{(U_i, \varphi_i)\}$. The maps $g_{ji} : x \mapsto D(\varphi_j \varphi_i^{-1})_{\varphi_i(x)}$ [viz., the Jacobian matrix of $\varphi_j \varphi_i^{-1}$ at the point $\varphi_i(x)$] have the following properties:

– g_{ji} is a differentiable map from $U_j \cap U_i$ into the group $\mathrm{Gl}(m, \mathbb{R}) = \mathrm{Gl}(\mathbb{R}^m)$.

– $g_{kj}(x)\, g_{ji}(x) = g_{ki}(x)$ for any $x \in U_k \cap U_j \cap U_i$.

They therefore define a differentiable cocycle (U_i, g_{ji}) on M^m with values in the group $\mathrm{Gl}(m, \mathbb{R})$.

4.2. Definition. – *Let M^m be an m-dimensional differentiable manifold that is defined by its maximal atlas $\hat{\mathcal{A}} = \{(U_i, \varphi_i)\}$. The* ***tangent bundle*** *to M^m is the differentiable vector bundle with base M^m and fiber $\mathbb{R}^m$ that is determined by the cocycle $(U_i, D(\varphi_j \varphi_i^{-1}))$.*

The tangent bundle to M^m can also be defined analogously by starting from an arbitrary atlas on M^m (Prop. 2.15, Chap. II).

One denotes the tangent fiber bundle to M^m by $\tau(M) = (T(M), p_M, M^m)$. The total space $T(M)$ is the *tangent bundle* to M^m, and the fiber $T_x(M)$ over x is the set of *tangent vectors* to M^m at the point x.

The associated bundle $\mathbf{A}^p(\tau(M))$ of exterior p-forms on $\tau(M)$ are also differentiable vector bundles. In particular, one denotes the dual bundle to $\tau(M)$ by $\tau^*(M) = (T^*(M), q_M, M^m)$: $\tau^*(M)$ is the *cotangent fiber bundle* to M^m, and $T^*(M)$ is the *cotangent bundle to* M^m.

4.3. Definition. – *A differentiable manifold is* ***parallelizable*** *if its tangent bundle is trivial.*

In that case, all of the bundles $\mathbf{A}^p(\tau(M))$ will also be trivial.

4.4. *Examples:*

i) The tangent bundle to $\mathbb{R}^m$ is trivial. One always chooses the trivialization of $\tau(\mathbb{R}^m)$ that is determined by the chart ($\mathbb{R}^m$, identity).

More generally, an isomorphism h of a vector space E onto $\mathbb{R}^m$ determines a trivialization $\Phi : T(E) \to E \times \mathbb{R}^m$ of $\tau(E)$. If $k : E \to \mathbb{R}^m$ is a second isomorphism, and if $\Psi : T(E) \to E \times \mathbb{R}^m$ is the corresponding trivialization of $\tau(E)$ then one will have $\Psi\,\Phi^{-1}$ = identity $\times\ k\,h^{-1}$.

Consequently, (identity $\times\ h^{-1}) \circ \Phi$ will determine an isomorphism of $\tau(E)$ onto the trivial bundle $(E \times E, p_1, E)$ that is independent of the choice of isomorphism h. That is why one will identify $T(E)$ with $E \times E$ in what follows.

ii) The tangent bundle to the sphere S^2 is differentiably isomorphic to the vector bundle in Example 1.6 of Chapter II.

iii) If U is an open set in a differentiable manifold M^m then the tangent bundle $\tau(U)$ will be the restriction to U of the tangent bundle $\tau(M)$.

iv) Let M^m and N^n be two differentiable manifolds. The tangent bundle $\tau(M \times N)$ is differentiably isomorphic to the product bundle $\tau(M) \times \tau(N)$.

4.5. – Let M^m and N^n be two differentiable manifolds of dimensions m and n, resp., and let h be a differentiable map of M^m into N^n. For any differentiable chart (U, φ) of M^m and any differentiable chart (V, ψ) on N^n such that $h(U) \cap V \neq \varnothing$, $g \mapsto D(\psi \circ h \circ \varphi^{-1})_{\varphi(x)}$ is a differentiable map of $h^{-1}(V) \times U$ in Hom $(\mathbb{R}^m, \mathbb{R}^n)$. Those maps verify the relations in Theorem 2.10 of Chapter II. They will then determine a differentiable homomorphism (h^{T}, h) of $\tau(M)$ into $\tau(N)$: (h^{T}, h) is the tangent homomorphism to h, and $h^{\mathrm{T}} : T(M) \to T(N)$ is the *tangent map* to h.

If h is the identity map on M^m then (h^{T}, h) will be the identity automorphism of $\tau(M)$.

If $h : M^m \to N^n$ and $k : N^n \to V^p$ are differentiable maps then $(k \circ h)^{\mathrm{T}} = k^{\mathrm{T}} \circ h^{\mathrm{T}}$.

Consequently, if h is a diffeomorphism of M^m onto N^n then (h^{T}, h) will be a differentiable isomorphism of $\tau(M)$ onto $\tau(N)$.

If $c : I \to M^m$ is a differentiable curve then $c'(t) = c^{\mathrm{T}}(t,1) \in T_{c(t)}(M)$ is called the *tangent vector to the curve c* at the point $c(t)$.

4.6. *Examples:*

i) If h is a linear map of E into F then h^{T} will be the product map $h \times h : E \times E \to F \times F$.

ii) If h is a bilinear map of $E_1 \times E_2$ into F then h^{T} will be the map $((x, u), (y, v)) \mapsto (h(x, y), h(u, y) + h(x, v))$.

iii) If $h : M^m \to N^n$ is a differentiable map and U is an open set of M^m then one will have $(h|_U)^{\mathrm{T}} = h^T|_{T(U)}$.

4.7. – Let M^m be a differentiable manifold. For any differentiable function f on M^m, one lets $df \in \mathcal{D}(T(M))$ be the second component of the tangent map $f^{\mathrm{T}} : T(M) \to T(\mathbb{R}) = \mathbb{R} \times \mathbb{R}$ [see Example *i*) in 4.4]. The following properties will then be verified:

i) $df = 0$ if f is a constant function.

ii) $d(f + g) = df + dg$ [Example *i*) of 4.6].

iii) $d(f\,g) = (df)\,g + f\,(dg)$ [Example *ii*) of 4.6],

and consequently:

iv) $d(a\,f) = a\,(df)$ for $a \in \mathbb{R}$.

One has, moreover:

v) $df(u + v) = df(u) + df(v)$.

vi) $df(\lambda\,u) = \lambda\,df(u)$.

vii) If U is an open set of M^m then $d(f|_U) = (df)|_U$ [Example *iii*) in 4.6].

viii) If $h : M^m \to N^n$ is a differentiable map then $d(f \circ h) = (df) \circ h^{\mathrm{T}}$.

One says that df is the *differential* of the function f.

If f is a differentiable function on an open set U in $\mathbb{R}^m$ then one will have $df(x, u) = D(f)_x u$, $(x, u) \in U \times \mathbb{R}^m$.

4.8. *Local expressions.* – For each differential chart (U, φ) on a manifold M^m, $(U, \varphi^{\mathrm{T}})$ [$(p_M^{-1}(U), \varphi^{\mathrm{T}})$, resp.] is the differentiable vector chart that corresponds to the tangent bundle $\tau(M)$ [the differentiable chart that corresponds to the manifold $T(M)$, resp.].

Consequently, if $(y_1, \ldots, y_m)$ is the system of local coordinates that is defined by φ on the open set U then $(y_1 \circ p_M, \ldots, y_m \circ p_M, dy_1, \ldots, dy_m)$ will be the system of local coordinates that is defined by φ^{T} on the open set $p_M^{-1}(U)$.

The local expression for the projection p_M in those systems of local coordinates will then be $y_i = y_i \circ p_M$, $1 \leq i \leq m$.

Let $h : M^m \to N^n$ be a differentiable map, and let $(z_1, \ldots, z_n)$ be a local coordinate system on an open set V in such that $h(U) \cap V \neq \emptyset$. If $z_i = h_i(z_1, \ldots, z_n)$, $i = 1, \ldots, n$ is the local expression for h then the local expression for h^{T} will be:

$$\begin{aligned} z_i \circ p_N &= h_i(y_1 \circ p_M, \ldots, y_m \circ p_M) \\ dz_i &= \sum_j \frac{\partial h_i}{\partial y_j}(y_1 \circ p_M, \ldots, y_m \circ p_M)\, dy_j . \end{aligned}$$

In particular:

If $N^n = \mathbb{R}$ then $dh = \sum_j \frac{\partial h_i}{\partial y_j} dy_j$,

If M^m is an interval in $\mathbb{R}$ then $h'(t) = (h_1'(t), \ldots, h_n'(t))$.

Exercise. – For any vector $v \in T_x(M)$, there exists a differentiable curve $c :]-\varepsilon, +\varepsilon[\to M^m$ such that $c(0) = x$ and $c'(0) = v$.

4.9. Definition. – *An m-dimensional differentiable manifold M^m is* ***orientable*** *if the bundle* $\mathbf{A}^m(\tau(M))$ *is (differentiably) trivial.*

A parallelizable manifold is therefore orientable.

An orientation on M is an orientation on its tangent bundle $\tau(M^m)$ (Chap. II, Def. 3.11).

When one has made a choice of orientation, one says that M^m is an *oriented manifold*.

Let M^m and N^n be two oriented differentiable manifolds. A diffeomorphism $h : M^m \to N^n$ is compatible with the orientations if that is true for the isomorphism $(h^{\mathrm{T}}, h) : \tau(M) \to \tau(N)$.

(Chap. II, Def. 3.13). One similarly defines the notations of diffeomorphism that preserve or reverse the orientation.

4.10. Theorem. – *Let M^m be an m-dimensional differentiable manifold. The following propositions are equivalent:*

i) *M^m is orientable.*

ii) *There exists a differential section with no zero of the bundle $\mathbf{A}^m(\tau(M))$ over M^m.*

iii) *There exists an atlas $\mathcal{A} = \{(U_i, \varphi_i)\}$ on M^m such that for any change of chart $\varphi_j\,\varphi_i^{-1}$ and for any point $x \in U_j \cap U_i$, the Jacobian $\det[D(\varphi_j\varphi_i^{-1})_{\varphi_i(x)}]$ will be positive.*

The proof of that theorem is identical (up to the qualifier "differentiable") to the proofs of the analogous results in Chapter II, **§ 3**.

4.11. *Examples:*

i) The vector space $\mathbb{R}^m$ is an orientable differentiable manifold. One always orients it by the choice of canonical orientation on each fiber $\{x\} \times \mathbb{R}^m$ of $T(\mathbb{R}^m) = \mathbb{R}^m \times \mathbb{R}^m$.

One proceeds similarly for the half-space H^m.

A diffeomorphism h of $\mathbb{R}^m$ (H^m, resp.) preserves the orientation if its Jacobian is positive; otherwise, it will reverse it.

ii) The sphere S^2 is an orientable differentiable manifold.

iii) An open set U of a orientable differentiable manifold M^m is orientable.

If M^m is oriented then an orientation on M^m will determine an orientation on U. One always endows U with that induced orientation.

iv) The tangent bundle $T(M)$ to a differentiable manifold M^m is orientable. Indeed, if $\mathcal{A} = \{(U_i, \varphi_i)\}$ is an atlas for M^m then $\mathcal{B} = \{(p_M^{-1}(U), \varphi_i^{\mathrm{T}}\}$ (see 4.8) will be an atlas for the manifold $T(M)$, and one will have:

$$\det[D(\varphi_j^{\mathrm{T}} \circ (\varphi_i^{\mathrm{T}})^{-1})_{\varphi_i^{\mathrm{T}}(x)}] = (\det[D(\varphi_j\varphi_i^{-1})_{\varphi_i\, p_M(x)}])^2 .$$

4.12. Proposition. – *If M^m is an orientable differentiable manifold then its boundary ∂M^m will also be orientable. Furthermore, an orientation on M^m will determine an orientation on ∂M^m.*

Proof: An orientation on M^m will permit one to choose an atlas $\mathcal{A} = \{(U_i, \varphi_i)\}$ that has the property *iii*) in Theorem 4.10 (Chap. II, Th. 3.14). Therefore, let $\mathcal{B}$ be the set of charts (V, φ) on ∂M^m for which there exists a chart (U, φ) of $\mathcal{A}$ such that $V = U \cap \partial M^m$ and $\psi = \varphi|_V$ (Prop. 1.10).

Let (U_i, φ_i) and (U_j, φ_j) be two charts on $\mathcal{A}$ such $U_i \cap U_j \cap \partial M^m \neq \emptyset$. If one writes:

$$\varphi_j \varphi_i^{-1}(x_1, \ldots, x_m) = (h_1 (x_1, \ldots, x_m), \ldots, h_1 (x_1, \ldots, x_m))$$

then one will have:

$$\frac{\partial h_m}{\partial x_i}(x_1, \ldots, x_{m-1}, 0) = 0 \qquad \text{for} \qquad 1 \leq i \leq m-1,$$

$$\frac{\partial h_m}{\partial x_m}(x_1, \ldots, x_{m-1}, 0) = a\,(x_1, \ldots, x_{m-1}) > 0\,.$$

Consequently, at a point $x \in V_j \cap V_i = U_j \cap U_i \cap \partial M^m$, one will have:

$$\det \cdot D(\psi_j \psi_i^{-1})_{\psi_i(x)} = \frac{1}{a(\varphi_i(x))} \det[D(\varphi_j \varphi_i^{-1})_{\varphi_i(x)}] > 0\,.$$

The atlas $\mathcal{B}$ of ∂M^m will also possess the property *iii*) of Theorem 4.9 then, and it will determine (Chapter II, Th. 3.14) an orientation of ∂M^m.

Q. E. D.

4.13. *Convention.* – If M^m has even (odd, resp.) dimension then one endows ∂M^m with the orientation that was determined in the proof of Theorem 4.11 (the opposite orientation to that orientation, resp.).

One finds the justification for that choice in Stokes's formula (Chap. IV, Th. 4.6).

§ 5. – Rank of a map. Submanifolds.

5.1. Definition. – *Let M^m and N^n be two differentiable manifolds, and let h be a differentiable map from M^m into N^n. The* ***rank*** *of h at a point x of M^m is the rank of the linear map $h_x^T : T_x(M) \to T_{h(x)}(M)$ that is the restriction of h^T to $T_x(M)$.*

The rank of h at a point x is therefore less than the dimensions of M^m and N^n. One says that h has *maximum rank* at x if its rank if equal to the smaller of those dimensions.

One also says that h is:

– an *immersion* if m is less than n and h has rank m at each point of M^m,
– a *submersion* if m is greater than n and h has rank n at each point of M^m.

5.2. Lemma. – *Let* (U, φ) *be a differentiable chart on* M^m *and let* (V, ψ) *be a differentiable chart on* N^n *such that* $h(U) \cap V \neq \varnothing$. *The rank of h at a point* $x \in h^{-1}(V) \cap U$ *is the rank of the Jacobian matrix* $D(\psi\, h\, \varphi^{-1})$ *at* $\varphi(x)$.

5.3. Corollary. – *The rank of a differentiable map is a lower-semicontinuous positive function with integer values*.

In other words, if h has rank p at a point x then it will have rank at least p at any point that is sufficiently-close to x.

We assume the following classical theorem (J. Dieudonné, [**5**]):

5.4. Theorem. (the rank theorem). – *Let* M^m *and* N^n *be two differentiable manifolds without boundary, and let* $h : M^m \to N^n$ *be a differentiable map of constant rank p. For any point x of* M^m, *there exists a local system* $(y_1, \ldots, y_m)$ *of differentiable coordinates on an open neighborhood of x, and a local system* $(z_1, \ldots, z_m)$ *of differentiable coordinates on an open neighborhood of h* (x) *such that the local expression for h is:*

$$\begin{aligned} z_i &= y_i \qquad &&\text{for} \qquad 1 \leq i \leq p\,, \\ z_i &= 0 \qquad &&\text{for} \qquad p < i \leq m\,. \end{aligned}$$

5.5 Definition. – *Let* $f_1, \ldots, f_p$ *be p differentiable functions on a differentiable manifold* M^m. *Those functions are* ***independent*** *at a point y of* M^m *if the map* $z \mapsto (f_1(z), \ldots, f_p(z))$ *of* M^m *into* $\mathbb{R}^p$ *has rank p at y*.

The functions $f_1, \ldots, f_p$ are then independent of the neighborhood of y and one will have $p \leq m$.

If $p = m$ then the functions $f_1, \ldots, f_p$ will form a local system of differentiable coordinates on a neighborhood of y.

5.6. Proposition. – *Let* $f_1, \ldots, f_p$ *be p differentiable functions on a differentiable manifold* M^m *that are independent at a point y of* M^m. *There exist* $m - p$ *differentiable functions* $f_{p+1}, \ldots, f_m$ *on* M^m *such that* $(f_1, \ldots, f_m)$ *is a local system of differentiable coordinates on a neighborhood of y*.

That proposition is an immediate consequence of the rank theorem (5.4) and proposition 2.11.

5.7. Corollary. – *In order for the functions $f_1, \ldots, f_p$ to be independent at a point y of M^m, it is necessary and sufficient that the differentials $df_1, \ldots, df_p$ should induce independent linear forms on $T_y(M^m)$.*

5.8. Definition. – *If M^m is an m-dimensional differentiable manifold without boundary. An n-dimensional* ***submanifold****, $n \leq m$, (or codimension $m - n$) of M^m is a subspace N of M^m that has the following property:*

For any point x of N, there exists a local system $(y_1, \ldots, y_m)$ of differentiable coordinates on an open neighborhood U of x in M^m such that $U \cap N$ is the subspace that is defined by $y_{n+1} = \ldots = y_m = 0$ or by $y_{n+1} = \ldots = y_m = 0$ and $y_n \geq 0$.

5.9. Proposition. – *Let N be an n-dimensional submanifold of a differentiable manifold without boundary M^m. There exists one and only one structure of an n-dimensional differentiable manifold for which the injection $i : N \to M$ is an immersion.*

In what follows, one will always endow a submanifold of a differentiable manifold with that structure of a differentiable manifold.

Proof: With the notations of 5.8, the local coordinates $y_1, \ldots, y_n$ define a chart on the open set $U \cap N$ of N (which is therefore a topological manifold of dimension n), and the set of charts, thus-defined, will determine a differentiable manifold structure on N for which the injection $i: N \to M$ will be an immersion.

If there exists a second differentiable structure on N for which i is also an immersion then one can deduce from the rank theorem that the identity map on N is a diffeomorphism, so those two structures are identical.

Q. E. D.

5.10. Corollary. – *Let M^m be an m-dimensional differentiable manifold without boundary, and let N^n be an n-dimensional differentiable manifold, $n \leq m$, and let h be an injective immersion of N^n into M^m such that the image $h(N^n)$ is an (n-dimensional) submanifold of M^m. Therefore, h is a diffeomorphism of N^n into the submanifold $h(N^n)$.*

Under those conditions, one says that h is an *embedding* of N^n into M^m.

Exercise. – A proper injective immersion is an embedding. (In particular, an injective immersion of a compact manifold is an embedding.)

5.11. *Examples:*

i) An interval in $\mathbb{R}$ is a submanifold of $\mathbb{R}$.

ii) The injection of H^m into $\mathbb{R}^m$ is an embedding.

iii) If *U* is an open subset of a differentiable manifold without boundary M^m then the injection of *U* into M^m will be an embedding.

iv) The injection of the sphere into $\mathbb{R}^3$ is an embedding (Example 1.9).

5.12. Definition. – *Let M^m and N^n be two differentiable manifolds of dimensions m and n, resp. If h is a differentiable map of M^m into N^n then a* ***regular value*** *of h is a point c in N^n such that h has rank n at each point $h^{-1}(c)$.*

In particular, if $h^{-1}(c)$ is vacuous then *c* will be a regular value of *h*.

One immediately deduces the following two propositions from the rank theorem:

5.13. Proposition. – *Let M^m and N^n be two differentiable manifolds without boundary of dimensions m and n, resp., with $m \geq n$. Let $h : M^m \to N^n$ be a differentiable map, and let c be a regular value of h, The subspace $h^{-1}(c)$ (if it is non-vacuous) is an n-codimensional submanifold of M^m.*

5.14. Proposition. – *Let M^m be a differentiable manifold without boundary and let h be a differentiable function on M^m. For any regular value c of $h(M^m)$, $h^{-1}(]-\infty, c])$ is a submanifold M^m that has the submanifold $h^{-1}(c)$ for its boundary.*

5.15. *Examples:*

i) If $\eta = (E, p, B)$ is a differentiable vector bundle then the projection $p : E \to B$ is a submersion. Consequently, every fiber $p^{-1}(x)$, $x \in B$, is a submanifold of *E*.

ii) The function $h = \sum_{1 \leq i \leq m} x_i^2$ is a differentiable function on $\mathbb{R}^m$ with maximum rank at every point $x \neq 0$. The set $D^m = h^{-1}([0,1])$ is therefore a compact submanifold of $\mathbb{R}^m$: D^m is the unit ball of dimension *m*, and its boundary S^{m-1} is the unit sphere of dimension $m - 1$.

§ 6. – Vector fields.

6.1. Definition. – *Let M^m be a differentiable manifold. A* ***vector field*** *on M^m is a differentiable section of that tangent bundle $\tau(M)$ over M^m.*

Let $\mathcal{A} = \{(U_i, \varphi_i)\}$ be an atlas for M^m. A vector field X on M^m is determined (Chap. II, **§ 2.7**) by a family of differentiable maps $X_i : U_i \to \mathbb{R}^m$ such that:

$$X_j(y) = [D(\varphi_j\, \varphi_i^{-1})_{\varphi_i(y)}]\, X_i(y) \qquad \text{for any} \qquad y \in U_j \cap U_i .$$

The set $\mathcal{T}(M)$ of vector fields on M^m is a module over the algebra $\mathcal{D}(M)$ of differentiable functions on M^m (Cor. 2.7, Chap. II). One also has the notion of a locally-finite family in $\mathcal{T}(M)$.

If M^m is parallelizable then $\mathcal{T}(M)$ will be a free module that has a basis of m elements.

If U is an open subset of M^m then the restriction to U of a vector field on M^m will be a vector field on U. The map $X \mapsto X|_U$ is then a homomorphism of the $\mathcal{D}(M)$-module $\mathcal{T}(M)$ into the $\mathcal{D}(U)$-module $\mathcal{T}(U)$. [It verifies $(f X)|_U = (f|_U)\,(X|_U)$.]

Exercise. – Let M^m and N^n be two differentiable manifolds such that $\partial M^m = \varnothing$. A vector field on $M^m \times N^n$ is written $X + Y$, in which X (Y, resp.) is a differentiable map from $M^m \times N^n$ into $T(M^m)$ [$T(N^n)$, resp.] such that $p_M \circ X(y, z) = y$ [$p_N \circ Y(y, z)$, resp.].

6.2. Proposition. – *Let M^m and N^m be two differentiable manifolds, and let h be a diffeomorphism of M^m onto N^n. If X is a vector field on M^m then $Y = h^{\mathrm{T}} X\, h^{-1}$ will be a vector field on N^m.*

Indeed, Y is a differentiable map of N^m into $T(N)$ such that:

$$p_N\, Y(y) = p_N\, h^{\mathrm{T}} X\, h^{-1}(y) = h\, p_M X\, h^{-1}(y) = h\, h^{-1}(y) = y \quad \text{for any } y \in N^m .$$

6.3. – Let X be a vector field on a differentiable manifold M^m. For any function $f \in \mathcal{D}(M)$, $X \cdot f : y \mapsto df(X(y))$ is a differentiable function on M^m: $X \cdot f$ is the *derivative of f with respect to X.*

The following properties are then verified:

i) If f is constant function on an open subset of M^m then $X \cdot f(y) = 0$ for any $y \in U$.

ii) $X \cdot (f + g) = X \cdot f + X \cdot g$.

iii) $X \cdot (f\, g) = (X \cdot f)\, g + f\, (X \cdot g)$,

and consequently:

iv) $X \cdot (a\, g) = a\, (X \cdot f)$, $a \in \mathbb{R}$.

One has, moreover:

v) $(X + Y) \cdot f = X \cdot f + Y \cdot f$.

vi) $(g\, X) \cdot f = g\, (X \cdot f)$.

vii) If U is an open set in M^m then $(X \cdot f)\big|_U = (X\,|_U) \cdot (f\,|_U)$.

viii) Let $h : M^m \to N^n$ be a differentiable map, let X be a vector field on M^m, and let Y be a vector field on N^n such that $h^{\mathrm{T}} X = Y\, h$. One will then have:

$$X \cdot (f \circ h) = (Y \cdot f) \circ h$$

for any $f \in \mathcal{D}\,(N)$. (Indeed:

$$X \cdot (f \circ h) = d\,(f \circ h)\, X = df\,(Y\, h) = (d\, f(Y)) \circ h = (Y \cdot f) \circ h\,.)$$

6.4. Definition. – *Let M^m be a differentiable manifold. A* ***derivation*** *of the algebra $\mathcal{D}\,(M)$ is a map $D : \mathcal{D}\,(M) \to \mathcal{D}\,(M)$ that has the following properties:*

i) $D\,(f + g) = D\,(f) + D\,(g)$.

ii) $D\,(f\, g) = D\,(f)\, g + f\, D\,(g)$.

iii) $D\,(f) = 0$, *if f is a constant function on M^m* .

Moreover, a derivation will then verify:

iv) $D\,(a\, f) = a\, D\,(f)$, $a \in \mathbb{R}$. As a result, it will be an endomorphism of the vector space $\mathcal{D}\,(M)$.

The set of derivations of the algebra $\mathcal{D}(M)$ is a module over $\mathcal{D}(M)$. A vector field X on M^m will determine a derivation $f \mapsto X \cdot f$ of $\mathcal{D}(M)$. That correspondence is linear and compatible (in a sense that is easy to explain) with diffeomorphisms [see 6.3 *iii*)].

Indeed, one has:

6.5. Theorem. – *The correspondence that associates a vector field X on M^m with the derivation $f \mapsto X \cdot f$ of $\mathcal{D}(M)$ is an isomorphism.*

The proof of that theorem utilizes the following lemma:

6.6. Lemma. – *Let f be a differentiable function on $\mathbb{R}^m$. There exist m differentiable functions $g_1, \ldots, g_m$ on $\mathbb{R}^m$ that have the following properties:*

i) $f(x) = f(0) + \sum_i x_i\, g_i(x)$.

ii) $g_i(0) = \dfrac{\partial f}{\partial x_i}(0)$.

Proof: One can write:

$$f(x) = f(0) + \int_0^1 \frac{df(t\,x)}{dt}\,dt = f(0) + \sum_i x_i \int_0^1 \frac{\partial f(t\,x)}{\partial x_i}\,dt .$$

The functions $g_i(x) = \displaystyle\int_0^1 \frac{\partial f(t\,x)}{\partial x_i}\,dt$ will then have the desired properties.

Q. E. D.

6.7. Lemma. – *The correspondence of Theorem 6.5 is an isomorphism for $M^m = \mathbb{R}^m$.*

Proof: Since $\tau(\mathbb{R}^m)$ is a trivial vector bundle, a vector field X on $\mathbb{R}^m$ will be determined by a differentiable map $x \mapsto (a_1(x), \ldots, a_m(x))$ of $\mathbb{R}^m$ to itself. One will then have $a_i = X \cdot x_i$, $1 \leq i \leq m$, and $X \cdot f = \sum_i a_i \dfrac{\partial f}{\partial x_i}$.

Now, let D be a derivation of the algebra $\mathcal{D}(\mathbb{R}^m)$, and let X be the vector field on $\mathbb{R}^m$ whose components are $a_i = D(x_i)$.

For a differentiable function f on $\mathbb{R}^m$ and any point y of $\mathbb{R}^m$, there exist m differentiable functions $g_1, \ldots, g_m$ on $\mathbb{R}^m$ such that:

$$f(z) = f(y) + \sum_i [x_i(z) - x_i(y)]\, g_i(z) ,$$
$$g_i(y) = \frac{\partial f}{\partial x_i}(y) .$$

Consequently:

$$D(f)(y) = \sum_i D(x_i)(y)\, g_i(y) = (X \cdot f)(y) .$$

The derivation that is associated with the vector field X is therefore D. That proves that the correspondence is bijective.

Q. E. D.

One likewise proves that:

6.8. Lemma. – *The correspondence in Theorem 6.5 is an isomorphism for* $M^m = H^m$.

6.9. Lemma. – *If D be a derivation of $\mathcal{D}(M)$. If f and g are two differential functions on M^m that are equal to each other on an open subset of M^m then the functions $D(f)$ and $D(g)$ will also be equal on U.*

Proof: Let y be a point of U and let θ be a differentiable function on M^m that is equal to 0 outside of U and to 1 on a neighborhood of y (Lemma 2.10). One will then have:

$$f - g = (f - g)(1 - \theta)$$

and:

$$D(f - g)(y) = D(f - g)(y)(1 - \theta(y)) + (f(y) - g(y))\, D(1 - \theta)(y) = 0 .$$

Q. E. D.

6.10. Lemma. – *Let D be a derivation of $\mathcal{D}(M)$, and let U be an open set of M^m. There exists one and only one derivation D_U of $\mathcal{D}(U)$ such that $D_U(f|U) = D(f)|_U$ for any $f \in \mathcal{D}(M)$.*

Proof: Let f be a differentiable function on U, and let y be a point of U. There exists a differentiable function g on M^m such that $g = f$ on a neighborhood of y (Prop. 2.11).

One then sets $D_U(f)(y) = D(g)(y)$. That definition is independent of the choice of g and determines a derivation D_U of $\mathcal{D}(U)$ that has the desired properties.

Q. E. D.

Proof of Theorem 6.5.: Let D be a derivation of $\mathcal{D}(M)$ and let $\mathcal{V} = (V_k)$ be a locally-finite open covering of M^m that has the properties required in Lemma 2.13. Let (θ_k) be a differentiable partition of unity subordinate to $\mathcal{V}$. The derivation $\theta_k D$ determines a derivation D_k of $\mathcal{D}(V_k)$.

Let X_k be the vector field on V_k that corresponds to the derivation D_k : X_k is zero outside of the support of θ_k. It will then extend by zeroes from a vector field on $M^m - V_k$ to a vector field on M^m, which is again denoted by X_k.

The (locally-finite) sum $X = \sum_k X_k$ is therefore a vector field on M^m such that $X \cdot f = \sum_k X_k \cdot f = \sum_k D_k(f) = D(f)$ for any function $f \in \mathcal{D}(M)$.

That field X is perfectly determined by the derivation D, which shows that the correspondence in Theorem 6.5 is an isomorphism.

Q. E. D.

In what follows, one will identify $\mathcal{T}(M)$ with the module of derivations of $\mathcal{D}(M)$ by means of that isomorphism.

The composition of two derivations is not generally a derivation. Meanwhile:

6.11. Lemma. – *Let X and Y be two vector fields on a differentiable manifold M^m. The map f $\to X \cdot (Y \cdot f) - Y \cdot (X \cdot f)$ is a derivation of the algebra $\mathcal{D}(M)$.*

That lemma (whose verification is a simple exercise) justifies the following definition:

6.12. Definition. – *Let X and Y be two vector fields on a differentiable manifold M^m. The* ***Lie bracket*** *of X and Y is the vector field $[X, Y] = XY - YX$.*

If U is an open set of M^m then one will have $[X, Y]|_U = [X|_U, Y|_U]$ [see 6.3, *iii*)].

6.13. Proposition. – *The Lie brackets have the following properties:*

i) $[X, Y + Z] = [X, Y] + [X, Z]$.

ii) $[X, f Y] = (X \cdot f)Y + f[X, Y], f \in \mathcal{D}(M^m)$.

iii) $[X, Y] = -[Y, X]$.

iv) $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$.

The latter equality is called the *Jacobi identity.*

The proof of Proposition 6.13 presents no difficulty.

6.14. Proposition. – *Let $h : M^m \to N^n$ be a differentiable map, and let X_1, X_2 be two vector fields on M^m, while Y_1, Y_2 are two vector fields on N^n such that $h^{\mathrm{T}} X_i = Y_i\, h$, $i = 1, 2$. One will then have $h^{\mathrm{T}}[X_1, X_2] = [Y_1, Y_2]\, h$.*

Proof: It suffices to show that for any $f \in \mathcal{D}(N^n)$, one will have:

$$df \circ (h^{\mathrm{T}}[X_1, X_2]) = df \circ ([Y_1, Y_2]\, h) .$$

Now:

$$\begin{aligned} df \circ (h^{\mathrm{T}}[X_1, X_2]) &= (df \circ h^{\mathrm{T}})\ ([X_1, X_2] \\ &= d(f \circ h)\ ([X_1, X_2] \\ &= X_1 \cdot (X_2 \cdot (f \circ h)) - X_2 \cdot (X_1 \cdot (f \circ h)) \\ &= X_1 \cdot ((Y_2 \cdot f) \circ h) - X_2 \cdot ((Y_1 \cdot f) \circ h) \qquad [6.3., \textit{viii})] \\ &= (Y_1 \cdot ((Y_2 \cdot f) \circ h - (Y_2 \cdot ((Y_1 \cdot f)) \circ h \\ &= ([Y_1, Y_2] \cdot f) \circ h \\ &= (df\,[Y_1, Y_2]) \circ h = df \circ ([Y_1, Y_2]h) . \end{aligned}$$

Q. E. D.

6.15. *Local expressions.* – Let ($y_1, \ldots, y_m$) be a local system of differentiable coordinates on an open subset U of M^m, and let X be a vector field on M^m.

If $a_i = dy_i\ (X \mid_U)$ then one says that $\sum_i a_i \frac{\partial}{\partial y_i}$ is the local expression for X (in the local coordinates $y_1, \ldots, y_m$). Under those conditions, for any function $f \in \mathcal{D}(M)$, one will have:

$$(X \cdot f)|_U = \sum_i a_i \frac{\partial f}{\partial y_i} .$$

Let Y be a second vector field on M^m whose local expression is $\sum_i b_i \frac{\partial}{\partial y_i}$. The local expression for the Lie bracket $[X, Y]$ will then be:

$$\sum_{i,j} \left(a_i \frac{\partial b_j}{\partial y_i} - b_i \frac{\partial a_j}{\partial y_i} \right) \frac{\partial}{\partial y_j} .$$

In particular, $\left[\frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_j} \right] = 0$.

6.16. Definition. – *Let N be a submanifold of a differentiable manifold without boundary* M^m, *and let i be the injection of N into* M^m. *A vector field X on* M^m *is* ***tangent to the submanifold*** *N if* $X(y) \in i^{\mathrm{T}}(T_y(N))$ *for any point* $y \in N$.

6.17. Proposition. – *Let Y be a vector field on a differentiable manifold* M^m *that is tangent to a submanifold N of* M^m. *There exists one and only one vector field X on N such that* $i^{\mathrm{T}} X = Y\, i$.

Proof: Since i^{T} is injective, for any point x of N, there exists one and only one tangent vector $X(x) \in T_x(N)$ such that $i^{\mathrm{T}} X(x) = Y(x)$. It remains to be verified that the map $x \mapsto X(x)$ is differentiable.

Let x be a point of N. There exists a local system $(y_1, \ldots, y_m)$ of differentiable coordinates on an open neighborhood U of x in M^m such that $U \cap N$ is the subspace that is defined by $y_{n+1} = \ldots = y_m = 0$ (and possibly $y_n \geq 0$).

If $Y = \sum_{i=1}^{m} a_i \frac{\partial}{\partial y_i}$ is the local expression for Y in U then one will have:

$$a_i(y_1, \ldots, y_m, 0, \ldots, 0) = 0 \qquad \text{for} \qquad i > n.$$

Consequently, $X \,|_{U \cap N} = \sum_{i=1}^{n} a_i(y_1, \ldots, y_n, 0, \ldots, 0) \frac{\partial}{\partial y_i}$ is differentiable.

Q. E. D.

More generally, one further proves the following result analogously (by using the rank theorem):

6.18. Proposition. – *Let* M^m *and* N^n *be two differentiable manifolds, let h be an injective immersion of* N^n *into* M^m, *and let X be a vector field on* M^m *such that for any point* $y \in N^n$, *one has* $X(h(y)) \in h^{\mathrm{T}}(T_y(N))$. *There will then exist one and only one vector field Y on* N^n *such that* $h^{\mathrm{T}} Y = X\, h$.

6.19. Corollary. – *Let N be a submanifold of a differentiable manifold without boundary* M^m. *If X and Y are two vector fields on* M^m *that are tangent to N then their Lie bracket* $[X, Y]$ *will also be tangent to N.*

6.20. Proposition. – *Let* M^m *be a differentiable manifold without boundary, and let* $h = (h_1, \ldots, h_n)$ *be a differentiable map of* M^m *into* $\mathbb{R}^n$, *while c is a regular value of h such that* $N = h^{-1}(c) \neq \emptyset$. *In order for a vector field X on* M^m *to be tangent to N, it is necessary and sufficient that one should have* $X \cdot h_1 = \ldots = X \cdot h_n = 0$ *on N.*

Indeed, for any point x of N, $T_x(N)$ is identified with the kernel of h_x^{T}.

§ 7. – Differential forms.

7.1. Definition. – *A **differential form of degree** p on a differentiable manifold* M^m *is a differentiable section of bundle* $\mathbf{A}^p$ $(\tau(M))$ *of exterior p-forms on* $\tau(M)$.

One also says that a differential form of degree 1 is a *Pfaff form* on M^m.

Any differential form of degree $p > m$ is zero.

The set $\Lambda^p(M)$ of differential forms of degree p on M^m is a module over the algebra $\mathcal{D}(M)$ of differentiable functions on M^m. One has the notion of a locally-finite family in $\Lambda^p(M)$, just as one does in $\mathcal{D}(M)$ and $\mathcal{T}(M)$,

If U is an open subset of M^m then the restriction to U of a differential form of degree p on M^m will be a differential form of degree p on U. The map $\alpha \mapsto \alpha_{|U}$ is then a homomorphism of the $\mathcal{D}(M)$-module $\Lambda^p(M)$ into the $\mathcal{D}(U)$-module $\Lambda^p(U)$.

7.2. Proposition. – *Let* M^m *be a differentiable manifold and let* $\varepsilon_p = (D^p, \pi, M^m)$ *be the Whitney sum of p exemplars of the tangent bundle* $\tau(M)$. *The module* $\Lambda^p(M)$ *of differential forms of degree p on* M^m *is isomorphic to the module of differentiable functions* $\sigma : D^p \to \mathbb{R}$ *whose restriction to each fiber* $(T_y(M))^p$, $y \in M^m$ *is an exterior p-form on* $T_y(M)$.

[The structure of $\mathcal{D}(M)$-module on $\mathcal{D}(D)$ is induced by the homomorphism $\pi^* : \mathcal{D}(M) \to \mathcal{D}(D)$.]

The proof of that proposition is analogous to that of Proposition 3.4 in Chapter II, up to the qualification that everything must be differentiable.

As in the continuous case (Chap. II, Remark 3.5), that isomorphism is compatible with restrictions.

In what follows, one denotes a differential form of degree p **on** M^m **and the corresponding differentiable function on** D^p **by the same symbol.**

7.3. Corollary. – *The differential df of a differentiable function* $f \in \mathcal{D}(M)$ *is a Pfaff form on* M^m.

Indeed, the Proposition 7.2 permits one to identify the module $\Lambda^1(M)$ with the module of differentiable functions on $T(M)$ whose restriction to each fiber $T_x(M)$ is linear.

7.4. *Local expression.* – Let (U, φ) be a differentiable chart on a manifold M^m, and let $(y_1, \ldots, y_m)$ be the system of local coordinates that is defined by φ on the open set U.

Proposition 7.2 then permits one to interpret the functions:

$$\left(y_1 \circ q_M, \ldots, y_m \circ q_M, \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_m}\right)$$

as a local system of coordinates on an open set $q_M^{-1}(U)$ of $T^*(M)$. The local expression for the projection q_M is then:

$$y_i = y_i \circ q_M, \qquad i = 1, \ldots, m.$$

7.5. – Let α be a differential form of degree p on a manifold M^m. If $X_1, \ldots, X_p$ are p vector field on M^m then:

$$\alpha(X_1, \ldots, X_p) : x \mapsto \alpha(X_1(x), \ldots, X_p(x))$$

will be a differentiable function on M^m.

One then associates α with an exterior p-form on $\mathcal{D}(M)$-module $\mathcal{T}(M)$, and that correspondence is compatible with the restrictions.

Indeed, one has:

7.6. Theorem. – *The correspondence that associates a differential form α of degree p on M^m with the exterior p-form $(X_1, \ldots, X_p) \mapsto \alpha(X_1, \ldots, X_p)$ on the module $\mathcal{T}(M)$ is an isomorphism.*

The proof of that theorem utilizes the following Lemma:

7.7. Lemma. – *The correspondence in Theorem 7.6 is an isomorphism when $M^m = \mathbb{R}^m$ or $M^m = H^m$.*

Proof: The proofs in both cases are analogous, so one will suppose that $M^m = \mathbb{R}^m$.

The vector fields $\partial / \partial x_1, \ldots, \partial / \partial x_m$ form a basis for $\mathcal{T}(\mathbb{R}^m)$ over $\mathcal{D}(\mathbb{R}^m)$. The Pfaff forms $dx_1, \ldots, dx_m$ form a basis of $\Lambda^1(\mathbb{R}^m)$ over $\mathcal{D}(\mathbb{R}^m)$, and one has $dx_i\left(\frac{\partial}{\partial x_j}\right) = \delta_{ij}$. The correspondence in 7.6. thus identifies the basis (dx_i) for $\Lambda^1(\mathbb{R}^m)$ with the dual basis to the basis $\left(\frac{\partial}{\partial x_j}\right)$ on $\mathcal{T}(\mathbb{R}^m)$. As a result, it will be an isomorphism when $p = 1$.

More generally, a differential form α of degree p on $\mathbb{R}^m$ is determined by its values $a_{i_1 \cdots i_p} = \alpha\left(\frac{\partial}{\partial x_{i_1}}, \ldots, \frac{\partial}{\partial x_{i_p}}\right) \in \mathcal{D}(\mathbb{R}^m)$, in which $1 \leq i_1 < \ldots < i_p \leq m$ (see 7.2). Consequently, the correspondence of 7.6 that associates α with the exterior p-form $\sum a_{i_1 \cdots i_p} dx_{i_1} \wedge \cdots \wedge dx_{i_p}$ [here, one identifies $\Lambda^1(\mathbb{R}^m)$ with $\mathcal{T}(\mathbb{R}^m)^*$] is an isomorphism.

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7.8. Lemma. – *Let a be an exterior p-form on $\mathcal{T}(M)$ and let U be an open subset of M^m. There exists one and only one exterior p-form α_{ij} on $\mathcal{T}(U)$ such that:*

$$\alpha_U(X_{1\mid U}, \ldots, X_{p\mid U}) = \alpha(X_1, \ldots, X_p)$$

for all $X_1, \ldots, X_p \in \mathcal{T}(M)$.

The proof of that Lemma, and that of Theorem 7.6, are now the same in appearance as those of Lemma 6.10 and Theorem 6.5.

In what follows, one will identify $\Lambda^p(M)$ with the module of exterior p-forms on $\mathcal{T}(M)$, and $\Lambda(M) = \sum_{p \geq 0} \Lambda^p(M)$ [$\Lambda^0(M) = \mathcal{D}(M)$], with the algebra of exterior forms on $\mathcal{T}(M)$: $\Lambda(M)$ is *the algebra differential forms on* M^m.

7.9. Proposition. – *Let $f_1, \ldots, f_p$ be p differentiable functions on a differentiable manifold M^m. In order for $f_1, \ldots, f_p$ to be independent at a point y of M^m, it is necessary and sufficient that the form $\alpha = df_1 \wedge \ldots \wedge df_p$ should not be zero at y.*

The result is an immediate consequence of Corollary 5.7.

7.10. – Let $h : M^m \to N^n$ be a differentiable map and let α be a differential form of degree p, with $p > 0$ on N^n. The map:

$$h^*\alpha : (v_1, \ldots, v_p) \mapsto \alpha(h^T v_1, \ldots, h^T v_p), \qquad (v_1, \ldots, v_p) \in D^p$$

determines a differential form of degree p on M^m (Prop. 7.2). That differential form is characterized by:

$$(h^*\alpha(X_1, \ldots, X_p))(x) \mapsto \alpha(h^T X_1(x), \ldots, h^T X_p(x)), \quad X_1, \ldots, X_p \in \mathcal{T}(M).$$

One says that is $h^*\alpha$ is the *reciprocal image form* of α under h.

For a differential form of degree 0, i.e., for a function $f \in \mathcal{D}(M)$, one sets $h^* f = f \circ h$ (see 2.8).

7.11. Proposition. – *Let $h : M^m \to N^n$ be a differentiable map, and let f be a differentiable function on N^n. One has $h^*(df) = d(h^* f)$.*

Indeed [see 4.7, *viii*)]:

$$h^*(df) = (df) \circ h^{\mathrm{T}} = d(h^* f) .$$

7.12. Proposition. – *Let h: $M^m \to N^n$ be a differentiable map. The map $h^* : \Lambda(N) \to \Lambda(M)$ is a homomorphism of algebras.*

The verification is immediate.

If h is the identity map of M^m then h^* will be the identity isomorphism of $\Lambda(M)$. If $h : M^m \to N^n$ and $k : N^n \to V^p$ are differentiable maps then one will have $(k \circ h)^* = h^* \circ k^*$.

7.13. *Local expression.* – Let h: $M^m \to N^n$ be a differentiable map whose local expression in the systems of differentiable coordinates $(y_1, \ldots, y_m)$ and $(z_1, \ldots, z_n)$ is $z_i = h_i(y_1, \ldots, y_m)$. If:

$$\sum_{1 \leq i_1 < \cdots < i_p \leq n} a_{i_1 \cdots i_p}(z_1, \ldots, z_n)\, dz_{i_1} \wedge \cdots \wedge dz_{i_p}$$

is the local expression for a differential form α of degree p on N^n then the local expression for $h^*\alpha$ will be:

$$\sum_{1 \leq i_1 < \cdots < i_p \leq n} a_{i_1 \cdots i_p}(h_1(z_1), \ldots, h_n(z_n))\, dh_{i_1} \wedge \cdots \wedge dh_{i_p} .$$

7.14. Definition. – *Let M^m be a differentiable manifold of dimension m. A* ***volume form*** *on M^m is a differential form ω of degree m on M^m such that $\omega(x) \neq 0$ for any $x \in M^m$.*

7.15. Proposition. – *In order for a differentiable manifold M^m to be orientable, it is necessary and sufficient that there should exist a volume form on M^m.*

That proposition is, in fact, only a partial reformulation of Theorem 4.10.

A volume form ω on M^m determines an orientation on $\tau(M)$ (Chap. II, 3.11). In that case, one also says that ω is an orientation on $\tau(M)$, and even more often, an orientation on M^m. In order for a diffeomorphism h of M^m to preserve (reverse, resp.) the orientation, it is necessary and sufficient that ω and $h^*\omega$ ($-\,h^*\omega$, resp.) should define the same orientation.

Appendix: Riemannian structures.

A.1. Definition. – *Let* $\eta = (E, \pi, B)$ *be a vector bundle with fiber F. A* ***Riemannian structure*** *on* η *is defined by a continuous function* $Q : E \to \mathbb{R}$ *such that the restriction of Q to each fiber* F_b *is a positive-definite quadratic form.*

If η is a differentiable vector bundle then one imposes the further condition on Q that it must be a differentiable function on E.

The given of Q is equivalent to that of a continuous (or differentiable) function $g : E \oplus E \to \mathbb{R}$, such that for any point $b \in B$, the restriction of g to the fiber $F_b \times F_b$ is the polar form of the restriction of Q to the fiber F_b of η. One says that g is a *Riemannian metric* of η.

When η is the tangent bundle to a differentiable manifold M^m, one also says that Q is a Riemannian structure on M^m and that M^m is a *Riemannian manifold*.

A.2. Theorem. – *If the space B is paracompact then there will exist a Riemannian structure on any vector bundle* $\eta = (E, p, B)$ *with base B.*

Proof: Let $\mathcal{U} = (U_\alpha)$ be a locally-finite open covering of B such that for any index α, there exists a trivialization Φ_α of $\eta|_{U_\alpha}$, and let (θ_α) be a partition of unity that is subordinate to the covering $\mathcal{U}$.

Let $q : F \to \mathbb{R}$, be a Riemannian structure on the fiber of η. For any α, $\hat{Q}_\alpha = q\, p_2\, \Phi_\alpha$ will be a Riemannian structure on $\eta|_{U_\alpha}$.

The function $\theta_\alpha \hat{Q}_\alpha$ then extends to a continuous function $Q_\alpha : E \to \mathbb{R}$, and one easily verifies that $Q = \sum_\alpha Q_\alpha$ is a Riemannian structure on η.

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When η is a differentiable vector bundle, one can similarly obtain a differentiable function Q: $E \to \mathbb{R}$.

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CHAPTER IV

DIFFERENTIAL AND INTEGRAL CALCULUS ON MANIFOLDS

From now on, all manifolds, charts, …, will be supposed to be differentiable. That qualification will be omitted in what follows when no possible confusion would arise.

§ 1. – Derivations and anti-derivations.

1.1. Definition. – *Let A be a unitary algebra over a commutative field K. A* ***gradation*** *of A is a denumerable family* $(A_p)_{p\in\mathbb{Z}}$ *of subspaces of A that has the following properties:*

i) *A is the direct sum of* A_p .

ii) $A_p\, A_q \in A_{p+q}$.

In particular, K is contained in A_0 (viz., it is identified with $K \cdot 1$).

One says that A is a *graded algebra* and that A_p is the set of *(homogeneous) elements of degree* p in A.

Let $(B_p)_{p\in\mathbb{Z}}$ be a gradation of an algebra B. A map $h : A \to B$ is compatible with the gradations if $h\,(A_p)$ is contained in B_p for every p.

Exercise. – The sub-modules A_0 and $\sum_p A_p$ are sub-algebras of A. Each A_p is a module over A_0.

1.2. Definition. – *A graded algebra* $A = \sum_{p\in\mathbb{Z}} A_p$ *is* ***anti-commutative*** *if one has:*

$$x_p\, x_q = (-1)^{pq}\, x_q\, x_p$$

for every element $x_p \in A_p$ *and every element* $x_q \in A_q$.

In this case, if A is an algebra with characteristic 2 then A will be commutative. If A is an algebra with characteristic not equal to 2 then any element of odd degree in A will have square zero.

The sub-algebra $\sum_p A_{2p}$ is contained in the center of A. It is then commutative.

1.3. *Example*. – For any manifold M^n, the algebra Λ (M) of differential forms on M^n will be a (real) graded anti-commutative algebra if one sets $\Lambda^p(M) = (0)$ for $p = 0$.

The homomorphism h^* : Λ (N) $\to$ Λ (M) that is associated with a differentiable map h : $M^m \to N^n$ is compatible with the gradations.

If M^m is a parallelizable manifold then the algebra Λ (M) will be generated by its elements of degree 0 and 1 (Chap. I, Th. 5.5). More generally:

1.4. Proposition. – *Any differential form on a manifold M^m is the sum of a locally-finite family of decomposable differential forms.*

Proof: Let $\mathcal{V} = (V_k)$ be a locally-finite open covering of M^m by parallelizable open sets.

Let α be differential form on M^m. Using a partition of unity subordinate to $\mathcal{V}$, one can find a locally-finite family (α_k) of differential forms that have the following properties:

i) The support F_k of α_k is contained in V_k.
ii) $\alpha = \sum \alpha_k$.

Each $\alpha_k |_U$ is a finite sum of decomposable forms that are zero outside of V_k . Consequently, α_k will be a finite sum of decomposable forms, which proves the proposition.

1.5. Corollary. – *If M^m is a compact manifold then the algebra* Λ (M) *will be generated by its elements of degree* 0 *and* 1.

In what follows, A will denote a graded anti-commutative algebra over a (commutative) field K.

1.6. Definition. – *Let p be an even integer. A* ***derivation*** *of degree q of A is an endomorphism d of the vector space A that has the properties:*

i) $dA_p \subset A_{p+q}$.
ii) *If* $x \in A_p$ *then* $d(xy) = (dx)\,y + (-1)^p\, x\,(dy)$.

Convention. – When A is the algebra of differential forms on a manifold M^m, one imposes the following additional condition upon a derivation (anti-derivation, resp.):

If (α_k) *is a locally-finite family of differential forms on* M^m *then* ($d\alpha_k$) *will also be a locally-finite family, and* $d\left(\sum_k \alpha_k\right) = \sum_k d\alpha_k$.

Exercises:

i) A derivation (anti-derivation, resp.) is zero on the base field.

ii) The set of derivations (anti-derivations, resp.) of A admits the structure of a module over the algebra A_0.

1.8. *Example.* – Let X be a vector field on a manifold M^m. Let $i_X : \Lambda(M) \to \Lambda(M)$ denote the interior product with X. i_X is an anti-derivation of degree – 1 on $\Lambda(M)$ (Chap. I, Prop. 6.3) that has the following properties:

i) $i_{X+Y} = i_X + i_Y$.

ii) $i_{fX} = f\, i_X$.

iii) Let $h : M^m \to N^n$ be a differentiable map, and let $X \in \mathcal{T}(M)$ and $Y \in \mathcal{T}(N)$ be two vector fields such that $h^T X = Y h$. For any form $\alpha \in \Lambda(N)$, one will have:

$$h^*(i_Y \alpha) = i_X(h^* \alpha) .$$

In particular, if U is an open set on M^m then $(i_X\, \alpha)|_U = (i_X|_U)\, \alpha|_U$.

1.9. Proposition. – *Let* d_1 (d_2, *resp.*) *be a derivation of degree* p_1 (p_2, *resp.*) *of* A*, and let* a_1 (a_2, *resp.*) *be an anti-derivation of degree* q_1 (q_2, *resp.*) *of* A*. One will then have:*

i) $a_1\, a_1$ *is a derivation of degree* $2q_1$ *of* A.

ii) $a_1\, a_2 + a_2\, a_1$ *is a derivation of degree* $q_1 + q_2$ *of* A.

iii) $[d_1, d_2] = d_1\, d_2 - d_2\, d_1$ *is a derivation of degree* $p_1 + p_2$ *of* A.

iv) $[a_1, d_1] = a_1\, d_1 - d_1\, a_1$ *is a derivation of degree* $p_1 + q_1$ *of* A.

1.10. Proposition. – *If* A *is generated by its elements of degree* 0 *and* 1 *then two derivations (anti-derivations, resp.) of* A *will be equal if and only if they coincide on* $A_0 \oplus A_1$.

The verification of those two propositions presents no difficulties.

1.11. Proposition. – *Let* M^m *be a manifold, and let* d *be a derivation (anti-derivation, resp.) of* $\Lambda(M)$*. For any open* U *of* M^m*, there exists one and only one derivation (anti-derivation, resp.)* d_U *of* $\Lambda(U)$ *such that* $(d\alpha)|_U = d_U(a|_U)$ *for any* $\alpha \in \Lambda(M)$.

One says that d_U is the *restriction* of d to the open set U.

The proof of that proposition is analogous to the proofs of Lemmas 6.9 and 6.10 in Chapter III.

1.12. Corollary. – *Two derivations (anti-derivations, resp.) of* Λ (M) *are equal if and only if they give the same values to* f *and* df *for any function* $f \in \mathcal{D}(M)$.

1.13. *Example.* – Let X be a vector field on M^m. The interior product i_X is characterized by the following relations:

i) $i_X f = 0$.
ii) $i_X\, df = X \cdot f, f \in \mathcal{D}(M)$.

§ 2. – Exterior differentiation.

2.1. Lemma. – *Let* α *be a differential form of degree* $p \geq 1$ *on a manifold* M^m :

$\boldsymbol{d}\alpha : (X_1, \ldots, X_{p+1}) \mapsto$

$$\sum_{i=1}^{p+1} (-1)^{i-1} X_i \cdot \alpha (X_1, \ldots, \hat{X}_i, \ldots, X_{p+1}) + \sum_{i<j} (-1)^{i+j} \alpha ([X_i, X_j] X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_{p+1})$$

(*in which the terms with "hats" are omitted) is a differential form of degree* $p + 1$ *on* M^m .

In particular, if α is a Pfaff form then:

$$\boldsymbol{d}\alpha (X, Y) = X \cdot \alpha (Y) - Y \cdot \alpha (X) - \alpha ([X, Y]) .$$

2.2. Lemma. – *Let* f *be a differentiable function on* M^m : $d\,(df) = 0$.

2.3. Lemma. – *Let* $f_1, \ldots, f_p$ *be differentiable functions on* M^m :

$$\boldsymbol{d}\,(g\, df_1 \wedge \ldots \wedge df_p) = dg \wedge df_1 \wedge \ldots \wedge df_p .$$

The verification of those three lemmas is a simple exercise in calculation.

Upon setting $\boldsymbol{d} = d$ on $\mathcal{D}(M)$, the maps $\boldsymbol{d}$ in Lemma permit one to define an endomorphism $\boldsymbol{d}$ of the vector space Λ (M) that has following properties:

i) $\boldsymbol{d}\Lambda^p(M) \subset \Lambda^{p+1}(M)$.

ii) If (α_k) is a locally-finite family of differential forms then $(\boldsymbol{d}\alpha_k)$ will also be a locally-finite family, and $\boldsymbol{d}\left(\sum_k \alpha_k\right) = \sum_k \boldsymbol{d}\alpha_k$.

iii) For any open subset U of M and any form $\alpha \in \Lambda(M)$:

$$(\boldsymbol{d}\alpha)|_U = \boldsymbol{d}(\alpha|_U) .$$

2.4. Proposition. – *The endomorphism* **d** *is an anti-derivation of degree* + 1 *on* $\Lambda(M)$.

Proof: Since **d** is compatible with the restrictions, it will suffice to prove that proposition when $M^m = \mathbb{R}^m$ and $M^m = H^m$.

In those cases, any differential form will be a finite sum of decomposable forms $f\, dx_{i_1} \wedge \ldots \wedge dx_{i_p}$. One can then restrict oneself to verifying the condition *ii*) of Definition 1.7 when $\alpha = f\, dx_{i_1} \wedge \ldots \wedge dx_{i_p}$ and $\beta = g\, dx_{j_1} \wedge \ldots \wedge dx_{j_q}$. One will then have (Lemma 2.3):

$$\boldsymbol{d}\alpha = df \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_p}$$

$$\boldsymbol{d}\beta = dg \wedge dx_{j_1} \wedge \ldots \wedge dx_{j_q}$$

$$\boldsymbol{d}(\alpha \wedge \beta) = (g\, df + f\, dg) \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_p} \wedge dx_{j_1} \wedge \ldots \wedge dx_{j_q}$$

$$(\boldsymbol{d}\alpha) \wedge \beta = g\, df \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_p} \wedge dx_{j_1} \wedge \ldots \wedge dx_{j_q}$$

$$\alpha \wedge (\boldsymbol{d}\beta) = (-1)^p\, g\, dg \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_p} \wedge dx_{j_1} \wedge \ldots \wedge dx_{j_q} .$$

Q. E. D.

2.5. Corollary. – *The anti-derivation* **d** *of* $\Lambda(M)$ *is characterized by the following relations:*

i) $\boldsymbol{d}f = df$.

ii) $\boldsymbol{d}(df) = 0, f \in \mathcal{D}(M)$.

2.6. Definition. – ***Exterior differentiation*** *on a manifold* M^m *is the anti-derivation* **d** *of degree* + 1 *of algebra* $\Lambda(M)$ *that is characterized by the following relations:*

i) $\boldsymbol{d}f = df$.

ii) $\boldsymbol{d}\,(df) = 0, f \in \mathcal{D}\,(M)$.

One says that $\boldsymbol{d}\alpha$ is the *exterior derivative* of the form α.

2.7. Proposition. – *Let* $h : M^m \to N^n$ *be a differentiable map. One has* $h^* \circ \boldsymbol{d} = \boldsymbol{d} \circ h^*$.

Proof: If $h^*\boldsymbol{d}\alpha_i = \boldsymbol{d}h^*\,\alpha_i$, $i = 1, 2$, then one will also have:

$$h^*\boldsymbol{d}(\alpha_1 \wedge \alpha_2) = \boldsymbol{d}h^*(\alpha_1 \wedge \alpha_2).$$

Consequently, since h and $\boldsymbol{d}$ are compatible with locally-finite sums, it will suffice to verify the relation $h^*\boldsymbol{d}\alpha = \boldsymbol{d}h^*\alpha$ when $\alpha = f$ and $\alpha = df, f \in \mathcal{D}\,(N)$. Now:

$$h^*\boldsymbol{d}f = h^*df = dh^* f = \boldsymbol{d}h^* f \,,$$

$$h^*\boldsymbol{d}(df) = 0 \qquad \text{and} \qquad \boldsymbol{d}h^*(df) = \boldsymbol{d}\, dh^* f = 0 \,.$$

Q. E. D.

2.8. Proposition. – *The exterior derivative is an anti-derivation of square zero.*

Proof: Since $\boldsymbol{d}^2$ is a derivation of degree 2, it would suffice to verify that $\boldsymbol{d}^2\alpha = 0$ when $\alpha = f$ and $\alpha = df, f \in \mathcal{D}\,(M)$. Now:

$$\boldsymbol{d}^2 f = \boldsymbol{d}\, df = 0 \,, \qquad \boldsymbol{d}^2 df = 0 \,.$$

Q. E. D.

2.9. Definition. – *A differential form* $\alpha \in \Lambda\,(M)$ *is a* ***closed*** *form if* $\boldsymbol{d}\alpha = 0$.

Any differential form of degree m on a manifold of dimension m is therefore closed.

2.10. Definition. – *A differential form* $\alpha \in \Lambda\,(M)$ *is an* ***exact*** *form if there exists a differential form* $\beta \in \Lambda\,(M)$ *such that* $\alpha = \boldsymbol{d}\beta$.

An exact differentiable form is a closed form. The converse is false: $\alpha = \dfrac{x\,dy - y\,dx}{x^2 + y^2}$ is a closed Pfaff form on $\mathbb{R}^2 - \{0\}$, but it is not exact.

Meanwhile, the Poincaré lemma is a local converse:

2.11. Theorem (Poincaré lemma). – *A closed differential form of degree* $p \geq 1$ *on* $\mathbb{R}^m$ (H^m, resp.) *is exact.*

The proof, which we will write out in the case of $\mathbb{R}^m$, utilizes the following lemma.

2.12. Lemma. – *Let* J_i, $i = 0, 1$, *be injections of* $\mathbb{R}^m \times \mathbb{R}$ *that are defined by* $J_i(x) = (x, i)$. *There will then exist a map* (*on* $\mathbb{R}$) $K : \Lambda(\mathbb{R}^m \times \mathbb{R}) \to \Lambda(\mathbb{R}^m)$ *that has the following properties:*

i) $K(\Lambda^{p+1}(\mathbb{R}^m \times \mathbb{R})) \subset \Lambda^p(\mathbb{R}^m)$.

ii) $\boldsymbol{d}K + K\boldsymbol{d} = J_1^* - J_0^*$.

Proof: Let $\imath$ denote the canonical coordinate of the factor $\mathbb{R}$ in $\mathbb{R}^m \times \mathbb{R}$ and define a linear map $K : \Lambda(\mathbb{R}^m \times \mathbb{R}) \to \Lambda(\mathbb{R}^m)$ by:

$$K f = 0 \qquad \text{if} \qquad f \in \mathcal{D}(\mathbb{R}^m \times \mathbb{R}),$$

$$K\alpha = 0 \qquad \text{if} \qquad \alpha = a\, dx_{i_1} \wedge \cdots \wedge dx_{i_p},$$

$$K\beta = \left(\int_0^1 b\, dt\right) dx_{j_1} \wedge \cdots \wedge dx_{j_{p-1}} \qquad \text{if} \qquad \beta = b\, dt \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_{p-1}}.$$

It remains to verify the condition *ii*). Now:

$$\boldsymbol{d}K f = 0,$$

$$K\,df = \int_0^1 \frac{\partial f}{\partial t}\, dt = (J_1^* - J_0^*) f,$$

$$d\,K\alpha = 0,$$

$$K\boldsymbol{d}\alpha = \left(\int_0^1 \frac{\partial a}{\partial t}\, dt\right) dx_{i_1} \wedge \cdots \wedge dx_{i_p} = (J_1^* - J_0^*)\alpha,$$

$$J_1^*\beta = J_0^*\beta = 0,$$

$$\boldsymbol{d}K\beta = \sum_{1 \leq i \leq m} \left(\int_0^1 \frac{\partial b}{\partial x_i}\, dt\right) dx_{i_1} \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_{p-1}}.$$

Q. E. D.

Proof of Theorem 2.11. – Let α be a closed differential form of degree $p \geq 1$ on $\mathbb{R}^m$.

Let H denote the differential map from $\mathbb{R}^m \times \mathbb{R}$ into $\mathbb{R}^m$ that is defined by $H(x_1, \ldots, x_m, t) = (t\,x_1, \ldots, t\,x_m)$. $H \circ J_1$ is the identity map on $\mathbb{R}^m$ and $H \circ J_0$ is the constant map $\mathbb{R}^m$ on 0. One will then have:

$$\begin{aligned}\alpha &= (J_1^* - J_0^*)\,H^*\alpha \\ &= \boldsymbol{d}K\,(H^*\alpha) + K\,\boldsymbol{d}H^*\alpha \\ &= \boldsymbol{d}K\,(H^*\alpha).\end{aligned}$$

Consequently, α will be an exact form.

Q. E. D.

Exercises:

i) Carry out the preceding calculation when α has degree 1.

ii) One can generalize Theorem 2.11 in the following way: Let M^m be a manifold (a manifold without boundary, resp.), and let $H : (x, t) \mapsto h_t(x)$ be differentiable map of $M^m \times \mathbb{R}$ ($M^m \times [0, 1]$, resp.) into M^m. If α is a closed form on M^m then $(h_1^* - h_0^*)\,\alpha$ will be an exact form.

§ 3. – The Lie derivative.

Let X be a vector field on a manifold M^m. $\mathbf{L}_X = \boldsymbol{i}_X\,\boldsymbol{d} + \boldsymbol{d}\,\boldsymbol{i}_X$ is a derivation of degree 0 on the algebra $\Lambda(M)$ (Prop. 1.9). That derivation is characterized by the following relations:

i) $\mathbf{L}_X f = X \cdot f$,
ii) $\mathbf{L}_X\, df = d(X \cdot f), f \in \mathcal{D}(M)$.

3.1. Definition. – *Let X be a field of vectors on a manifold M^m. The* ***Lie derivative*** *with respect to the vector field X is the derivation of degree* 0 *of* $\Lambda(M)$ *that is defined by* $\mathbf{L}_X = \boldsymbol{i}_X\,\boldsymbol{d} + \boldsymbol{d}\,\boldsymbol{i}_X$.

Exercise. – If α is a differential form of degree $p \geq 1$ then one will have:

$$(\mathbf{L}_X\,\alpha)(Y_1, \ldots, Y_p) = X \cdot \alpha(Y_1, \ldots, Y_p) - \sum_i \alpha(Y_1, \ldots, [X, Y_i], \ldots, Y_p).$$

In particular, if α is a Pfaff form then $(\mathbf{L}_X\,\alpha)(Y) = X \cdot \alpha(Y) - \alpha([X, Y])$.

3.2. Proposition. – *The Lie derivative commutes with exterior derivation.*

Indeed, $\mathbf{L}_X \boldsymbol{d} = \boldsymbol{d}\, \boldsymbol{i}_X \boldsymbol{d} = \boldsymbol{d}\, \mathbf{L}_X$.

3.3. Proposition. – *Let X and Y be two vector fields on a manifold* M^m *, and let* $f \in \mathcal{D}(M)$ *and* $\alpha \in \Lambda(M)$*. One has:*

i) $\mathbf{L}_{X+Y} = \mathbf{L}_X + \mathbf{L}_Y$.

ii) $\mathbf{L}_{fX}\, \alpha = f\, \mathbf{L}_X + df \wedge \boldsymbol{i}_X\, \alpha$.

The proof of that proposition is immediate.

3.4. Proposition. – *Let X and Y be two vector fields on a manifold* M^m *. One has:*

i) $[\mathbf{L}_X, \boldsymbol{i}_Y] = \boldsymbol{i}_{[X,\, Y]}$.

ii) $[\mathbf{L}_X, \mathbf{L}_Y] = \mathbf{L}_{[X,\, Y]}$.

Proof. – Since $[\mathbf{L}_X, \boldsymbol{i}_Y]$ and $\boldsymbol{i}_{[X,\, Y]}$ ($[\mathbf{L}_X, \mathbf{L}_Y]$ and $\mathbf{L}_{[X,\, Y]}$, resp.) are two anti-derivations of degree – 1 (two derivations of degree 0, resp.), it will suffice to verify that they take the same values on the forms $\alpha = f$ and $\alpha = df$, $f \in \mathcal{D}(M)$. Now:

$$[\mathbf{L}_X, \boldsymbol{i}_Y] f = \mathbf{L}_X\, \boldsymbol{i}_Y f - \boldsymbol{i}_Y\, \mathbf{L}_X f = 0 \,,$$
$$\boldsymbol{i}_{[X,\, Y]} f = 0 \,,$$

$$[\mathbf{L}_X, \boldsymbol{i}_Y]\, df = \mathbf{L}_X (Y \cdot f) - \boldsymbol{i}_Y\, \boldsymbol{d}\, (X \cdot f) = X \cdot (Y \cdot f) - Y \cdot (X \cdot f) = [X, Y] \cdot f \,,$$
$$\boldsymbol{i}_{[X,\, Y]}\, df = [X, Y] \cdot f \,,$$

$$[\mathbf{L}_X, \mathbf{L}_Y] f = X \cdot (Y \cdot f) - Y \cdot (X \cdot f) = [X, Y] \cdot f \,,$$
$$\mathbf{L}_{X,\, Y]} f = [X, Y] \cdot f \,,$$

$$[\mathbf{L}_X, \mathbf{L}_Y]\, df = \mathbf{L}_X\, \mathbf{L}_Y\, df - \mathbf{L}_Y\, \mathbf{L}_X\, df = d\, ([\mathbf{L}_X, \mathbf{L}_Y] f) = d\, ([X, Y] \cdot f)$$
$$\mathbf{L}_{X,\, Y]}\, df = d\, ([X, Y] \cdot f)$$

Q. E. D.

Exercise. – Let $h : M^m \to N^n$ be a differentiable map, and let $X \in \mathcal{T}(M)$, $Y \in \mathcal{T}(N)$ be two vector fields such that $h^{\mathrm{T}} X = Y h$. For any form $\alpha \in \Lambda(N)$:

$$h^*(\mathbf{L}_Y \alpha) = \mathbf{L}_Y (h^* \alpha)\,.$$

In particular, if U is an open subset of M^m then $(\mathbf{L}_X\,\alpha)|_U = (\mathbf{L}_X|_U)\,(\alpha|_U)$.

3.5. *Local expression*. – Let $(x_1, \ldots, x_m)$ be a system of local coordinates on an open subset U of a manifold M^m, and let:

$$\sum_{1 \le i_1 < \cdots < i_p \le m} a_{i_1 \cdots i_p}\, dx_{i_1} \wedge \cdots \wedge dx_{i_p}$$

be the local expression for a form $\alpha \in \Lambda\,(M)$. One has, for example:

$$\mathbf{L}_{\partial/\partial x_1}\alpha = \sum_{1 \le i_1 < \cdots < i_p \le m} \frac{\partial a_{i_1 \cdots i_p}}{\partial x_1}\, dx_{i_1} \wedge \cdots \wedge dx_{i_p} .$$

§ 4. – Integration of differential forms.

For any manifold M^m, let $\Lambda_c^p(M)$ denote the sub-module (over the algebra $\mathcal{D}\,(M)$) of differential forms of degree p on M^m that have compact support. One has:

$$\boldsymbol{d}\,(\Lambda_c^p(M)) \subset \Lambda_c^{p+1}(M).$$

If M^m is compact then $\Lambda_c^p(M) = \Lambda^p\,(M)$.

4.1. – Let $\alpha = f\, dx_1 \wedge \ldots \wedge dx_m$ be a differential form with compact support of degree m on an open subset U of $\mathbb{R}^m$ (H^m, resp.). The number $\int_U \alpha = \int_U f\, d\mu$ (in which μ is the Lebesgue measure $dx_1 \ldots dx_m$ on U) is called the *integral* of the form α on U. One thus defines a linear form on the vector space $\Lambda_c^m(U)$.

Let $\mathcal{V} = (V_i)$ one a locally-finite open covering of U and let (θ_i) be a partition of unity that is subordinate to $\mathcal{V}$. The support of a differential form α meets only a finite number of open subsets V_i, and one will have:

$$\int_U \alpha = \sum_i \int_{V_i} \theta_i\, \alpha .$$

Let V be an open subset of $\mathbb{R}^m$ (H^m, resp.) and let $h = (h_1, \ldots, h_m) : V \to U$ be a diffeomorphism that is compatible with the orientation. One will then have $\det\left(\frac{\partial h_i}{\partial x_j}\right) > 0$. In that case:

$$\int_U f\,d\mu = \int_V (f \circ h)\det\left(\frac{\partial h_i}{\partial x_j}\right) d\mu\,,$$

and consequently, $\int_U \alpha = \int_V h^* \alpha$.

4.2. Theorem. – *Let* M^m *be an oriented manifold. There exists one and only one linear form* $\alpha \mapsto \int_{M^m} \alpha$ *on the vector space* $\Lambda_c^m(U)$ *that has the following property:*

(I) *If h is a diffeomorphism of an open subset U of* $\mathbb{R}^m$ (H^m, *resp.*) *onto an open subset V of* M^m *that is compatible with the orientations*, *and if* $\alpha \in \Lambda_c^m(U)$ *has its support contained in V then one will have:*

$$\int_{M^m} \alpha = \int_U h^* \alpha\,.$$

One says that $\int_{M^m} \alpha$ is *the* ***integral*** *of the form* α *on* M^m .

Proof: Let $\mathcal{V} = (V_i)$ be a locally-finite open covering of M^m such that for any *i*, there exists a diffeomorphism h_i of an open subset *U* of $\mathbb{R}^m$ (H^m, *resp.*) onto V_i (Chap. III, Th. 4.10) that is compatible with the orientations, and let (θ_i) be a partition of unity that is subordinate to $\mathcal{V}$.

The support of a differential form $\alpha \in \Lambda_c^m(M)$ meets only a finite number of open subsets V_i . One must then have:

$$\int_{M^m} \alpha = \int_{M^m}\left(\sum_i \theta_i\right)\alpha = \sum_i \int_{M^m} \theta_i\,\alpha = \sum_i \int_{U_i} h_i^*(\theta_i\,\alpha)\,.$$

That shows the uniqueness of the integral.

Conversely, the preceding equality determines a linear form on the vector space $\Lambda_c^m(M)$. It remains for us to verify that condition (I) is satisfied.

Let *h* be a diffeomorphism of an open subset *U* of $\mathbb{R}^m$ (H^m, *resp.*) onto an open subset *V* of M^m that is compatible with the orientations. The diffeomorphisms $h_i^{-1} \circ h$ are compatible with the orientations, and one has:

$$\begin{aligned}\int_{M^m} \alpha &= \sum_i \int_{U_i \cap h_i^*(V)} h_i^*(\theta_i\,\alpha) \\ &= \sum_i \int_{U_i \cap h^{-1}(V_i)} (h_i^{-1} \circ h)^* h_i^*(\theta_i\,\alpha) \\ &= \sum_i \int_{U \cap h^{-1}(V_i)} (\theta_i \circ h)\,h^*\,\alpha\end{aligned}$$

$$= \int_U h^* \alpha .$$

Q. E. D.

4.3. Corollary. – *Let M^m and N^m be two oriented manifold, and let $h : M^m \to N^m$ be a diffeomorphism that is compatible with the orientations. One has:*

$$\int_{N^m} \alpha = \int_{M^m} h^* \alpha \qquad \text{for any } \alpha \in \Lambda_c^m(N).$$

That corollary is an immediate consequence of the uniqueness of the integral. In particular, if h is a diffeomorphism of M^m that preserves the orientation then:

$$\int_{M^m} h^* \alpha = \int_{M^m} \alpha \qquad \text{for any } \alpha \in \Lambda_c^m(M).$$

4.4. Corollary. – *If M^m is a compact manifold that is oriented by a volume form α then the integral of α over M^m will be strictly positive.*

Exercise. – If one changes the orientation of a connected manifold then the integral will change sign.

Consequently, if h is a diffeomorphism of M^m that reverses the orientation then:

$$\int_{M^m} h^* \alpha = -\int_{M^m} \alpha \quad \text{for any } \alpha \in \Lambda_c^m(M).$$

4.5. *Remark.* – Let M^m be an oriented manifold and let $\mathcal{D}_c(M)$ be the ideal of $\mathcal{D}(M)$ that consists of differentiable functions with compact support. For any differential form $\alpha \in \Lambda^m(M)$, $\mu_\alpha : f \mapsto \int_{M^m} f \alpha$ is a linear form on the vector space $\mathcal{D}_c(M)$ that determines a Radon measure on M^m in a unique manner.

4.6. – Let $\alpha = \sum_i a_i \, dx_1 \wedge \ldots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \ldots \wedge dx_m$ be a differential form with compact support of degree $m - 1$ on an open subset U of H^m. If $U \cap \partial H^m = \varnothing$ then one will have:

$$\int_U \boldsymbol{d}\alpha = \sum_i (-1)^{i-1} \int_U \frac{\partial a_i}{\partial x_i} \, d\mu = 0 .$$

Now suppose that $V = U \cap \partial H^m \neq \varnothing$, and let j be the canonical injection of V into U. In that case:

$$\begin{aligned}\int_U \boldsymbol{d}\alpha &= \sum_i (-1)^{i-1} \int_U \frac{\partial a_i}{\partial x_i}\, d\mu \\ &= -(-1)^{m-1} \int_V a_i(x_1,\dots,x_{m-1},0)\, d\mu' \\ &= \int_V j^*\alpha \qquad\qquad \text{(convention 4.13 of Chapter III).}\end{aligned}$$

4.7. Theorem (Stokes's formula): *Let M^m be an oriented manifold, and let j be the canonical injection of the oriented manifold ∂H^m (convention* 4.13 *of Chapter III) into M^m. For any differential form $\alpha \in \Lambda_c^{m-1}(M)$, one will have:*

$$\int_{M^m} \boldsymbol{d}\alpha = \int_{\partial M^m} j^*\alpha\,.$$

Proof: With the same notations as in the proof of Theorem, 4.2., one will have:

$$\int_{M^m} \boldsymbol{d}\alpha = \int_{M^m} \boldsymbol{d}\left(\sum_i \theta_i\,\alpha\right) = \sum_i \int_{U_i} h_i^*\, \boldsymbol{d}\left(\theta_i\,\alpha\right).$$

Let j be the canonical injection of ∂H^m into $\mathbb{R}^m$ and let k_i denote the restriction of h_i to $U \cap \partial H^m$. When one takes 4.5 into account, one will have:

$$\int_{U_i} h_i^*\, \boldsymbol{d}\left(\theta_i\,\alpha\right) = 0 \qquad\qquad \text{if } U \cap \partial H^m = \varnothing,$$

while:

$$\begin{aligned}\int_{U_i} h_i^*\, \boldsymbol{d}\left(\theta_i\,\alpha\right) &= \int_{U_i \cap \partial H^m} j'^*\, h_i^*\left(\theta_i\,\alpha\right) \\ &= \int_{U_i \cap \partial H^m} k_i^*\left((\theta_i \circ j)\, j^*\alpha\right) \qquad \text{if } U \cap \partial H^m \neq \varnothing.\end{aligned}$$

Consequently:

$$\int_{M^m} \boldsymbol{d}\alpha = \int_{\partial M^m} j^*\alpha\,.$$

Q. E. D.

4.8. Corollary. – *Let M^m be a manifold without boundary. For any differential form $\alpha \in \Lambda_c^{m-1}(M)$, one will have:*

$$\int_{M^m} \boldsymbol{d}\alpha = 0\,.$$

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CHAPTER V

DIFFERENTIAL EQUATIONS AND DIFFERENTIAL SYSTEMS ON MANIFOLDS

Unless stated to the contrary, the manifolds that will be considered in the rest of this book will be manifolds without boundary.

§ 1. Integrating vector fields.

1.1. Definition. – *Let X be a vector field on a manifold M^m. An* ***integral curve*** *of X is a differentiable curve $c : I \to M^m$ such that for any $t \in I$, $c'(t) = X(c(t))$ (Chap. III, §* ***4.5****).*

If y is a singular point of X [viz., $X(y) = 0$] then the constant map $t \mapsto y$ of $\mathbb{R}$ into M^m will be an integral curve of X.

Let $(y_1, \ldots, y_m)$ be a local coordinate system on an open set U of M^m. If $\sum_i a_i \frac{\partial}{\partial y^i}$ is the local expression for X then the integral curves of X in U will be the solutions of the differential equation $x_i' = a_i(x)$, $i = 1, \ldots, m$. That is why one says that X is a *differential equation* or a *dynamical system* on M^m. The integral curves of X are also called the *solutions* or *trajectories* of X.

If one reformulates the local existence and uniqueness theorem for solutions of a differential equation (H. Cartan [**4**], J. Dieudonné [**5**]) then one will get:

1.2 Theorem. – *Let X be a vector field on a manifold M^m. For any point $y \in M^m$ and any $\tau \in \mathbb{R}$, there will exist:*

– *An open neighborhood U of y,*
– *A number $\varepsilon > 0$,*
– *A differentiable map $\Phi : (t, z) \mapsto \varphi_t(z)$ of $(t - \varepsilon, t + \varepsilon) \times U$ into M^m that has the following properties for any $z \in U$:*

i) $t \mapsto \varphi_t(z)$ *is an integral curve of X.*
ii) $\varphi_\tau(z) = z$.

Furthermore:

iii) *If* V_i , η_i , Ψ_i , $i = 1, 2$ *are analogous givens that have properties i*) *and ii*), *and if* ζ = inf. (η_1, η_2) *then* Ψ_1 *and* Ψ_2 *will coincide on* $(t - \zeta, t + \zeta) \times (V_1 \cap V_2)$.

In particular, two integral curves that are defined on the same interval I in $\mathbb{R}$ will be equal if they take the same value at a point of I.

1.3 Corollary. – *Let X be a vector field on a manifold* M^m . *There exists an open neighborhood U of* $\{0\} \times M^m$ *in* $\mathbb{R} \times M^m$ *and a differentiable map* $\Phi : (t, y) \mapsto \varphi_t(y)$ *of U into* M^m *that has the following properties for any* $y \in M^m$:

i) $\mathbb{R} \times \{y\} \cap U$ *is connected.*

ii) $t \mapsto \varphi_t(y)$ *is an integral curve of X.*

iii) $\varphi_0(y) = y$.

iv) *If* (t', y), $(t + t', y)$, *and* $(t, \varphi_{t'}(y))$ *are in U then* $\varphi_{t+t'}(y) = \varphi_t(\varphi_{t'}(y))$.

Furthermore:

v) *If* V_i , Ψ_i , $i = 1, 2$ *are analogous givens that have the properties i*), *ii*), *and iii*) *then they will also have property iv*), *and* $\Psi_1 = \Psi_2$ *on* $V_1 \cap V_2$.

Proof. – One can find an open covering (U_i) of M^m , a family (ε_i) of strictly-positive numbers, and a family (Φ_i) of differentiable maps $\Phi_i : (-\varepsilon_i , + \varepsilon_i) \times U_i \to M^m$ that has properties *i*) and *ii*) of Theorem 1.2.

Let $U = \bigcup_i (-\varepsilon_i , + \varepsilon_i) \times U_i \subset \mathbb{R} \times M^m$ and let Φ be the differentiable map of U into M^m that is equal to Φ_i on $(-\varepsilon_i , + \varepsilon_i) \times U_i$. [That choice is justified by property *iii*) of 1.2.] The open set U and the map Φ will then have properties *i*), *ii*), and *iii*) of the statement.

Under the hypotheses of property *iv*), $\tau \mapsto \varphi_{\tau+t'}(y)$ and $\tau \mapsto \varphi_t(\varphi_{t'}(y))$, $0 \le \tau \le t$ are two integral curves of X that take the same value $\varphi_{t'}(y)$ for $\tau = 0$; consequently, $\varphi_{\tau+t'}(y) = \varphi_t(\varphi_{t'}(y))$.

Property *v*) is proved in an analogous fashion.

Q. E. D.

1.4 – Let W be an open set of M^m such that $\{t\} \times W$ are contained in U, as well as $\{-t\} \times \varphi_t(W)$. The map $y \mapsto \varphi_t(y)$ is a diffeomorphism of W onto $\varphi_t(y)$ that has $z \mapsto \varphi_{-t}(z)$ for its inverse; in particular, $\varphi_t(W)$ is an open subset of M^m .

Moreover, if $\{t'\} \times \varphi_t(W)$, $\{-t'\} \times \varphi_t(\varphi_{t'}(y))$, $\{t' + t\} \times W$ and $\{-t - t'\} \times \varphi_{t'+t}(y)$ are in U then $\varphi_{t'+t}(y) = \varphi_{t'}(\varphi_t(y))$ for any $y \in W$; one will then have $\varphi_{t'+t} = \varphi_{t'} \circ \varphi_t$ on W.

Those remarks justify the following terminology:

1.5 Definition. – *A* ***local one-parameter group of diffeomorphisms*** *of a manifold* M^m *is a pair* (U, Φ) *in which:*

– U is an open neighborhood of $\{0\} \times M^m$ in $\mathbb{R} \times M^m$,

– $\Phi : (t, y) \mapsto \varphi_t(y)$ is a differentiable map of U into M^m that has the following properties:

i) For any $y \in M^m$, $\mathbb{R} \times \{y\} \cap U$ is connected.

ii) $y \mapsto \varphi_0(y)$ is the identity map of M^m.

iii) If (t', y), $(t + t', y)$, and $(t, \varphi_{t'}(y))$ are in U then $\varphi_{t+t'}(y) = \varphi_t(\varphi_{t'}(y))$.

One also denotes that local one-parameter group of diffeomorphisms by φ_t (without specifying the domain of definition).

A vector field X on a manifold M^m permits one to construct a local group (U, Φ) of diffeomorphisms of M^m: One says that (U, Φ) is *generated* by X. In that case [property *v*) of **1.3**], the germ of Φ at $\{0\} \times M^m$ will be determined by X.

1.6. – When $U = \mathbb{R} \times M^m$, one says that (U, Φ) (or φ_t) is a *(global) one-parameter group of diffeomorphisms on* M^m. The following properties will then be satisfied:

i) For any $t \in \mathbb{R}$, $\varphi_t : y \mapsto \varphi_t(y)$ is a diffeomorphism of M^m.

ii) φ_0 is the identity map of M^m.

iii) $\varphi_{t+t'} = \varphi_t \circ \varphi_{t'}$.

iv) $\varphi_{-t} = (\varphi_t)^{-1}$.

Example. – For any manifold M^m, $h_t : (t, v) \mapsto e^t\, v$ is a one-parameter group of diffeomorphisms of $T(M)$: h_t is the *one-parameter group of homotheties of* $T(M)$.

1.7. Lemma. – *Let X be a vector field on a manifold* M^m. *The set of local one-parameter group of diffeomorphisms of* M^m *that is generated by X will possess one and only one maximal element when it is ordered by inclusion.*

That lemma is a direct consequence (thanks to Zorn's lemma) of property *v*) of **1.3**.

In general, that maximal local group is not a global one-parameter group of diffeomorphisms of M^m.

Example. – The maximal local group (U, Φ) that is generated by the vector field $x^2 \dfrac{\partial}{\partial x}$ on $\mathbb{R}$ is given by:

$$U = \{(t, x) \in \mathbb{R} \times \mathbb{R} \mid 1 - t\,x > 0\},$$

$$\Phi\,(t, x) = \frac{x}{1 - tx}.$$

1.8. Definition. – *A vector field on a manifold M^m is **complete** if it is generated by a global one-parameter group of diffeomorphisms of M^m.*

1.9. Theorem. – *A vector field on a compact manifold is complete.*

That theorem is a consequence of the following proposition:

1.10. Proposition. – *Let X be a vector field on a manifold M^m and let (U, Φ) be the largest local one-parameter group of diffeomorphisms of M^m that is generated by X. For any $y \in M^m$, let (α_y, ω_y) denote the interval of $\mathbb{R}$ that is defined by $\mathbb{R} \times \{y\} \cap U = (\alpha_y, \omega_y) \times \{y\}$ and let $c_y^+ : [0, \omega_y) \to M^m$ [$c_y^- : (\alpha_y, 0] \to M^m$, resp.] denote the integral curve $t \mapsto \varphi_t(y)$ of X. If the image of c_y^+ (c_y^-, resp.) is relatively compact then one will have $\omega_y = +\infty$ ($\alpha_y = -\infty$, resp.).*

Proof. – Suppose that the image of c_y^+ is relatively compact and that ω_y is finite (the second case can be deduced by changing X into $-X$).

Let z be an accumulation point for the curve c_y^+ for $t \to \omega_y$, let W be an open subset of z, let ε be a strictly-positive number, and let $\Psi : (-\varepsilon, +\varepsilon) \times W \to M^m$ be a differentiable map that has properties *i*) and *ii*) of Theorem **1.2**.

Let τ be in the interval $(\omega_y - \varepsilon, \omega_y)$ such that $\varphi_t(y) \in W$. One can find an open neighborhood V of y such that $\{\tau\} \times V$ is contained in U and $\varphi_t(V)$ is contained in W.

Then let $U' = U \cup (\omega_y - \varepsilon, \omega_y + \varepsilon) \times V$. One can prolong Φ into a local group (U', Φ') by setting:

$$\Phi'(t, x) = \Psi\,(t - \tau, \Phi\,(\tau, x)), \quad x \in V, \text{ and } \quad |\,t - \omega_y\,| < \varepsilon,$$

which is absurd, since (U, Φ) is maximal.

Q.E.D.

1.11 Corollary. – *A vector field with compact support is complete.*

1.12. With the same notations as in **1.10**, let y be a point of M^m such that $(\alpha_y, \omega_y) = \mathbb{R}$, and let $\gamma = \Phi\,(\mathbb{R} \times \{y\})$. For any point $z \in \gamma$, one will have $(\alpha_z, \omega_z) = \mathbb{R}$.

The set $G = \{t \in \mathbb{R} \mid \varphi_t(y) = y\}$ is a closed subgroup of $\mathbb{R}$ that independent of the choice of point y on γ. Three cases must then be considered according whether $G = \{0\}$, $G = \mathbb{R}$, or $G = \mathbb{Z}\omega$, $\omega \neq 0$:

– If $G = \{0\}$ then $t \mapsto \varphi_t(y)$ is an injective immersion of $\mathbb{R}$ in M^m.

– If $G = \mathbb{R}$ then y is a singular point of X (and conversely).

– If $G = \mathbb{Z}\omega$, $\omega \neq 0$ then one says that $t \mapsto \varphi_t(y)$ is a *periodic solution* of X of *period* ω. In that case, γ will be compact and a submanifold of M^m that is diffeomorphic to the circle S^1.

Exercise. – A trajectory $c : \mathbb{R} \to M^m$ is periodic iff its image is compact.

1.13. Proposition. – *Let X be a vector field on a manifold M^m. There exists a strictly-positive differentiable function f on M^m such that the vector field $Y = f X$ is complete.*

The maximal solutions (in the sense of **1.7**) of X and Y that pass through a point y of M^m have the same images then.

The proof of this proposition uses the following lemma:

1.14. Lemma. – *There exists a proper differentiable function on any manifold.*

Proof:

Let $\mathcal{U} = (U_i)_{i \in \mathbf{N}}$ be a locally-finite open covering of a manifold M^m that is indexed by the set of strictly-positive integers (any local-finite open covering of a manifold is denumerable) and let (θ_i) be a partition of unity that is subordinate to $\mathcal{U}$.

The family $(i\ \theta_i)_{i \in \mathbf{N}}$ is locally finite, and $g = \sum_{i \in \mathbf{N}} i\,\theta_i$ is a proper differentiable function on M^m. (If K is compact in $\mathbb{R}$ then $g^{-1}(K)$ will be compact in M^m]. Q.E.D.

Proof of proposition 1.13 ([2]): Let $g : M^m \to \mathbb{R}$ be a proper differentiable function on M^m and let $f = e^{-(X\cdot g)^2}$. If $Y = f X$ then one will have $dg(Y) = (X \cdot g)\, e^{-(X\cdot g)^2} \leq 1$ on M^m.

Let $c : (a, b) \to M$ be a solution of Y that is defined on a bounded interval of $\mathbb{R}$. One has:

([2]) I must thank A. Dold for the idea behind this proof.

$$\frac{dg \circ c}{dt} = [(X \cdot g)\ e^{-(X \cdot g)^2}] \circ c$$

and

$$\left|\frac{dgc(t)}{dt}\right| \leq 1, \quad t \in (a, b).$$

The set $gc\ (a, b)$ is then bounded, and consequently the image of c will be relatively-compact in M^m.

One then deduces from Proposition **1.10** that Y is complete.

Q.E.D.

1.15. Proposition. – *Let X be a vector field on a manifold M^m and let y be a point of M^m such that $X(y) \neq 0$. There exists a local coordinate system $(x_1, \ldots, x_m)$ on an open neighborhood of y in which the local expression for X is $\partial / \partial x_1$.*

Proof:

Since the property has a local character, one can suppose that X is a vector field on $\mathbb{R}^m$ such that $X(0) = \partial / \partial x_1$.

Let (U, Φ) be a local one-parameter group of diffeomorphisms of $\mathbb{R}^m$ that is generated by X: $\Phi(t, x_1, \ldots, x_m) = (h_1(t, x_1, \ldots, x_m), \ldots, h_m(t, x_1, \ldots, x_m))$. Let $k = (k_1, \ldots, k_m)$ be the differentiable map that is defined in a neighborhood of 0 by:

$$k_i(x_1, \ldots, x_m) = k_i(x_1, 0, x_2, \ldots, x_m), \qquad i = 1, \ldots, m.$$

Since $X(0) = \partial / \partial x_1$, the Jacobian matrix $\left(\frac{\partial k_i}{\partial x_j}(0,0)\right)$ will be the identity matrix. The map k will then possess an inverse map $l = (l_1, \ldots, l_m)$ on a neighborhood of 0 that defines local coordinates $y_i = l_i(x_1, \ldots, x_m)$ on that neighborhood.

In that local coordinate system, the trajectories of X are curves $t \mapsto (t + y_1, y_2, \ldots, y_m)$, and consequently the local expression for X is $\partial / \partial y_1$.

Q.E.D.

1.16. Proposition. – *Let $h: M^m \to N^n$ be a differentiable map and let $X \in \mathcal{T}(M)$, $Y \in \mathcal{T}(N)$ be two vector fields such that $h^{\mathrm{T}} X = Y h$. If φ_t and ψ_t are local one-parameter groups that are generated by X and Y then one will have $\psi_t \circ h = h \circ \varphi_t$.*

The proof of that property is immediate.

Similarly:

1.17. Proposition. – *Let X be a vector field on a manifold M^m that is tangent to a submanifold N of M^m. Any integral curve of X that passes through a point $y \in N$ will be contained in N.*

§ 2. One-parameter groups and derivations

2.1 Theorem. – *Let φ_t be a local one-parameter group of diffeomorphisms on a manifold M^m. There exists one and only one vector field X on M^m such that φ_t is a local one-parameter group that is generated by X. That vector field is characterized by the relations:*

$$(X \cdot f)\,(y) = \lim_{t\to 0}\frac{f(\varphi_t(y)) - f(y)}{t}, \qquad f \in \mathcal{D}\,(M).$$

The proof of that theorem uses the following lemma:

2.2. Lemma. – *Let U be an open neighborhood of* $\{0\} \times \mathbb{R}^m$ *in* $\mathbb{R} \times \mathbb{R}^m$ *such that for any* $x \in \mathbb{R}^m$, $\mathbb{R} \times \{x\} \cap U$ *is connected, and let* $f(t, x)$ *be a differentiable function on U such that* $f(0, x) = 0$ *for any* $x \in \mathbb{R}^m$. *There exists a differentiable function* $g(t, x)$ *on U such that* $f(t, x) = t\,g(t, x)$.

Indeed, one has $f(t, x) = \int_0^1 t\frac{\partial f}{\partial t}(ts, x)\,ds$. It will then suffice to take $g(t, x) = \int_0^1 \frac{\partial f}{\partial t}(ts, x)\,ds$. $g(0, x)$ will then be equal to $\frac{\partial f}{\partial t}(0, x)$.

Proof of theorem 2.1:

Suppose that there exists a vector field X on M^m that generates the local one-parameter group φ_t. For any point $y \in M^m$, $X(y)$ is the vector tangent to the curve $t \mapsto \varphi_t(y)$ at $y = \varphi_0(y)$. Such a vector field is then unique.

We shall now show that for any function $f \in \mathcal{D}(M)$ and any point $y \in M^m$, $\frac{f(\varphi_i(y)) - f(y)}{t}$ has a limit $(Df)(y)$ when t tends to 0 and that Df is a differentiable function on M^m.

Since that result has a local character, one can suppose that $M^m = \mathbb{R}^m$. There will then exist a differentiable function $g(t, y)$ such that:

$$f(\varphi_t(y)) - f(y) = t\, g(t, y) .$$

One then verifies, in a classical fashion, that $f \mapsto Df$ is a derivation of the algebra $\mathcal{D}(M)$. It then determines a vector field X on M^m.

Finally (Chap. III, § **4.8**), for any point $y \in M^m$, $X(y)$ is the tangent vector to the curve $t \mapsto \varphi_t(y)$ at $y = \varphi_0(y)$. If one recalls condition *iii*) of Definition **1.5** then $X((y))$ will also be the tangent vector to the curve $t \mapsto \varphi_t(y)$ at $\varphi_t(y)$, which shows (Corollary **1.3**) that φ_t is a local one-parameter group that is generated by X.

Q.E.D.

2.3 Definition. – *Let X be a vector field on a manifold M^m. A **first integral** of X is a differentiable function f on M^m such that $X \cdot f = 0$.*

Proposition **1.15** then ensures the existence of $m - 1$ independent first integrals in the neighborhood of a point y of M^m such that $X(y) \neq 0$.

If f is a first integral of X then one will have $i_X(df) = X \cdot f = 0$. That is why one says, more generally, that a *first integral* of X is a closed Pfaffian form α on M^m such that $i_X\,\alpha = 0$. (When one takes the Poincaré lemma into account, those two notions will be locally equivalent.)

2.4 Proposition. – *Let X be a vector field on a manifold M^m. In order for a differentiable function on M^m to be a first integral of X, it is necessary and sufficient that it should be constant on the trajectories of M^m.*

Proof:

That proposition (briefly) expresses the following property: Let φ_t be a local one-parameter group of diffeomorphisms of M^m that is generated by X. In order for $f \in \mathcal{D}(M)$ to be a first integral of X, it is necessary and sufficient that for any $y \in M^m$, $t \mapsto f(\varphi_t(y))$ should be constant.

Now, if one lets f_y denote the function $t \mapsto f(\varphi_t(y))$ then one will have (Th. **2.1**):

$$\frac{df_y(t)}{dt} = \lim_{\varepsilon \to 0} \frac{f(\varphi_{t+\varepsilon}(y)) - f(\varphi_t(y))}{\varepsilon} = (X \cdot f)\,((\varphi_t(y)).$$

Q.E.D.

2.3 Proposition. – *Let X be a vector field on a manifold M^m and let φ_t be a local one-parameter group of diffeomorphisms of M^m that is generated by X. For any differential form $\alpha \in \Lambda(M)$, one will have:*

$$\begin{aligned} \mathrm{L}_X\, \alpha &= \lim_{t\to 0} \frac{\varphi_t^* \alpha - \alpha}{t} \\ &= \lim_{t\to 0} \frac{\alpha - \varphi_{-t}^* \alpha}{t}. \end{aligned}$$

Proof:

Since this proposition has a local character, one can suppose that $M^m = \mathbb{R}^m$. In that case, it will suffice to verify it for $\alpha = f$ and $\alpha = df, f \in \mathcal{D}(\mathbb{R}^m)$. Now:

$$\mathrm{L}_X f = \lim_{t\to 0} \frac{\varphi_t^* f - f}{t} \quad \text{(Th. } \mathbf{2.1}\text{)},$$

$$\begin{aligned} \mathrm{L}_X\, df = d\,(X \cdot f) &= d\left(\lim_{t\to 0} \frac{\varphi_t^* f - f}{t} \right) \\ &= \lim_{t\to 0} \frac{\varphi_t^* df - df}{t}. \end{aligned}$$

One deduces the second relation from the first one by switching X with $-X$ and t with $-t$.

Q.E.D.

2.6 Proposition. – *Let X and Y be two vector fields on a manifold M^m and let φ_t be a local one-parameter group of diffeomorphisms of M^m that is generated by X. One has:*

$$\begin{aligned} [X, Y] &= \lim_{t\to 0} \frac{\varphi_{-t}^{\mathrm{T}}\, Y\, \varphi_t - Y}{t} \\ &= \lim_{t\to 0} \frac{Y - \varphi_t^{\mathrm{T}}\, Y\, \varphi_{-t}}{t}. \end{aligned}$$

Proof:

Since $\Lambda^1(M)$ and $\mathcal{T}(M)$ are locally dual to each other, it will suffice to show that for any Pfaffian form α:

$$\alpha([X, Y]) = \lim_{t\to 0} \alpha\left(\frac{\varphi_{-t}^{\mathrm{T}}\, Y\, \varphi_t - Y}{t} \right).$$

Now:

$$\lim_{t\to 0} \alpha\left(\frac{\varphi_{-t}^{\mathrm{T}}\, Y\, \varphi_t - Y}{t} \right) = \lim_{t\to 0} \frac{1}{t}[(\varphi_{-t}^{\mathrm{T}}\alpha)(Y) \circ \varphi_t - \alpha(Y)]$$

$$= \lim_{t\to 0}\frac{(\varphi_{-t}^{\mathrm{T}}\alpha-\alpha)(Y)}{t}\circ\varphi_t+\lim_{t\to 0}\frac{\alpha(Y)\circ\varphi_t-\alpha(Y)}{t}$$

$= -(\mathrm{L}_X\,\alpha)(Y) + X\cdot\alpha\,(Y)$ (Prop. **2.5** and Th. **2.1**)

$= a\,([X, Y])$ (Chap. IV, § **3.1**) .

The second relations are obtained as before by changing X into $-X$ and t into $-t$.

Q.E.D.

2.7. Corollary. – *Let X and Y be two vector fields on a manifold M^m and let φ_t and ψ_t be local one-parameter groups of diffeomorphisms of M^m that are generated by X and Y. The following properties are equivalent:*

i) $[X, Y] = 0$.

ii) *φ_t and ψ_t commute.*

Under those conditions, one says that the vector fields X and Y commute.

That corollary is a direct consequence of Propositions **1.16** and **2.6**.

2.8 Remark. – When y is a point of M^m such that $X(y) \neq 0$, one can verify Propositions **2.5** and **2.6** more simply by using a local coordinate system $(y_1, \dots, y_m)$ in the neighborhood of y such that the local expression for X is $\partial / \partial y_1$ (Prop. **1.15**).

§ 3. – Differential systems

3.1 Definition: *A* ***p-dimensional differential system*** *on a manifold M^m is a sub-module $\mathcal{X}$ of $\mathcal{T}(M)$ that has the following properties:*

i) $\mathcal{X}$ is stable under locally-finite sums.

ii) For any point y of M^m, $\mathcal{X}_y = \{X(y), X \in \mathcal{X}\}$ is a p-dimensional subspace of $T_y(M)$.

A vector field with no singularity on a manifold M^m will then generate a one-dimensional differential system on M^m.

If U is an open subset of M^m then let $\mathcal{X}_U$ denote the sub-module of $\mathcal{T}(U)$ that is generated by the restrictions of the vector fields on $\mathcal{X}$ to U. One will then have:

3.2 Lemma. – *The sub-module $\mathcal{X}_U$ is a p-dimensional differential system on U.*

The proof of this lemma presents no difficulties.

3.3. Definition. – *Let $\mathcal{X}$ be a p-dimensional differential system on a manifold M^m. An* ***integral manifold*** *of $\mathcal{X}$ is a pair* (V^p, h) *in which V^p is a p-dimensional manifold and h is an injective immersion of V^p in M^m such that form any point $y \in V^p$, one will have $h^{\mathrm{T}}(T_y(V)) = \mathcal{X}_{h(y)}$.*

3.4 Definition. – *A differential system $\mathcal{X}$ on a manifold is* ***integrable*** *if there exists an integral manifold of $\mathcal{X}$ for every point y of M^m whose image contains y.*

3.5 Theorem. – *Let $\mathcal{X}$ be a p-dimensional differential system on a manifold M^m. In order for* X *to be integrable, it is necessary and sufficient that it should be stable under the Lie bracket.* (*That is, if X and Y are in $\mathcal{X}$ then* $[X, Y]$ *will also be in $\mathcal{X}$.*)

The necessity of that condition is a consequence of Proposition **6.18** of Chapter III. The proof of the converse uses the following two lemmas:

3.6 Lemma. – *Let $\mathcal{X}$ be a differential system on M^m that is stable for the Lie bracket. For any open subset U of M^m, $\mathcal{X}_U$ will also be stable under Lie bracket.*

Proof:

Let X and Y be in $\mathcal{X}_U$. One can find:

– Two locally-finite families (f_i) and (g_j) of differentiable functions on M^m.
– Two locally-finite families (X_i) and (Y_j) of vector fields in $\mathcal{X}$, such that $X = \sum_i (f_i X_i)\big|_U$ and $Y = \sum_j (g_j Y_j)\big|_U$.

If one writes out $[X, Y]$ explicitly then one will find that the bracket belongs to $\mathcal{X}_U$.

Q.E.D.

3.7. Lemma (Frobenius's theorem). – *Let $\mathcal{X}$ be a p-dimensional differential system on a manifold M^m. If $\mathcal{X}$ is stable under Lie brackets then there will exist a local coordinate system* $(z_1, \ldots, z_m)$ *on a neighborhood U of any point y in M^m such that $\mathcal{X}_U$ is generated by* $\partial / \partial z_1, \ldots, \partial / \partial z_p$.

Proof:

Let $X_1, \ldots, X_p$ be p vector fields in $\mathcal{X}$ such that $X_1(y), \ldots, X_p(y)$ generate $\mathcal{X}_y$. One can find an open neighborhood V of y and a local coordinate system $(y_1, \ldots, y_m)$ on V that has the following properties:

i) $X_1, \ldots, X_p$ generate $\mathcal{X}_U$.

ii) $y_1(y) = 0$.

iii) The local expression for X_1 in V is $\partial / \partial z_1$ (Prop. **1.15**).

If $p = 1$ then the lemma will be proved. If $p > 1$ then one will proceed by recurrence. Let $Y_1, \ldots, Y_p$ be the vector fields on $\mathcal{X}_V$ that are defined $Y_1 = X_1$ and $Y_i = X_i - (X_i \cdot y_1) X_1$, $i = 2, \ldots, p$.

Those vector fields have the following properties:

i) $Y_1, \ldots, Y_p$ generate $\mathcal{X}_V$.

ii) $[Y_i, Y_j] \in \mathcal{X}_V$.

iii) $Y_i \cdot y_1 = 0$ for $i \geq 2$.

Let N^{m-1} be the submanifold of V that is defined by $y_1 = 0$. The vector fields $Y_2, \ldots, Y_p$ are tangent to N^{m-1}. They generate a $(p-1)$-dimensional differential system on N^{m-1} that is stable under Lie bracket. One can then find a local coordinate system $(\zeta_2, \ldots, \zeta_m)$ on a neighborhood W of y in N^{m-1} such that $\mathcal{X}'_W$ is generated by $\partial / \partial\zeta_1, \ldots, \partial / \partial\zeta_p$.

Let $z_1, \ldots, z_m$ be differentiable functions on a neighborhood of y in M^m that are defined by:

$$z_1 = y_1,$$
$$z_i = \zeta_i(y_1, \ldots, y_m), \qquad i = 2, \ldots, m.$$

Those functions, which are independent of y, for a local coordinate system in the neighborhood of y, and one will have:

$$Y_1 = \frac{\partial}{\partial z_1},$$

$$\frac{\partial}{\partial z_1}(Y_i \cdot z_j) = [Y_1, Y_i] \cdot z_p$$
$$= \sum_k a_{ij}^k (Y_k \cdot z_j) \qquad \text{for} \qquad j \geq 2.$$

For each $j > p$, the functions Y_i, z_j, $i = 1, \ldots, m$ are then the solutions to a linear, homogeneous differential system. Now, they are annulled for $z_1 = 0$. As a result, they will be identically zero on a neighborhood U of y. Consequently, $Y_i = \sum_{j\leq p} b_{ij} \dfrac{\partial}{\partial z_j}$ on U, which will show that the vector fields $\partial / \partial z_1, \ldots, \partial / \partial z_p$ generate $\mathcal{X}_U$.

Q.E.D.

Proof of Theorem 3.5: The proof is now immediate, because with the notations of Lemma **3.7**, the submanifolds of U that are defined by z_i = const., $i = p + 1, \ldots, m$ will be integral manifolds of $\mathcal{X}$.

3.8. Corollary. – *A one-dimensional differential system is integrable.*

§ 4. – Pfaffian systems.

4.1 Definition: *A* ***Pfaffian system of rank p*** *on a manifold M^m is a sub-module $\mathcal{P}$ of $\Lambda^1(M)$ that has the following properties:*

i) *$\mathcal{P}$ is stable under locally-finite sums.*

ii) *For any point y of M^m, $\mathcal{P}_y = \{\alpha(y), \alpha \in \mathcal{P}\}$ is a p-dimensional subspace of $T_y^*(M)$*

.

If U is an open subspace of M^m then let $\mathcal{P}_U$ denote the sub-module of $\Lambda^1(U)$ that is generated by the restrictions of Pfaffian forms in $\mathcal{P}$ to U. As in the case of differential systems (Lemma **3.2**), one has:

4.2 Lemma. – *The sub-module $\mathcal{P}_U$ is a Pfaffian system of rank p on U.*

4.3 Proposition. – *Let $\mathcal{X}$ be a p-dimensional differential system on a manifold M^m. The orthogonal complement $\mathcal{X}^\perp$ to $\mathcal{X}$ is a Pfaffian system of rank $m - p$ on M^m such that $\mathcal{X} = \{X \in \mathcal{T}(M) \mid \alpha(X) = 0 \;\; \forall \alpha \in \mathcal{X}^\perp\}$.*

Proof:

One can use a partition of unity argument to reduce to the case in which there exist m vector fields $X_1, \ldots, X_m$ on M^m that define a basis for $\mathcal{T}(M)$ and are such that $X_1, \ldots, X_m$ generate $\mathcal{X}$.

If (α_i) denotes the basis that is dual to the basis (X_i) then the Pfaffian forms α_{p+1} , …, α_m will generate $\mathcal{X}^\perp$, and one will have $\mathcal{X} = \{X \in \mathcal{T}(M) \mid \alpha(X) = 0 \ \forall \alpha \in \mathcal{X}^\perp\}$.

Q.E.D.

One likewise proves that:

4.4. Proposition. – *Let $\mathcal{P}$ be a Pfaffian system of rank p on a manifold M^m. The sub-module $\mathcal{P}^0 = \{X \in \mathcal{T}(M) \mid \alpha(X) = 0 \ \forall \alpha \in \mathcal{P}\}$ will be an (m – p)-dimensional differential system on M^m such that $(\mathcal{P}^0)^\perp = \mathcal{P}$.*

Propositions **3.5** and **3.6** then show that $\mathcal{X} \mapsto \mathcal{X}^\perp$ is a bijective correspondence between p-dimensional differential systems on M^m and Pfaffian systems of rank $m-p$. (Of course, that is not true for the set of all sub-modules of $\mathcal{T}(M)$ and $\Lambda^1(M)$}.

4.5 Definition: *Let $\mathcal{P}$ be a Pfaffian system of rank p on a manifold M^m. An **integral manifold** of $\mathcal{P}$ is a pair (N^{m-p}, h), in which N^{m-p} is an $(m-p)$–dimensional manifold and h is an injective immersion of N^{m-p} in M^m such that for any Pfaffian form $\alpha \in \mathcal{P}$, one will have $h^*\alpha = 0$.*

In other words, in order for (N^{m-p}, h) to be an integral manifold of the Pfaffian system $\mathcal{P}$, it is necessary and sufficient that it should be an integral manifold of the differential system $\mathcal{P}^0$.

4.6. Definition: *A Pfaffian system $\mathcal{P}$ on a manifold M^m is **integrable** if there exists an integral manifold of $\mathcal{P}$ for any point y of M^m whose image contains y.*

4.7. Proposition. – *In order for a differential system $\mathcal{X}$ on M^m to be integrable, it is necessary and sufficient that $\mathcal{X}^\perp$ should also be so.*

The proof is immediate.

4.8. Theorem. – *Let $\mathcal{P}$ be a Pfaffian system on a manifold M^m. In order for $\mathcal{P}$ to be integrable, it is necessary and sufficient that $d\mathcal{P}$ should be contained in the ideal of $\Lambda(M)$ that is generated by $\mathcal{P}$.*

Proof:

Let $\mathcal{X}$ be the differential system on M^m such that $\mathcal{X}^\perp = \mathcal{P}$. For any α in $\mathcal{P}$ and X, Y in $\mathcal{X}$, one will have:

$$\begin{aligned} d\alpha(X, Y) &= X \cdot \alpha(Y) - Y \cdot \alpha(X) - \alpha([X, Y]) \\ &= -\alpha([X, Y]) . \end{aligned}$$

First suppose that $\mathcal{P}$ is integrable. One will then have that $d\alpha(X, Y) = 0$ for any $\alpha \in \mathcal{P}$ and any $X, Y \in \mathcal{X}$. That will put one into a situation that is analogous to the one in the proof of Proposition **4.3**, so one will deduce that $d\mathcal{P}$ is contained in the ideal in $\Lambda(M)$ that is generated by $\mathcal{P}$.

Conversely, if that property is verified then one will have $\alpha([X, Y]) = -d\alpha(X, Y) = 0$ for all X, Y in $\mathcal{X}$ and all α in $\mathcal{P}$. Consequently, the bracket $[X, Y]$ will be in $\mathcal{X}$, and $\mathcal{X}$ will then be integrable.

Q.E.D.

4.9. Corollary. – *Let $\mathcal{P}$ be a Pfaffian system that is generated by a Pfaffian form α with no singularities. In order for $\mathcal{P}$ to be integrable, it is necessary and sufficient that there should exist a Pfaffian form β such that $d\alpha = \alpha \wedge \beta$.*

When one translates Lemma **3.7** into the language of Pfaffian systems, one will get:

4.10. Lemma (Frobenius's theorem). – *Let $\mathcal{P}$ be a Pfaffian system of rank p on a manifold M^m. In order for $\mathcal{P}$ to be integrable, it is necessary and sufficient that for any point y of M^m there should exist a local coordinate system $(z_1, \ldots, z_m)$ on an open neighborhood U of y such that $\mathcal{P}_U$ is generated by $dz_1, \ldots, dz_p$.*

Exercise. – In Lemma 4.10, one can choose the system $(z_1, \ldots, z_m)$ in such a way that $z_1 = f|_U$, in which f is a differentiable function on M^m that have the following properties:

i) $df \in \mathcal{P}$.

ii) $df(x) \neq 0$.

4.11. Proposition. – *Let $\mathcal{P}$ be a Pfaffian system of rank p on a manifold M^m. In order for $\mathcal{P}$ to be integrable, it is necessary and sufficient that for any forms $\alpha, \alpha_1, \ldots, \alpha_p$ in $\mathcal{P}$, one should have $d\alpha \wedge \alpha_1 \wedge \ldots \wedge \alpha_p = 0$.*

If one takes **4.8** into account then this proposition will be a consequence of the following result:

4.12. Proposition. – *Let* $\alpha_1, \ldots, \alpha_p$ *be p independent Pfaffian forms on a manifold* M^m. *In order for a differential form* $\alpha \in \Lambda(M)$ *to be in the ideal that is generated by* $\alpha_1, \ldots, \alpha_p$, *it is necessary and sufficient that one should have* $\alpha \wedge \alpha_1 \wedge \ldots \wedge \alpha_p = 0$.

Proof:

One can suppose that M^m is parallelizable and that there exist $m - p$ Pfaffian forms $\alpha_{p+1}, \ldots, \alpha_m$ such that $\alpha_1, \ldots, \alpha_m$ generate $\Lambda(M)$.

Since the ideal I that is generated by $\alpha_1, \ldots, \alpha_p$ is the direct sum of sub-modules of $I \cap \Lambda^q(M)$, one can restrict oneself to the case in which α is homogeneous of degree q :

$$\alpha = \sum_{1 \le i_1 < \cdots < i_q \le m} a_{i_1 \cdots i_q} \alpha_{i_1} \wedge \cdots \wedge \alpha_{i_q} .$$

In that case, in order for α to be in I, it is necessary and sufficient that $a_{i_1 \cdots i_q} = 0$ for $i_1 > p$. Now, since:

$$\alpha \wedge \alpha_1 \wedge \ldots \wedge \alpha_p = \sum_{p < i_1 < \cdots < i_q \le m} a_{i_1 \cdots i_q} \alpha_{i_1} \wedge \cdots \wedge \alpha_{i_q} \wedge \alpha_1 \wedge \ldots \wedge \alpha_p ,$$

that condition is equivalent $\alpha \wedge \alpha_1 \wedge \ldots \wedge \alpha_p = 0$.

Q.E.D.

CHAPTER VI

CHARACTERISTIC SYSTEM AND CLASS OF A DIFFERENTIAL FORM

§ 1. – Characteristic system and class.

Let α be a differential form of degree $p \geq 1$ on a manifold M^m.

1.1. Definition. – *The* ***characteristic subspace*** *of* α *at a point* y *of* M^m *is the subspace* $C_y(a)$ *of* $T_y(M)$ *that is the intersection of the associated subspaces* $A(\alpha(y))$ *and* $A(d\alpha(y))$. (Chap. I, Def. **7.4**)

1.2. Definition. – *The* ***characteristic system*** *of* α *at a point* y of M^m *is the subspace* $C_y^*(\alpha)$ of $T_y^*(M)$ *that is orthogonal to the characteristic subspace* $C_y(\alpha)$.

The characteristic system of α at y is then the sum of the associated systems $A^*(\alpha(y))$ and $A^*(d\alpha(y))$. (Chap. I, Def. **7.7**)

1.3. Definition. – *The* ***class*** *of* α *at a point* y *of* M^m *is the dimension of the characteristic system* $C_y^*(\alpha)$ [*or the codimension of the characteristic subspace* $C_y(a)$].

The class of α at y is therefore greater than the rank of $\alpha(x)$, and as a result, it will be greater than the degree p of α when $\alpha(x) \neq 0$.

Example. The form $\alpha = (x_1^2 + x_2^2)\ dx^2$ on $\mathbb{R}^2$ has:

– Class 2 if $x_1 \neq 0$.
– Class 1 if $x_1 = 0$ and $x_2 \neq 0$.
– Class 0 if $x_1 = x_2 = 0$.

If α has degree p and class p at y then $d\alpha(y) = 0$.

If α is a closed form [in which $d\alpha(y) = 0$] then the class α of y will be equal to the rank of $\alpha(y)$. Consequently, (Chap. I, Prop. **8.3**, **8.4**, and **8.5**).

1.4. Proposition. – *A closed form of degree* 2 *has even class at each point.*

1.5. Proposition. – *Let* α *be a closed form of degree* 2 *on* M^m. *In order for* α *to have class* 2s *at a point* y *of* M^m, *it is necessary and sufficient that one should have* $\alpha^s(y) \neq 0$ *and* $\alpha^{s+1}(y) = 0$.

Furthermore, under those conditions, one will have:

$$C_y^*(\alpha) = C_y^*(\alpha^2) = \ldots = C_y^*(\alpha^s) = A^*(\alpha(y)).$$

1.6. Proposition. – *Let* α *be a Pfaffian form on* M^m. *In order for* α *to be have class* $2s + 1$ *at a point* y *of* M^m, *it is necessary and sufficient that one should have* $(\alpha \wedge (d\alpha)^s)(y) \neq 0$ *and* $(d\alpha)^{s+1}(y) = 0$.

Moreover, under those conditions, one will have:

$$C_y^*(\alpha) = C_y^*(\alpha \wedge d\alpha) = C_y^*(\alpha \wedge (d\alpha)^s) = (\alpha(y)) + A^*(d\alpha(y)).$$

Proof. – In order for α to have class $2s + 1$ at y, it is necessary and sufficient that $d\alpha$ should have class $2s$ at y and that $C_y^*(\alpha)$ should be the direct sum of:

$$C_y^*(d\alpha) = A^*(d\alpha(y))$$

and the subspace that is generated by $\alpha(y)$, or rather that there should exist a basis $(\varepsilon_i)_{1\leq i\leq m}$ for $T_y^*(M)$ such that:

$$\alpha(y) = \varepsilon_1,$$
$$d\alpha(y) = \varepsilon_1 \wedge \varepsilon_2 + \ldots + \varepsilon_{2s} \wedge \varepsilon_{2s+1}.$$

Q.E.D.

1.7. Proposition. – *Let* α *be a Pfaff form on* M^m. *In order for* α *to have class* 2s *at a point* y *of* M^m, *it is necessary and sufficient that one should have:*

$$(d\alpha)^s(y) \neq 0 \quad \text{and} \quad (a \wedge (d\alpha)^s)(y) = (d\alpha)^{s+1}(y) = 0.$$

Moreover, one has:

$$C_y^*(\alpha) = C_y^*(d\alpha) = C_y^*(\alpha \wedge d\alpha) = \ldots = C_y^*(d\alpha^s) = A^*(d\alpha(y))$$

under those conditions.

Indeed, in order for α to have class 2s at y, it is necessary and sufficient that $d\alpha$ should have class 2s at y and that $\alpha(y)$ should belong to the characteristic system $C_y^*(d\alpha)$, or rather that $d\alpha$ should have class 2s at y and that $(a \wedge (d\alpha)^s)(y) = 0$.

When y is not a singularity of α [so $\alpha\,(y) \neq 0$], one can make Proposition **1.7** more precise in the following way:

1.8. Proposition. – *Let α be a Pfaffian form on M^m and let y be a point of M^m such that $\alpha\,(y) \neq 0$. In order for α to have class $2s$ at y, it is necessary and sufficient that one should have:*

$$(d\alpha)^s\,(y) \neq 0 \qquad \text{and} \qquad (\alpha \wedge (d\alpha)^s)\,(y) = 0.$$

Proof: One first points out that if α has class $2s$ at y then one will have $(\alpha \wedge (d\alpha)^{s-1})(y) \neq 0$. Indeed, one can find a basis $(\varepsilon_i)_{1 \leq i \leq m}$ for $T_y^*(M)$ such that:

$$\alpha\,(y) = \varepsilon_1\,,$$

$$d\alpha\,(y) = \varepsilon_1 \wedge \varepsilon_2 + \ldots + \varepsilon_1 \wedge \varepsilon_2 \qquad \text{(Chap. I, Cor. } \mathbf{8.2}\text{)}\,.$$

Now suppose that one has:

$$(d\alpha)^s\,(y) \neq 0 \quad \text{and} \quad (\alpha \wedge (d\alpha)^s)\,(y) = 0.$$

The class of α at y is then greater than s. Now, α cannot have class $s' > s$ because one would then have (Prop. **1.6**, or the preceding remark):

$$(\alpha \wedge (d\alpha)^s)\,(y) \neq 0.$$

Q.E.D.

1.9. Rules: Let α be a Pfaffian form on M^m, and let $\omega_1 = \alpha$, $\omega_2 = d\alpha$, $\omega_3 = \alpha \wedge d\alpha$, $\omega_4 = (d\alpha)^2$, …

The class of α at a point y of M^m such that $\alpha\,(y) \neq 0$ is the smallest integer r such that:

$$\omega_{r+1}\,(y) = \omega_{r+2}\,(y) = 0,$$

The class of α at a point y of M^m such that $a\,(y) \neq 0$ is the smallest integer r such that $\omega_{r+1}\,(y) = 0$.

1.10. Local study. – Let $(y_1, \ldots, y_m)$ be a local coordinate system on an open subset U of M^m. The characteristic system of $\alpha \in \Lambda^p\,(M)$ at a point y of U is generated (Chap. I, Prop. **7.9**) by the linear forms:

$$i\left(\frac{\partial}{\partial y_{i_1}}\right), \ldots, i\left(\frac{\partial}{\partial y_{i_{p-1}}}\right)\alpha(y), \qquad 1 \leq i_1 < \ldots < i_{p-1} \leq m,$$

and

$$i\left(\frac{\partial}{\partial y_{i_1}}\right), \ldots, i\left(\frac{\partial}{\partial y_{i_p}}\right)d\alpha(y), \qquad 1 \leq j_1 < \ldots < j_{p-1} \leq m .$$

Consequently:

1.11. Proposition. – *The class of a differential form $\alpha \in \Lambda^p(M)$ is a positive lower-semi-continuous function with integer values.*

In other words, if α has class q at a point y of M^m then it will have a class that is greater than q at any point that is sufficient close to y.

§ 2. Characteristic vector fields and forms.

Let α be a differential form of degree $p \geq 1$ on a manifold M^m.

2.1. Definition. – *A **characteristic vector field** for α is a vector field X on M^m such that $X(y) \in C_y(\alpha)$ for any $y \in M^m$.*

The set $\mathcal{C}(\alpha)$ of characteristic vector fields of α is a sub-module of $\mathcal{T}(M)$ that is stable under locally-finite sums.

2.2. Theorem. – *In order for a vector field X on M^m to be a characteristic vector field of α, it is necessary and sufficient that one should have $i_X\,\alpha = i_X(d\alpha) = 0$.*

That theorem is a direct consequence of Proposition **7.5** in Chapter I.

2.3. Corollary. – *In order for X to be characteristic vector field of α, it is necessary and sufficient that one should have $i_X\,\alpha = \mathrm{L}_X\,\alpha = 0$.*

Indeed, $\mathrm{L}_X\,\alpha = i_X\,d\alpha + d\,i_X\,\alpha$.

2.4. Corollary. – *If X and Y are characteristic vector fields of α then their Lie bracket $[X, Y]$ will also be a characteristic vector field of α.*

Indeed (Chap. IV, Prop. **3.4**):

$$i_{[X,Y]}\,\alpha = \mathrm{L}_X\,i_Y\,\alpha - i_Y\,\mathrm{L}_X\,\alpha = 0,$$

$$\mathrm{L}_{[X,Y]}\,\alpha = \mathrm{L}_X\,\mathrm{L}_Y\,\alpha - \mathrm{L}_Y\,\mathrm{L}_X\,\alpha = 0\ .$$

2.5. Definition. – *A* ***characteristic Pfaffian form*** *of α is a Pfaffian form ω on M^m such that $\omega(y) \in C_y^*(\alpha)$ for any $y \in M^m$.*

The set $\mathcal{C}^*(\alpha)$ of characteristic Pfaffian forms of α is a sub-module of $\Lambda^1(M)$ that is stable under locally-finite sums.

2.6. Proposition. – *For any point y of M^m, the set of covectors $\omega(y) \in T_y^*(M^m)$, $\omega \in \mathcal{C}^*(\alpha)$ is equal to the characteristic system $C_y^*(\alpha)$ of α at y.*

Proof: Let ε_y be element of $C_y^*(\alpha)$. With the notations of **1.10**, there exists a Pfaffian form ε on U that has the following properties:

i) $\varepsilon(y) = \varepsilon_y$.
ii) $\varepsilon(y) \in C_y^*(\alpha)$ for any $z \in U$.

Let θ be a differentiable function on U that is equal to 1 at y and zero outside of a neighborhood of y. The Pfaffian form $\theta\varepsilon$ extends by zeroes on $M^m - U$ to a Pfaffian form ω that belongs to $\mathcal{C}^*(\alpha)$ and is such that $\omega(y) = \varepsilon_y$. Q.E.D.

Exercises:

i) There is no analogue of Proposition **2.6** for the module of characteristic vector fields of α.

ii) The sub-module $\mathcal{C}^*(\alpha)$ is contained in the orthogonal complement to $\mathcal{C}(\alpha)$, but it will generally be distinct from it.

iii) If ω is a characteristic Pfaffian form for α then $d\omega$ will not necessarily belong to the ideal of $\Lambda(M)$ that is generated by $\mathcal{C}^*(\alpha)$.

iv) The form α does not necessarily belong to the sub-algebra of $\Lambda(M)$ that is generated by $\mathcal{C}^*(\alpha)$ (contrary to the linear case: Chap. I, Prop. **7.8**).

§ 3. – Differential forms with constant classes.

In this section, suppose that α is a differential form on a manifold M^m of degree p and constant class $q \geq p$.

The sub-module $\mathcal{C}^*(\alpha)$ of characteristic Pfaffian forms of α is a Pfaffian system of rank q on M^m in this case (Prop. **2.6**).

The sub-module $\mathcal{C}(\alpha)$ of characteristic vector fields of α is then equal to the differential system $\mathcal{X}$ on M^m such that $\mathcal{X}^\perp = \mathcal{C}^*(\alpha)$ (Chap. V, Prop. **4.5** and **4.6**).

Consequently, $\mathcal{C}(\alpha)$ will be an $(m - q)$-dimensional differential system on M^m, and one will deduce the following theorem from Corollary **2.4**:

3.1. Theorem. – *Let α be a differential form of constant class q on a manifold M^m. The sub-module $\mathcal{C}(\alpha)$ of characteristic vector fields of α is an integrable $(m - q)$-dimensional differential system on M^m. The sub-module $\mathcal{C}^*(\alpha)$ of characteristic Pfaffian forms of α is the orthogonal system to $\mathcal{C}(\alpha)$.*

3.2. Proposition. – *Let α be a differential form of degree p and constant class q on a manifold M^m. For any point y of M^m, there exists a local coordinate system $(y_1, \ldots, y_m)$ on an open neighborhood U of y such that local expression for α is:*

$$\sum_{1 \leq i_1 < \cdots < i_p \leq q} a_{i_1 \cdots i_p}(y_1, \ldots, y_q)\, dy_{i_1} \wedge \cdots \wedge dy_{i_p} .$$

Proof: For any point y of M^m, Frobenius's theorem (Chap. V, Lemma **4.10**) insures the existence of a local coordinate system $(y_1, \ldots, y_m)$ on an open neighborhood U of y such that the Pfaffian system $\mathcal{C}^*(\alpha|_U) = \mathcal{C}^*(\alpha)|_U$ is generated by the forms $dy_1, \ldots, dy_q$.

One will then have:

$$\alpha = \sum_{1 \leq i_1 < \cdots < i_p \leq q} a_{i_1 \cdots i_p}(y_1, \ldots, y_q)\, dy_{i_1} \wedge \cdots \wedge dy_{i_p} ,$$

$$d\alpha = \sum_{1 \leq i_1 < \cdots < i_p \leq q} \sum_j \frac{\partial a_{i_1 \cdots i_p}}{\partial y_j}\, dy_j \wedge dy_{i_1} \wedge \cdots \wedge dy_{i_p}$$

on U. Consequently, $\partial a_{i_1 \cdots i_p} / \partial y_j = 0$ for $j > q$.

One can then suppose, after possibly restricting U, that the functions $a_{i_1 \cdots i_p}$ are independent of y_j for $j > q$.

Q.E.D.

3.3. Remark: If α admits a local expression that is analogous to the one in Proposition 3.2 then α will have a class that is less than q. Consequently, if α has class q then that expression will include each of the functions $y_1, \ldots, y_p$ explicitly. One can then say (E. Cartan [**3**]) that:

> "The class of a form α (of constant class) is the minimal number of independent functions that are necessary for expressing α."

3.4. Corollary. – *Let α be a differential form of degree p and constant class p on a manifold M^m. For any point y of M^m, there exists a local coordinate system $(y_1, \ldots, y_m)$ on an open neighborhood U of y such that local expression for α on U is $dy_1 \wedge \ldots \wedge dy_q$.*

Proof: One can find a local coordinate system $(z_1, \ldots, z_m)$ on a neighborhood V of y such that:

$$\alpha = a\,(z_1, \ldots, z_p)\, dz_1 \wedge \ldots \wedge dz_p\,, \qquad a \neq 0$$

on V.

Let $A = A\,(z_1, \ldots, z_p)$ be a differentiable function on V such that $\partial A / \partial z_1 = a$, and let $y_1, \ldots, y_m$ be functions that are defined by:

$$y_1 = A\,(z_1, \ldots, z_p),$$
$$y_i = z_i \qquad \text{for} \qquad i \geq 2.$$

Those functions define a local coordinate system on a neighborhood U of x and one will have $\alpha = dy_1 \wedge \ldots \wedge dy_q$.

Q.E.D.

§ 4. – Local models for differential forms of degrees 1 and 2.

When α is a differential form of constant class and degree 1, or of degree 2 and closed, one can make Proposition **3.2** more precise:

4.1 Theorem (Darboux). – *Let α be a Pfaffian form with no singularities on a manifold M^m of constant class $2s + 1$ ($2s$, resp.). For any point y of M^m, there exist $2s + 1$ ($2s$, resp.) differentiable functions $y_1, \ldots, y_{2s+1}$ ($y_1, \ldots, y_{2s}$, resp.) on a neighborhood U of y that are zero at y and are such that:*

$$\alpha\,|_U = dy_1 + y_2\, dy_3 + \ldots + y_{2s}\, dy_{2s+1}\,,$$

$$[\alpha\,|_U = (1 + y_1)\, dy_2 + y_3\, dy_4 + \ldots + y_{2s-1}\, dy_{2s}\,, \text{ resp.}].$$

The proof of that theorem uses the following two lemmas ([1]):

4.2. Lemma. – *Let α be a Pfaffian form without singularities and constant class* $2s + 1 > 0$ *on* M^m. *For any point y of* M^m *there exists a differentiable function f on an open neighborhood V of y that is zero at y and such that* $\alpha_1 = \alpha\,|_V - df$ *has no singularity for* $s > 0$ *and constant class* $2s$ *on V.*

4.3. Lemma. – *Let α be a Pfaffian form without singularities and constant class* $2s > 0$ *on* M^m. *For any point y of* M^m *there exists a differentiable function g on an open neighborhood W of y that is zero at y and such that* $\alpha_2 = (1 + g)\,(\alpha\,|_W)$ *has constant class* $2s - 1$ *on W.*

Proof of Lemma 4.2. – The sub-module of characteristic forms $\mathcal{C}^*(\alpha \wedge (d\alpha)^s) = \mathcal{C}^*(\alpha)$ [$\mathcal{C}^*((d\alpha)^s) = \mathcal{C}^*(d\alpha)$, resp.] is an integrable Pfaffian system of rank $2s + 1$ ($2s$, resp.) on M^m, and one has $\mathcal{C}^*(d\alpha) \subset \mathcal{C}^*(\alpha)$.

One can then find a local coordinate system $(z_1, \ldots, z_m)$ on an open neighborhood V of y that is zero for y and such that:

i) $\quad (d\alpha)^s\,|_V = dy_2 \wedge \ldots \wedge dy_{2s+1}$,

ii) $\quad \alpha \wedge (d\alpha)^s\,|_V = dy_1 \wedge \ldots \wedge dy_{2s+1}$ (Corollary **3.4**),

iii) $\quad \alpha\,|_V = dy_1 + \sum_{i=2}^{2s+1} a_i\,dy_i$ with $\sum a_i(z)^2 \neq 0$ for any $z \in V$.

The form $\alpha_1 = \alpha\,|_V - df$ has no singularities on V then and possesses the following properties:

i) $\quad (d\alpha_1)^s = (d\alpha)^s \neq 0$,

ii) $\quad \alpha_1 \wedge (d\alpha_1)^s = 0$.

It then has class $2s$ on V. Q.E.D.

Proof of Lemma 4.3. – The sub-module of characteristic forms $\mathcal{C}^*((d\alpha)^s) = \mathcal{C}^*(d\alpha)$ is an integrable Pfaffian system of rank $2s$ on M^m. Let $\mathcal{A}^*$ be the set of Pfaffian forms ω on M^m such that $\omega(y) \in \mathcal{A}^*((\alpha \wedge (d\alpha)^{s-1})(y))$ for any $y \in M^m$. $\mathcal{A}^*$ is then a sub-module of $\Lambda^1(M)$ that is stable for locally-finite sums. Since $\alpha \wedge (d\alpha)^{s-1}$ is a form of constant rank

([1]) The principle of this proof is due to J. Martinet.

$2s - 1$ (proof of Proposition **1.8**), one shows, as in **2.6**, that $\mathcal{A}^*$ is a Pfaffian system of rank $2s - 1$ on M^m. One has, moreover, $\mathcal{A}^* \subset \mathcal{C}^*((d\alpha)^s)$.

The Pfaffian system $\mathcal{A}^*$ is integrable. Indeed, it is the orthogonal complement to the differential system:

$$\mathcal{X} = \{X \in \mathcal{T}(M) \mid i_X(\alpha \wedge (d\alpha)^{s-1}) = 0\},$$

and if X and Y are in $\mathcal{X}$ then one will have:

$$i_{[X,Y]}(\alpha \wedge (d\alpha)^{s-1}) = \mathrm{L}_X\, i_Y(\alpha \wedge (d\alpha)^{s-1}) - i_Y\, \mathrm{L}_X(\alpha \wedge (d\alpha)^{s-1})$$

$$= -\, i_Y\, d i_X(\alpha \wedge d\alpha)^{s-1}) - i_Y\, i_X(d\alpha)^s$$

$$= -\, i_Y\, i_X(d\alpha)^s = 0 \quad [\mathcal{C}((d\alpha)^s) \subset \mathcal{X}] .$$

One can then find a local coordinate system $(z_1, \ldots, z_m)$ on an open neighborhood W of y that is zero at y and such that:

i) $(d\alpha)^s\,|_W = dz_1 \wedge \ldots \wedge dz_{2s}$,

ii) $(\alpha \wedge (d\alpha)^{s-1})\,|_W = b\, dz_2 \wedge \ldots \wedge dz_{2s}$, with $b(z) \neq 0$ for any $z \in W$.

If h is a differentiable function on W and if $\alpha_2 = h\,\alpha\,|_W$ then one will have:

$$\alpha_2 \wedge (d\alpha_2)^{s-1} = h^s\,(\alpha \wedge (d\alpha)^{s-1})\,|_W,$$

$$(d\alpha_2)^s = h^{s-1}\,[s\, dh \wedge (\alpha \wedge (d\alpha)^{s-1}) + h\,(d\alpha_2)^s\,|_W] .$$

Consequently, if $g = e^{-B/s} - 1$, in which $B = \int_0^{z_1} \frac{dz_1}{b}$ then the form $\alpha_2 = (1 + g)\,(\alpha\,|_W)$ will have class $2s - 1$ on W. Q.E.D.

*Proof of Theorem 4.*1. – One achieves that proof by recurrence on the class of α, while noting that a form of constant class zero is identically zero.

First of all, suppose that α has constant class $2s + 1$. There will then exist a differentiable function f on an open neighborhood V of y that is zero at y and such that $\alpha_1 = \alpha\,|_V - df$ has no singularity for $s > 0$ and has constant class $2s$ on V. One can then find $2s$ differentiable functions $g_1, \ldots, g_{2s}$ on an open neighborhood $U \subset V$ of y that are zero at y and such that:

$$\alpha_1\,|_U = (1 + g_1)\, dg_2 + g_3\, dg_4 + \ldots + g_{2s-1}\, dg_{2s} .$$

Set $y_1 = f + g_2$ ($y_1 = f$ if $s = 0$), $y_i = g_{i-1}$ for $i = 2, \ldots, 2s + 1$. Those functions are annulled at y, and one will have:

$$\alpha\,|_U = dy_1 + y_2\, dy_3 + \ldots + y_{2s}\, dy_{2s+1} .$$

Now suppose that α has no singularity and constant class $2s + 2$. There will then exist a differentiable function g on an open neighborhood W of y that is zero at y and is such that $\alpha_2 = (1 + g)\,(\alpha\,|_W)$ has constant class $2s + 1$ on W. One can then find $2s + 1$ differentiable functions $f_1, \ldots, f_{2s+1}$ on an open neighborhood $U \subset W$ of y that are zero at y and such that:

$$\alpha_2\,|_U = df_1 + f_2\, df_3 + \ldots + f_{2s}\, df_{2s+1} .$$

Set $y_1 = -\dfrac{g}{1+g}$, $y_i = \dfrac{f_{i-1}}{1+g}$ for $i = 3, 5, \ldots, 2s + 1$ and $y_i = f_{i-1}$ for $i = 2, 4, \ldots, 2s + 2$. Those functions are zero at y, and one will have:

$$\alpha\,|_U = (1 + y_1)\, dy_2 + y_3\, dy_4 + \ldots + y_{2s-1}\, dy_{2s} .$$

4.4. Remarks:

i) The functions (y_i) that enter into the statement of Theorem **4.1** are independent at y (Remark **3.3**).

ii) If α is a Pfaffian form of constant odd class then α will have no singularity on M^m .

By contrast, if α has constant even class then it can have singularities. In that case (as in the one where α does not have constant class), one cannot exhibit a general local model.

4.5. Theorem. – *Let ω be a closed differential form of degree 2 and constant class $2s$ on a manifold M^m. For any point y of M^m, there exist $2s$ differentiable functions $y_1, \ldots, y_{2s}$ on an open neighborhood U of y that are zero at y and such that:*

$$\omega\,|_U = dy_1 \wedge dy_2 + \ldots + dy_{2s-1} \wedge dy_{2s} .$$

Proof. – The Poincaré Lemma (Chap. IV, Th. 2.11) insures the existence of a Pfaffian form α on an open neighborhood V of y such that $d\alpha = \omega\,|_V$. The class of α at y is either $2s$ or $2s + 1$ then.

First suppose that $2s < m$. After possibly adding the differential of a function f in $\mathcal{D}$ (M), one can suppose that α has class $2s + 1$ at y. Hence, it will have constant class $2s + 1$ on an open neighborhood $W \subset V$ of y. One can then find $2s + 1$ differentiable functions $y_1, \ldots, y_{2s+1}$ on an open neighborhood $U \subset W$ of y that are zero at y and such that:

$$\alpha\,|_U = y_1\, dy_2 + \ldots + y_{2s-1}\, dy_{2s} + dy_{2s+1}\,.$$

One will then have:

$$(d\alpha)\,|_U = dy_1 \wedge dy_2 + \ldots + dy_{2s-1} \wedge dy_{2s}\,.$$

When $2s = m$, one can likewise suppose that α has no singularity on a neighborhood $W' \subset V$ of y. One can then find $2s$ differentiable functions on an open neighborhood $U' \subset W'$ of y that are zero at y and such that:

$$\alpha\,|_{U'} = (1 + z_1)\, dz_2 + z_3\, dz_4 + \ldots + z_{2s-1}\, dz_{2s}\,,$$

$$(d\alpha)\,|_{U'} = dz_1 \wedge dz_2 + \ldots + dz_{2s-1} \wedge dz_{2s}\,.$$

Q.E.D.

4.6. Remark: In Theorem **4.5**, one can take y_1 to be the restriction to U of a differentiable function f on M^m such that df is a characteristic Pfaffian form of ω that is not zero at y.

The verification of that assertion is left as an exercise.

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CHAPTER VII

HAMILTONIAN SYSTEMS AND CONTACT STRUCTURES

§ 1. Symplectic manifolds.

1.1. Definition – *Let M^{2n} be a manifold of even dimension $2n$. A* ***symplectic structure*** *on M^{2n} is defined when one is given a closed differential form $\omega \in \Lambda^2(M)$ of degree 2 and constant class $2n$.*

One also says that (M^{2n}, ω) (or M^{2n}) is a *symplectic manifold*, and that ω is a *symplectic form* on M^{2n}.

If U is an open subset of M^{2n} then (U, $\omega|_U$) will be a symplectic manifold.

For any point y of M^{2n}, ($T_y(M)$, $\omega(y)$) is a symplectic vector space (Chap. I, § **8**).

1.2. Proposition. – *Let ω be a closed differential form of degree 2 on a manifold M^{2n}. In order for ω to be a symplectic form, it is necessary and sufficient that ω^n should be a volume form* (Chap. III, Def **7.14**).

This assertion is a direct consequence of Proposition **1.5** in Chapter VI.

1.3. Corollary. – *A symplectic manifold (M^{2n}, ω) is orientable.*

One can then orient M^{2n} by the form volume $(-1)^{n(n-1)/2}\omega^n$ (see example **1.4**). Conversely, any orientable manifold of dimension 2 is symplectic; however, that result will no longer be true in even dimensions that are greater than 2.

1.4. Example. – The differential form:

$$\omega = dx_1 \wedge dx_{n+1} + dx_2 \wedge dx_{n+2} + \ldots + dx_n \wedge dx_{2n}$$

is a symplectic form on $\mathbb{R}^{2n}$; indeed:

$$\omega^n = (-1)^{n(n-1)/2} n!\, dx_1 \wedge \ldots \wedge dx_{2n}.$$

The orientation that is associated with ω is the canonical orientation on $\mathbb{R}^{2n}$.

The following theorem will permit us to construct some symplectic structures that are fundamental in analytical mechanics.

Let $\tau^*(M) = (T^*(M), q_M, M^m)$ be the cotangent bundle to a manifold M^m. For a point $y \in M^m$ and any cotangent vector $\alpha \in T_y^*(M)$, the tangent map q_M^{T} sends $T_\alpha(T^*(M))$ to $T_y(M)$. One can then define a linear form on $T_\alpha(T^*(M))$ by:

$$\mathrm{u} \mapsto \langle q_M^T(\mathrm{u}), \alpha \rangle = \langle \mathrm{u}, (q_M^T)^*(\alpha) \rangle .$$

1.5. Theorem. – *The correspondence* $\alpha \mapsto (q_M^T)^*(\alpha)$ *defines a Pfaffian form* λ *of constant class* $2m$ *on* $T^*(M)$.

Proof. – Let $(y_1, \ldots, y_m)$ be a coordinate system on an open subset U of M^m. The functions $q_i = y_i \circ q_M$ and $p_i = \partial / \partial y_i$, $i = 1, \ldots, m$, define a local coordinate system on an open subset $V = q_M^{-1}(U)$ of $T^*(M)$.

If $u = \sum_i \left(a_i \dfrac{\partial}{\partial q_i} + b_i \dfrac{\partial}{\partial p_i} \right)$ is a tangent vector on V then one will have $q_M^{\mathrm{T}}(u) = \sum_i a_i \dfrac{\partial}{\partial y_i}$. Consequently, if $\alpha = \sum_i c_i \, dy_i$ then $\langle u, \lambda(\alpha) \rangle = \sum_i c_i \, a_i$. The local expression for λ in V is then $\sum_i p_i \, dq_i$, which shows that λ is a Pfaffian form on $T^*(M)$.

One also deduces that λ has constant class $2m$ on $T^*(M)$ from this local expression.

Q.E.D.

1.6. Definition. – *The* ***Liouville form*** on $T^*(M)$ *is the Pfaff form* λ *that is defined by* $\lambda(\alpha) = (q_M^{\mathrm{T}})^*(\alpha)$.

1.7. Corollary. – *The exterior differential* $\Lambda = \boldsymbol{d}\lambda$ *of the Liouville form determines a symplectic structure on the cotangent space* $T^*(M)$.

1.8. Corollary. – *The cotangent space* $T^*(M)$ *to a manifold* M^m *is an orientable manifold.*

1.9. Definition. – Let (M^{2n}, ω) and (N^{2n}, ω') be two symplectic manifolds. A differentiable map $h : M^{2n} \to N^{2n}$ is *symplectic* if one has $h^* \omega' = \omega$.

In that case, for any point y of M^{2n}, h^{T} will be a symplectic isomorphism of $(T_y(M), \omega(y))$ onto $(T_{h(y)}(N), \omega'(h(y))$. Consequently, h will have constant rank $2n$ (h is then a local diffeomorphism).

If h is a symplectic diffeomorphism then it will be compatible with the orientation.

1.10. Proposition. – *Let* (M^{2n}, ω) *be a symplectic manifold. For any point y of* M^{2n}, *there exists an open neighborhood U of y and a symplectic diffeomorphism h of* (U, $\omega|_U$) *onto an open subset of* $\mathbb{R}^{2n}$ *(that is endowed with the symplectic structure of* **1.4**).

Indeed (Chap. VI, Th. **4.5**), one can find a local coordinate system (y_1, …, y_{2n}) on an open neighborhood U of y such that:

$$\omega|_U = dy_1 \wedge dy_{n+1} + \ldots + dy_n \wedge dy_{2n} .$$

Exercise. – Let M^{2n} be a manifold of even dimension $2n$. In order for M^{2n} to admit a symplectic structure, it is necessary and sufficient that there should exist an atlas $\{(U, \varphi_i)\}$ on M^{2n} such that the changes of charts $\varphi_j\, \varphi_i^{-1}$ are symplectic diffeomorphisms (for the structure of **1.4**).

1.11. Proposition. – *Let* (M^{2n}, ω) *be a symplectic manifold. The map* $\Omega_y : u \mapsto i\,(u)\, \omega\,(y)$ *of* $T_y\,(M)$ *to* $T_y^*(M)$ *determines a differentiable homomorphism* Ω *(over* M^{2n}*) of the tangent bundle* $\tau\,(M)$ *to the cotangent bundle* $\tau^*\,(M)$.

The rank of Ω_y *is equal to the rank of* $\omega\,(y)$.

Proof. – Let (y_1, …, y_m) be a local coordinate system on an open subset U of M^{2n} . The functions $r_i = y_i \circ p_M$ and $\dot{r}_i = dy_i$ ($q_i = y_i \circ p_M$ and $p_i = \partial / \partial y_i$, resp.), $i = 1, \ldots, m$, define a local coordinate system on the open subset $p_M^{-1}(U)$ of $T\,(M)$ [$q_M^{-1}(U)$ of $T^*(M)$, resp.].

Let $\sum_{i,j} a_{ij}\, dy_i \wedge dy_j$, with $a_{ji} = -\, a_{ij}$, be the local expression for ω in U. The map Ω is then determined on $p_M^{-1}(U)$ by:

$$\begin{aligned} q_i &= r_i\,, \\ p_i &= 2\sum_j a_{ij}\, \dot{r}_j\,, \qquad\qquad i = 1, \ldots, m\,; \end{aligned}$$

it is then differentiable.

Since Ω_y is a linear map of $T_y\,(M)$ to $T_y^*(M)$, Ω will be a homomorphism of $\tau\,(M)$ with $\tau^*(M)$ (Chap. II, Prop. **2.11**). Finally (Chap. I, Prop. **7.9**), the rank of Ω_y is equal to the rank of $\omega\,(y)$.

Q.E.D.

1.13. Corollary. – *Under the hypotheses of the proposition* **1.11**, $X \mapsto \boldsymbol{i}_X\omega$ *will be an isomorphism of* $\mathcal{T}(M)$ *with* $\Lambda^1(M)$.

§ 2. Poisson brackets.

Let (M^{2n} , ω) be a symplectic manifold. For a Pfaffian form α on M^{2n} , one lets X_α be the vector field on M^{2n} such that $\alpha = i_{X_\alpha}\omega$ (Cor. **1.13**).

2.1. Definition. – *The* ***Poisson bracket*** *(relative to the symplectic structure of* M^{2n} *) of two Pfaffian forms* α *and* β on M^{2n} *is the Pfaffian form* $(\alpha, \beta) = \boldsymbol{i}_{(X_\alpha, X_\beta)}\omega$.

The Poisson bracket is then obtained by transporting the Lie bracket on $\mathcal{T}(M)$ to $\Lambda^1(M)$ by means of the isomorphism $X \mapsto \boldsymbol{i}_X\omega$. Consequently:

2.2. Proposition:

i) $(\alpha, \beta + \gamma) = (\alpha, \beta) + (\alpha, \gamma)$.

ii) $(\alpha, \lambda\beta) = \lambda\,(\alpha, \beta)$, $\lambda \in \mathbb{R}$.

iii) $(\beta, \alpha) = -\,(\alpha, \beta)$.

iv) $(\alpha, (\beta, \gamma)) + (\beta, (\gamma, \alpha)) + (\gamma, (\alpha, \beta)) = 0$ (*Jacobi identity*) .

v) $(\alpha, f\beta) = (X_\alpha \cdot f)\,\beta + f\,(\alpha, \beta)$, $f \in \mathcal{D}(M)$.

2.3. Proposition. – *If* α *and* β *are two closed Pfaffian forms on* M^{2n} *then one will have* $(\alpha, \beta) = -\,d\,(\omega\,(X_\alpha , X_\beta))$.

Indeed:

$$
\begin{aligned}
(\alpha, \beta) &= \boldsymbol{i}_{(X_\alpha, X_\beta)}\omega \\
&= \mathbf{L}_{X_\alpha}\boldsymbol{i}_{X_\beta}\omega - \boldsymbol{i}_{X_\beta}\mathbf{L}_{X_\alpha}\omega \\
&= \mathbf{L}_{X_\alpha}\beta - \boldsymbol{i}_{X_\beta}\boldsymbol{d}\boldsymbol{i}_{X_\alpha}\omega - \boldsymbol{i}_{X_\beta}\boldsymbol{i}_{X_\alpha}\boldsymbol{d}\omega \\
&= \mathbf{L}_{X_\alpha}\beta - \boldsymbol{i}_{X_\beta}\boldsymbol{d}\alpha \\
&= \boldsymbol{d}_{X_\alpha}\beta + \boldsymbol{i}_{X_\beta}\boldsymbol{d}\beta \\
&= \boldsymbol{d}(\boldsymbol{i}_{X_\beta}\boldsymbol{i}_{X_\alpha}\boldsymbol{d}\omega) = -\,d\,(\omega\,(X_\alpha , X_\beta)) .
\end{aligned}
$$

2.4. Definition. – *Let f and g be two differentiable functions on* M^{2n} *, and let* α *and* β *be the differentials of f and g. The* ***Poisson bracket*** *(relative to the symplectic structure on* M^{2n} *) of the functions f and g is the differentiable function:*

$$(f, g) = -\,\omega\,(X_\alpha\,, X_\beta) = X_\alpha \cdot g = -\,X_\beta \cdot f\,.$$

One will then have (Prop. **2.3**) $d\,(f, g) = (df, dg)$.

2.5. Proposition. – *The Poisson bracket in* $\mathcal{D}\,(M)$ *has the following properties:*

i) $\quad (f, g + h) = (f, g) + (f, h)\,.$

ii) $\quad (f, \lambda\, g) = \lambda\,(f, g), \quad \lambda \in \mathbb{R}\,.$

iii) $\quad (g, f) = -\,(f, g)\,.$

iv) $\quad (f, (g, h)) + (g, (h, f)) + (h, (f, g)) = 0 \qquad$ *(Jacobi identity).*

v) $\quad (f, gh) = h\,(f, g) + g\,(f, h)\,.$

Proof. – Let α, β, and γ be the differentials of f, g, and h, resp. One has:

i) $\quad (f, g + h) = -\,\omega\,(X_\alpha\,, X_\beta + X_\gamma) = (f, g) + (f, h)\,.$

ii) $\quad (f, \lambda\, g) = -\,\omega\,(X_\alpha\,, \lambda\, X_\beta) = \lambda\,(f, g)\,.$

iii) $\quad (g, f) = -\,\omega\,(X_\beta\,, X_\alpha) = -\,(f, g)\,.$

iv)
$$\begin{aligned}(f, (g, h)) &= X_\alpha \cdot (g, h) = X_\alpha \cdot (X_\beta \cdot h)\,,\\ (g, (h, f)) &= X_\beta \cdot (h, f) = -\,X_\beta \cdot (X_\alpha \cdot h)\,,\\ (h, (f, g)) &= -\,[X_\alpha\,, X_\beta] \cdot h \qquad \text{[because } d\,(f, g) = (\alpha, \beta)]\,,\end{aligned}$$

so

$$(f, (g, h)) + (g, (h, f)) + (h, (f, g)) = 0\,.$$

v) $\quad (f, gh) = -\,\omega\,(X_\alpha\,, h\,X_\beta + g\,X_\gamma) = h\,(f, g) + g\,(f, h)$

[because $d\,(gh) = h\,(dg) + g\,(dh)$]

Q.E.D.

2.6. Local expression. – Let $(q_1\,, \ldots, q_n\,, p_1\,, \ldots, p_n)$ be a local coordinate system on an open subset U of such that $\omega\,|_U = \sum_i dp_i \wedge dq_i$.

If $\alpha = \sum_i (a_i\, dq_i + b_i\, dp_i)$ then one will have $X_\alpha = \sum_i \left(-b_i \frac{\partial}{\partial q_i} + a_i \frac{\partial}{\partial p_i} \right)$. Consequently,

$$(f, g) = -\sum_i \left(\frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} \right).$$

One will then recover the classical expression for the Poisson bracket, up to sign.

Exercise. – In order for a diffeomorphism on a symplectic manifold to be symplectic, it is necessary and sufficient that it must be compatible with the Poisson bracket.

2.7. Definition. – *Two Pfaffian forms α and β on a symplectic manifold (M^{2n}, ω) are* ***in involution*** *when one has $\omega(X_\alpha, X_\beta) = 0$.*

Two differentiable functions are *in involution* if their differentials *df* and *dg* are.

Consequently, if α and β are two closed Pfaffian forms in involution then their Poisson bracket (α, β) will be zero. Conversely:

2.8. Proposition. – *In order for two differentiable functions f and g to be in involution, it is necessary and sufficient that their Poisson bracket (f, g) should be zero.*

2.9. Proposition. – *In order for two closed Pfaffian forms α and β to be in involution, it is necessary and sufficient that α (β, resp.) should be a first integral of X_β (X_α, resp.). Indeed:*

$$\omega(X_\alpha, X_\beta) = -\beta(X_\alpha) = \alpha(X_\beta).$$

2.10 Proposition. – *Let α, β, and γ be three closed Pfaffian forms. If α is in involution with β and γ then it will also be in involution with the Poisson bracket (β, γ).*

Indeed, $X_{(\beta,\gamma)} = [X_\beta, X_\gamma]$, and:

$$\begin{aligned} \omega(X_\alpha, X_{(\beta,\gamma)}) &= \alpha([X_\beta, X_\gamma]) \\ &= \alpha(X_\beta) - \beta(X_\alpha) \qquad (d\alpha = 0) \\ &= 0. \end{aligned}$$

§ 3. Hamiltonian systems (E. Cartan [**3**]).

3.1. Definition. – *A* ***Hamiltonian*** *(dynamical)* ***system*** *on a symplectic manifold (M^{2n}, ω) is a vector field X on M^{2n} such that $\boldsymbol{i}_X\omega$ is a closed Pfaffian form.*

If $\boldsymbol{i}_X\omega$ is an exact form then a *Hamiltonian* for X is a differentiable function H on M^{2n} such that $\boldsymbol{i}_X\omega = - \alpha$ ($\boldsymbol{i}_X\omega = - dH$, resp.). One says that X is the Hamiltonian system that is associated with α (H, resp.).

3.2. Proposition. – *In order for a vector field X on a symplectic manifold (M^{2n}, ω) to be a Hamiltonian system, it is necessary and sufficient that one must have* $\mathbf{L}_X\omega = 0$.

Indeed, $\mathbf{L}_X\omega = \boldsymbol{d}\boldsymbol{i}_X\omega$.

Let X be the Hamiltonian system that is associated with a closed Pfaffian form α on a symplectic manifold (M^{2n}, ω). One immediately has:

3.3. Proposition. – *In order for a point y of M^{2n} to be a zero of X, it is necessary and sufficient that it should be a singular point of α.*

3.4. Proposition. – *The Pfaffian form α is a first integral of X.*

In particular, if $\alpha = dH$ then H will be a first integral of X : H is the *energy integral.*

Let U be the set of points y of M^{2n} such that $\alpha(y) \neq 0$; U is an open subset of M^{2n}, and the Pfaffian form α generates a Pfaff system (α) on U that is integrable of rank 1.

3.5. Proposition. – *Let (N^{2n-1}, h) be an integral manifold of (α). Hence:*

i) *The vector field X is tangent to $h(N^{2n-1})$.*

ii) *$h^*\omega$ is a closed differential form of degree* 2 *and constant class* $2n - 2$ *on* N^{2n-1}.

iii) *The differential system $\mathcal{C}(h^*\omega)$ is generated by the vector field Y that is induced by X on N^{2n-1}.*

Proof. – The first property is immediate, since $\alpha(X) = 0$.

Let x be a point of N^{2n-1} and let $(e_1, \ldots, e_{2n})$ be a basis for $T_{h(x)}(M)$ such that if $(\varepsilon_1, \ldots, \varepsilon_{2n})$ is the dual basis on $T^*_{h(x)}(M)$ then one will have (Chap. I, Cor. **8.2**):

$$\alpha(h(x)) = \varepsilon_{2n},$$
$$\omega(h(x)) = \varepsilon_1 \wedge \varepsilon_2 + \ldots + \varepsilon_{2n-1} \wedge \varepsilon_{2n},$$
$$\mathrm{X}(h(x)) = \varepsilon_{2n-1}.$$

The linear forms $\eta_i = (h_x^{\mathrm{T}})^*\varepsilon_i$, $1 \leq i \leq 2n - 1$ form a basis for $T_x^*(N)$, and one will have:

$$(h^*\omega)(x) = \eta_1 \wedge \eta_2 + \ldots + \eta_{2n-3} \wedge \eta_{2n-2}.$$

That shows that $h^*\omega$ has constant class $2n-2$ on N^{2n-1} and that the characteristic subspace of $h^*\omega(x)$ is generated by $Y(x)$.

Q.E.D.

In particular, if $\alpha = dH$, and if c is a regular value of H then one can take N^{2n-1} to be the submanifold $H^{-1}(c)$.

3.6. Proposition. – *There exists a differential form π of degree $2n-1$ on U such that $(\omega|_U)^* = \alpha\wedge\pi$. One will then have $\mathbf{L}_X\pi = \alpha\wedge\rho$, $\rho\in\Lambda^{2n-2}(U)$.*

If π is a second differential form on U such that $(\omega|_U)^n = \alpha\wedge\pi'$ then one can write $\pi' = \pi + \alpha\wedge\sigma$, $\sigma\in\Lambda^{2n-2}(U)$.

Proof. – Since $\alpha\wedge\omega^n = 0$, there exists a differential form $\pi\in\Lambda^{2n-2}(U)$ such that $(\omega|_U)^n = \alpha\wedge\pi$ (Chap. V, Prop. **4.12**). One will then have:

$$0 = \mathbf{L}_X(\omega|_U)^n = \mathbf{L}_X(\alpha\wedge\pi) = \alpha\wedge(\mathbf{L}_X\pi),$$

and consequently $\mathbf{L}_X\pi = \alpha\wedge\rho$, $\rho\in\Lambda^{2n-2}(U)$.

Finally, if π' is a second differential form on U such that:

$$(\omega|_U)^n = \alpha\wedge\pi' \qquad \text{then one will have} \qquad \alpha\wedge(\pi-\pi') = 0.$$

Hence, $\pi - \pi' = \alpha\wedge\sigma$, $\sigma\in\Lambda^{2n-2}(U)$.

Q.E.D.

3.7. Corollary. – *Let (N^{2n-1}, h) be an integral manifold of α, and let $\pi\in\Lambda^{2n-2}(U)$ be a differential form such that $(\omega|_U)^n = \alpha\wedge\pi$. The form $\Pi = h^*\omega$ will then possess the following properties:*

i) Π *is independent of the choice of the differential form $\pi\in\Lambda^{2n-2}(U)$ such that* $(\omega|_U) = \alpha\wedge\pi$.

ii) Π *is a volume form on* N^{2n-1}.

iii) *If Y is the vector field on N^{2n-1} that is induced by X then* $\mathbf{L}_Y\Pi = 0$.

3.8. Local expression. – Let $(q_1, \ldots, q_n, p_1, \ldots, p_n)$ be a local coordinate system on an open subset U of M^{2n} such that $\omega|_U = \sum_i dp_i\wedge dq_i$. If $\alpha = dH$ is a closed Pfaffian form on U then the local expression for the Hamiltonian system X that is associated with α will be $\sum_i \left(\frac{\partial H}{\partial p_i} \frac{\partial}{\partial q_i} - \frac{\partial H}{\partial q_i} \frac{\partial}{\partial p_i} \right)$. The integral curves of X will then be the solutions to the *Hamiltonian equations:*

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \qquad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i} .$$

3.9. *Examples.* – Let Λ be the canonical symplectic form (Corollary **1.7**) on the cotangent bundle $T^*(M)$ to a manifold M^m .

i) *Finsler structure.* – That is defined when one is given a differentiable function H on $T^*(M)$ that has the following properties:

– H is positively homogeneous of degree $p > 0$.
– $H^{-1}(0)$ is the image of the zero section of $T^*(M)$.
– Any number $\lambda > 0$ is a regular value of H.

The Hamiltonian system X that is associated with H is called the *geodesic field* of the Finsler structure, and the projections of the integral curves of X onto M^{2n} are the *geodesics* of that structure.

ii) *Riemannian structure.* – That is determined by a Riemannian metric T on the cotangent bundle $\tau^*(M)$.

One can verify that T is a Finsler structure on $T^*(M)$ (with $p = 2$).

iii) *Classical Hamiltonian structure.* – That is determined by a differentiable function H on $T^*(M)$ that has the form $T - U \circ q_M$, where:

– T is a Riemannian metric on $\tau^*(M)$.
– U is a differentiable function on M^m .

One can generalize the Hamiltonian systems in the following fashion:

3.10. Proposition (E. Cartan [**3**]). – *Let* (M^{2n} , ω) *be a symplectic manifold, and let H be a differential function on* $M^{2n} \times \mathbb{R}$ *. There exists one and only one vector field Y on* $M^{2n} \times \mathbb{R}$ *that has the following properties:*

i) $Y(x, t) = X_t(x) + \frac{\partial}{\partial t}$ *in* $T_{(x, t)}(M^{2n} \times \mathbb{R}) = T_x(M) \oplus T_t(\mathbb{R})$.

ii) $\boldsymbol{i}_{\mathrm{Y}}(p_1^* \omega - dH \wedge dt) = 0$.

Proof. – Let H_t be the restriction of H to $M^{2n} \times \{t\}$. Equation *ii*) becomes:

$$\boldsymbol{i}_{\mathrm{X}_t} \omega - (X_t \cdot H_t)\, dt + dH_t = 0,$$

or

$$\boldsymbol{i}_{X_t} \omega = - dH_t \qquad \text{and} \qquad X_t \cdot H_t = 0.$$

One must then take X_t to be a Hamiltonian on $M^{2n} \times \mathbb{R}$ that is associated with H_t .

The local expressions for X_t (§ **3.8**) then show that:

$$(x, t) \to Y(x, t) = X_t(x) + \frac{\partial}{\partial t}$$

is a vector field on $M^{2n} \times \mathbb{R}$.

Q.E.D.

Remark. – When H is independent of t, X_t will also be independent of t, and it will be equal to the Hamiltonian system X that is associated with H.

3.11. Corollary. – *One has* $Y \cdot H = \dfrac{\partial H}{\partial t}$.

The function H is not generally a first integral of Y then.

§ 4. First integrals of Hamiltonian systems.

Let X be a Hamiltonian system on a symplectic system (M^{2n}, ω), and let $\alpha = - \boldsymbol{i}_X \omega$. When one reformulates Propositions **2.9** and **2.10**, one will get:

4.1. Proposition. – *In order for a closed Pfaffian form β on M^{2n} to be a first integral of X, it is necessary and sufficient that α and β should be in involution.*

Consequently, if Y is a Hamiltonian system on M^{2n}, in order for $\boldsymbol{i}_X \omega$ to be a first integral of X, it is necessary and sufficient that $[X, Y] = 0$.

4.2. Proposition. – *If β and γ are two first integrals of X then their Poisson bracket (β, γ) will also be a first integral.*

4.3. Proposition (Gallisot [**7**]). – *Let $\beta_1, \dots, \beta_{n-1}$ be $n - 1$ first integrals of X that have the following properties:*

i) *The forms α, $\beta_1, \dots, \beta_{n-1}$ are independent on an open subset U of M^{2n}.*

ii) *The forms β_i are pair-wise in involution.*

There will then exist n Pfaffian forms $\gamma, \gamma_1, \dots, \gamma_{n-1}$ on U that have the following property:

i) $\omega|_U = \alpha \wedge \gamma + \sum_i \beta_i \wedge \gamma_i$.

ii) *The differentials $\boldsymbol{d}\gamma$ and $\boldsymbol{d}\gamma_i$ belong to the ideal of* Λ (U) *that is generated by α, β_1, …, β_{n-1} .*

Proof. – Let $Y_1, \dots, Y_{n-1}$ be the vector fields on M^{2n} that are defined by $\boldsymbol{i}_{Y_i}\omega = \beta_i$. One has:

$$\beta_i(X) = \beta_i(Y_j) = 0,$$
$$\alpha_i(X) = \alpha_i(Y_j) = 0, \qquad i, j = 1, \dots, n-1.$$

Consequently:

$$\boldsymbol{i}_X\, \boldsymbol{i}_{Y_{n-1}} \dots \boldsymbol{i}_{Y_i}\omega^q = \pm\, q\,(q-1) \dots (q-n+1)\, \beta_1 \wedge \dots \wedge \beta_{n-1} \wedge \alpha \wedge \omega^{q-n} .$$

If one takes $q = n + 1$ then one will get $\beta_1 \wedge \dots \wedge \beta_{n-1} \wedge \alpha \wedge \omega = 0$, which shows (Chap. V, Prop **4.12**) that ω belongs to the ideal of Λ (U) that is generated by α, $\beta_1, \dots, \beta_{n-1}$. There will then exist n Pfaffian forms $\gamma, \gamma_1, \dots, \gamma_{n-1}$ on U such that:

$$\omega|_U = \alpha \wedge \gamma + \sum_i \beta_i \wedge \gamma_i .$$

The forms α, $\beta_1, \dots, \beta_{n-1}$, $\gamma_1, \dots, \gamma_{n-1}$ are, in turn, independent on U.

One has $(d\omega)|_U = -\,\alpha \wedge d\gamma - \sum_i \beta_i \wedge \boldsymbol{d}\gamma_i$. If one multiplies that by:

$$\alpha \wedge \beta_1 \wedge \dots \wedge \beta_{i-1} \wedge \beta_{i+1} \wedge \dots \wedge \beta_{n-1} \qquad (\beta_1, \dots, \beta_{n-1}, \text{resp.})$$

then one will get:

$$\beta_1 \wedge \dots \wedge \beta_{n-1} \wedge \alpha \wedge \boldsymbol{d}\gamma_i = 0, \qquad i = 1, \dots, n-1$$
$$(\beta_1 \wedge \dots \wedge \beta_{n-1} \wedge \alpha \wedge \boldsymbol{d}\gamma = 0, \ \text{resp.})$$

That proves property *ii*).

Q.E.D.

The Pfaffian forms $\alpha, \beta_1, \ldots, \beta_{n-1}$ generate a Pfaff system $\mathcal{P}$ on U that is integrable of rank n. If (N^n, h) is an integral manifold of $\mathcal{P}$ then X will be tangent to $h(N^n)$, and it will induce a vector field Z on N^n.

Therefore, let $\pi = h^*(\gamma)$ and $\pi = h^*(\gamma_i)$, $i = 1, \ldots, n-1$. One has:

4.4. Theorem (Liouville-Cartan integrability theorem). – *Under the hypotheses above, the following properties will be verified:*

i) $\pi, \pi_1, \ldots, \pi_{n-1}$ *are independent on* N^n.

ii) $\pi_1, \ldots, \pi_{n-1}$ *are* $n-1$ *first integrals of* Z : $\boldsymbol{d}\pi_i = 0$ *and* $\pi_i(Z) = 0$.

iii) $\boldsymbol{d}\pi = 0$ *and* $\pi(Z) = 1$.

That theorem then expresses the idea that the vector field Z on N^n admits $n-1$ independent first integrals; Z is therefore "integrable by quadratures."

Proof. – One deduces from Proposition **4.3** that $\pi, \pi_1, \ldots, \pi_{n-1}$ are independent and that $\boldsymbol{d}\pi = \boldsymbol{d}\pi_i = 0$, $i = 1, \ldots, n-1$. On the other hand, one has $\pi(Z) = \gamma(X)$ and $\pi_i(Z) = \gamma_i(X)$. Now, $\boldsymbol{i}_X\omega = -\gamma(X)\,\alpha - \sum \gamma_i(X)\beta_i = -\alpha$. Consequently, $\gamma(X) = 1$ and $\gamma_i(X) = 0$, $i = 1, \ldots, n-1$.

Q.E.D.

In the case where X is a Hamiltonian system on the cotangent bundle to a manifold M^m (that is endowed with its canonical symplectic structure), the "symmetry groups" determine the first integrals of X:

4.3. Proposition. – *Let* φ_t *be a one-parameter group of diffeomorphisms of* M^m. *There exists a one-parameter group* ψ_t *of diffeomorphisms of* $T^*(M)$ *that has the following properties:*

i) $q_M \circ \psi_t = \varphi_t \circ q_M$.

ii) $(\psi_t)^*\lambda = \lambda$ [λ *is the Liouville form on* $T^*(M)$].

Proof. – Let $\{(U_i, \varphi_i)\}$ be an atlas on M^m; $(U_i, [D(\varphi_i\varphi_j^{-1})]^*)$ is a cocycle that defines $\tau^*(M)$.

If φ_t is the one-parameter group of diffeomorphisms of M^m that is generated by X then the maps $(\psi_t)_{ij} = [D(\varphi_i\,\varphi_i^{-1}\varphi_j^{-1})]^*$ will determine (Chap. II, Th. **2.10**) a one-parameter group

ψ_t of diffeomorphisms of $T^*(M)$ such that $q_M \circ \psi_t = \varphi_t \circ q_M$. That one-parameter group is characterized by the relations:

$$< \varphi_i^{\mathrm{T}} \mathrm{u} , \psi_t\, \alpha > = < \mathrm{u}, \alpha >, \qquad \mathrm{u} \in T_y(M) \quad \text{and} \quad \alpha \in T_y^*(M) .$$

If $\alpha \in T^*(M)$ and $u \in T_\alpha(T^*(M))$ then one will have that:

$$\begin{aligned} < u, (\psi_i^* \lambda)\alpha > &= < \psi_i^{\mathrm{T}} u , \lambda(\psi_t\, \alpha) > = < q_M^{\mathrm{T}}\, \psi_i^{\mathrm{T}} u , \psi_t\, \alpha > \\ &= < q_i^{\mathrm{T}}\, q_M^{\mathrm{T}} u , \psi_t\, \alpha > = < q_M^{\mathrm{T}} u , \alpha > = < u, \lambda(\alpha) >. \end{aligned}$$

Consequently, $\psi_i^* \lambda = \lambda$ for any t.

Q.E.D.

Exercises:

i) Proposition **4.5** will remain valid when φ_t is a local one-parameter group.

ii) Let h be a diffeomorphism of M^m. The map $\alpha \mapsto [(h_x^T)^{-1}]^*(\alpha)$ of $T_x^*(M)$ to $T_{h(x)}^*(M)$ determines a diffeomorphism $\overline{h}$ of $T^*(M)$, and $(\overline{h}, h)$ is an automorphism of $\tau^*(M)$.

4.6. Proposition. – *Let Y be a vector field on a manifold M^m. There exists one and only one vector field Z on the cotangent bundle $T^*(M)$ that has the following properties:*

i) $q_M^{\mathrm{T}} Z = Y \circ q_M$.

ii) $\mathbf{L}_X \lambda = 0$ [λ *is the Liouville form on* $T^*(M)$] .

Proof. – Let φ_t be a local one-parameter group of diffeomorphisms of M^m that is generated by X and let ψ_t be the correspond local group on $T^*(M)$ (Prop. **4.5**). The vector field Z that generates ψ_t will then have the desired properties.

The proof of the uniqueness of Z is carried out locally.

Let $(y_1, \ldots, y_m)$ be a local coordinate system on an open subset U of M^m, and let $\sum_i a_i \dfrac{\partial}{\partial y_i}$ be the local expression for X on U.

The functions $q_i = y_i \circ q_M$ and $p_i = \partial / \partial y_i$ define a local coordinate system on $q_M^{-1}(U)$. Writing $Z = \sum_i \left(b_i \frac{\partial}{\partial q_i} + c_i \frac{\partial}{\partial p_i} \right)$, one must then have:

$$q_M^{-1} Z = \sum_i b_i \frac{\partial}{\partial y_i} = Y \circ q_M = \sum_i (a_i \circ q_M) \frac{\partial}{\partial y_i}, \qquad \text{namely,} \qquad b_i = a_i \circ q_M,$$

and

$$\begin{aligned} \mathbf{L}_Z \lambda &= \boldsymbol{i}_Z\, d\lambda + \boldsymbol{d}\, \boldsymbol{i}_Z \lambda \\ &= \sum_i c_i\, dq_i - \sum_i b_i\, dp_i + \sum_i b_i\, dp_i + \sum_i p_i\, db_i \\ &= \sum_i c_i\, dq_i + \sum_i p_i \left(\frac{\partial a_i}{\partial q_j} \circ q_M \right) dq_i = 0, \end{aligned}$$

when one lets $c_i = -\sum_j p_i \left(\frac{\partial a_j}{\partial q_j} \circ q_M \right)$.

Q.E.D.

For example, if $Y = \partial / \partial y_i$, $Z = \partial / \partial q_i$ in the preceding local coordinate system then one will have:

4.7. Corollary. – *The vector field Z is a Hamiltonian system on* $(T^*(M), \Lambda = d\lambda)$.

Indeed, $\boldsymbol{i}_Z \Lambda = -\boldsymbol{d}\, (\lambda\, (Z))$.

4.8. Theorem. – *Let X be a Hamiltonian system on a cotangent space $T^*(M)$, and let $\alpha = -\boldsymbol{i}_X \Lambda$. If Z is a vector field on M^m such that $\alpha(Z) = 0$ (with the notations of* **4.5**) *then $\lambda(Z)$ will be a first integral of X.*

Indeed:

$$\begin{aligned} \boldsymbol{d}\lambda\, (X, Z) &= -\alpha\, (Z) \\ &= X \cdot \lambda\, (Z) - Z \cdot \lambda\, (X) - \lambda\, ([X, Z]) \\ &= X \cdot \lambda\, (Z) - (\mathbf{L}_Z\, \lambda)\, (X) = X \cdot \lambda\, (Z)\,. \end{aligned}$$

In this case, when Z generates a one-parameter group φ_t of diffeomorphisms of M^m, one says that φ_t is a *symmetry group* of X.

4.9. *Examples:*

i) *Classical Hamiltonian systems.* – Let Y be a vector field on M^m, let Z be the corresponding vector field on $T^*(M)$, and let $H = T - U \circ p_M$ be a differentiable function on $T^*(M)$ that defines a classical Hamiltonian structure [Example *iii*) of **3.9.**].

If $Y \cdot U = Z \cdot T = 0$ then $\lambda(Z)$ will be a first integral of X.

The following example is an explicit case of that situation:

ii) *Motion of a body.* – Let M^{3n} be the set of points:

$$(x_1, y_1, z_1, \ldots, x_n, y_n, z_n)$$

of $\mathbb{R}^{3n}$ such that $(x_i, y_i, z_i) \neq (x_j, y_j, z_j)$ for $i \neq j$; M^{3n} is an open subset of $\mathbb{R}^{3n}$, and the maps:

$$\begin{aligned}
(x_i, y_i, z_i) &\mapsto (x_i + t, y_i, z_i) ,\\
(x_i, y_i, z_i) &\mapsto (x_i, y_i + t, z_i) ,\\
(x_i, y_i, z_i) &\mapsto (x_i, y_i, z_i + t) ,\\
(x_i, y_i, z_i) &\mapsto (x_i \cos t - y_i \sin t, x_i \sin t + y_i \cos t, z_i) ,\\
(x_i, y_i, z_i) &\mapsto (x_i, y_i \cos t - z_i \sin t, y_i \sin t + z_i \cos t, z_i) ,\\
(x_i, y_i, z_i) &\mapsto (x_i \cos t + z_i \sin t, y_i, - x_i \sin t + z_i \cos t, z_i)
\end{aligned}$$

define six one-parameter groups of diffeomorphisms of M^{3n}.

The function:

$$H = \tfrac{1}{2} \sum_i \frac{1}{m_i} \left[\left(\frac{\partial}{\partial x_i} \right)^2 + \left(\frac{\partial}{\partial y_i} \right)^2 + \left(\frac{\partial}{\partial z_i} \right)^2 \right] + \left[\sum_{i<j} \frac{k\, m_i\, m_j}{\sqrt{(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2}} \right] \circ q_M$$

is a differentiable function on the cotangent bundle $T^*(M) = M^{3n} \times \mathbb{R}^{3n}$ that determines a classical Hamiltonian system X on $T^*(M)$.

Each of the six preceding one-parameter groups is then a group of symmetries of X, and the corresponding first integrals are the kinetic resultant and kinetic moment.

§ 5. Contact structures (G. Reeb [**13**]).

5.1. Definition. – Let M^{2n+1} be a manifold of odd dimension $2n + 1$. A *contact structure* on M^{2n+1} is defined when one is given a Pfaffian form $\alpha \in \Lambda^1(M)$ of constant class $2n + 1$.

In particular, α is a Pfaffian form without singularities. One also says that α is a *contact form* on M^{2n+1}.

If U is an open subset M^{2n+1} then $\alpha|_U$ will be a contact form on U.

5.2. Proposition. – *Let α be a Pfaffian form on a manifold M^{2n+1}. In order for α to be a contact form, it is necessary and sufficient that $\alpha \wedge (\boldsymbol{d}\alpha)^n$ should be a volume form.*

That assertion is a direct consequence of Proposition **1.6** of Chapter VI.

5.3. Corollary. – *If a manifold admits a contact structure then it will be orientable.*

In that case, one can orient M^{2n+1} by way of the volume form $\alpha \wedge (\boldsymbol{d}\alpha)^n$.

5.4. *Example*. – The Pfaffian form:

$$\alpha = dx_1 + x_2\, dx_3 + \ldots + x_{2n}\, dx_{2n+1}$$

is a contact form on $\mathbb{R}^{2n+1}$; indeed:

$$\alpha \wedge (\boldsymbol{d}\alpha)^n = n\,!\; dx_1 \wedge \ldots \wedge dx_{2n+1}\ .$$

The orientation that is associated with α is the canonical orientation on $\mathbb{R}^{2n+1}$.

5.5. Theorem (G. Reeb [**13**]). – *Let α be a contact form on a manifold M^{2n+1}. There exists one and only one vector field $Y \in \mathcal{T}(M)$ such that:*

$$\alpha(\mathrm{Y}) = 1 \qquad \text{and} \qquad \boldsymbol{i}_{\mathrm{Y}}(\boldsymbol{d}\alpha) = 0.$$

One says that Y is a dynamical system on M^{2n+1} that is associated with the contact form α.

Proof. – For any point y of $(y_1, \ldots, y_{2n+1})$, there exists a local coordinate system on an open neighborhood U of y such that:

$$\alpha|_U = dy_1 + y_2\, dy_3 + \ldots + y_{2n}\, dy_{2n+1}\ .$$

One will then have:

$$(\alpha|_U)\left(\frac{\partial}{\partial y_1}\right) = 1 \qquad \text{and} \qquad \boldsymbol{i}_{\partial/\partial y_1}(\boldsymbol{d}\alpha) = 0\ .$$

Since the associated system to $(\boldsymbol{d}\alpha)|_U$ is generated by $\partial / \partial y_1$, that will show the existence and uniqueness of the vector field Y.

Q.E.D.

5.6. Corollary. – *The vector field Y that is associated with the contact form α has no singularities.*

5.7. Corollary. – *One has* $\mathbf{L}_Y(\alpha) = \mathbf{L}_Y(\boldsymbol{d}\alpha) = 0$.

Indeed:

$$\begin{aligned}\mathbf{L}_Y(\alpha) &= \boldsymbol{d}\,\boldsymbol{i}_Y\alpha + \boldsymbol{i}_Y\,\boldsymbol{d}\alpha = 0\,,\\ \mathbf{L}_Y(\boldsymbol{d}\alpha) &= \boldsymbol{d}\,\mathbf{L}_Y(\alpha) = 0\,.\end{aligned}$$

More generally:

5.8. Corollary. – *Let f be a differentiable function on* M^{2n+1}. *One has* $\mathbf{L}_{fY}(\boldsymbol{d}\alpha) = 0$ *and* $\mathbf{L}_{fY}(\alpha) = 0$.

Indeed:

$$\begin{aligned}\mathbf{L}_{fY}(\alpha) &= f\,\mathbf{L}_Y(\alpha) + df \wedge \boldsymbol{i}_Y\alpha = df,\\ \mathbf{L}_{fY}(\boldsymbol{d}\alpha) &= \boldsymbol{d}\,(\mathbf{L}_{fY}\alpha) = 0.\end{aligned}$$

5.9. Theorem. – *Let α be a Pfaffian form on a manifold M^{2n} such that $\omega = \boldsymbol{d}\alpha$ is a symplectic form, and let X be a Hamiltonian system with no singularities on (M^{2n}, ω). If (N^{2n-1}, h) is an integral manifold of the Pfaff system that is generated by $\boldsymbol{i}_X\omega$, and if $\alpha(X)$ is not annulled on $h(N^{2n-1})$ then the following properties will be verified:*

i) *$h^*\alpha$ is a contact form on N^{2n-1}.*

ii) *If Y is the vector field that is induced by X on N^{2n-1} then the associated dynamical system to $h^*\alpha$ will be* $\dfrac{Y}{h^*\alpha(Y)} = \dfrac{Y}{\alpha(X)\circ h}$.

Proof. – Let x be a point of N^{2n-1}, and let $(e_1, \ldots, e_{2n})$ be a basis for $T_{h(x)}(M)$ such that $(\varepsilon_1, \ldots, \varepsilon_{2n})$ is the dual basis for $T^*_{h(x)}(M)$. One will have:

$$\begin{aligned}\boldsymbol{i}_X\omega\,(h(x)) &= \varepsilon_{2n}\,,\\ \omega\,(h(x)) &= \varepsilon_1 \wedge \varepsilon_2 + \ldots + \varepsilon_{2n-1} \wedge \varepsilon_{2n}\,,\\ X\,(h(x)) &= \varepsilon_{2n-1}\,,\\ \omega\,(h(x)) &= \sum_{i=1}^{2n} a_i\,\varepsilon_i\,, \qquad \text{with} \quad a_{2n-1} \neq 0.\end{aligned}$$

The linear forms $\eta_i = (h_x^{\mathrm{T}})^* \varepsilon_i$, $1 \leq i \leq 2n - 1$ define a basis for $T_x^*(M)$, and one will have:

$$h^* \omega(x) = \eta_1 \wedge \eta_2 + \ldots + \eta_{2n-1} \wedge \eta_{2n},$$

$$h^* \alpha(x) = \sum_{i=1}^{2n-1} a_i \, \eta_i, \qquad \text{with} \qquad a_{2n-1} \neq 0.$$

Those expressions show that $h^* \alpha$ has class $2n - 1$ at x.

Finally, since $h^* \alpha(Y) = \alpha(Y) \circ h$ is not annulled on N^{2n-1}, the dynamical system that is associated with the contact form $h^* \alpha$ is $Y / h^* \alpha(Y)$.

Q.E.D.

5.10. *Examples:*

If the notations are the ones in **3.9** then consider a local coordinate system $(y_1, \ldots, y_m)$ on an open subset V of M^m, and the corresponding local coordinate system on $q_M^{-1}(V)$ that is defined by $q_i = y_i \circ q_M$ and $p_i = \partial / \partial y_i$.

i) *Finsler structure.* – Let $h > 0$ and let N^{2n-1} be the submanifold of $T^*(M)$ that is defined by $H = h$. One locally writes:

$$\begin{aligned}
\lambda &= \sum_i p_i \, dq_i \,, \\
X &= \sum_i \left(\frac{\partial H}{\partial p_i} \frac{\partial}{\partial q_i} - \frac{\partial H}{\partial q_i} \frac{\partial}{\partial p_i} \right) \\
&= \sum_i \left(\frac{\partial T}{\partial p_i} \frac{\partial}{\partial q_i} - \frac{\partial H}{\partial q_i} \frac{\partial}{\partial p_i} \right), \\
\lambda(X) &= \sum_i p_i \frac{\partial T}{\partial p_i} = 2T.
\end{aligned}$$

Consequently, if $\bar{N}^{2n-1}$ is the complement in N^{2n-1} to the image of the zero section of $T^*(M)$ ($\bar{N}^{2n-1}$ is an open submanifold of N^{2n-1}) then the Liouville form λ will induce a contact structure on $\bar{N}^{2n-1}$ that has $X / 2T$ for its associated dynamical system.

Let W be open subset of M^m that is defined by $U \neq -h$. $q_M^{-1}(W)$ is an open subset of $T^*(M)$ that contains $\bar{N}^{2n-1}$, and $T' = \dfrac{T}{U \circ q_M + h}$ is a Riemannian metric on the cotangent bundle $\tau^*(W)$. The submanifold $\bar{N}^{2n-1}$ is then characterized by $T' = 1$.

Let Y be geodesic field of T'. Y is defined by the relation $\boldsymbol{i}_Y \boldsymbol{d}\lambda = -\boldsymbol{d}T'$; it will then be tangent to $\bar{N}^{2n-1}$. Furthermore, one has $\lambda(Y) = 2T'$ [Example *i*)].

Consequently, since λ induces a contact structure on $\bar{N}^{2n-1}$, one will have $X / T = Y / T'$, or rather $X = (U \circ q_M + h) \, Y = T \, Y$.

5.11. Proposition (Maupertuis's principle). – *On the constant-energy submanifold that is defined by* $H = h$ *and* $T \neq 0$, *the Hamiltonian system* X *is equal to* $T\,Y$, *where* Y *is the geodesic field of the Riemannian structure* $\dfrac{T}{U \circ q_M + h}$.

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CHAPTER VIII

INVARIANT FORMS. INTEGRAL INVARIANTS.

§ 1. – Invariant forms.

1.1 Definition. – *Let X be a vector field on a manifold M^m. A differential form $\alpha \in \Lambda(M)$ is* ***invariant under*** *X if one has* $\mathrm{L}_X\alpha = 0$.

A function $f \in \mathcal{D}(M)$ that is invariant under X is therefore a first integral for X.

1.2 Proposition. *The set of differential forms that are invariant under X is a subalgebra of $\Lambda(M)$ that is stable under d.*

Indeed:

$$\mathrm{L}_X(\alpha \wedge \beta) = (\mathrm{L}_X \alpha) \wedge \beta + \alpha \wedge (\mathrm{L}_X \beta),$$
$$\mathrm{L}_X(d\alpha) = d(\mathrm{L}_X \alpha).$$

1.3 *Local expression.* – If the point y is not a zero of X then one can find a system $(y_1, \ldots, y_m)$ of local coordinates on an open neighborhood U of y such that $X \mid_U = \partial / \partial y_1$.

If:

$$\alpha = \sum_{1 \le i_1 < \cdots < i_p \le m} a_{i_1 \cdots i_p} dy_{i_1} \wedge \cdots \wedge dy_{i_p}$$

is a form of degree p on U then one will have:

$$\mathrm{L}_X \alpha = \sum_{1 \le i_1 < \cdots < i_p \le m} \frac{\partial a_{i_1 \cdots i_p}}{\partial y_1} dy_{i_1} \wedge \cdots \wedge dy_{i_p} .$$

Consequently, in order for any α to be invariant under X, it is necessary and sufficient that each of the functions $a_{i_1 \cdots i_p}$ must be independent of y_1 in a neighborhood of y.

1.4 *Examples:*

i) *Hamiltonian systems.* If X is a Hamiltonian on a symplectic manifold (M^{2n}, ω) then ω will be a form that is invariant under X (Chap. VII, prop 3.2).

Consequently, ω^n will be a volume form that is invariant under X; this result is the expression for *Liouville's theorem* (see prop 2.2) in terms of differentials. If (N^{2n-1}, h) is an integral manifold of the Pfaff system that is generated by $i_X\omega$, and if Y is the vector field that is induced by X on N^{2n-1} then there exists a volume form on N^{2n-1} that is invariant under Y (Chap. VII, Cor. **3.7**).

Finally, under the hypotheses of Proposition **3.10** of Chapter VII, $p_1^*\omega - dH \wedge dt$ will be a form that is invariant under Y.

ii) *Contact structures.* If Y is the dynamical system that is associated with a contact form α on a manifold M^{2n+1} then α and $d\alpha$ will be forms that are invariant under Y (Chap. VII, Cor. **5.7**)

Consequently, $\alpha \wedge (d\alpha)^n$ is a volume form that is invariant under Y.

Let N^n be a compact, orientable manifold (possibly with boundary) of dimension n, and let h be a differentiable map from N^n to a manifold M^m Since h (N) is compact, a local one-parameter group φ_t of diffeomorphisms of M^m [which is generated by the vector field $X \in \mathcal{T}(M)$] will be defined on a neighborhood $U \times I$ of $h(N) \times \{0\}$ in $M^m \times \mathbb{R}$.

Under those conditions:

1.5. Theorem. – *If α is a differential form of degree n on M^m that is invariant under X then the integral:*

$$I(t) = \int_{N^n} (\varphi_t \circ h)^* \alpha$$

will be independent of t.

That theorem is a consequence of the following proposition:

1.6. Proposition. – *Let α be a differential form of degree n on M^m and let $I(t) = \int_{N^n} (\varphi_t \circ h)^* \alpha$. One has:*

$$\frac{dI(t)}{dt} = \int_{N^n} (\varphi_t \circ h)^* \mathrm{L}_X \alpha .$$

Proof. – One has:

$$\begin{aligned} I(t+\varepsilon) - I(t) &= \int_{N^n} [(\varphi_{t+\varepsilon} \circ h)^* \alpha - (\varphi_t \circ h)^* \alpha] \\ &= \int_{N^n} (\varphi_t \circ h)^* (\varphi_t^* \alpha - \alpha) . \end{aligned}$$

Since N^n is compact, one can then write:

$$\begin{aligned} \frac{dI(t)}{dt} &= \lim_{\varepsilon \to 0} \frac{I(t+\varepsilon) - I(t)}{\varepsilon} , \\ &= \int_{N^n} (\varphi_t \circ h)^* \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} (\varphi_t^* \alpha - \alpha) , \\ &= \int_{N^n} (\varphi_t \circ h)^* \mathrm{L}_X \alpha \qquad \text{(Chap. V, Prop. } \mathbf{2.5}\text{)} \end{aligned}$$

Q. E. D.

Conversely, one has, moreover:

1.7. Proposition. – *If the integral* $I(t) = \int_{N^n} (\varphi_t \circ h)^* \alpha$ *is independent of t for any compact manifold* N^n *and any differentiable map h*: $N^n \to M^n$ *then the form* α *will be invariant under X.*

Proof. – If $X(y) \neq 0$ then one can find a system $(y_1, \ldots, y_m)$ of local coordinates on an open neighborhood U of y such that $X \mid_U = \partial / \partial y_1$.

If one writes:

$$\alpha \mid_U = \sum_{1 \le i_1 < \cdots < i_n \le m} a_{i_1 \cdots i_n} dy_{i_1} \wedge \cdots \wedge dy_{i_n}$$

then one will have:

$$\mathrm{L}_X \alpha = \sum_{1 \le i_1 < \cdots < i_n \le m} \frac{\partial a_{i_1 \cdots i_n}}{\partial y_1} dy_{i_1} \wedge \cdots \wedge dy_{i_n} \, .$$

The integral $\int_{D^n} \frac{\partial a_{i \cdots n}}{\partial y_1} dy_1 \wedge \cdots \wedge dy_n$ will be zero (Prop **1.6**) for any closed ball D^n in the subspace whose equations are:

$$y_{n+1} = \ldots = y_m = 0;$$

consequently, $\frac{\partial a_{i \cdots n}}{\partial y_1}(y) = 0$.

One thus proves that $\mathrm{L}_X \alpha$ is zero at y, and consequently, that $\mathrm{L}_X \alpha$ is zero on the support of X.

However, if X is zero on an open subset U of M^m then $\mathrm{L}_X\alpha = i_X d\alpha + d i_X \alpha$ is zero on U. One thus obtains $\mathrm{L}_X\alpha = 0$ on M^m .

Q. E. D.

§ 2. – Invariant volume forms.

In this paragraph, suppose that X is a vector field on a manifold M^m that generates a global one-parameter group φ_t of diffeomorphisms on M^m (which is the case when M^m is compact, in particular). One also supposes that there exists a volume form ω on M^m that is invariant under X.

2.1. Lemma. – *One has* $\varphi_t^* \omega = \omega$ *for any t.*

Indeed, since $L_X\omega = \lim_{t\to 0} (\varphi_t^*\omega - \omega)$ for each point y of M, $(\varphi_t^*\omega)(y)$ will be the solution of the differential equation $z' = 0$ in $T_y^*(M)$ that takes the value $\omega(y)$ for $t = 0$.

2.2 Proposition. – *The Radon measure μ_ω that is associated with ω on M^m is invariant for φ_t: For any Borel set A in M^m and for any $t \in \mathbb{R}$, one will have:*

$$\mu_\omega(\varphi_t A) = \mu_\omega(A) .$$

Proof. – One can find a locally-finite open covering $\mathcal{U} = (U_i)$ of M^m such that for any i there exists a diffeomorphism h_i of U_i onto an open subset of $\mathbb{R}^m$ that verifies $h_i^*(dx_1 \wedge \ldots \wedge dx_n) = \omega\big|_{U_i}$ (Chap. VI, Cor. **3.4**).

If μ_i is the Radon measure on U_i that is obtained by transporting the Lebesgue measure μ on $h(U_i)$ by way of h_i then one will have $\mu_i = \mu_j$ on $U_i \cap U_j$. The Radon measure μ_ω will then be the measure on M^m such that $\mu_\omega\big|_{U_i} = \mu_i$.

One can suppose that the Borel set A is contained in an open subset U_i and that $\varphi_t(A)$ is contained in an open subset U_j, moreover; in this case, one will have $\mu_\omega(\varphi_t A) = \mu(h_j\,\varphi_t A) = \mu(h_j A) = \mu_\omega(A)$ (i.e., μ is invariant under $h_j\,\varphi_t h_i^{-1}$).

Q. E. D.

This proposition has some important consequences in regard to the geometric nature of the dynamical system.

2.3. Definition. – *The dynamical system X is **recurrent** if for any open subset U of M^m and any $T \geq 0$ there exists a $t \geq T$ such that $U \cap \varphi_t(U) \neq \emptyset$.*

Under those conditions, for any open subset U of M^m and for any $T \leq 0$ there exists a $t \leq T$ such that $U \cap \varphi_t(U) \neq \emptyset$.

2.4. Theorem. (Poincaré recurrence theorem). – *If X is a vector field that leaves a volume form ω on a compact manifold M^m invariant then X will be a recurrent dynamical system.*

Proof. – Let U be an open subset of M^m, and let $\mu_\omega(U)$ be a finite number $m > 0$ such that one has $\mu_\omega(\varphi_t U) = \mu_\omega(U)$ for any $t \in \mathbb{R}$.

Let $T > 0$. If the open subsets $\varphi_{iT}(U)$, $i = 1, \dots, k$ are pair-wise disjoint then one will have $\mu_\omega\left(\bigcup_i \varphi_{iT}(U)\right) = km$.

Consequently, if k is greater than $\dfrac{\mu(M)}{m} = \dfrac{1}{m}\int_{M^m}\omega$ then there will exist two integers i and j, $1 \le i \le j \le k$, such that $\varphi_{iT}(U) \cap \varphi_{jT}(U) \neq \emptyset$; thus, $U \cap \varphi_{(j-i)T}(U) \neq \emptyset$.

Q. E. D.

2.5. Definition. A point y of M^m is *stable in the Poisson sense* for the dynamical system **X** if for any neighborhood U of x and for any $T \ge 0$ there exist $t_1 \ge T$ and $t_2 \le -T$ such that $\varphi_{t_1}(y)$ and $\varphi_{t_2}(y)$ are in U.

In this case, any point of the trajectory of **X** that passes through y is also stable in the Poisson sense.

2.6. Theorem. *If* **X** *is a vector field that leaves a form* ω *on a compact* M^m *invariant then almost all points of* M^m *will be stable in the Poisson sense.*

In other words, the set of points that are unstable in the Poisson sense has measure zero for μ_ω .

2.7. Definition. – *A point* y *of* M^m *is a* ***wandering point*** *for the dynamical system* X *if for any compact subset* K *in* M^m *there exists a* $T \ge 0$ *such that* $\varphi_t(y) \notin K$ *for* $|t| \ge T$.

In that case, any point of the trajectory of X that passes through y will also be wandering.

2.8. Theorem. (E. Hopf). – *If* X *is a vector field on a non-compact manifold* M^m *that leaves a volume form* ω *invariant then almost all points of* M^m *will be either wandering or stable in the Poisson sense.*

One will find proofs of Theorems 2.6 and 2.8 in the treatise of V. Nemitskii and V. Stepanov [**12**].

§ 3. – Absolute integral invariants (E. Cartan [3]).

3.1. Definition. – *Let* X *be a vector field on a manifold* M^m. *A differentiable form* $\alpha \in \Lambda(M)$ *is an* ***absolute integral invariant*** *of* X *if one has* $i_X\,\alpha = i_X\,d\alpha = 0$.

This is equivalent to saying that α is an absolute integral invariant of X if X is a characteristic vector field for α (Chap. VI, Th. **2.2**)

In particular, in order for a closed differential form $\alpha \in \Lambda(M)$ to be an absolute integral invariant for X, it is necessary and sufficient that $i_X \alpha = 0$.

3.2. Proposition. – *In order for a differential form $\alpha \in \Lambda(M)$ to be an absolute integral invariant for X, it is necessary and sufficient that one must have $i_X\alpha = \mathrm{L}_X\alpha = 0$.*

Indeed, $\mathrm{L}_X\alpha = i_X d\alpha + d i_X\alpha$.

3.3. Corollary. – *If α is an absolute integral invariant for X then α will be an invariant form for X.*

3.4. Proposition. – *The set of absolute integral invariants for X is a subalgebra of $\Lambda(M)$ that is stable under d.*

The verification of that proposition is a simple exercise in calculation.

3.5. Proposition. – *If a differential form α is an absolute integral invariant for X then it will also be an absolute integral invariant for $f X$ for any function $f \in \mathcal{D}(M)$.*

Indeed:

$$i_{fX}\alpha = f\, i_X\alpha = 0,$$
$$i_{fX}d\alpha = f\, i_X d\alpha = 0.$$

3.6. Corollary. – *Let α be an absolute integral invariant for X. There exists a strictly positive function $f \in \mathcal{D}(M)$ that has the following properties:*

i) *α is an absolute integral invariant for $f X$.*

ii) *$f X$ generates a global one-parameter group of diffeomorphisms of M^m.*

This corollary is a direct consequence of Proposition **1.13** of Chapter V.

When α is a volume form on M^m, moreover, one can apply the conclusions of the preceding paragraph to the dynamical system $f X$ (whose trajectories have the same images as the trajectories of X).

3.7. *Local expression.* – If the point y is not a zero of X then one can find a system $(y_1, \ldots, y_m)$ of local coordinates on an open neighborhood U of y such that $X|_U = \partial / \partial y_1$.

If:

$$\alpha = \sum_{1 \le i_1 < \cdots < i_p \le m} a_{i_1 \cdots i_p}\, dy_{i_1} \wedge \cdots \wedge dy_{i_p}$$

is a form of degree p on U then one will have:

$$i_X\alpha = \sum_{2\le i_1<\cdots<i_p\le m} a_{1i_2\cdots i_p}\, dy_{i_2}\wedge\cdots\wedge dy_{i_p}\,,$$

$$\mathrm{L}_X\alpha = \sum_{1\le i_1<\cdots<i_n\le m} \frac{\partial a_{i_1\cdots i_n}}{\partial y_1}\, dy_{i_1}\wedge\cdots\wedge dy_{i_n}\,.$$

Consequently, in order for α to be an absolute integral invariant of X, it is necessary and sufficient that $a_{1i_2\cdots i_p} = \dfrac{\partial a_{i_1\cdots i_n}}{\partial y_1} = 0$; i.e., that the local expression for α should contain neither y_1 nor dy_1 in a neighborhood of y.

3.8. *Examples:*

i) *Hamiltonian systems.* Let X be a Hamiltonian system on a symplectic manifold (M^{2n}, ω), and let (N^{2n-1}, h) be an integral manifold of the Pfaff system that is generated by $i_X\,\omega$. The differential form $h^*\omega$ is an absolute integral invariant of the vector field Y that is induced by X on N^{2n-1} (Chap. VII, Prop. **3.5**).

Under the hypotheses of Proposition **3.10** in Chap. VII, the vector field $Y : (x, t) \mapsto X_t(x) + \partial / \partial t$ on $M^{2n} \times \mathbb{R}$ is characterized by the property that it must admit the differential form $p_1^*\omega - dH \wedge dt$ as an absolute integral invariant (E. Cartan [**3**]).

ii) *Contact structures.* If Y is the dynamical system that is associated with a contact form α on a manifold M^{2n+1} then $d\alpha$ will be an absolute integral invariant for Y (Chap. VII, Th. **5.5**).

When α is an absolute integral invariant of degree n of X, one can generalize Theorem **1.5** in the following fashion:

There exists a differentiable function $f \in \mathcal{D}$ ($\mathbb{R} \in N$) such that:

$$H^T_{(t,y)}\left(\frac{\partial}{\partial t}\right) = f(t, y)\, X(H(t, y))\,.$$

[Geometrically, one can say that h_t is a deformation of h_0 (N^n) "along the tube of trajectories of X that issue from h_0 (N^n)."] The situation in Theorem **1.5** then corresponds to the case in which $f(t, y) = 1$.

Under these conditions:

3.9. Theorem. – *If α is an absolute integral invariant of degree n for X then the integral:*

$$I(t) = \int_{N^n} h_t^* \alpha$$

will be independent of t.

Proof. – One has:

$$i_{\partial / \partial t}\, H^* \alpha = i_{\partial / \partial t}\, dH^* \alpha = 0 \qquad \text{(Prop. } \mathbf{3.5}\text{)}$$

in $\mathbb{R} \times N^n$. Consequently, $H^* \alpha$ is an invariant form for the vector field $\partial / \partial t$ on $\mathbb{R} \times N^n$, which is the vector field that is associated with the one-parameter group $\theta_t : (t, y) \mapsto (t + \tau, y)$ of diffeomorphisms of $\mathbb{R} \times N^n$.

Let j be the canonical diffeomorphism of N^n onto $\{0\} \times N^n$; one has:

$$I(t) = \int_{N^n} h_t^* \alpha = \int_{N^n} (H \circ \theta_t \circ j)^* \alpha,$$
$$= \int_{N^n} (\theta_t \circ j)^* H^* \alpha.$$

One then deduces from Theorem **1.5** that $I(t)$ independent of t.

Q. E. D.

Conversely, one proves the following property in an analogous fashion to 1.7:

3.10. Proposition. – *If the integral:*

$$I(t) = \int_{N^n} h_t^* \alpha$$

is independent of t for any pair (N^n, H) that has the foregoing properties then α will be an absolute integral invariant for X.

§ 4. – Relative integral invariants (E. Cartan [3]).

4.1. Definition. – *Let X be a vector field on a manifold M^m. A differential $\alpha \in \Lambda(M)$ is a* ***relative integral invariant*** *of X if one has $i_X\, d\alpha = 0$.*

It is then equivalent to say that $d\alpha$ is an absolute integral invariant of X, or even that X is a characteristic vector field of $d\alpha$.

An absolute integral invariant is also a relative integral invariant.

4.2. *Examples:*

i) *Contact structure.* If Y is a dynamical system that is associated with a contact form α on a manifold M^{2n+1} then α will be a relative integral invariant of Y [example *ii*) of **3.8**].

ii) *Finsler structure.* With the notations of example *i*) of **5.10**, Chapter VII, the Liouville form λ induces a relative integral invariant of the geodesic field on N^{2n-1}.

iii) *Classical Hamiltonian structure.* [example *ii*) of **5.10**, Chap. VII]. One has an analogous result.

With the same hypotheses as in Theorem **3.9**, suppose, moreover, that ∂N^n is non-vacuous, and let k_t denote the restriction of h_t to ∂N^n.

Under these conditions, one will deduce the following result from **3.9** and Stokes's theorem:

4.3. Theorem. – *If α is a relative integral invariant of degree $n - 1$ of X then the integral:*

$$I(t) = \int_{\partial N^n} k_t^* \alpha$$

will be independent of t.

Conversely, just as one has for absolute integral invariants:

4.4. Proposition. – *If the integral:*

$$I(t) = \int_{\partial N^n} k_t^* \alpha$$

is independent of t for any pair (N^n, H) that has the preceding properties then α will be a relative integral invariant of X.

4.5. Definition. – *Let X be a vector field on a manifold M^m. A **transversal** to X is a submanifold N^{m-1} (possibly with boundary) of codimension 1 of M^m such that for any point y* of N^{m-1} *the tangent vector $X(y)$ does not belong to the subspace $T_y(N)$ of $T_y(M)$.*

In particular, X is not annulled on N^{m-1}.

As H. Poincaré and G. Birkhoff have shown, knowing a transversal of X can be very interesting in the geometrical study of the dynamical system X.

Meanwhile:

4.6. Theorem. (G. Reeb [**13**]). – *If the vector field X possesses a relative integral invariant α of degree m – 2 such that $d\alpha$ has no singularity then X will possess no compact transversal without boundary.*

Proof. – Suppose the N^{m-1} is a compact transversal without boundary of *X*. The exterior differential $d\alpha$ is a differential form of degree $m - 1$ and constant class $m - 1$ whose characteristic system at each point $y \in N$ is generated by $X(y)$. Consequently (transversality hypothesis), $i^* d\alpha$ will be a volume form on N^{m-1}. One thus has that $\int_{N^{m-1}} i^*(d\alpha) \neq 0$ (Chap. IV, Cor. **4.4**).

Now, one deduces that this integral is zero from Stokes's theorem (Chap. IV, Cor. **4.8**). That is a contradiction.

Q. E. D.

4.7. Corollary. – *The dynamical system Y that is associated with a contact form α on a manifold M^{2m+1} does not possess a compact transversal without boundary.*

Indeed, $\alpha \wedge (d\alpha)^{n-1}$ is a relative integral invariant of *Y* of degree $2n - 1$, and $d(\alpha \wedge (d\alpha)^{n-1}) = (d\alpha)^n$ has no singularity.

This corollary applies, in particular, to the cases of Finsler structures and classical Hamiltonian structures that were studied in examples **5.10** of Chapter VII (see example **4.2**).

§ 5. – Integral invariance relations (A. Lichnérowicz [11]).

5.1. Definition. – *Let X be a vector field on a manifold M^m. A differential form $\alpha \in$* $\Lambda(M)$ is an ***integral invariance relation*** for *X* if one has $i_X \alpha = 0$.

Consequently:

i) The set of integral invariance relations for *X* is a subalgebra of $\Lambda(M)$.

ii) If α is an integral invariance relation for *X* then the same thing will be true for fX, $f \in \mathcal{D}(M)$.

iii) In order for $\alpha \in \Lambda(M)$ to be a relative integral invariant of *X*, it is necessary and sufficient that $d\alpha$ must also be an integral invariance relation for *X*.

iv) In order for $\alpha \in \Lambda(M)$ to be an absolute integral invariant of *X*, it is necessary and sufficient that α and $d\alpha$ must both be integral invariance relations for *X*.

5.2. *Example: Time-dependent Hamiltonian system.* With the notations of proposition 3.10 of Chapter VII, the vector field **Y**: $(x, t) \mapsto \mathbf{X}_t(x) + \partial / \partial t$ is characterized by the property that it admits the form $p_1^* \omega - dH \wedge dt$ as an integral invariance relation.

5.3. Theorem. – *Under the hypotheses of Theorem* **3.9**, *if* α *is an integral invariance relation of degree* $n + 1$ *for X then one will have* $H^* \alpha = 0$.

Indeed, $H^* \alpha$ is a form of degree $n + 1$ on $\mathbb{R} \times N^n$ such that $i_{\partial / \partial t} (H^* \alpha) = 0$.

Conversely:

5.4. Proposition. – *If one has* $H^* \alpha = 0$ *for any pair* (N^n, H) *that has the properties in* **3.9** *then one will have an integral invariance relation for X.*

____

CHAPTER IX

SECOND TANGENT BUNDLE

§ 1. – Tangent bundle to a vector bundle.

Let $\eta = (E, p, M^m)$ be an n-dimensional differentiable vector bundle whose base M^m is an m-dimensional manifold. The space E is then an $(m + n)$-dimensional manifold, and the diagram:

$$\begin{array}{ccc} T(E) & \xrightarrow{p^{\mathrm{T}}} & T(M) \\ {\scriptstyle p_E}\downarrow & & \downarrow{\scriptstyle p_M} \\ E & \xrightarrow[p]{} & M^m \end{array}$$

commutes.

One supposes (and this is no restriction) that the fiber of η is $\mathbb{R}^n$. If (U, Φ) is a differentiable chart on the bundle h then Φ^{T} will be a diffeomorphism of $p_E^{-1}p^{-1}(U)$ = $(p^{\mathrm{T}})^{-1}p_M^{-1}(U)$ onto:

$$T(U \times \mathbb{R}^n) = T(U) \times \mathbb{R}^n \times \mathbb{R}^n = p_M^{-1}(U) \times \mathbb{R}^n \times \mathbb{R}^n.$$

Consequently, $\tau(\eta) = (T(E), p^{\mathrm{T}}, T(M))$ is a locally-trivial bundle whose fiber is $\mathbb{R}^{2n}$. Indeed:

1.1. Proposition. – *If* $\{(U_i, \Phi_i)\}$ *is a differentiable atlas on the vector bundle* $\eta = (E, p, M^m)$ *then the set* $\{(p^{-1}(U_i), \Phi_i^{\mathrm{T}})\}$ *will be an atlas that defines a structure of a* $2n$*-dimensional differentiable vector bundle on* $\tau(\eta) = (T(E), p^{\mathrm{T}}, T(M))$.

One then says that $\tau(\eta)$ is the *tangent vector bundle* to the bundle η.

The proof of that proposition utilizes the following two lemmas, whose verifications present no difficulties:

1.2. Lemma. – *Let G be the linear group* Gl $(n, \mathbb{R})$. *The tangent maps to the maps* $(g, h) \mapsto gh$ *and* $g \mapsto g^{-1}$ *definition a group structure on the tangent space* $T(G) = G \times \mathbb{R}^{n^2}$.

1.3. Lemma. – *The tangent map to the canonical map* $(g, f) \mapsto g(f)$ *of* $G \times \mathbb{R}^n$ *into* $\mathbb{R}^n$ *permits one to identify* $T(G)$ *with the subgroup of* $\mathrm{Gl}(2n, \mathbb{R})$ *of matrices of the form* $\begin{pmatrix} A & 0 \\ B & A \end{pmatrix}$, *in which A and B are square matrices of order n and A is invertible.*

Proof of Proposition 1.1: Let (U, Φ) and (V, Ψ) be two differentiable charts of η such that $U \cap V \neq \emptyset$. One can write:

$$\Psi\,\Phi^{-1}(y, f) = (y, g(y) f), \qquad (y, f) \in (U \cap V) \times \mathbb{R}^n,$$

in which g is a differentiable map from $U \cap V$ into $G = \mathrm{Gl}(n, \mathbb{R})$.

One will then have:

$$\Psi^{\mathrm{T}}(\Phi^{\mathrm{T}})^{-1}(u, v) = (u, g^{\mathrm{T}}(u) v), \qquad (u, v) \in p_M^{-1}(U \cap V) \times \mathbb{R}^{2n},$$

in which g^{T} is the tangent map to the change of chart g.

Q. E. D.

1.4. Corollary. – *The pair* (p_E, p_M) *is a differentiable homomorphism of* $\tau(\eta)$ *into* η.

Indeed (Chap. II, Prop. **2.11**), since:

$$p_E(\Phi^{\mathrm{T}})^{-1}(u, (v, w)) = \Phi^{-1}(p_M(u), w),$$
$$(u, (v, w)) \in T(U \times \mathbb{R}^n) = p_M^{-1}(U) \times \mathbb{R}^n \times \mathbb{R}^n,$$

p_E will be linear on each fiber of $\tau(\eta)$.

1.5. Definition. – *The* ***tangent bundle to the fibers*** *of* η *is the bundle* $p^*\eta$ *over E that is the reciprocal image of* η *by the projection* $p : E \to M^m$.

The tangent bundle to the fibers of η is therefore an n-dimensional differentiable vector bundle over E. One denotes it by $p^*\eta = (p^*E, \pi, E)$.

The total space to p^*E to $p^*\eta$ is identified with the subspace $\bigcup_{y \in M^m} p^{-1}(y) \times p^{-1}(y)$ of $E \times E$, and the map π is the restriction of the first projection of $E \times E$ onto E. If π' denotes the restriction of p^*E to the second projection of $E \times E$ onto E then one will have the following commutative diagram:

$$\begin{array}{ccc} p^*E & \xrightarrow{\pi'} & E \\ \pi \downarrow & & \downarrow p \\ E & \xrightarrow[p]{} & M^m \end{array}$$

The term "tangent bundle to the fibers of η" is justified by the following construction:

For any point y of M^m, the fiber $p^{-1}(y)$ is an n-dimensional vector space. The tangent map to the canonical injection $p^{-1}(y) \to E$ then determines an injective map H of $p^*E = \bigcup_{y\in M^m} p^{-1}(y)\times p^{-1}(y)$ into $T(E)$ such that $\pi = p_E \circ H$. Indeed:

1.6. Proposition. – *The map* $H: p^*E \to T(E)$ *is a differentiable homomorphism (over* E*) of* $p^*\eta$ *into the tangent bundle* $\tau(E)$.

Proof: It suffices to verify that proposition locally.

Therefore, let (U, φ) be a chart on M^m for which there exists a differentiable trivialization Φ of $\eta|_U$, and let $(y_1, \dots, y_m)$ be the local coordinate system that is determined by φ on U.

The functions $z_i = y_i \circ p$, $i = 1, \dots, m$, and $\alpha_j = x_i \circ p_2 \circ \Phi$, $j = 1, \dots, n$, form a local coordinate system on the open subset $p^{-1}(U)$ of E. Consequently, the functions $u_i = z_i \circ p_E$, $v_j = \alpha_j \circ p_E$, dz_i, $d\alpha_j$ will form a local coordinate system on the open subset $p_E^{-1}\,p^{-1}(U)$ of $T(E)$.

Finally, let $(w_1, \dots, w_m, \beta_1, \dots, \beta_n, \gamma_1, \dots, \gamma_n)$ denote the local coordinate system on the open subset $\pi^{-1}p^{-1}(U)$ of p^*E that is obtained by starting from the trivialization:

$$(e, e') \mapsto (p(e), p_2\,\Phi(e), p_2\,\Phi(e')) \quad \text{of} \qquad (p^*\eta)|_{p^{-1}(U)}\,.$$

The local expressions for the maps π and H in those local coordinate systems will then be:

$$\begin{array}{llll} z_i = w_i\,, & \alpha_j = \beta_j\,, & & \\ u_i = w_i\,, & v_j = \beta_j\,, & dz_i = 0\,, & d\alpha_j = \gamma_j\,, \end{array}$$

respectively. Those expressions then show that H is differentiable and linear over each fiber of $p^*\eta$.

Q. E. D.

1.7. Definition. – *The **transverse bundle** to the fibers of η is the bundle $p^* \tau(M)$ over E that is the reciprocal image of the tangent bundle $\tau(M)$ by the projection p: $E \to M^m$.*

The transverse bundle to the fibers of η is therefore an m-dimensional differentiable vector bundle over E. One denotes it by $p^* \tau(M) = (p^* T(M), \varpi, E)$.

The total space $p^* T(M)$ of $p^* \tau(M)$ is identified with the subspace $\bigcup_{y \in M^m} p^{-1}(y) \times T_y(M)$ of $E \times T(M)$, and the map ϖ is identified with the restriction of the projection of $E \times T(M)$ onto E. If ϖ' denotes the restriction of the projection of $E \times T(M)$ onto $T(M)$ to $p^* T(M)$ then one will have the following commutative diagram:

$$\begin{tikzcd} p^* T(M) \arrow[r, "\varpi'"] \arrow[d, "\varpi"'] & T(M) \arrow[d, "p_M"] \\ E \arrow[r, "p"'] & M^m \end{tikzcd}$$

One also has this commutative diagram:

$$\begin{tikzcd} T(E) \arrow[r, "p^{\mathrm{T}}"] \arrow[d, "p_E"'] & T(M) \arrow[d, "p_M"] \\ E \arrow[r, "p"'] & M^m \end{tikzcd}$$

Consequently (Chap. II, Th. **1.7**):

1.8. Proposition. – *There exists a differentiable homomorphism K (over E) of $\tau(E)$ into $p^* \tau(M)$ such that the following diagram commutes:*

$$\begin{tikzcd} T(E) \arrow[dr, "K"] \arrow[drr, "p^{\mathrm{T}}"] \arrow[ddr, "p_E"'] & & \\ & p^* T(M) \arrow[r, "\varpi'"] \arrow[d, "\varpi"] & T(M) \arrow[d, "p_M"] \\ & E \arrow[r, "p"'] & M^m \end{tikzcd}$$

1.9. Theorem. – *The sequence* $0 \to p^*\eta \xrightarrow{H} \tau(E) \xrightarrow{E} p^*\tau(M) \to 0$ *is exact.*

Proof: With the same notations as in the proof of Proposition **1.6**, let $(l_1, \ldots, l_m, \delta_1, \ldots, \delta_n, \varepsilon_1, \ldots, \varepsilon_n)$ denote the local coordinate system on the open subset $\varpi^{-1}\, p^{-1}(U)$ of $p^*T(M)$ that is obtained by starting from the trivialization $(\varepsilon, v) \mapsto (p\,(\varepsilon), p_2\;\Phi\;(\varepsilon), p_2\,\varphi^{\mathrm{T}}(v))$ of $(p^*T(M))|_{p^{-1}(U)}$.

The local expressions for the maps ϖ and K will then be:

$$z_i = l_i\,, \qquad \alpha_j = \delta_j\,,$$

$$z_i = u_i\,, \qquad \delta_j = v_j\,, \qquad \varepsilon_i = dz_i\,,$$

respectively. That shows the exactness of the sequence in **1.9**.

Q. E. D.

§ 2. – Second tangent bundle.

2.1. – We now specialize the preceding situation to the case in which η is the tangent bundle $\tau(M)$ to M^m. We will then get the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[dr] & & & & \\
& p^*T(M) \arrow[dr, "K"] \arrow[ddddr, "\pi"'] & & & \\
& & T(T(M)) \arrow[rr, "p^{\mathrm{T}}\,T(M)"] \arrow[dr] \arrow[ddd, "p_{T(M)}"'] & & T(M) \arrow[ddd, "p_M"] \\
& & & p^*T(M) \arrow[ur, "\pi'"] \arrow[ddl, "\pi"'] \arrow[dr] & \\
& & & & 0 \\
& & T(M) \arrow[rr, "p_M"'] & & M^m
\end{tikzcd}$$

2.2. – Let U be an open subset of M^m on which there exists a system $(q_1, \ldots, q_m)$ of local coordinates. By abuse of notation, one lets $q_1, \ldots, q_m$ (instead of $q_i \circ p_M$), $\dot{q}_1 = dq_1, \ldots, \dot{q}_m = dq_m$ denote the corresponding system of local coordinates on the open subset $p_M^{-1}(U)$ of $T(M)$. One also lets $q_1, \ldots, q_m$ (instead of $q_i \circ p_M \circ p_{T(M)}$), $\dot{q}_1, \ldots, \dot{q}_m$ (instead of $\dot{q}_i \circ p_{T(M)}$), $dq_1, \ldots, dq_m$, $d\dot{q}_1, \ldots, d\dot{q}_m$ denote the local coordinate system on the open subset $p_{T(M)}^{-1}\, p_M^{-1}(U) = (p_M^{\mathrm{T}})^{-1}\, p_M^{-1}(U)$ of $T(T(M))$.

Furthermore, let $q_1, \ldots, q_m$ (instead of $q_i \circ p_M \circ \pi$), $\dot{q}_1, \ldots, \dot{q}_m$ (instead of $\dot{q}_i \circ \pi$), $dq_1, \ldots, dq_m$ (instead of $\dot{q}_i \circ \pi'$) denote the local coordinate system on the open subset $\pi^{-1} p_M^{-1}(U)$ of $p_M^* T(M)$.

With those systems of local coordinates, the expressions for p_M, $p_{T(M)}$, p_M^{T}, π, π', H, K will be:

$$
\begin{aligned}
p_M &: (q_i, \dot{q}_i) \mapsto q_i , \\
p_{T(M)} &: (q_i, \dot{q}_i, dq_i, d\dot{q}_i) \mapsto (q_i, \dot{q}_i) , \\
p_M^{\mathrm{T}} &: (q_i, \dot{q}_i, dq_i, d\dot{q}_i) \mapsto (q_i, dq_i) , \\
\pi &: (q_i, \dot{q}_i, d\dot{q}_i) \mapsto (q_i, \dot{q}_i) , \\
\pi' &: (q_i, \dot{q}_i, d\dot{q}_i) \mapsto (q_i, dq_i) , \\
H &: (q_i, \dot{q}_i, d\dot{q}_i) \mapsto (q_i, \dot{q}_i, 0, dq_i) , \\
K &: (q_i, \dot{q}_i, dq_i, d\dot{q}_i) \mapsto (q_i, \dot{q}_i, dq_i) .
\end{aligned}
$$

Exercise. – Verify that those local expressions are actually compatible with the changes of charts.

2.3. Theorem. – *There exists a diffeomorphism s of T (T (M)) onto itself that has just one of the following properties:*

i) *s is an involution of T (T (M))* (viz., s^2 = identity).

ii) *s is a differentiable isomorphism* (*over T* (*M*)) *of the bundle* $\tau\,(T\,(M))$ *onto the bundle* $\tau\,(\tau\,(M))$.

iii) *For any differentiable function f on* M^m, *one will have* $(d\,(df))\circ s = d\,(df)$.

The condition *ii*) expresses the idea that the diagram:

$$
\begin{tikzcd}
T(T(M)) \arrow[rr, shift left, "s"] \arrow[dr, "p_{T(M)}"'] & & T(T(M)) \arrow[ll, shift left, "s"] \arrow[dl, "p_M^{\mathrm{T}}"] \\
& T(M) &
\end{tikzcd}
$$

commutes.

One says that s is the *canonical involution of the second tangent space T* (*T* (*M*)).

Proof: If s is such an involution of $T(T(M))$ then one will have:

$$s(p_{T(M)}^{-1}\,p_M^{-1}(U)) = (p_M^{\mathrm{T}})^{-1}\,p_M^{-1}(U) = p_{T(M)}^{-1}\,p_M^{-1}(U)$$

for any open subset U in M^m.

Moreover, if U is an open subset of M^m of type **2.2** then the conditions *ii*) and *iii*) will imply [since $d\dot{q}_i = d(dq_i)$] that the local expression for s will necessarily be:

$$(q_i, \dot{q}_i, dq_i, d\dot{q}_i) \mapsto (q_i, dq_i, \dot{q}_i, d\dot{q}_i) \;.$$

Conversely, that local expression determines a diffeomorphism s_U of $p_{T(M)}^{-1}\,p_M^{-1}(U)$ onto itself that verifies the conditions *i*) and *ii*).

If f is a differentiable function on U then one will have:

$$d(df) = \sum_{j,k} \frac{\partial^2 f}{\partial q_j\,\partial q_k}\,\dot{q}_j\,dq_k + \sum_j \frac{\partial f}{\partial q_j}\,d\dot{q}_j \,.$$

Consequently (Schwarz's theorem), s_U also verifies the condition *iii*).

The existence and uniqueness of those diffeomorphisms s_U for any open subset U of M^m of type **2.2** will then permit one to obtain a unique diffeomorphism ι of $T(T(M))$ that verifies the Properties of **2.3** by gluing them together.

Q. E. D.

Exercises:

i) The involution s exchanges the image of H and the restriction of s $(T(M))$ to the zero section of $\tau(M)$.

ii) Let X be a vector field on M^m and let φ_t be a local one-parameter group of diffeomorphisms of M^m that is generated by X.

a) φ_t^{T} is a local one-parameter group of diffeomorphisms of $T(M)$.

b) X^{T} is a section of the bundle $\tau(\tau(M)) = (T(T(M)),\ p_M^{\mathrm{T}},\ T(M))$.

c) φ_t^{T} is generated by $s \in X^{\mathrm{T}}$.

Recall (Chap. V, § **1.6**) that the map $(t, u) \mapsto h_t(u) = e^t u$ is a one-parameter group of diffeomorphisms of $T(M)$. h_t is the one-parameter group of homotheties of $T(M)$. One can set:

2.4. Definition. – *The **Liouville field** on $T(M)$ is the vector field V that is generated by the one-parameter group of homotheties of $T(M)$.*

2.5. *Local expression.* – Let U be an open subset of M^m of type **2.2**. The local expression for h_t in $p_M^{-1}(U)$ is $(q_i, \dot{q}_i) \mapsto (q_i, e^t \dot{q}_i)$.

Consequently, the local expression for V is $\sum_i \dot{q}_i \frac{\partial}{\partial \dot{q}_i}$.

That local expression justifies the following construction of the Liouville field (see **2.2**):

2.6. Proposition. – *Let σ section of the bundle $p^* T(M)$ over $T(M)$ that is defined $\sigma(u) = (u, v)$. One has $V = H \circ s$.*

Exercise. – Let λ be a differentiable function on M^m. The map $(t, u) \mapsto e^{t\lambda(p_M u)} u$ is a one-parameter group of diffeomorphisms of $T(M)$ that is generated by the vector field $(\lambda \circ p_M)\, V$.

§ 3. – Second-order differential equations.

3.1. Definition. – *A **second-order differential equation** on a manifold M^m is a differentiable map $X : T(M) \to T(T(M))$ that is simultaneously a section of the tangent bundle $\tau(T(M))$ and a section of the bundle $\tau(\tau(M))$.*

In other words, $p_{T(M)} \circ X$ and $p_M^{\mathrm{T}} \circ X$ must be equal to the identity map on $T(M)$. In particular, X must be a vector field on $T(M)$.

A *solution* to the second-order differential equation X on M^m is a differentiable curve $c : I \to M^m$ such that $c' : I \to T(M)$ is an integral curve of X.

3.2. *Local expression.* – With the same notations as in **2.2**, the local expression for a second-order differential equation on M^m has the form:

$$\sum_i \left(\dot{q}_i \frac{\partial}{\partial q_i} + a_i(q_j, \dot{q}_j) \frac{\partial}{\partial \dot{q}_i} \right) .$$

The integral curves of X in are then the solutions to the differential system:

$$\frac{dq_i}{dt} = \dot{q}_i , \qquad \frac{d\dot{q}_i}{dt} = a_i(q_j, \dot{q}_j),$$

or rather, the second-order differential system:

$$\frac{d^2\dot{q}_i}{dt^2} = a_i\left(q_j, \frac{dq_j}{dt}\right), \qquad i = 1, \ldots, m.$$

One can deduce the following proposition from that local expression (see the proof of **2.3**):

3.3. Proposition. – *In order for a vector field X on T (M) to be second-order differential equation on M^m, it is necessary and sufficient that one should have $s \circ X = X$.*

One now introduces a type of second-order differential equation that is very important in differential geometry and analytical mechanics. (One can consult the treatise by S. Lang [**10**] for a more detailed study of this.)

3.4. Definition. – *A **spray** on a manifold M^m is a second-order differential equation X on M^m such that [V, X] = X (in which V denotes the Liouville field on T(M)).*

3.5. *Local expression.* – With the notations of **2.2**, one can write:

$$V = \sum_i \dot{q}_i \frac{\partial}{\partial q_i},$$

$$X = \sum_i \left(\dot{q}_i \frac{\partial}{\partial q_i} + a_i \frac{\partial}{\partial \dot{q}_i}\right),$$

$$[V, X] = \sum_i \dot{q}_i \frac{\partial}{\partial q_i} + \sum_{i,j} \left(\dot{q}_i \frac{\partial a_j}{\partial q_i} - a_j\right) \frac{\partial}{\partial \dot{q}_i}.$$

Consequently, in order for X to be a spray on M^m, it is necessary and sufficient that one should have:

$$\sum_i \dot{q}_i \frac{\partial a_j}{\partial q_i} = 2\, a_j, \qquad j = 1, \ldots, m,$$

or rather that the functions a_j should be homogeneous of degree 2 in the $\dot{q}_i$.

Exercises:

i) Let X be a second-order differential equation on a manifold M^m, and let (U, Φ) be the maximal local one-parameter group of diffeomorphisms of T (M) that is generated by X. The following properties are equivalent:

a) X is a spray.

b) The point (t, u) is in U if and only if the point $(1, tu)$ is in U, and then $t\,\varphi_t(u) = \varphi_1(tu)$.

c) $X(h_t(u)) = e^t\,h_t^{\mathrm{T}}(u)$.

ii) The constructions that were made in this chapter are "functorial," i.e., they are compatible with differentiable homomorphisms (in a sense that one must specify).

CHAPTER X

DIFFERENTIAL CALCULUS ON TANGENT BUNDLES

When a manifold is a tangent bundle, its differential calculus is enriched by certain operators that play a fundamental role in Lagrangian mechanics. Those operators have been studied by J. Klein [**9**] and, in a more general context, by A. Fröhlicher and A. Nijenhuis [**6**].

In this chapter, we will consider an m-dimensional manifold M^m. As before (Chap. IX, **§ 2.2**), if (q_1, …, q_m) is a local coordinate system on an open subset U of M^m then we will let $(q_i, \dot{q}_i)$ $[(q_i, \dot{q}_i, dq_i, d\dot{q}_i)$, resp.] denote the local coordinate system on the open subset $p_M^{-1}(U)$ of T (M) $[p_{T(M)}^{-1} p_M^{-1}(U)$ of T (T (M)), resp.].

Recall the exact sequence in Chapter IX:

$$0 \to p_M^* \, \tau(M) \xrightarrow{H} \tau(T(M)) \xrightarrow{K} p_M^* \, \tau(M) \to 0$$

§ 1. – Vertical endomorphism.

1.1. Definition. – *The endomorphism* $v = H \circ K$ *of* τ (T (M)) *is called the* ***vertical endomorphism*** *of the second tangent bundle.*

One will then have:

1.2. Proposition. – *The vertical endomorphism of* τ (T (M)) *is a differentiable endomorphism of constant rank m and square zero.*

1.3. *Local expression.* – The vertical endomorphism is given locally by (Chap. IX, Prop. **1.6** and Th. **1.9**):

$$v : (q_i, \dot{q}_i, dq_i, d\dot{q}_i) \mapsto (q_i, \dot{q}_i, 0, d\dot{q}_i) \ .$$

Since v is an endomorphism of the tangent bundle τ (T (M)), it determines an endomorphism (which is once more denoted by v) of the module $\mathcal{T}$ (T (M)) of vector fields on T (M). That endomorphism is compatible with the restrictions, and one can write:

$$v\left(\sum_i \left(a_i \frac{\partial}{\partial q_i} + b_i \frac{\partial}{\partial \dot{q}_i}\right)\right) = \sum_i a_i \frac{\partial}{\partial \dot{q}_i} \ ,$$

locally. Consequently (Chap. IX, **§ 2.4**):

1.4. Proposition. – *If V denotes the Liouville field of $T(M)$ then one will have $v\,V = 0$.*

The endomorphism v of $\mathcal{T}(T(M))$ is not compatible with the Lie bracket. Indeed:

1.3. Proposition. – *If X and Y are two vector fields on $T(M)$ then one will have:*

$$[v\,X, v\,Y] = v\,[v\,X, Y] + v\,[X, v\,Y]\,.$$

Proof: Locally one can write:

$$X = \sum_i \left(a_i \frac{\partial}{\partial q_i} + b_i \frac{\partial}{\partial \dot{q}_i} \right),$$

$$Y = \sum_i \left(c_i \frac{\partial}{\partial q_i} + d_i \frac{\partial}{\partial \dot{q}_i} \right),$$

$$v\,X = \sum_i a_i \frac{\partial}{\partial \dot{q}_i},$$

$$v\,Y = \sum_i c_i \frac{\partial}{\partial \dot{q}_i},$$

$$[v\,X, v\,Y] = \sum_{i,j} \left(a_i \frac{\partial c_j}{\partial \dot{q}_i} - c_i \frac{\partial a_j}{\partial \dot{q}_i} \right) \frac{\partial}{\partial \dot{q}_j},$$

$$[v\,X, Y] = \sum_{i,j} a_i \frac{\partial c_j}{\partial \dot{q}_i} \frac{\partial}{\partial q_j},$$

$$[X, v\,Y] = -\sum_{i,j} c_i \frac{\partial a_j}{\partial \dot{q}_i} \frac{\partial}{\partial q_j},$$

$$v\,[v\,X, Y] = \sum_{i,j} a_i \frac{\partial c_j}{\partial \dot{q}_i} \frac{\partial}{\partial \dot{q}_j},$$

$$v\,[X, v\,Y] = -\sum_{i,j} c_i \frac{\partial a_j}{\partial \dot{q}_i} \frac{\partial}{\partial \dot{q}_j}.$$

Q. E. D.

1.6. Proposition. – *Let X be a vector field on $T(M)$, and let V be the Liouville field. One has:*

$$v\,X = v\,[V, X] + [v\,X, V]\,.$$

Proof: Locally one can write:

$$X = \sum_i \left(a_i \frac{\partial}{\partial q_i} + b_i \frac{\partial}{\partial \dot{q}_i} \right),$$

$$V = \sum_i \dot{q}_i \frac{\partial}{\partial \dot{q}_i},$$

$$v\,X = \sum_i a_i \frac{\partial}{\partial \dot{q}_i},$$

$$v\,[V, X] = \sum_{i,j} \dot{q}_i \frac{\partial a_j}{\partial \dot{q}_i} \frac{\partial}{\partial q_j} + \cdots,$$

$$v\,[V, X] = \sum_{i,j} \dot{q}_i \frac{\partial a_j}{\partial \dot{q}_i} \frac{\partial}{\partial \dot{q}_j},$$

$$[v\,X, V] = \sum_i a_i \frac{\partial}{\partial \dot{q}_i} - \sum_{i,j} \dot{q}_i \frac{\partial a_j}{\partial \dot{q}_i} \frac{\partial}{\partial \dot{q}_j}.$$

Q. E. D.

1.7. Definition. – *The* ***vertical operator*** *in* $\Lambda\,(T\,(M))$ *is the endomorphism* v^* *of the exterior algebra* $\Lambda\,(T\,(M))$ *that is determined by the endomorphism v of* $\mathcal{T}\,(T\,(M))$.

1.8. Proposition. – *The vertical operator is an endomorphism with square zero of the algebra* $\Lambda\,(T\,(M))$ *that is compatible with locally-finite sums.*

1.9. Proposition. – *If X is a vector field on T (M) then one will have* $i_X v^* = v^* i_{vX}$.

In particular:

1.10. Corollary. – *One has* $i_V v^* = 0$.

1.11. *Local expression.* – The endomorphism v^* is determined locally by:

$$v^* f = f, \qquad f \in \mathcal{D}\,(T\,(M)),$$

$$v^*(dq_i) = 0,$$

$$v^*(d\dot{q}_i) = dq_i.$$

Those local expressions show, in particular, that v^* does not commute with the exterior derivative ***d***.

§ 2. – Vertical differentiation.

If ω is a differential form of degree p, $p \geq 1$, on $T\,(M)$ then:

$$\boldsymbol{i}_v\,\omega : (X_1, \ldots, X_p) \mapsto \sum_i \omega(X_1, \ldots, v\,X_i, \ldots, X_p)$$

will also be a differential form of degree p on $T(M)$.

If one agrees to set $\boldsymbol{i}_v f = 0$ for $f \in \mathcal{D}(T(M))$ then one will get an endomorphism of the vector space (over $\mathbb{R}$) $\Lambda(T(M))$ that is compatible with locally-finite sums. Indeed, one verifies the following result with no difficulty:

2.1 Proposition. – *The map $\omega \mapsto \boldsymbol{i}_v\,\omega$ is a derivation of degree* 0 *of the algebra $\Lambda(T(M))$.*

That derivation is characterized (Chap. IV, Cor. 1.12) by the relations:

$$\boldsymbol{i}_v f = 0\,,$$
$$\boldsymbol{i}_v (df) = v^*(df)\,, \qquad f \in \mathcal{D}(T(M)).$$

2.2. Definition. – *The* ***vertical derivative*** *of $\Lambda(T(M))$ is the derivation $\boldsymbol{i}_v$ of degree* 0 *on the algebra $\Lambda(T(M))$ that is characterized by the relations:*

i) $\boldsymbol{i}_v f = 0$,

ii) $\boldsymbol{i}_v (df) = v^*(df)$, $\qquad f \in \mathcal{D}(T(M))$.

2.3. *Local expression.* – The derivation $\boldsymbol{i}_v$ is determined locally by:

$$\boldsymbol{i}_v f = 0\,,$$
$$\boldsymbol{i}_v (dq_i) = 0\,,$$
$$\boldsymbol{i}_v (d\dot{q}_i) = dq_i\,.$$

2.4. Proposition. – *Let ω be a form of degree p on $T(M)$. One has:*

i) $(\boldsymbol{i}_v)^p\,\omega = p!\,v^*\omega$,

ii) $(\boldsymbol{i}_v)^q\,\omega = 0$ *for* $q > p$.

The verification of that property is immediate.

2.5. Corollary. – *One has* $\boldsymbol{i}_v\,v^* = v^*\,\boldsymbol{i}_v = 0$.

2.6. Proposition. – *Let X be a vector field on $T(M)$, and let V be the Liouville field. One has:*

i) $[\boldsymbol{i}_X, \boldsymbol{i}_v] = \boldsymbol{i}_X\, \boldsymbol{i}_v - \boldsymbol{i}_v\, \boldsymbol{i}_X = \boldsymbol{i}_{vX}$.

ii) $[\boldsymbol{i}_v, \mathbf{L}_V] = \boldsymbol{i}_v\, \mathbf{L}_V - \mathbf{L}_V\, \boldsymbol{i}_v = \boldsymbol{i}_v$.

Proof: It suffices to verify that those expressions take the same values for $\alpha = f$ and $\alpha = df, f \in \mathcal{D}(T(M))$. Now:

$$
\begin{aligned}
&[\boldsymbol{i}_X, \boldsymbol{i}_v]\, f = 0\ , \quad \boldsymbol{i}_{vX}\, f = 0\ ,\\
&[\boldsymbol{i}_X, \boldsymbol{i}_v]\, df = \boldsymbol{i}_X v^* df = \boldsymbol{i}_{vX}\, df\ ,\\
&[\boldsymbol{i}_v, \mathbf{L}_V]\, f = 0\ , \quad \boldsymbol{i}_v\, f = 0\ ,\\
([\boldsymbol{i}_v, \mathbf{L}_V]\, df)\, (Y) &= (\boldsymbol{i}_v\, d\, (V \cdot f) - \mathbf{L}_V v^*(df))(Y)\ ,\\
&= v\, Y \cdot (V \cdot f) - V \cdot (vY \cdot f) + v\, [V, Y] \cdot f\\
&= ([v\, Y, V] + v\, [V, Y]) \cdot f\\
&= (v\, Y) \cdot f \qquad \text{(Prop. 1.6)},\\
(\boldsymbol{i}_v\, df)\, (Y) &= df\, (v\, Y) = (v\, Y) \cdot f\ .
\end{aligned}
$$

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Hence, by recurrence:

2.7. Corollary. – *One has* $[(\boldsymbol{i}_v)^p, \mathbf{L}_V] = p\, (\boldsymbol{i}_v)^p$.

§ 3. – Vertical differentiation.

The bracket $\boldsymbol{d}_v = [\boldsymbol{i}_v, \boldsymbol{d}] = \boldsymbol{i}_v\, \boldsymbol{d} - \boldsymbol{d}\, \boldsymbol{i}_v$ is an antiderivation of degree 1 on the algebra $\Lambda\, (T\, (M))$ (Chap. IV, Prop. **1.9**):

3.1. Definition. – *The* ***vertical derivative*** *on* $\Lambda\, (T\, (M))$ *is the antiderivation of degree* 1 *on the algebra* $\Lambda\, (T(M))$ *that is defined by* $\boldsymbol{d}_v = [\boldsymbol{i}_v, \boldsymbol{d}]$.

One then has:

3.2. Proposition. – *The vertical derivative* $\boldsymbol{d}_v$ *is the antiderivation of degree* 1 *on* $\Lambda\, (T\, (M))$ *that is characterized by the relations:*

i) $\boldsymbol{d}_v\, f = v^*(df)$,

ii) $\boldsymbol{d}_v\, (df) = -\, d\, (v^*(df))$, $\qquad f \in \mathcal{D}\, (T\, (M))$.

3.3. *Local expression.* – The antiderivation $\boldsymbol{d}_v$ is determined locally by:

$$\boldsymbol{d}_v f = \sum_i \frac{\partial f}{\partial \dot{q}_i} dq_i \,,$$

$$\boldsymbol{d}_v\,(dq_i) = \boldsymbol{d}_v(d\dot{q}_i) = 0 \,.$$

3.4. Proposition. – *The exterior derivative commutes with the vertical derivative on* $\Lambda\,(T\,(M))$.

Indeed, $\boldsymbol{d}\,\boldsymbol{d}_v = \boldsymbol{d}\,\boldsymbol{i}_v\,\boldsymbol{d} = -\,\boldsymbol{d}_v\,\boldsymbol{d}$.

3.5. Proposition. – *The vertical derivative is an antiderivation of square zero.*

The proof of that proposition uses the following lemma:

3.6. Lemma. – *One has* $\boldsymbol{i}_v\,\boldsymbol{d}v^* df = 0$ *for any* $f \in \mathcal{D}\,(T\,(M))$.

Proof: Let X and Y be two vector fields on $T\,(M)$.

$$\begin{aligned}
&(\boldsymbol{i}_v\,\boldsymbol{d}v^* df)(X,Y) \\
&\qquad = (dv^* df)(v\,X, Y) + (dv^* df)(X, v\,Y) \\
&\qquad = v\,X \cdot (v\,Y \cdot f) - (v\,[v\,X,\,Y]) \cdot f - v\,Y \cdot (v\,X \cdot f) - (v\,[X,\,v\,Y] \cdot f \\
&\qquad = ([v\,X,\,v\,Y] - v\,[v\,X,\,Y] - v\,[X,\,v\,Y]) \cdot f \\
&\qquad = 0 \qquad\qquad \text{(Prop. 1.5).}
\end{aligned}$$

Q. E. D.

Proof of Proposition 3.5: It suffices to verify that $\boldsymbol{d}_v\,\boldsymbol{d}_v\,\alpha = 0$ for $\alpha = f$ and $\alpha = df, f \in \mathcal{D}\,(T\,(M))$. Now:

$$\begin{aligned}
\boldsymbol{d}_v\,\boldsymbol{d}_v f &= \boldsymbol{d}_v v^* df \\
&= \boldsymbol{i}_v\,\boldsymbol{d}v^* df \qquad\qquad \text{(Cor. 2.5)} \\
&= 0\,, \\
\boldsymbol{d}_v\,\boldsymbol{d}_v\,df &= \boldsymbol{d}\,\boldsymbol{d}_v\,\boldsymbol{d}_v f = 0 \qquad\qquad \text{(Prop. 3.4)}
\end{aligned}$$

Q. E. D.

3.7. Proposition. – *Let* V *be the Liouville field on* $T\,(M)$. *One has:*

i) $[\boldsymbol{i}_v, \boldsymbol{d}_v] = \boldsymbol{i}_v\,\boldsymbol{d}_v - \boldsymbol{d}_v\,\boldsymbol{i}_v = 0$,

ii) $\boldsymbol{i}_v\,\boldsymbol{d}_v + \boldsymbol{d}_v\,\boldsymbol{i}_v = \boldsymbol{i}_v$,

iii) $[\boldsymbol{d}_v, \mathbf{L}_v] = \boldsymbol{d}_v\,\mathbf{L}_v - \mathbf{L}_v\,\boldsymbol{d}_v = \boldsymbol{d}_v$.

Proof: It suffices to verify that those expressions take the same values for $\alpha = f$ and $\alpha = df$, $f \in \mathcal{D}(T(M))$. Now:

$$[\boldsymbol{i}_v, \boldsymbol{d}_v]\, f = \boldsymbol{i}_v\, \boldsymbol{d}_v\, f = \boldsymbol{i}_v v^* df = 0 \qquad \text{(Cor. } \mathbf{2.5}\text{)}$$

$$[\boldsymbol{i}_v, \boldsymbol{d}_v]\, df = -\, \boldsymbol{i}_v\, \boldsymbol{d}\, \boldsymbol{i}_v\, df - \boldsymbol{i}_v\, \boldsymbol{d}\, \boldsymbol{i}_v\, df + \boldsymbol{d}\, (\boldsymbol{i}_v)^2\, df$$

$$= 0 \qquad \text{(Lemma } \mathbf{3.6} \text{ and Prop. } \mathbf{2.4}\text{)}$$

$$(\boldsymbol{i}_v\, \boldsymbol{d}_v + \boldsymbol{d}_v\, \boldsymbol{i}_v)\, f = \boldsymbol{i}_v v^* df = 0 \qquad \text{(Cor. } \mathbf{1.10}\text{)},$$

$$\begin{aligned}
(\boldsymbol{i}_v\, \boldsymbol{d}_v + \boldsymbol{d}_v\, \boldsymbol{i}_v)\, df &= -\, \boldsymbol{i}_V\, \boldsymbol{d}\, \boldsymbol{i}_v\, df + v^* d\,(V \cdot f) \\
&= -\, \mathbf{L}_V\, \boldsymbol{i}_v\, df + \boldsymbol{i}_v\, \mathbf{L}_V\, df && \text{(Cor. } \mathbf{1.10}\text{)}, \\
&= [\boldsymbol{i}_v, \mathbf{L}_V]\, df = \boldsymbol{i}_v\, df && \text{(Prop. } \mathbf{2.6}\text{)} \\
[\boldsymbol{d}_v, \mathbf{L}_V]\, f &= \boldsymbol{i}_v\, \mathbf{L}_V\, df - \mathbf{L}_V\, \boldsymbol{i}_v\, df \\
&= [\boldsymbol{i}_v, \mathbf{L}_V]\, df = \boldsymbol{i}_v\, df = \boldsymbol{d}_v\, f, \\
[\boldsymbol{d}_v, \mathbf{L}_V]\, df &= \boldsymbol{d}_v\, d\,(V \cdot f) - \mathbf{L}_V\, \boldsymbol{d}_v\, df \\
&= -\, \boldsymbol{d}\, \boldsymbol{d}_v\, (V \cdot f) + \boldsymbol{d}\, \mathbf{L}_V\, \boldsymbol{d}_v\, f \\
&= -\, \boldsymbol{d}\, ([\boldsymbol{d}_v, \mathbf{L}_V]\, f) \\
&= -\, \boldsymbol{d}\, \boldsymbol{d}_v\, f = \boldsymbol{d}_v\, (df)\, .
\end{aligned}$$

Q. E. D.

3.8. Corollary. – *One has* $v^* \boldsymbol{d}_v = 0$.

Proof: If ω is a differential form of degree p on $T(M)$ then one can write (Prop. **2.4**):

$$v^* \boldsymbol{d}_v\, \omega = \frac{1}{(p+1)!} (\boldsymbol{i}_v)^{p+1} \boldsymbol{d}_v\, \omega = \frac{1}{(p+1)!}\, \boldsymbol{d}_v\, (\boldsymbol{i}_v)^{p+1}\, \omega = 0\, .$$

3.9. Corollary. – *One has* $\boldsymbol{d}_v v^* = v^* \boldsymbol{d}$.

Proof: One first verifies the relation:

$$(\boldsymbol{i}_v)^p\, \boldsymbol{d} = p\, \boldsymbol{d}_v\, (\boldsymbol{i}_v)^{p-1} + \boldsymbol{d}_v\, (\boldsymbol{i}_v)^p$$

by recurrence. Now, if ω is a differential form of degree p on $T(M)$ then one can write:

$$v^* \boldsymbol{d}\omega = \frac{1}{(p+1)!} (\boldsymbol{i}_v)^{p+1} \boldsymbol{d}\omega = \frac{1}{p!}\, \boldsymbol{d}_v (\boldsymbol{i}_v)^p\, \omega + \frac{1}{(p+1)!}\, \boldsymbol{d}\, (\boldsymbol{i}_v)^{p+1} \omega = \boldsymbol{d}_v\, v^* \omega\, .$$

Q. E. D.

§ 4. – Semi-basic differential forms.

4.1. Definition. – *A **semi-basic differential form** on T (M) is a differential form on T (M) belongs to the image of the vertical operator* v^* .

Consequently:

4.2. Proposition. – *The set $\mathcal{B}$ of semi-basic forms on T (M) is a graded sub-algebra of* $\Lambda(T(M))$ $\left(\mathcal{B} = \sum_i \mathcal{B} \cap \Lambda^i(T(M))\right)$ *that is stable under locally-finite sums and contains the algebra $\mathcal{D}$ (T (M))* of *differentiable functions on T (M).*

4.3. Proposition. – *The algebra of semi-basic differential forms on T (M) is stable under vertical differentiation.*

Indeed (Cor. **3.9**), $\boldsymbol{d}_v v^* = v^* \boldsymbol{d}$.

4.4. Corollary. – *If f is a differentiable function on T (M) then* $\boldsymbol{d}_v f$ *will be a semi-basic Pfaff form on T (M).*

4.5. Proposition. – *The endomorphisms* $\boldsymbol{i}_V$ *(V is the Liouville field) and* $\boldsymbol{i}_v$ *are zero on the algebra $\mathcal{B}$ of semi-basic differential forms on T (M).*

4.6. *Local expression.* – Since $v^*(dq_i) = 0$ and $v^*(d\dot{q}_i) = dq_i$, the algebra $\mathcal{B}$ will be generated locally by differentiable functions and the differentials dq_i . A semi-basic differential form of degree p can then be written locally as:

$$\sum_{1 \le i_1 < \cdots < i_p \le m} a_{i_1 \cdots i_m}(q_1, \ldots, q_m, \dot{q}_1, \ldots, \dot{q}_m)\, dq_{i_1} \wedge \cdots \wedge dq_{i_p} .$$

One immediately deduces the following results from those local expressions:

4.7. Proposition. – *If α is a differential form on M^m then $\beta = p_M^* \alpha$ will be a semi-basic differential form on T (M) such that* $\boldsymbol{d}_v \beta = 0$.

This proposition justifies the terms “semi-basic differential form on *T* (*M*),” in particular.

4.8. Proposition. – *In order for a Pfaff form on T (M) to be a semi-basic differential form, it is necessary and sufficient that it should be zero on the image of v.*

4.9. Corollary. – *In order for a Pfaff form α on $T(M)$ to be a semi-basic differential form, it is necessary and sufficient that there should exist a differentiable function σ on $p_M^* T(M)$ that has the following properties:*

i) *σ is linear on each fiber of $p_M^* \tau(M)$.*

ii) $\alpha = \sigma \circ K$.

Indeed, the image of v is equal to the image of H and $p_M^* \tau(M)$ is the quotient bundle of $\tau(T(M))$ that is associated with H.

One can also express Corollary **4.9** in the following fashion (Chap. II, Prop. **3.4**):

4.10. Corollary. – *In order for a Pfaff form α on $T(M)$ to be a semi-basic differential form, it is necessary and sufficient that there should exist a differentiable section σ of the dual bundle $(p_M^* \tau(M))^*$ over $T(M)$ such that $\alpha(u) = \langle K(u), \sigma(p_{T(M)}(u)) \rangle$, $u \in T(T(M))$.*

Conversely, if σ is a differentiable section of $(p_M^* \tau(M))^*$ over $T(M)$, moreover, then $\alpha = \langle K, \sigma \circ p_{T(M)} \rangle$ will be a semi-basic Pfaff form over $T(M)$.

4.11. Definition. – *The* ***bundle of semi-basic forms*** *over $T(M)$ is the bundle $p_M^* \tau^*(M)$ that is the reciprocal image of the cotangent bundle to M^m by the projection p_M.*

The bundle of semi-basic forms is therefore an m-dimensional differentiable vector bundle over $T(M)$. One denotes it by:

$$p_M^* \tau^*(M) = \langle p_M^* T^*(M), \chi, T(M) \rangle.$$

The total space $p_M^* T^*(M)$ of $p_M^* \tau^*(M)$ is identified with the subspace $\bigcup_{y \in M^m} p_M^{-1}(y) \times q_M^{-1}(y)$ of $T(M) \times T^*(M)$, and the map χ is identified with the restriction of the projection of $T(M) \times T^*(M)$ onto $T(M)$. If χ' denotes the restriction of the projection of $T(M) \times T^*(M)$ onto $T^*(M)$ to $p_M^* T^*(M)$ then one will have the commutative diagram:

$$\begin{tikzcd} p^* T^*(M) \arrow[r, "\chi'"] \arrow[d, "\chi"'] & T^*(M) \arrow[d, "q_m"] \\ T(M) \arrow[r, "p_M"'] & M^m \end{tikzcd}$$

Let h be the map of $p_M^* T(M) \oplus p_M T^*(M)$ to $\mathbb{R}$ that is defined by $((u, v), (u, a)) \mapsto < v, \alpha >$. The restriction of h to each fiber of $p_M^* \tau(M) \oplus p_M \tau^*(M)$ is a non-degenerate bilinear form. Consequently (Chap. II, Prop. **4.15**):

4.12. Proposition. – *The bundle* $p_M^* \tau(M)$ *of semi-basic forms on* $T(M)$ *is equivalent to the dual of* $p_M^* \tau(M)$.

One can then state Corollary **4.9** in the following form:

4.13. Theorem. – *The relation* $\alpha(u) = < p_M^*(u), D(p_{T(M)}(u) >$, $u \in T(T(M))$, *establishes a bijective correspondence between the semi-basic Pfaff forms* α *on* $T(M)$ *and the differentiable maps* $D : T(M) \to T^*(M)$ *such that* $q_M \circ D = p_M$.

4.14. *Local expression.* – By abuse of notation, let $q_1, \ldots, q_m$ (instead of $q_i \circ q_M$) and $p_1 = \frac{\partial}{\partial q_1}$, $\ldots, p_m = \frac{\partial}{\partial q_m}$ denote the local coordinate system on the open subset $q_M^{-1}(U)$ of $T^*(M)$.

If $\alpha = \sum_i a_i \, dq_i$ is a semi-basic Pfaff form on $T(U)$ and of $u = \sum_i \left(x_i \frac{\partial}{\partial q_i} + y_i \frac{\partial}{\partial \dot{q}_i} \right)$ then one will have $\alpha(u) = \sum_i a_i x_i$. Consequently, the local expression for the corresponding map D will be:

$$q_i = q_i, \qquad p_i = a_i.$$

One then deduces that:

4.15. Proposition. – *If* λ *is the Liouville form on* $T^*(M)$ *then* $D^*\lambda = \alpha$.

Exercise. – There exists one and only one antiderivation $\boldsymbol{j}$ of degree – 1 of the algebra $\mathcal{B}$ of semi-basic differential forms on $T(M)$ that has the following properties:

i) $\boldsymbol{j} f = 0$.

ii) $\boldsymbol{j}\, \boldsymbol{d}_v f = V \cdot f, f \in \mathcal{D}(T(M))$.

One will then have:

i) $\boldsymbol{j} \circ \boldsymbol{j} = 0$.

ii) $(\boldsymbol{j}\, \boldsymbol{d}_v + \boldsymbol{d}_v f) \circ v^* = v^* \mathbf{L}_V$.

§ 5. – Homogeneous differential forms.

Let $h_t : u \mapsto e^t u$ be the one-parameter group of homotheties of $T(M)$ (Chap. IX, **§ 2.4**).

5.1. Definition. – *A differential form ω on $T(M)$ is **homogeneous of degree** k if one has:*

$$(h_t^*)\omega = e^{kt}\omega .$$

5.2. Proposition. – *Let V be the Liouville field on $T(M)$. In order for a differential form ω on $T(M)$ to be homogeneous of degree k, it is necessary and sufficient that one must have:*

$$\mathbf{L}_V\, \omega = k\, \omega .$$

Proof: Let ω be a homogeneous differential form of degree k on $T(M)$. Since h_t is the one-parameter of diffeomorphisms of $T(M)$ that is generated by V, one will have:

$$\mathbf{L}_V\, \omega = \lim_{t \to 0} \frac{1}{t}(h_t^* \omega - \omega) = \lim_{t \to 0} \frac{e^{kt} - 1}{t} \omega = k\, \omega .$$

Conversely, if $\mathbf{L}_V\, \omega = k\, \omega$ for every point u of $T(M)$ then $h_t\, \omega(u)$ will be the solution to the differential equation [on $T_u(T(M))$]: $dz / dt = k\, z$ that makes $z(0) = \omega(u)$. Consequently, $h_t^* \omega = e^{kt}\omega$.

Q. E. D.

5.3. Corollary. – *Let ω be a semi-basic form on $T(M)$. In order for ω to be homogeneous of degree k, it is necessary and sufficient that one should have $\boldsymbol{i}_V\, \boldsymbol{d}\omega = k\, \omega$.*

Indeed (Prop. **4.5**), $\boldsymbol{i}_V\, \omega = 0$.

5.4. Proposition. – *Let ω be a differential form of degree p on $T(M)$ that is homogeneous of degree k. The differential forms $\boldsymbol{d}\omega$, $\boldsymbol{i}_V\,\omega$, $\boldsymbol{i}_v\,\omega$, $\boldsymbol{d}_v\,\omega$ are homogeneous of degree k, k, $k-1$, and $k-1$, respectively.*

5.5. *Local expressions.*

i) *Functions.* – One has $\mathbf{L}_V f = \sum_i \dot{q}_i \frac{\partial f}{\partial \dot{q}_i}$. Consequently, in order for f to be homogeneous of degree k, it is necessary and sufficient that it should be homogeneous of degree k in the $\dot{q}_i$.

ii) *Pfaff forms.* – Let $\alpha = \sum_i (a_i\, dq_i + b_i\, d\dot{q}_i)$. One has:

$$\mathbf{L}_V\,\alpha = \sum_{i,j} \dot{q}_j \frac{\partial a_i}{\partial \dot{q}_j} dq_i + \sum_{i,j} \dot{q}_j \frac{\partial b_i}{\partial \dot{q}_j} d\dot{q}_i + \sum_i b_i\, d\dot{q}_i\ .$$

Consequently, in order for α to be homogeneous of degree k, it is necessary and sufficient that a_i and b_i, $i = 1, \ldots, m$ should be homogeneous of degree k and $k-1$ in the $\dot{q}_i$, respectively.

Exercise. – Let ω be a differential form on $T(M)$ of degree p that is semi-basic and homogeneous of degree k such that $\boldsymbol{d}_v\,\omega = 0$. Then (see the exercise in **§ 4**):

$$\boldsymbol{d}_v\,\boldsymbol{j}\omega = (p + k)\ \omega\ .$$

CHAPTER XI

ANALYTICAL MECHANICS

§ 1. Mechanical systems (J. Klein [**6**]).

1.1. Definition. – *A* ***mechanical system*** $\mathfrak{M}$ *is a triplet* (M^m, T, π) *in which:*

M^m	is an m-dimensional manifold
T	is a differentiable function on $T(M)$
π	is a semi-basic form on $T(M)$.

One says that:

M^m	is the *configuration manifold*
m	is the number of *degrees of freedom*
$T(M)$	[or $T^*(M)$] is the *phase space*
T	is the *kinetic energy*
π	is the *force field*.

The closed form of degree two $\omega = \boldsymbol{dd}_v T$ on $T(M)$ is called the *fundamental form* of the mechanical system $\mathfrak{M}$.

1.2. Definition. – *A mechanical system* (M^m, T, π) *is* ***regular*** *if its fundamental form* $\omega = \boldsymbol{dd}_v T$ *is a symplectic form on* $T(M)$.

1.3. Local expression. – With the notations of Chapter X, one can write, locally:

$$\boldsymbol{d}_v T = \sum_j \frac{\partial T}{\partial \dot{q}_j} dq_j ,$$

$$\omega = \boldsymbol{dd}_v T = \sum_{i,j} \left(\frac{\partial T}{\partial q_i \, \partial \dot{q}_j} dq_i \wedge dq_j + \frac{\partial T}{\partial \dot{q}_i \, \partial \dot{q}_j} d\dot{q}_i \wedge dq_j \right),$$

$$\omega^m = \pm\, m!\ \det\left(\frac{\partial T}{\partial \dot q_i\, \partial \dot q_j}\right) d\dot q_1 \wedge \ldots \wedge d\dot q_m \wedge dq_1 \wedge \ldots \wedge dq_m\,.$$

Consequently, in order for $\mathfrak{M}$ to be a regular mechanical system, it is necessary and sufficient that one must have $\det\left(\frac{\partial T}{\partial \dot q_i\, \partial \dot q_j}\right) \neq 0$.

Only regular mechanical systems will be considered in what follows; hence, that fact will not be specified.

1.4. Proposition. – *Let* $\mathfrak{M} = (M^m, T, \pi)$ *be a mechanical system. There exists one and only one vector field X on T (M) such that:*

$$i_X\, \omega = d\,(T - V \cdot T) + \pi$$

[*where V is the Liouville field on T (M)*].

Indeed, such a vector field X is the dynamical system on the symplectic manifold $(T(M), \omega)$ that is associated with the Pfaff form $d\,(T - V \cdot T) + \pi$ (Chap. VII, Cor. **1.13**). One says that X is the dynamical system that is associated with the mechanical system $\mathfrak{M}$.

One will then have $d\,(T - V \cdot T) + \pi$, and as a result:

1.5. Corollary (*Vis viva* theorem). – *Let* $c: I \to T(M)$ *be an integral curve of X*, and let *a and b be two numbers in I. One has:*

$$\int_a^b c^*\pi = [V \cdot T - T]_{c(a)}^{c(b)}\ .$$

1.6. Theorem. – *The dynamical system associated with a mechanical system* $\mathfrak{M} = (M^m, T, \pi)$ *is a second-order differential equation on* M^m.

Proof: With the notations of Chapter X, one can write, locally:

$$X = \sum_i \left(a_i \frac{\partial}{\partial q_i} + b_i \frac{\partial}{\partial \dot q_i} \right),$$

$$\pi = \sum_j X_j\, dq_j\,,$$

$$\boldsymbol{i}_X\omega = \sum_{i,j}\frac{\partial^2 T}{\partial q_i\,\partial \dot q_j}a_i\,dq_j - \sum_{i,j}\frac{\partial^2 T}{\partial q_i\,\partial \dot q_j}a_j\,dq_i + \sum_{i,j}\frac{\partial^2 T}{\partial \dot q_i\,\partial \dot q_j}b_i\,dq_j - \sum_{i,j}\frac{\partial^2 T}{\partial \dot q_i\,\partial \dot q_j}a_j\,d\dot q_i\,,$$

$$dT = \sum_i\left(\frac{\partial T}{\partial q_i}dq_j + \frac{\partial T}{\partial \dot q_i}d\dot q_j\right),$$

$$V\cdot T = \sum_i\frac{\partial T}{\partial \dot q_i}d\dot q_j\,,$$

$$d\,(V\cdot T) = \sum_{i,j}\frac{\partial^2 T}{\partial q_i\,\partial \dot q_j}\dot q_i\,dq_j + \sum_{i,j}\frac{\partial^2 T}{\partial \dot q_i\,\partial \dot q_j}\dot q_j\,d\dot q_i + \sum_j\frac{\partial T}{\partial \dot q_j}d\dot q_j\,.$$

The equation $\boldsymbol{i}_X\,\omega = d\,(T - V\cdot T) + p$ then leads to the following two equations:

a) $$\sum_j\frac{\partial^2 T}{\partial \dot q_i\,\partial \dot q_j}a_j = \sum_j\frac{\partial^2 T}{\partial \dot q_i\,\partial \dot q_j}\dot q_j\,,$$

b) $$\sum_j\frac{\partial^2 T}{\partial \dot q_i\,\partial \dot q_j}b_j = -\sum_j\frac{\partial^2 T}{\partial q_i\,\partial \dot q_j}\dot q_j + \frac{\partial T}{\partial q_i} + X_i\,.$$

When one takes the regularity hypothesis for the system $\mathfrak{M}$ into account, equations *a*) will then give $a_j = \dot q_j$, $j = 1, \ldots, m$.

Q.E.D.

One also deduces the following results from these local expressions:

1.7. Proposition. – *Let $s : M^m \to T(M)$ be the zero section of $\tau(M)$. The singular points of X are the points $y = s(x)$ in the image of s for which $s^*(\pi)(x) = -s^*(dT)(x)$.*

Indeed, those points are characterized in local coordinates $\dot q_j = 0$ and $\dfrac{\partial T}{\partial q_i} = -X_i$.

1.8. Proposition. – *The integral curves of X are locally solutions of the "Lagrange equations":*

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot q_i}\right) - \frac{\partial T}{\partial q_i} = X_i\,,\quad i = 1, \ldots, m\,.$$

Indeed, the integral curves of X verify:

$$\sum_j \frac{\partial^2 T}{\partial q_i\, \partial \dot{q}_j} q_j''(t) + \sum_j \frac{\partial^2 T}{\partial q_j\, \partial \dot{q}_i} q_j'(t) = \frac{\partial T}{\partial q_i} + X_i .$$

1.9. Theorem. – *The dynamical system X that is associated with the mechanical system $\mathfrak{M} = (M^m, T, \pi)$ is characterized by the following property:*

The differential form $\Omega = p_1^* \omega + [d\,(T - V \cdot T) + \pi] \wedge dt \in \Lambda^2\,(T\,(M) \times \mathbb{R})$ *is an integral invariance relation for the vector field* $X + \partial / \partial t$.

Proof: Since the form Ω has constant class $2m$ on $T\,(M) \times \mathbb{R}$, there exists one and only one tangent vector $u \in T_y\,(T\,(M))$ such that $\boldsymbol{i}\left(u + \dfrac{\partial}{\partial t}\right) \Omega\,(y, t) = 0$.

Now, one has:

$$\boldsymbol{i}_{X + \partial / \partial t}\, \Omega = \boldsymbol{i}_X \omega + [X \cdot (T - V \cdot T) + \pi\,(X)]\, dt - [d\,(T - V \cdot T) + \pi] = 0 .$$

Q.E.D.

1.10. Remark. – That theorem shows how one can generalize (in a fashion that is analogous to the argument in Proposition **3.10** of Chapter VII) the notion of a mechanical system to the case in which the force field π depends differentiably on a parameter t .

1.11. Proposition. – *Let $\mathfrak{M} = (M^m, T, \pi)$ be a mechanical system for which the kinetic energy T is a homogeneous function of degree k. Hence:*

i) *The dynamical system X associated with $\mathfrak{M}$ is characterized by the relation* $\boldsymbol{i}_X \omega = (1 - k)\, dT + \pi$.

ii) *If k is non-zero then the zeroes of X will be the points of $T\,(M)$ that belong to the image of the zero section and annul π .*

Proof: The first property is immediate. Indeed, if T is homogeneous of degree k then $V \cdot T = k\, T$.

As for the second one, one already knows that the singular points of X are characterized locally by:

$$\dot{q}_i = 0 \qquad \text{and} \qquad X_i = -\frac{\partial T}{\partial q_i} \qquad \text{(Prop. } \mathbf{1.7}\text{)}.$$

Now, if T is homogeneous of degree k then $\partial T / \partial q_i$ will also be homogeneous of degree k, so it will be zero on the image of the zero section of $\tau(M)$ in the case of $k \neq 0$.

Q.E.D.

In particular:

1.12. Corollary (A. Lichnerowicz [**11**]). – *If T is homogeneous of degree two then the dynamical system X will be characterized by the following property: The differential form $\Omega = p_1^* \omega - (dT - \pi) \wedge dt \in \Lambda^2(T(M) \times \mathbb{R})$ will be an integral invariance relation for the vector field $X + \partial / \partial t$.*

In this corollary, one can possibly suppose that π depends upon a parameter t (Remark **1.10**).

1.13. Proposition. – *Let $\mathfrak{M} = (M^m, T, \pi)$ be a mechanical system for which the kinetic energy T and the force field π are homogeneous of degree k. The dynamical system X that is associated to $\mathfrak{M}$ will then be a spray on M^m.*

Proof: One already knows (Th. **1.6**) that X is a second-order differential equation, so it then remains to verify the equality $[V, X] = X$. Now:

$$\begin{aligned}
\boldsymbol{i}_{[V,X]}\,\omega &= \mathbf{L}_V\, \boldsymbol{i}_X \omega - \boldsymbol{i}_X\, \mathbf{L}_V\, \omega, \\
\mathbf{L}_V\, \boldsymbol{i}_X \omega &= \mathbf{L}_V ((1-k)\, dT + \pi), \\
&= k\,(1-k)\, dT + k\,\pi, \\
\mathbf{L}_V\, \omega &= \mathbf{L}_V\, \boldsymbol{d}\boldsymbol{d}_v T = \boldsymbol{d}\, \mathbf{L}_V\, \boldsymbol{d}_v T, \\
&= \boldsymbol{d}\,((k-1)\, \boldsymbol{d}_v T) \qquad \text{(Chap. X, Prop } \mathbf{5.4}\text{)}, \\
&= (k-1)\,\omega, \\
\boldsymbol{i}_X\, \mathbf{L}_V\, \omega &= (k-1)(1-k)\, dT + (k-1)\,\pi.
\end{aligned}$$

Consequently:

$$\boldsymbol{i}_{[V,X]}\,\omega = (1-k)\, dT + \pi = \boldsymbol{i}_X \omega,$$

which shows (Prop. **1.11**) that $[V, X] = X$.

Q.E.D.

1.14 *Examples:*

i) *Riemannian structure.* – A Riemannian structure on M^m is defined when one is given a Riemannian metric $T : T(M) \to \mathbb{R}$ on the tangent bundle $\tau(M)$. One also says that T is a Riemannian metric on M^m (the relationship between that notion and example **3.9** in Chap. VII will be pointed out in § **3**).

Such a structure defines a regular mechanical system $\mathfrak{M} = (M^m, T, 0)$. Indeed, the regularity of $\mathfrak{M}$ is a consequence of the non-degeneracy hypothesis on the quadratic form that is induced by T on the fibers $T_x(M)$.

The dynamical system X that is associated with $\mathfrak{M}$ is called the *geodesic field* of T. That dynamical system is the spray on M^m that is defined by $\boldsymbol{i}_X\omega = -dT$ (Prop. **1.11** and **1.13**).

The projections of the integral curves of X onto M^m are called the *geodesics* of the Riemannian structure.

ii) *Motion of a material point.* That mechanical system is defined by:

– $M = \mathbb{R}^3$,

– $T = \frac{1}{2}mg : T(\mathbb{R}^3) = \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, in which m is a positive number, and g is the canonical constant Riemannian metric on $\mathbb{R}^3$.

– $p = \sum_{i=1}^{3} X_i \, dq_i$.

The associated dynamical system X is then:

$$\sum_{i=1}^{3}\left(\dot{q}_i \frac{\partial}{\partial q_i} + \frac{X_i}{m}\frac{\partial}{\partial \dot{q}_i}\right)$$

Its integral curves are then the solutions to the second-order differential equation:

$$\frac{d^2 q_i}{dt^2} = \frac{X_i}{m}, \quad i = 1, 2, 3.$$

Here, one recovers the fundamental equation of point mechanics: $F = m\,\gamma$.

iii) *Harmonic oscillator.* The mechanical system of m independent harmonic oscillators is defined by:

– $M^m = \mathbb{R}^m$,

– $T = g$, where g is the canonical constant Riemannian metric on $\mathbb{R}^m$,

– $\pi = -d\left(\sum_{i=1}^{m} \omega_i^2 q_i^2\right)$, where the ω_i are positive numbers called the *pulsations* (†) of the oscillator.

The associated dynamical system X is then:

$$\sum_{i=1}^{m}\left(\dot{q}_i \frac{\partial}{\partial q_i} - \omega_i^2 q_i \frac{\partial}{\partial \dot{q}_i}\right).$$

Its integral curves are therefore solutions to the second-order differential equation:

$$\frac{d^2 q_i}{dt^2} = -\,\omega_i^2 q_i, \qquad i = 1, \ldots, m\,.$$

The functions $h_i = \dot{q}_i^2 + \omega_i^2 q_i^2$, $i = 1, \ldots, m$, and $H = \sum_{i=1}^{m} h_i$ are first integrals of X. Consequently, the submanifolds of $T(M^m)$ that are defined by $h_i = a_i > 0$, $i = 1, \ldots, m$ will be invariant under X (X is tangent to those submanifolds). Those submanifolds are diffeomorphic to the torus $T^m = (S^1)^m$.

If the pulsations ω_i are rationally-independent then X will induce an ergodic dynamical system on each of those tori (V. Arnold and A. Avez [**2**]). Conversely, V. Arnold shows ([**1**]) showed that this is the generic geometric situation for Hamiltonian systems that satisfy the hypotheses of the Liouville-Cartan integrability theorem (Chap. VII, Th. **4.4**).

By contrast, if all of the pulsations are equal (to 1, for example) then all of the integral curves of X will be periodic with period 1. The sphere $S^{2m-1} \subset \mathbb{R}^{2m} = T(\mathbb{R}^m)$ whose equation in $H = \sum_i (q_i^2 + \dot{q}_i^2) = 1$ will then be an invariant submanifold of X, and the trajectories of X define the Hopf fibration of S^{2m-1} over the complex projective space $\mathrm{P}_{m-1}(\mathbb{C})$.

(†) Translator: Also called the *frequencies*.

§ 2. – Lagrangian systems.

2.1. Definition. – *A mechanical system $\mathfrak{M} = (M^m, T, \pi)$ is* ***conservative*** *if the force field π is a closed semi-basic Pfaff form.*

2.2. Proposition. – *If $\mathfrak{M} = (M^m, T, \pi)$ is a conservative mechanical system then the dynamical system X that is associated with $\mathfrak{M}$ will be the Hamiltonian system on the symplectic manifold $(T(M), \omega)$ that is associated with the closed Pfaff system $\varepsilon = d(V \cdot T - T) - \pi$.*

One can then apply all of the results of Chapters VII and VIII that are concerned with first integrals, integral invariants, etc., … of Hamiltonian systems to the dynamical systems that are associated with conservative mechanical systems. In particular:

2.3. Proposition (*vis viva* integral). – *Let $\mathfrak{M} = (M^m, T, \pi)$ be a conservative mechanical system. The Pfaff form $\varepsilon = d(V \cdot T - T) - \pi$ is a first integral of the dynamical system that is associated with $\mathfrak{M}$.*

2.4. Definition. – *A mechanical system $\mathfrak{M} = (M^m, T, \pi)$ is a* ***Lagrangian*** *system if there exists a differentiable function U on M^m such that:*

$$\pi = p_M^* = d(U \circ p_M) .$$

Under those conditions, one denotes $\mathfrak{M}$ by (M^m, T, U) and says that the force field π is *derived from the force function U*.

A Lagrangian system is then a conservative system (Chap. X, Prop. **4.7**).

2.5. Definition. – *Let $\mathfrak{M} = (M^m, T, U)$ be a Lagrangian mechanical system. The function $H = V \cdot T - T - U \circ p_M$ is called the* ***Hamiltonian*** *of the system $\mathfrak{M}$.*

In particular, if T is a homogeneous function of degree k then:

$$H = (k-1)\,T - U \circ p_M , \quad (H = T - U \circ p_M \text{ for } k = 2).$$

2.6. Local expression. – With the notations of Chapter X, one will have:

$$H = \sum_i \dot{q}_i \frac{\partial T}{\partial \dot{q}_i} - T - U .$$

When one reformulates **2.2**, one will get:

2.7. Proposition. – *Let* $\mathfrak{M} = (M^m, T, U)$ *be a Lagrangian system. The dynamical system* X *that is associated with* $\mathfrak{M}$ *is characterized by the relations:*

$$\boldsymbol{i}_X \omega = - dH .$$

2.8. Corollary (Painlevé integral). – *The Hamiltonian:*

$$H = V \cdot T - T - U \circ p_M$$

is a first integral of X.

2.9. Corollary (E. Cartan [**3**]). – *The dynamical system* X *is characterized by the following property: The differential form* $\Omega = \omega - dH \wedge dt \in \Lambda^2 \in (T(M))$ *is an absolute integral invariant for the vector field* $X + \partial / \partial t$.

2.10. Corollary (E. Cartan [**3**]). – *The dynamical system* X *is characterized by the following property: The Pfaff form* $\alpha = d_v T - H\, dt \in \Lambda^1 \in (T(M))$ *is an relative integral invariant for the vector field* $X + \partial / \partial t$.

2.11. Definition. – *Let* $\mathfrak{M} = (M^m, T, U)$ *be a Lagrangian mechanical system. The function* $L = T + U \circ p_M$ *is called the* ***Lagrangian*** *of the system* $\mathfrak{M}$.

One will then have $H = V \cdot L - L$.

In the case of Lagrangian mechanical systems, Proposition **1.8** will become:

2.12. Proposition. – *The integral curves of the dynamical system* X *that is associated with a Lagrangian system* $\mathfrak{M} = (M^m, T, U)$ *are locally solutions to the "Lagrange equation":*

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0, \qquad 1 \le i \le m.$$

§ 3. Legendre transformation.

Let $\mathfrak{M} = (M^m, T, \pi)$ be a (regular) mechanical system. The Pfaff form $\boldsymbol{d}_v T$ is a semi-basic form on the tangent space $T(M)$. Consequently, when one reformulates Theorem **4.13** and Proposition **4.13** of Chapter X, one will get:

3.1. Theorem. – *There exists a differentiable map* $D : T(M) \to T^*(M)$ *that has the following properties:*

i) $q_M \cdot D = p_M$.

ii) *D has constant rank* $2m$.

iii) $D^*\lambda = \boldsymbol{d}_v T$ [*where* λ *is the Liouville form on* $T^*(M)$].

That differentiable map $D : T(M) \to T^*(M)$ is called the *Legendre transformation* of the mechanical system $\mathfrak{M}$.

Recall (§ **4.13** in Chap. X) that the local expression for D is:

$$q_i = q_i\,, \qquad p_i = \frac{\partial T}{\partial \dot{q}_i}\,.$$

D is then the classical transformation that permits one to pass from the Lagrange equations to the Hamiltonian ones.

3.2 Remark. – Although the Legendre transformation verifies $q_M \circ D = p_M$, it is not generally a homomorphism of $\tau(M)$ into $\tau^*(M)$; indeed, it is not (generally) linear on the fibers.

3.3. Definition. – *If the Legendre transformation* $D : T(M) \to T^*(M)$ *is a diffeomorphism then the mechanical system* $\mathfrak{M} = (M^m, T, \pi)$ *satisfies the* ***Lagrange-Hamilton duality hypothesis.***

3.4. Lemma. – *In order for D to be a diffeomorphism, it is necessary and sufficient that it should be a bijection of* $T_y(M)$ *onto* $T_y^*(M)$ *for any point y in* M^m .

Indeed (cf., the rank theorem), in order for D to be a diffeomorphism, it is necessary and sufficient that D should be a bijection. Now, since $q_M \circ D = p_M$, the latter condition is equivalent to the one that D should be a bijection on each fiber of $\tau(M)$.

3.5. Theorem. – *Let* $\mathfrak{M} = (M^m, T, \pi)$ *be a mechanical system that satisfies the Lagrange-Hamilton duality hypothesis, and let X be the dynamical system that is associated with* $\mathfrak{M}$*. Therefore,* $Y = D^{\mathrm{T}} X D^{-1}$ *will be the vector field on* $T^*(M)$ *that is characterized by the relation:*

$$\boldsymbol{i}_Y\, \boldsymbol{d}\lambda = (D^{-1})^*\,[d\,(T - V \cdot T) + \pi].$$

Proof: Indeed, one has (Chap. III, Prop. **6.2**):

$$\boldsymbol{i}_Y\, \boldsymbol{d}\lambda = \boldsymbol{i}_Y\, (D^{-1})^*\omega\,,$$

$= (D^{-1})^* \, \boldsymbol{i}_X \omega ,$
$= (D^{-1})^* \, [d \, (T - V \cdot T) + \pi]$

Q.E.D.

3.6. Corollary. – *If $c : I \to T(M)$ is an integral curve of X then $\gamma = D \circ c$ will be an integral curve of Y, and one will have $p_M \circ c = q_M \circ \gamma$.*

Moreover, if $\mathfrak{M}$ is a Lagrangian system with the Hamiltonian:

$$H = V \cdot T - T - U \circ p_M$$

then one will have:

3.7. Proposition. – *The vector field Y on $T^*(M)$ is the Hamiltonian system on the symplectic manifold $(T^*(M), d\lambda)$ that is characterized by the relation:*

$$\boldsymbol{i}_Y \, \boldsymbol{d}\lambda = - \, d \, (H \circ D^{-1}) \, .$$

Under those conditions, one further says that $H \circ D^{-1} : T^*(M) \to \mathbb{R}$ is the *Hamiltonian* of the Lagrangian system.

In fact, a good number of mechanical systems that classically show up in differential geometry or analytical mechanics satisfy the Lagrange-Hamilton duality hypothesis. Indeed:

3.2. Theorem. – *Let $\mathfrak{M} = (M^m, T, \pi)$ be a mechanical system for which the kinetic energy T is a Riemannian metric on M^m. The following properties will then be verified:*

i) *$\mathfrak{M}$ satisfies the Lagrange-Hamilton duality hypothesis.*

ii) *D is an isomorphism (in the vector bundle sense) of the tangent bundle $\tau(M)$ onto the cotangent bundle $\tau^*(M)$.*

iii) *$T \circ D^{-1}$ is a Riemannian metric on $\tau^*(M)$.*

Indeed, if one writes $T = \frac{1}{2} \sum a_{ij} \, \dot{q}_i \, \dot{q}_j$ locally, with $a_{ij} = a_{ji}$, then the expression of D will be $q_i = q_i$, $p_i = \sum_j a_{ij} \, \dot{q}_j$.

3.9. Corollary. – *Under the hypotheses of Proposition **3.7**, and if T is a Riemannian metric on M^m, moreover, then the dynamical system Y will be the Hamiltonian system on $T^*(M)$ that is* *associated with the classical Hamiltonian structure (in the sense of Chap. VII) that is defined by the* $H = T \circ D^{-1} - U \circ q_M$.

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The reader whose desires to go deeper into some other modern aspects of mechanics can consult the following references:

1) Abstract dynamical systems and ergordic theory: [**2**] and [**12**].
2) Qualitative study of the trajectories of a dynamical system:

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