"Sur les invariants intégraux," J. math. pures et appl. (6) **4** (1908), 331-365.

On integral invariants

By E. GOURSAT

Translated by D. H. Delphenich

The general theory of integral invariants was developed by Poincaré in Tome III of his book *Méthodes Nouvelles de la Mécanique céleste*. The goal of this article is to contribute to the study of the following general question: If one knows an integral invariant, whether absolute or relative, of arbitrary order for a system of differential equations, what will that imply for the integration of that system? I will show that from any integral invariant, one can deduce at least one system of differential equations whose integrals all belong to the proposed system and whose integration is generally a simple problem. In the case where the two systems are equivalent, one can determine a multiplier ([1]).

I.

1. – I shall first recall the main results of the theory of multiple integrals that will be utilized in this article, as well as the precise significance of the notations that will be employed ([2]).

Let $x_1, x_2, \ldots, x_n$ be a system of n independent variables, and let:

$$A_{\alpha_1 \alpha_2 \cdots \alpha_p} \qquad\qquad (p \leq n)$$

be a system of functions of those n variables, each of which is endowed with p different indices $\alpha_1, \alpha_2, \ldots, \alpha_p$ that are taken from the first n numbers. Each arrangement of the first n numbers taken p at a time will then correspond to a well-defined function of the n variables $x_1, x_2, \ldots, x_n$. The functions for which some indices differ will be completely independent, but all of the functions whose indices differ by only their order are *equal, up to sign*. Therefore, let $(\alpha_1', \alpha_2', \ldots, \alpha_p')$ be a new permutation of the indices $(\alpha_1, \alpha_2, \ldots, \alpha_p)$. One has:

([1]) The main results of this article have been summarized in a note that was presented to the Academy of Sciences (C. R. Acad. Sci. Paris, 2 June 1907).

([2]) In addition to the cited work of Poincaré, one can consult the following articles by the same author: "Sur les résidus des intégrales doubles" (Acta Mathematica, t. IX); "Analysis Situs" (Journal de l'École Polytechnique, 1895); "Complément à l'*Analysis Situs*" (Rendiconti del Circolo matematico do Palermo). One will also find some bibliographic information on invariants in two papers by de Donder (Rendiconti, 1901 and 1902).

(1) $$A_{\alpha'_1\alpha'_2\cdots\alpha'_p} = A_{\alpha_1\alpha_2\cdots\alpha_p}$$

if the two permutations ($\alpha_1, \alpha_2, \ldots, \alpha_p$) and $(\alpha'_1, \alpha'_2, \ldots, \alpha'_p)$ belong to the same class, and:

(2) $$A_{\alpha'_1\alpha'_2\cdots\alpha'_p} = -A_{\alpha_1\alpha_2\cdots\alpha_p}$$

when the two permutations belong to different classes. When two indices are equal, the function will necessarily be zero.

We observe, in passing, that the two permutations:

$$(\alpha_1, \alpha_2, \ldots, \alpha_p) \text{ and } (\alpha'_1, \alpha'_2, \ldots, \alpha'_p)$$

will belong to the same class if p is odd and to different classes if p is even. Indeed, in the two cases, one passes from the first permutation to the second one by $p - 1$ exchanges of two consecutive elements.

Suppose that the n variables $x_1, x_2, \ldots, x_n$ are expressed by means of p independent variables $u_1, u_2, \ldots, u_p$, and consider the multiple integral of order p :

(I) $$I_p = \iint\cdots\int \sum A_{\alpha_1\alpha_2\cdots\alpha_p} \frac{\partial x_{\alpha_1}}{\partial u_1}\frac{\partial x_{\alpha_2}}{\partial u_2}\cdots\frac{\partial x_{\alpha_p}}{\partial u_p} du_1\cdots du_p,$$

which is extended over a certain domain (e_p) in the space ($u_1, u_2, \ldots, u_p$), and the summation that is indicated by the symbol $\sum$ extends over all arrangements of the first n numbers taken p at a time. From formulas (1) and (2), that multiple integral can be further written:

(II) $$I_p = \iint\cdots\int \sum A_{\alpha_1\alpha_2\cdots\alpha_p} \frac{D(x_{\alpha_1}, x_{\alpha_2}, \ldots, x_{\alpha_p})}{D(u_1, u_2, \ldots, u_p)} du_1\cdots du_p,$$

in which the symbol $\sum$ in that formula extends over all *combinations* of the first n numbers taken p at a time. In each combination, one can take the indices in an arbitrary order, but one must be careful to take the variables x_i in each functional determinant in the order that is indicated by the order of indices of the corresponding coefficient.

When the point with the coordinates ($u_1, u_2, \ldots, u_p$) describes the domain (e_p) in p-dimensional space, the point whose coordinates are ($x_1, x_2, \ldots, x_n$) will describe a p-dimensional *continuum* (E_p) in n-dimensional space. The form (II) of the integral I_p, which resembles the formula for the change of variables in a multiple integral, shows immediately that the value of I_p does not depend upon the choice of auxiliary variables $u_1, u_2, \ldots, u_p$, but only on the domain (E_p). Meanwhile, it must be pointed out that the integral might change sign when one exchanges any of those variables. That fact is analogous to the one that presents itself for a surface integral in three-dimensional space, in which the sign of the integral will change at the same time as the surface on which one takes the integral.

Most often, we shall write the multiple integral I_p in the abbreviated form:

$$\text{(III)} \qquad I_p = \iint \cdots \int \sum A_{\alpha_1 \alpha_2 \cdots \alpha_p} \, dx_{\alpha_1} dx_{\alpha_2} \cdots dx_{\alpha_p} ,$$

in which the symbol $\sum$ extends over all combinations of the first n numbers take p at a time. However, in order to get the precise significance of that symbol, one must always refer to the expression (I) or (II).

Remark. – The order in which one writes the differentials in products such as $dx_{\alpha_1} dx_{\alpha_2} \cdots dx_{\alpha_p}$ is not irrelevant, as one sees. For example, if one is dealing with a surface integral then the symbols:

$$\iint A\,dy\,dz + \iint B\,dz\,dx + \iint C\,dx\,dy ,$$
$$\iint A\,dy\,dz + \iint B\,dz\,dx + \iint C\,dy\,dx ,$$
$$\iint A\,dy\,dz + \iint B\,dx\,dz + \iint C\,dx\,dy$$

will not all have the same meaning. The notation (I) has the advantage of eliminating any ambiguity. A surface integral will be written with that notation as:

$$I_2 = \iint \left[A_{12} \frac{D(x_1, x_2)}{D(u,v)} + A_{23} \frac{D(x_2, x_3)}{D(u,v)} + A_{31} \frac{D(x_3, x_1)}{D(u,v)} \right] du\,dv ,$$

in which x_1, x_2, x_3 are supposed to be expressed in terms of the two auxiliary variables.

2. – One will find proofs of the following theorems in the cited article by Poincaré (Acta mathematica). The functions that one considers are supposed to be uniform and continuous, at least within the domain of integration.

Any integral I_p that is extended over a closed multiplicity (E_p) in the n-dimensional space ($p < n$) can be replaced with an integral I_{p+1} that is extended over a multiplicity (E_{p+1}) in the n-dimensional space that is bounded by the first p-dimensional multiplicity:

$$\text{(IV)} \qquad I_{p+1} = \iint \cdots \int \sum A_{\alpha_1 \alpha_2 \cdots \alpha_p \alpha_{p+1}} dx_{\alpha_1} dx_{\alpha_2} \cdots dx_{\alpha_{p+1}} .$$

The symbol $\sum$ extends over all combinations of the first n numbers taken $p + 1$ at a time.

The coefficient:

$$\mathcal{A}_{\alpha_1 \alpha_2 \cdots \alpha_p \alpha_{p+1}}$$

can have two different expressions according to the parity of p.

If p is even then one will have:

(3) $$\mathcal{A}_{\alpha_1 \alpha_2 \cdots \alpha_{p+1}} = \frac{\partial \mathcal{A}_{\alpha_1 \alpha_2 \cdots \alpha_p}}{\partial x \alpha_{p+1}} + \frac{\partial \mathcal{A}_{\alpha_2 \cdots \alpha_p \alpha_{p+1}}}{\partial x \alpha_1} + \cdots + \frac{\partial \mathcal{A}_{\alpha_{p+1} \alpha_1 \cdots \alpha_p}}{\partial x \alpha_p},$$

with only + signs in the right-hand side. If p is odd then one will have:

(4) $$\mathcal{A}_{\alpha_1 \alpha_2 \cdots \alpha_{p+1}} = \frac{\partial \mathcal{A}_{\alpha_1 \alpha_2 \cdots \alpha_p}}{\partial x \alpha_{p+1}} - \frac{\partial \mathcal{A}_{\alpha_2 \cdots \alpha_p \alpha_{p+1}}}{\partial x \alpha_1} + \cdots - \frac{\partial \mathcal{A}_{\alpha_{p+1} \alpha_1 \cdots \alpha_p}}{\partial x \alpha_p},$$

in which the + and – signs alternate.

Those formulas provide the answer to the following question:

What are the necessary and sufficient conditions for the integral I_p, which is extended over a p-dimensional multiplicity, to depend upon only the p – 1-dimensional multiplicity that bounds that domain?

In order for that to be true, it is necessary and sufficient that the integral I_p should be zero when it is extended over an arbitrary *closed* p-dimensional multiplicity or that the integral I_{p+1} (which is equal to it) that extends over an arbitrary $p + 1$-dimensional multiplicity should be zero, i.e., that one must have:

$$\mathcal{A}_{\alpha_1 \alpha_2 \cdots \alpha_{p+1}} = 0$$

for all combinations of indices.

We say, to abbreviate, that the expression:

(5) $$\sum \mathcal{A}_{\alpha_1 \alpha_2 \cdots \alpha_p \alpha_{p+1}} \, dx_{\alpha_1} \, dx_{\alpha_2} \cdots dx_{\alpha_{p+1}}$$

is an *exact total differential*, and we can state the following proposition:

In order for the expression (5) *to be an exact differential, it is necessary and sufficient that one should have:*

(6) $$\frac{\partial \mathcal{A}_{\alpha_1 \alpha_2 \cdots \alpha_p}}{\partial x \alpha_{p+1}} + \frac{\partial \mathcal{A}_{\alpha_2 \cdots \alpha_p \alpha_{p+1}}}{\partial x \alpha_1} + \cdots + \frac{\partial \mathcal{A}_{\alpha_{p+1} \alpha_1 \cdots \alpha_p}}{\partial x \alpha_p} = 0$$

for all combinations of indices if p is even, and:

$$(6') \qquad \frac{\partial \mathcal{A}_{\alpha_1 \alpha_2 \cdots \alpha_p}}{\partial x \alpha_{p+1}} - \frac{\partial \mathcal{A}_{\alpha_2 \cdots \alpha_p \alpha_{p+1}}}{\partial x \alpha_1} + \cdots - \frac{\partial \mathcal{A}_{\alpha_{p+1} \alpha_1 \cdots \alpha_p}}{\partial x \alpha_p} = 0$$

if p is odd. The total number of those conditions is equal to the number of combinations of n objects taken p + 1 *at a time.*

If the expression (5) is not an exact total differential then the analogous expression:

$$(7) \qquad \sum_{\alpha_1 \alpha_2 \cdots \alpha_{p+1}} \mathcal{A}_{\alpha_1 \alpha_2 \cdots \alpha_{p+1}} \, dx_{\alpha_1} \, dx_{\alpha_2} \cdots dx_{\alpha_{p+1}} ,$$

in which the coefficients $\mathcal{A}_{\alpha_1 \alpha_2 \cdots \alpha_{p+1}}$ are given by formulas (3) or (4), will be an exact total differential. Indeed, one will very easily deduce that the relations that are analogous to (6) and (6′) are verified identically from those expressions for the coefficients.

It follows from this that any multiple integral I_p of order p that extends over a closed multiplicity (E_p) can be replaced with an integral of an exact total differential I_{p+1} that is extended over a multiplicity (E_{p+1}) that is bounded by the multiplicity (E_p) (viz., the generalized Stokes theorem).

Conversely, if the expression (5) is an exact total differential then the integral I_p, which is extended over a non-closed multiplicity (E'_p), can be replaced with an integral I_{p-1} that is extended over the closed multiplicity (E_{p-1}) that bounds (E'_p). In order for that to be true, it would suffice to show that one can define an integral I_{p-1} :

$$I_{p-1} = \iint \cdots \int \sum_{\alpha_1 \alpha_2 \cdots \alpha_{p-1}} C_{\alpha_1 \alpha_2 \cdots \alpha_{p-1}} \, dx_{\alpha_1} \, dx_{\alpha_2} \cdots dx_{\alpha_{p-1}} ,$$

such that I_p can be deduced from I_{p-1} in the same way that we have deduced I_{p+1} from I_p . One will then have a certain number of partial differential equations for determining the coefficients $C_{\alpha_1 \alpha_2 \cdots \alpha_{p-1}}$, and those expressions will be compatible as long as the relations (6) or (6′)

3. – Let us apply those generalities to the simplest cases.

If p = 1 then one has the simple integral that is analogous to a curvilinear integral:

$$(8) \qquad I_1 = \int A_1 \, dx_1 + A_2 \, dx_2 + \cdots + A_n \, dx_n \; .$$

When that integral I_1 is extended over a closed line L, it will be equal to the double integral:

(9) $$I_2 = \iint \sum_{i,k} \left(\frac{\partial A_i}{\partial x_k} - \frac{\partial A_i}{\partial x_k} \right) dx_i \, dx_k$$

that extends over a two-dimensional multiplicity that is bounded by the line L. In order for I_1 to be an integral of an exact differential, it is necessary and sufficient that the integral I_2 should be identically zero, which will then give the well-known conditions:

(10) $$\frac{\partial A_i}{\partial x_k} = \frac{\partial A_i}{\partial x_k} \qquad (i, k = 1, 2, \ldots, n).$$

Now, let I_2 be an arbitrary double integral:

(11) $$I_2 = \iint \sum_{i,k} A_{i,k} \, dx_i \, dx_k \, .$$

That double integral I_2, which extends over a closed multiplicity (E_2), is equal to a triple integral I_3 that extends over a three-dimensional multiplicity that is bounded by (E_2):

(12) $$I_3 = \iiint \sum_{i,k,l} \left(\frac{\partial A_{ik}}{\partial x_l} + \frac{\partial A_{kl}}{\partial x_i} + \frac{\partial A_{li}}{\partial x_k} \right) dx_i \, dx_k \, dx_l$$

In order for I_2 to be an integral of an exact differential, it is necessary that one should have:

(13) $$\frac{\partial A_{ik}}{\partial x_l} + \frac{\partial A_{kl}}{\partial x_i} + \frac{\partial A_{li}}{\partial x_k} = 0 \qquad (i, k, l = 1, 2, \ldots, n)$$

for any indices i, k, l.

If those conditions are satisfied then one can identify the expressions (9) and (11). In other words, one can determine n functions $A_1, A_2, \ldots, A_n$ that are satisfied by the relations:

(14) $$\frac{\partial A_i}{\partial x_k} - \frac{\partial A_i}{\partial x_k} = A_{i,k} \qquad (i, k = 1, 2, \ldots, n).$$

4. – Recall once more the definition of integral invariants.

Let:

(15) $$\frac{dx_1}{X_1} = \frac{dx_2}{X_2} = \ldots = \frac{dx_n}{X_n} = dt$$

be a system of differential equations. We suppose that the functions X_i are uniform and continuous, as well as their derivatives, and do not include t, and we say *characteristic* to mean any one-dimensional multiplicity Γ_1 that is represented by the equations:

$$x_1 = f_1(t), \qquad x_2 = f_2(t), \qquad \ldots, \qquad x_n = f_n(t),$$

when $f_1(t), \ldots, f_n(t)$ form a system of solutions to equations (15). A characteristic Γ that is described by the point $(x_1, x_2, \ldots, x_n)$ when t varies starts from each point $(x_1^0, x_2^0, \ldots, x_n^0)$ in n-dimensional space.

If the initial value of t is supposed to be zero then consider an arbitrary p-dimensional multiplicity (E_p^0) in n-dimensional space. A characteristic starts from each point $(x_1^0, \ldots, x_n^0)$ of that multiplicity, and after a length of time t, the point $(x_1^0, \ldots, x_n^0)$ will arrive at the point $(x_1, \ldots, x_n)$. The locus of those different points is another p-dimensional multiplicity (E_p). If the multiple integral:

$$(16) \qquad I_p = \iint \cdots \int \sum A_{\alpha_1 \alpha_2 \cdots \alpha_p} \, dx_{\alpha_1} dx_{\alpha_2} \cdots dx_{\alpha_p}$$

has the same value for two multiplicities (E_p^0) and (E_p) for any t then one says that I_p is an ***absolute** integral invariant of order p* of the system (15).

It might happen that this invariance property is true for only *closed* multiplicities. One then says that one has a ***relative** integral invariant of order p*, and one denotes it by the symbol J_p.

As far as absolute invariants are concerned, we shall once more make the following distinction: An absolute invariant can be an integral of an exact total differential. In that case, we represent it by I_p^d. There is no reason to make that distinction for relative invariants since the integral of an exact total differential will always be zero when it is extended over a closed multiplicity.

From the foregoing, an integral invariant J_p or I_p will immediately give an integral invariant I_{p+1}^d. Conversely, an integral invariant I_p^d is equivalent to a relative integral invariant J_{p-1}.

5. – We shall now look for the conditions that the coefficients $A_{\alpha_1 \alpha_2 \cdots \alpha_p}$ must satisfy in order for I_p to be an absolute invariant in a general manner. In order to do that, it will suffice to consider I_p to be a function of t and to write that its derivative is zero:

$$\frac{dI_p}{dt} = 0 .$$

In order to obtain dI_p / dt, suppose that one gives an increment of δt to t and calculate the coefficient of δt in the difference $I_p(t + \delta t) - I_p(t)$.

Let $(x_1, x_2, \ldots, x_n)$ be the coordinates of an arbitrary point of the multiplicity (E_p) at the time t, and let $(x_1', x_2', \ldots, x_n')$ be the coordinates of the corresponding point on the multiplicity at time $t + \delta t$. One has:

$$x_i' = x_i + \delta t \, X_i + \ldots \qquad (i = 1, 2, \ldots, n),$$

in which the unwritten terms are infinitely-small of second order in δt. We write the two integrals $I_p(t)$ and $I_p(t + \delta t)$ in the explicit form (I):

$$I_p(t) = \iint \cdots \int \sum A_{\alpha_1 \alpha_2 \cdots \alpha_p} \frac{\partial x_{\alpha_1}}{\partial u_1} \cdots \frac{\partial x_{\alpha_p}}{\partial u_p} du_1\, du_2 \cdots du_p ,$$

$$I_p(t + \delta t) = \iint \cdots \int \sum A'_{\alpha_1 \alpha_2 \cdots \alpha_p} \frac{\partial x'_{\alpha_1}}{\partial u_1} \cdots \frac{\partial x'_{\alpha_p}}{\partial u_p} du_1\, du_2 \cdots du_p .$$

$A'_{\alpha_1 \alpha_2 \cdots \alpha_p}$ denotes what $A_{\alpha_1 \alpha_2 \cdots \alpha_p}$ will become when one replaces x_i with x'_i, and the two integrals are extended over the same domain for the auxiliary variables $u_1, u_2, \ldots, u_p$.

Let $A'_{\beta_1 \beta_2 \cdots \beta_p} \frac{\partial x'_{\beta_1}}{\partial u_1} \frac{\partial x'_{\beta_2}}{\partial u_2} \cdots \frac{\partial x'_{\beta_p}}{\partial u_p}$ be an arbitrary term in the second integral. One will have:

$$\begin{aligned}
A'_{\beta_1 \beta_2 \cdots \beta_p} &= A_{\beta_1 \beta_2 \cdots \beta_p} + \delta t\, X(A_{\beta_1 \beta_2 \cdots \beta_p}) + \cdots \\
\frac{\partial x'_{\beta_1}}{\partial u_1} &= \frac{\partial x_{\beta_1}}{\partial u_1} + \delta t \sum_h \frac{\partial X_{\beta_1}}{\partial x_h} \frac{\partial x_h}{\partial u_1} + \cdots, \\
\frac{\partial x'_{\beta_2}}{\partial u_2} &= \frac{\partial x_{\beta_2}}{\partial u_2} + \delta t \sum_h \frac{\partial X_{\beta_2}}{\partial x_h} \frac{\partial x_h}{\partial u_2} + \cdots, \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\frac{\partial x'_{\beta_p}}{\partial u_p} &= \frac{\partial x_{\beta_p}}{\partial u_p} + \delta t \sum_h \frac{\partial X_{\beta_p}}{\partial x_h} \frac{\partial x_h}{\partial u_p} + \cdots
\end{aligned}$$

Let us seek the coefficient of $\frac{\partial x_{\alpha_1}}{\partial u_1} \frac{\partial x_{\alpha_2}}{\partial u_2} \cdots \frac{\partial x_{\alpha_p}}{\partial u_p}$ in the new integral. In order for the product:

$$A'_{\beta_1 \beta_2 \cdots \beta_p} \frac{\partial x'_{\beta_1}}{\partial u_1} \frac{\partial x'_{\beta_2}}{\partial u_2} \cdots \frac{\partial x'_{\beta_p}}{\partial u_p}$$

to give a term of that type, two and only two hypotheses are possible:

1. One can have:

$$\beta_1 = \alpha_1, \qquad \beta_2 = \alpha_2, \qquad \ldots, \qquad \beta_p = \alpha_p,$$

which will give the term:

$$\delta t\, X(A_{\alpha_1 \alpha_2 \cdots \alpha_p}) \frac{\partial x_{\alpha_1}}{\partial u_1} \frac{\partial x_{\alpha_2}}{\partial u_2} \cdots \frac{\partial x_{\alpha_p}}{\partial u_p},$$

in which one has set:

$$X(f) = X_1 \frac{\partial f}{\partial x_1} + \cdots + X_n \frac{\partial f}{\partial x_n}.$$

2. One can once more obtain a product of the desired form by supposing that all of the p equalities $\beta_i = \alpha_i$ are verified, except for one. For example, if one has:

$$\beta_1 = \alpha_1, \qquad \ldots, \qquad \beta_{i-1} = \alpha_{i-1}, \qquad \beta_{i+1} = \alpha_{i+1}, \qquad \ldots, \qquad \beta_p = \alpha_p,$$

in which β_i is arbitrary, then one will have the product:

$$\delta t\, A_{\alpha_1 \cdots \alpha_{i-1} \beta_i \alpha_{i+1} \cdots \alpha_p} \frac{\partial X_{\beta_i}}{\partial x_{\alpha_1}} \frac{\partial x_{\alpha_1}}{\partial u_1} \frac{\partial x_{\alpha_2}}{\partial u_2} \cdots \frac{\partial x_{\alpha_p}}{\partial u_p},$$

and the sum of the terms that are obtained from it by varying β_i can be written:

$$\delta t \sum_h A_{\alpha_1 \cdots \alpha_{i-1} h \alpha_{i+1} \cdots \alpha_p} \frac{\partial X_h}{\partial x_{\alpha_1}} \frac{\partial x_{\alpha_1}}{\partial u_1} \frac{\partial x_{\alpha_2}}{\partial u_2} \cdots \frac{\partial x_{\alpha_p}}{\partial u_p}.$$

Since the variable index β_i can replace any one of the indices $\alpha_1, \alpha_2, \ldots, \alpha_p$, one sees that, by definition, the coefficient of $\delta t \frac{\partial x_{\alpha_1}}{\partial u_1} \cdots \frac{\partial x_{\alpha_p}}{\partial u_p}$ in the second integral will have the expression:

$$B_{\alpha_1 \alpha_2 \cdots \alpha_p} = X(A_{\alpha_1 \alpha_2 \cdots \alpha_p}) + \sum_h \left(A_{h \alpha_2 \cdots \alpha_p} \frac{\partial X_h}{\partial x_{\alpha_1}} + A_{\alpha_1 h \cdots \alpha_p} \frac{\partial X_h}{\partial x_{\alpha_2}} + \cdots + A_{\alpha_1 \cdots \alpha_{p-1}, h} \frac{\partial X_h}{\partial x_{\alpha_p}} \right), \tag{17}$$

and the derivative dI_p / dt will be represented by a multiple integral of the same form as I_p :

$$\frac{dI_p}{dt} = \iint \cdots \int \sum_h B_{\alpha_1 \alpha_2 \cdots \alpha_p} \frac{\partial x_{\alpha_1}}{\partial u_1} \frac{\partial x_{\alpha_2}}{\partial u_2} \cdots \frac{\partial x_{\alpha_p}}{\partial u_p} du_1\, du_2 \cdots du_p, \tag{18}$$

in which that integral is extended over the same domain as the original one.

In order for I_p to be an absolute integral invariant, it is necessary that dI_p / dt must be zero for any domain of integration, i.e., all of the coefficients $B_{\alpha_1 \alpha_2 \cdots \alpha_p}$ must be zero separately. Therefore, *in order for I_p to be an absolute integral invariant, it is necessary and sufficient that one should:*

$$X(A_{\alpha_1 \alpha_2 \cdots \alpha_p}) + \sum_h \left(A_{h \alpha_2 \cdots \alpha_p} \frac{\partial X_h}{\partial x_{\alpha_1}} + A_{\alpha_1 h \cdots \alpha_p} \frac{\partial X_h}{\partial x_{\alpha_2}} + \cdots + A_{\alpha_1 \cdots \alpha_{p-1}, h} \frac{\partial X_h}{\partial x_{\alpha_p}} \right) = 0 \tag{19}$$

for all combinations of the indices $\alpha_1, \alpha_2, \ldots, \alpha_p$.

In order for $I_p(t)$ to be a *relative* integral invariant, it would suffice that the multiple integral (18) that expresses dI_p / dt is zero when extended over an arbitrary closed multiplicity, i.e., that the expression:

$$\sum_{\alpha_1\cdots\alpha_p} B_{\alpha_1\alpha_2\cdots\alpha_p} dx_{\alpha_1}\cdots dx_{\alpha_p}$$

should be an exact total differential. One will obtain the same conditions by expressing the idea that the multiple integral I_{p+1}^d of order $p+1$ that one deduced from I_p in the way that was explained above is an *absolute* integral invariant of order $p+1$ that provides equations of the same form as equations (19):

$$X(\mathcal{A}_{\alpha_1\alpha_2\cdots\alpha_p\alpha_{p+1}}) + \sum_h \left(\mathcal{A}_{h\alpha_2\cdots\alpha_{p+1}} \frac{\partial X_h}{\partial x_{\alpha_1}} + \mathcal{A}_{\alpha_1 h\cdots\alpha_{p+1}} \frac{\partial X_h}{\partial x_{\alpha_2}} + \cdots + \mathcal{A}_{\alpha_1\cdots\alpha_p h} \frac{\partial X_h}{\partial x_{\alpha_{p+1}}} \right) = 0 , \tag{20}$$

in which $\mathcal{A}_{\alpha_1\alpha_2\cdots\alpha_{p+1}}$ are given by equations (3) or (4) according to the parity of p.

II.

6. – Poincaré has also indicated (*Méthodes Nouvelles de la Mécanique céleste*, t. III, pp. 33) a procedure that will permit one to deduce an absolute invariant of lower order I_{p-1} from an absolute invariant I_p.

Let:

$$I_p = \iint\cdots\int \sum A_{\alpha_1\alpha_2\cdots\alpha_p} dx_{\alpha_1} dx_{\alpha_2}\cdots dx_{\alpha_p}$$

be an absolute invariant of order p of the system (15). *If one sets:*

$$C_{\alpha_1\alpha_2\cdots\alpha_{p-1}} = \sum_{i=1}^n A_{\alpha_1\alpha_2\cdots\alpha_{p-1}i} X_i \tag{21}$$

then one will have a new relative invariant of order $p-1$:

$$I_{p-1} = \iint\cdots\int \sum C_{\alpha_1\alpha_2\cdots\alpha_{p-1}} dx_{\alpha_1} dx_{\alpha_2}\cdots dx_{\alpha_{p-1}} . \tag{22}$$

That is the general form of the proposition that Poincaré deduced from the link that exists between the integral invariants and the equations of variations. It is easy to verify that by means of the conditions (19). Indeed, it suffices for one to show that the relations:

(23) $$X(C_{\alpha_1\alpha_2\cdots\alpha_{p-1}})+\sum_{h=1}^{n}\left(C_{h\alpha_2\cdots\alpha_{p-1}}\frac{\partial X_h}{\partial x_{\alpha_1}}+\cdots+C_{\alpha_1\alpha_2\cdots\alpha_{p-2}h}\frac{\partial X_h}{\partial x_{\alpha_{p-1}}}\right)=0$$

are consequences of equations (19). Now, one has:

$$X(C_{\alpha_1\alpha_2\cdots\alpha_{p-1}}) = \sum_i X_i X(A_{\alpha_1\alpha_2\cdots\alpha_{p-1}i})+\sum_i\sum_k A_{\alpha_1\alpha_2\cdots\alpha_{p-1}i}\, X_k\frac{\partial X_i}{\partial x_k}.$$

Replace $X(A_{\alpha_1\alpha_2\cdots\alpha_{p-1}i})$ with its value that one infers from formula (19). That will give:

$$\begin{aligned} &X(C_{\alpha_1\alpha_2\cdots\alpha_{p-1}}) \\ &= \sum_i\sum_h A_{\alpha_1\alpha_2\cdots\alpha_{p-1}i}\, X_h\frac{\partial X_i}{\partial x_h}-\sum_i X_i\sum_h\left(A_{h\,\alpha_2\cdots\alpha_{p-1}i}\frac{\partial X_h}{\partial x_{\alpha_1}}+\cdots+A_{\alpha_1\cdots\alpha_{p-2}hi}\frac{\partial X_h}{\partial x_{\alpha_{p-1}}}+A_{\alpha_1\cdots\alpha_{p-1}h}\frac{\partial X_h}{\partial x_i}\right).\end{aligned}$$

Upon likewise replacing the C with their values, the relation to be verified (22) will become:

$$\begin{aligned}&\sum_i\sum_h A_{\alpha_1\alpha_2\cdots\alpha_{p-1}i}\, X_h\frac{\partial X_i}{\partial x_h}-\sum_i X_i\sum_h\left(A_{h\,\alpha_2\cdots\alpha_{p-1}i}\frac{\partial X_h}{\partial x_{\alpha_1}}+\cdots+A_{\alpha_1\cdots\alpha_{p-1}h}\frac{\partial X_h}{\partial x_i}\right)\\ &\qquad+\sum_h\left(\frac{\partial X_h}{\partial x_{\alpha_1}}\sum_i A_{h\,\alpha_2\cdots\alpha_{p-1}i}X_i+\cdots+\frac{\partial X_h}{\partial x_{\alpha_{p-1}}}\sum_i A_{\alpha_1\cdots\alpha_{p-1}h}\, X_i\right)=0,\end{aligned}$$

or, upon suppressing the terms that cancel immediately:

$$\sum_i\sum_h A_{\alpha_1\alpha_2\cdots\alpha_{p-1}i}\, X_h\frac{\partial X_i}{\partial x_h} = \sum_i\sum_h A_{\alpha_1\alpha_2\cdots\alpha_{p-1}h}\, X_i\frac{\partial X_h}{\partial x_i}.$$

In order to perceive the identity of the two sides, it would suffice to permute the indices *i* and *h*.

7. – In order to state the results that follow more simply, I shall first present a certain number of expressions and notations that will be employed. I shall call the operation by which one passes from an absolute invariant I_p or a relative invariant J_p to an absolute invariant I_{p+1}^d (**§ 4**) the operation (*D*). When that operation is applied to an invariant I_p^d, it will lead to an invariant that is identically zero, as one has seen before. The operation that was defined in the preceding section by which one deduces an absolute invariant I_{p-1} of lower order from an absolute invariant I_p will be called the operation (*E*), to abbreviate. That operation will lead to an invariant that is identically zero when the invariant I_p to which applies it satisfies the relations:

(24) $$\sum_{i=1}^{n} A_{\alpha_1\alpha_2\cdots\alpha_{p-1}i} X_h = 0$$

for any indices α_1, α_2, …, α_{p-1}. We then say that the invariant I_p is *exceptional*, and we represent it by the notation I_p^e. *When it is applied to an invariant that is not exceptional, the operation* (*E*) *will lead to an exceptional invariant* I_{p-1}^e. Indeed, we have:

$$\sum_{h=1}^{n} C_{\alpha_1\alpha_2\cdots\alpha_{p-2}h} X_h = \sum_h \sum_i A_{\alpha_1\alpha_2\cdots\alpha_{p-2}hi} X_i X_h ,$$

and the coefficient of $X_i X_h$ in the right-hand side is:

$$\sum_{i,h} A_{\alpha_1\alpha_2\cdots\alpha_{p-2}hi} + \sum_{i,h} A_{\alpha_1\alpha_2\cdots\alpha_{p-2}ih} = 0 .$$

That property shows the close relationship between (*E*) and (*D*), and by definition, we will be led to consider four types of absolute invariants:

1. The invariants that are neither I_p^d nor I_p^e. We shall represent them by the notation I_p^0 when there is good reason to make that property obvious.

2. The invariants I_p^d for which the expression under the integral sign is an exact total differential.

3. The exceptional invariants I_p^e that were just defined.

4. An invariant I_p can be both I_p^d and I_p^e. We shall represent them by $I_p^{(d,e)}$.

The results that we have acquired up to now can be summarized as follows:

1. *When the operation* (*D*) *is applied to an invariant* I_p^0 *or* I_p^e, *it will lead to an invariant* I_p^d *or* $I_p^{(d,e)}$. *When it is applied to an invariant* I_p^d *or* $I_p^{(d,e)}$, *it will lead to an invariant that is identically zero.*

2. *When the operation* (*E*) *is applied to an invariant* I_p^0 *or* I_p^d, *it will lead to an invariant* I_p^e *or* $I_p^{(d,e)}$. *When it is applied to an invariant* I_p^e *or* $I_p^{(d,e)}$, *it will lead to an invariant that is identically zero.*

Those two operations parallel each other by the following property then: If one applies one of them twice in a row then one will always arrive at an invariant that is identically zero.

8. – We shall complete the preceding statements.

A. *The operation* (E) *applied to an invariant* I_p^d *leads to an invariant* $I_{p-1}^{(d,e)}$.

It will suffice to prove that if the functions $A_{\alpha_1\alpha_2\cdots\alpha_p}$ verify the relations (19) and the relations (6) or (6′) then the functions $C_{\alpha_1\alpha_2\cdots\alpha_{p-1}}$ that are defined by the formulas (21) will satisfy relations that are analogous to the relations (6) or (6′).

Suppose, to fix ideas, that p is *odd*. $p - 1$ will then be even, and one must prove that the equation:

$$\frac{\partial C_{\alpha_1\cdots\alpha_{p-1}}}{\partial x_{\alpha_p}} + \frac{\partial C_{\alpha_2\cdots\alpha}}{\partial x_{\alpha_1}} + \frac{\partial C_{\alpha_3\cdots\alpha_p\alpha_1}}{\partial x_{\alpha_2}} + \cdots + \frac{\partial C_{\alpha_p\alpha_1\cdots\alpha_{p-2}}}{\partial x_{\alpha_{p-1}}} = 0 \tag{25}$$

is a consequence of equations (6′) and (19).

Replace the C with their values. The relation to be verified will then be:

$$\sum_i \frac{\partial X_i}{\partial x_{\alpha_p}} A_{\alpha_1\alpha_2\cdots\alpha_{p-1}i} + \sum_i \frac{\partial X_i}{\partial x_{\alpha_1}} A_{\alpha_2\cdots\alpha_p i} + \cdots + \sum_i \frac{\partial X_i}{\partial x_{\alpha_{p-1}}} A_{\alpha_p\alpha_2\cdots\alpha_{p-1}i}$$
$$+ \sum_i X_i \left(\frac{\partial A_{\alpha_1\alpha_2\cdots\alpha_{p-1}i}}{\partial x_{\alpha_p}} + \frac{\partial A_{\alpha_2\cdots\alpha_p i}}{\partial x_{\alpha_1}} + \cdots + \frac{\partial A_{\alpha_p\alpha_2\cdots\alpha_{p-1}i}}{\partial x_{\alpha_{p-1}}} \right) = 0 .$$

However, since p is odd, a circular permutation of p indices will be equivalent to an even number of transpositions, and that relation can once more be written:

$$\sum_i \left(A_{i\alpha_2\cdots\alpha_p} \frac{\partial X_i}{\partial x_{\alpha_1}} + A_{\alpha_1 i\alpha_3\cdots\alpha_p} \frac{\partial X_i}{\partial x_{\alpha_2}} + \cdots + A_{\alpha_1\alpha_2\cdots\alpha_{p-1}i} \frac{\partial X_i}{\partial x_{\alpha_p}} \right)$$
$$+ \sum_i X_i \left(\frac{\partial A_{\alpha_1\alpha_2\cdots\alpha_{p-1}i}}{\partial x_{\alpha_p}} + \frac{\partial A_{\alpha_2\cdots\alpha_{p-1}i\alpha_p}}{\partial x_{\alpha_1}} + \cdots + \frac{\partial A_{i\alpha_p\alpha_1\cdots\alpha_{p-2}}}{\partial x_{\alpha_{p-1}}} \right) = 0 ,$$

or, upon taking the relation (6′) into account:

$$X(A_{\alpha_1\alpha_2\cdots\alpha_p}) + \sum_i \left(A_{i\alpha_2\cdots\alpha_p} \frac{\partial X_i}{\partial x_{\alpha_1}} + \cdots + A_{\alpha_1\cdots\alpha_{p-1}i} \frac{\partial X_i}{\partial x_{\alpha_p}} \right) = 0$$

One will get back to equations (19) precisely.

One will get some analogous calculations for the case in which p is even ([1]).

B. *When the operation* (*D*) *is applied to an invariant* I_p^e, *it will lead to an invariant* $I_{p+1}^{(d,e)}$.

Let I_p^e be an absolute invariant of order p :

$$I_p^e = \iint \cdots \int \sum A_{\alpha_1\alpha_2\cdots\alpha_p} dx_{\alpha_1}\, dx_{\alpha_2} \cdots dx_{\alpha_p} \;,$$

in which the functions $A_{\alpha_1\alpha_2\cdots\alpha_p}$ verify the relations:

$$(19) \qquad X(A_{\alpha_1\alpha_2\cdots\alpha_p}) + \sum_i \left(A_{i\alpha_2\cdots\alpha_p} \frac{\partial X_i}{\partial x_{\alpha_1}} + \cdots + A_{\alpha_1\cdots\alpha_{p-1}i} \frac{\partial X_i}{\partial x_{\alpha_p}} \right) = 0 \,,$$

$$(24) \qquad \sum_i A_{\alpha_1\cdots\alpha_{p-1}i} X_i = 0 \,.$$

If one supposes, for example, that p is even then the corresponding invariant I_{p+1}^d will have the expression:

$$I_{p+1}^d = \iint \cdots \int \sum \mathcal{A}_{\alpha_1\cdots\alpha_{p+1}} dx_{\alpha_1}\, dx_{\alpha_2} \cdots dx_{\alpha_{p+1}} \;,$$

in which one has set:

$$\mathcal{A}_{\alpha_2\cdots\alpha_p i} = \frac{\partial A_{\alpha_1\alpha_2\cdots\alpha_p}}{\partial x_i} + \frac{\partial A_{\alpha_2\cdots\alpha_p i}}{\partial x_{\alpha_1}} + \cdots + \frac{\partial A_{i\alpha_1\cdots\alpha_{p-1}}}{\partial x_{\alpha_p}} \;.$$

The problem is to show that the relations:

$$\sum_{i=1}^n \mathcal{A}_{\alpha_1\alpha_2\cdots\alpha_p i} X_i = 0$$

or

$$\sum_i X_i \left(\frac{\partial A_{\alpha_1\alpha_2\cdots\alpha_p}}{\partial x_i} + \frac{\partial A_{\alpha_2\cdots\alpha_p i}}{\partial x_{\alpha_1}} + \cdots + \frac{\partial A_{i\alpha_1\cdots\alpha_{p-1}}}{\partial x_{\alpha_p}} \right) = 0$$

([1]) It can happen that when the operation (*E*) is applied to an integral invariant I_p^0 will also lead to an invariant $I_p^{(d,e)}$.

are consequences of the relations (19) and (24).

The relation to be verified can be written:

$$X(A_{\alpha_1\alpha_2\cdots\alpha_p}) + \sum_i X_i \left(\frac{\partial A_{\alpha_1\alpha_2\cdots\alpha_p}}{\partial x_i} + \frac{\partial A_{\alpha_2\cdots\alpha_p i}}{\partial x_{\alpha_1}} + \cdots + \frac{\partial A_{i\alpha_1\cdots\alpha_{p-1}}}{\partial x_{\alpha_p}} \right) = 0\,,$$

or, upon replacing $X(A_{\alpha_1\alpha_2\cdots\alpha_p})$ with its value that is inferred from (19):

$$\sum_i X_i \frac{\partial A_{\alpha_2\cdots\alpha_p i}}{\partial x_{\alpha_1}} + \cdots + \sum_i X_i \frac{\partial A_{i\alpha_1\cdots\alpha_{p-1}}}{\partial x_{\alpha_p}} - \sum_i \frac{\partial X_i}{\partial x_{\alpha_1}} A_{\alpha_2\cdots\alpha_p i} - \cdots - \sum_i \frac{\partial X_i}{\partial x_{\alpha_p}} A_{\alpha_1\alpha_2\cdots\alpha_{p-1} i}\ .$$

However, since p is *even*, one will have:

$$A_{i\alpha_2\cdots\alpha_p} = -A_{\alpha_2\cdots\alpha_p i}\,, \quad \ldots, \qquad A_{\alpha_1\cdots\alpha_{p-1} i} = -A_{i\alpha_1\cdots\alpha_{p-1}}\,,$$

and what will remain is:

$$\frac{\partial}{\partial x_{\alpha_1}} \left(\sum_i A_{\alpha_2\cdots\alpha_p i} X_i \right) + \cdots + \frac{\partial}{\partial x_{\alpha_p}} \left(\sum_i A_{\alpha_1\cdots\alpha_{p-1} i} X_i \right) = 0\,.$$

In that form, one will see immediately that the relation to which one will be led is a result of the relations (24).

The same calculation proves that one will be led to an invariant $I_{p+1}^{(d,e)}$ upon applying the operation (D) to an invariant I_p^0 for which all of the sums:

$$\sum_i A_{\alpha_1\cdots\alpha_{p-1} i} X_i$$

are constants.

C. *The operations* (D) and (E) *commute.*

Upon applying the operations (E) and (D) to an invariant I_p in succession, one will be led to an invariant $I_p^{(d,e)}$ (which can be identically zero). Upon applying the same operations in the opposite order to the same invariant I_p, one will again obtain an invariant $I_p'^{(d,e)}$. *The two invariants* $I_p^{(d,e)}$ *and* $I_p'^{(d,e)}$ *are identical, up to sign.*

Let:

$$I_p = \iint \cdots \int \sum A_{\alpha_1\alpha_2\cdots\alpha_p}\, dx_{\alpha_1} \cdots dx_{\alpha_p}$$

be an arbitrary absolute invariant of order p. Upon applying the operation (E) to it, one will get an invariant I_{p-1}^{e}:

$$I_{p-1}^{e} = \iint\cdots\int\sum C_{\alpha_1\alpha_2\cdots\alpha_{p-1}}\,dx_{\alpha_1}\cdots dx_{\alpha_{p-1}}\,,$$

in which:

$$C_{\alpha_1\alpha_2\cdots\alpha_{p-1}} = \sum_i A_{\alpha_1\alpha_2\cdots\alpha_{p-1}i}X_i\,.$$

One then deduces the invariant $I_p'^{(d,e)}$ from I_{p-1}^{e}:

$$I_p'^{(d,e)} = \iint\cdots\int\sum \mathcal{A}_{\alpha_1\alpha_2\cdots\alpha_p}\,dx_{\alpha_1}\cdots dx_{\alpha_p}\,,$$

in which one has set:

$$\begin{aligned}\mathcal{A}_{\alpha_1\alpha_2\cdots\alpha_p} &= \frac{\partial C_{\alpha_1\alpha_2\cdots\alpha_{p-1}}}{\partial x_{\alpha_p}} + \frac{\partial C_{\alpha_1\cdots\alpha_{p-1}\alpha_p}}{\partial x_{\alpha_1}} + \cdots + \frac{\partial C_{\alpha_p\alpha_2\cdots\alpha_{p-1}}}{\partial x_{\alpha_{p-1}}}\\ &= \sum_i \frac{\partial}{\partial x_{\alpha_p}}(X_iA_{\alpha_1\alpha_2\cdots\alpha_{p-1}i}) + \sum_i \frac{\partial}{\partial x_{\alpha_1}}(X_iA_{\alpha_1\cdots\alpha_p i}) + \cdots + \sum_i \frac{\partial}{\partial x_{\alpha_{p-1}}}(X_iA_{\alpha_p\alpha_1\cdots\alpha_{p-1}i})\end{aligned}$$

upon supposing that p is *odd*, to fix ideas.

On the other hand, upon applying the operation (D) to I_p first, one will get an invariant I_{p+1} :

$$I_{p+1}^{d} = \iint\cdots\int\sum \mathcal{A}_{\alpha_1\alpha_2\cdots\alpha_{p+1}}\,dx_{\alpha_1}\cdots dx_{\alpha_{p+1}}\,,$$

in which one has set:

$$\mathcal{A}_{\alpha_1\alpha_2\cdots\alpha_p i} = \frac{\partial A_{\alpha_1\cdots\alpha_p}}{\partial x_i} - \frac{\partial A_{\alpha_2\cdots\alpha_p i}}{\partial x_{\alpha_1}} + \frac{\partial A_{\alpha_3\cdots\alpha_p i\alpha_1}}{\partial x_{\alpha_2}} - \cdots - \frac{\partial A_{i\alpha_1\cdots\alpha_{p-1}}}{\partial x_{\alpha_p}}\,,$$

since p is *odd*, which one can further write as:

$$\mathcal{A}_{\alpha_1\alpha_2\cdots\alpha_p i} = \frac{\partial A_{\alpha_1\cdots\alpha_p}}{\partial x_i} - \frac{\partial A_{\alpha_2\cdots\alpha_p i}}{\partial x_{\alpha_1}} - \frac{\partial A_{\alpha_3\cdots\alpha_p\alpha_1 i}}{\partial x_{\alpha_2}} - \cdots - \frac{\partial A_{\alpha_1\cdots\alpha_{p-1}i}}{\partial x_{\alpha_p}}\,,$$

from an earlier remark (**§ 1**).

Finally, one deduces the invariant $I_p'^{(d,e)}$ from I_{p+1}^{d} by means of the operation (E):

$$I_p'^{(d,e)} = \iint\cdots\int\sum C'_{\alpha_1\cdots\alpha_p}\,dx_{\alpha_1}\cdots dx_{\alpha_p}\,,$$

in which one sets:

$$C'_{\alpha_1\cdots\alpha_p} = \sum_i \mathcal{A}_{\alpha_1\alpha_2\cdots\alpha_p i} X_i$$

$$= X\,(A_{\alpha_1\alpha_2\cdots\alpha_p}) + \sum_i X_i \frac{\partial A_{\alpha_2\cdots\alpha_p i}}{\partial x_{\alpha_1}} - \cdots \sum_i X_i \frac{\partial A_{\alpha_1\cdots\alpha_{p-1} i}}{\partial x_{\alpha_p}}.$$

Upon adding the expressions for $\mathcal{A}'_{\alpha_1\cdots\alpha_p}$ and $C'_{\alpha_1\cdots\alpha_p}$, one will get:

$$\mathcal{A}'_{\alpha_1\cdots\alpha_p} + C'_{\alpha_1\cdots\alpha_p} = X\,(A_{\alpha_1\alpha_2\cdots\alpha_p}) + \sum_i \left(A_{\alpha_1\cdots\alpha_{p-1} i} \frac{\partial X_i}{\partial x_{\alpha_p}} + \cdots + A_{\alpha_2\cdots\alpha_p i} \frac{\partial X_i}{\partial x_{\alpha_1}} \right),$$

or rather:

$$\mathcal{A}'_{\alpha_1\cdots\alpha_p} + C'_{\alpha_1\cdots\alpha_p} = X\,(A_{\alpha_1\alpha_2\cdots\alpha_p}) + \sum_i \left(A_{i\alpha_1\cdots\alpha_p} \frac{\partial X_i}{\partial x_{\alpha_1}} + \cdots + A_{\alpha_1\cdots\alpha_{p-1} i} \frac{\partial X_i}{\partial x_{\alpha_p}} \right) = 0,$$

since p is odd.

One will then have, by definition:

$$I_p^{(d,e)} = -\, I_p'^{(d,e)}.$$

9. – Having established those properties, suppose that one knows an absolute integral invariant I_p^0 of equation (15) ($p > 1$).

Upon applying the operation (E) to it, we will get an invariant I_{p-1}^e that will not be an invariant $I_{p-1}^{(d,e)}$, *in general*. Thus, upon applying the operation (D) to I_{p-1}^e, one will get an invariant $I_p^{(d,e)}$ *that is not identically zero*.

We just saw that one will get the same result by proceeding in the opposite order. The links between the four invariants I_p^0, I_{p-1}^e, I_{p+1}^d, $I_p^{(d,e)}$ is represented by the following diagram (Fig. 1):

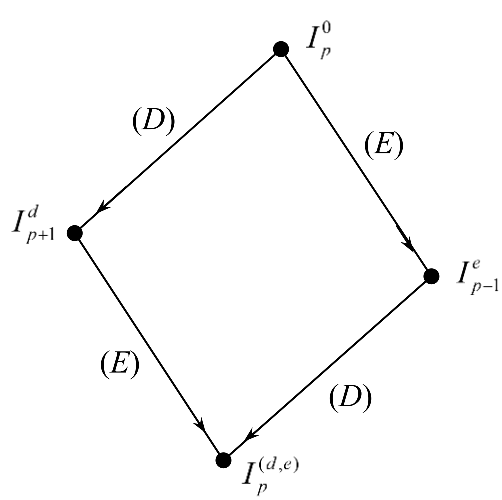

Figure 1.

It can happen that the cycle is incomplete. As always, start from an invariant I_p^0. If the operation (E) leads to an invariant $I_{p-1}^{(d,e)}$ then the invariant $I_p^{(d,e)}$ will be identically zero, and the invariant I_{p+1}^d that is deduced from I_p^0 by the operation (D) will be an invariant $I_{p+1}^{(d,e)}$. If one starts from an invariant I_p^d then the operation (E) will lead to an invariant $I_{p-1}^{(d,e)}$. If one starts from an invariant I_p^e then the operation (D) will lead to an invariant $I_{p+1}^{(d,e)}$.

We can then summarize all of the preceding results into the following statement:

One can always deduce at least one invariant $I_p^{(d,e)}$ *that is not identically zero from any absolute invariant* I_p ($p > 1$) *or any relative invariant* J_p *by additions, multiplications, and differentiations*.

The conclusion can break down for an absolute invariant I_1; that case will be treated separately.

III.

10. – We are now led to examine the following question:

If one knows an integral invariant $I_p^{(d,e)}$ *of the differential equations* (15) *then what can one infer from that knowledge that will help one integrate the system?*

Let $I_p^{(d,e)}$ be an integral invariant of order p :

(26) $$I_p^{(d,e)} = \iint \cdots \int \sum A_{\alpha_1 \alpha_2 \cdots \alpha_p} dx_{\alpha_1} dx_{\alpha_2} \cdots dx_{\alpha_p} .$$

The coefficients $A_{\alpha_1 \alpha_2 \cdots \alpha_p}$ verify the relations:

$$\sum_{i=1}^{n} A_{\alpha_1 \alpha_2 \cdots \alpha_{p-1} i} X_i = 0$$

for any indices $\alpha_1, \alpha_2, \ldots, \alpha_{p-1}$, so equations (15) will imply the following ones:

(27) $$\sum_{i=1}^{n} A_{\alpha_1 \alpha_2 \cdots \alpha_{p-1} i} \, dx_i = 0 .$$

Equations (27), which are linear and homogeneous in $dx_1, dx_2, \ldots, dx_n$, will then reduce to m distinct equations, where m is less than or equal to $n - 1$. If $m = n - 1$ then the two systems (15) and (27) will be equivalent. However, when m is less than $n - 1$, the system (27) will be more general than then proposed system (15), and any integral of the system (27):

$$F(x_1, x_2, \ldots, x_n) = \text{const.}$$

will also be a first integral of the system (15). In that case, knowing the invariant $I_p^{(d,e)}$ will permit one to simplify the problem of integration. Indeed, we shall show that *the m distinct equations to which the system* (27) *reduces will define a completely-integrable system*.

First recall the following result of Frobenius ([1]). If one is given k equations:

$$A_{\mu 1}\, dx_1 + \ldots + A_{\mu n}\, dx_n = 0 \qquad (\mu = 1, 2, \ldots, k) \tag{28}$$

that reduce to m distinct equations ($m < n$), in order for those m equations to define a completely-integrable system, it is necessary and sufficient that the relations:

$$\sum_{i=1}^{n}\sum_{j=1}^{n} u_i\, v_j \left(\frac{\partial A_{\mu i}}{\partial x_j} - \frac{\partial A_{\mu j}}{\partial x_i}\right) = 0 \tag{29}$$

should be consequences of the relations:

$$\sum_{i=1}^{n} A_{\mu i}\, u_i = 0\,, \qquad \sum_{j=1}^{n} A_{\mu j}\, v_j = 0\,. \tag{30}$$

For the system that is considered here, the coefficients $A_{\mu i}$ in equations (28) have the form $A_{\alpha_1 \alpha_2 \cdots \alpha_{p-1} i}$, in which $\alpha_1\ \alpha_2 \ldots \alpha_{p-1}$ denotes a combination of the first n numbers taken $(p - 1)$ at a time. The difference:

$$\frac{\partial A_{\alpha_1 \alpha_2 \cdots \alpha_{p-1} i}}{\partial x_{\alpha_j}} - \frac{\partial A_{\alpha_1 \alpha_2 \cdots \alpha_{p-1} j}}{\partial x_{\alpha_i}}$$

is a linear combination of the derivatives:

$$\frac{\partial A_{\alpha_1 \alpha_2 \cdots \alpha_{p-1} ij}}{\partial x_{\alpha_1}}, \quad \frac{\partial A_{\alpha_2 \cdots \alpha_{p-1} ij \alpha_1}}{\partial x_{\alpha_2}}, \ldots,$$

from the condition equations (6) or (6′), which express the idea that:

([1]) **Frobenius**, Crelle's Journal **82** (1877), pp. 276. *See* also **Forsyth**, *Theory of differential equations*, Part I, pp. 51. Frobenius supposed that $k < n$. However, one can also suppose that $k > n$, which is the case in the example that is treated here.

$$\sum A_{\alpha_1 \alpha_2 \cdots \alpha_p} dx_{\alpha_1} dx_{\alpha_2} \cdots dx_{\alpha_p}$$

is an exact differential.

It will then suffice to verify that the relations:

$$\sum_{i=1}^{n} \sum_{j=1}^{n} u_i v_j \frac{\partial A_{\alpha_2 \alpha_3 \cdots \alpha_{p-1} ij}}{\partial x_{\alpha_1}} = 0 \tag{31}$$

are consequences of the relations (30). We can write equation (31) as:

$$\sum_{i=1}^{n} u_i \sum_{j=1}^{n} \frac{\partial A_{\alpha_2 \alpha_3 \cdots \alpha_{p-1} ij}}{\partial x_{\alpha_1}} v_j = 0 .$$

On the other hand, from the relation (30):

$$\sum_{i=1}^{n} A_{\alpha_2 \cdots \alpha_{p-1} ij} v_j = 0 ,$$

one will deduce that:

$$\sum_{j=1}^{n} \frac{\partial A_{\alpha_2 \cdots \alpha_{p-1} ij}}{\partial x_{\alpha_1}} v_j + \sum_{j=1}^{n} A_{\alpha_2 \cdots \alpha_{p-1} ij} \frac{\partial v_j}{\partial x_{\alpha_1}} = 0 .$$

and the relation to be verified can be further written as:

$$\sum_{i=1}^{n} \sum_{j=1}^{n} A_{\alpha_2 \alpha_3 \cdots \alpha_{p-1} ij} \frac{\partial v_j}{\partial x_{\alpha_1}} u_i = 0$$

or

$$\sum_{j=1}^{n} \frac{\partial v_j}{\partial x_{\alpha_1}} \sum_{i=1}^{n} A_{\alpha_2 \alpha_3 \cdots \alpha_{p-1} ij} u_i = 0 ,$$

and the latter condition is obviously a consequence of the relations (30).

When $m = n - 1$, it seems that the method gives no simplification. However, one can then find a multiplier ([1]), as one will see later (no. **11**) in a special case.

11. – We shall treat the simplest cases in detail, namely $p = 1$ and $p = 1$.

Let $I_1^{(d,e)}$ be a first-order invariant of the system (15):

([1]) The proof in the general case will be given in another work that will be dedicated to the study of the systems (27), in particular.

$$I_1^{(d,e)} = \int A_1\, dx_1 + A_2\, dx_2 + \cdots + A_n\, dx_n .$$

The coefficients $A_1, A_2, \ldots, A_n$ must verify the relations:

$$A_1 X_1 + A_2 X_2 + \ldots + A_n X_n = 0 ,$$

$$\frac{\partial A_i}{\partial x_k} = \frac{\partial A_k}{\partial x_i} .$$

It will then follow that $A_1\, dx_1 + A_2\, dx_2 + \ldots + A_n\, dx_n$ is an exact differential du, and $u = c$ is a first integral of equations (15). *One will then deduce an integral combination of equations* (15) *from any first-order invariant* $I_1^{(d,e)}$.

Let us go on to the case of $p = 2$. Let:

$$I_2^{(d,e)} = \iint \sum A_{ik} dx_i\, dx_k$$

be a second-order invariant $I_2^{(d,e)}$. The coefficients A_{ik} verify the relations:

$$\frac{\partial A_{ik}}{\partial x_j} + \frac{\partial A_{kl}}{\partial x_i} + \frac{\partial A_{li}}{\partial x_k} = 0 , \tag{32}$$

$$A_{i1} X_1 + A_{i2} X_2 + \ldots + A_{in} X_n = 0 . \tag{33}$$

$(i, j, k = 1, 2, \ldots, n)$

The n relations:

$$\left\{\begin{array}{l} A_{11}\, dx_1 + A_{12}\, dx_2 + \cdots + A_{1n}\, dx_n = 0, \\ A_{21}\, dx_1 + A_{22}\, dx_2 + \cdots + A_{2n}\, dx_n = 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ A_{n1}\, dx_1 + A_{n2}\, dx_2 + \cdots + A_{nn}\, dx_n = 0 \end{array}\right. \tag{34}$$

can be considered to be linear combinations of equations (15). By virtue of the relations (32), one can determine n functions $B_1, B_2, \ldots, B_n$ such that one has:

$$A_{ik} = \frac{\partial B_i}{\partial x_k} - \frac{\partial B_k}{\partial x_i} ,$$

and the system of differential equations (34) is a covariant of the Pfaff form ([1]):

([1]) **Darboux**, "Sur le problème de Pfaff," Bull. Sci. math. (2) **6** (1882), 14-36 and 49-68.

$$(35) \qquad B_1\,dx_1 + B_2\,dx_2 + \ldots + B_n\,dx_n\,.$$

Furthermore, that system is always compatible since it admits all solutions to the proposed system (15). It will then result that the corresponding Pfaff determinant:

$$\Delta = \begin{vmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{vmatrix}$$

is always zero. Having said that, one can distinguish two cases according to the parity of n.

Suppose, to begin with, that n is even. Since Δ is zero, the same thing will be true for all of its first-order minors, and in reality, the n equations (34) reduce to $n - p - 1$ distinct equations ($p > 0$). Since those equations form a completely-integrable system, they will admit $(n - p - 1)$ distinct integrals:

$$\varphi_1 = C_1\,, \qquad \varphi_2 = C_2\,, \qquad \ldots, \qquad \varphi_{n-p-1} = C_{n-p-1}$$

that one will obtain by integrating a complete system or a system of $n - p - 1$ differential equations. Since those integrals $\varphi_1, \varphi_2, \ldots, \varphi_{n-p-1}$ also belong to the system (15), one sees that the problem of integration has been simplified since one can obtain $(n - p - 1)$ first integrals by integrating a system of $(n - p - 1)$ differential equations.

If n is odd then Δ will always be zero. If all of its first-order minors are also zero then the system (34) will again reduce to $(n - p - 1)$ distinct equations ($p > 0$), and the conclusion will be the same as before. However, if all of the first-order minors of Δ are non-zero then the system (34) will be comprised of $(n - 1)$ distinct equations, and it will be entirely equivalent to the system (15). In that case, *one can find a multiplier for the proposed system* (15).

It would suffice to recall the following properties of skew-symmetric determinants. Let $\alpha, \beta, \ldots, \lambda$ be a system of $2r$ whole numbers that are chosen from the first n numbers. The expressions $(\alpha, \beta, \ldots, \lambda)$ are defined step-by-step by means of the recurrence relation:

$$(\alpha, \beta, \ldots, \lambda) = (\alpha, \beta)\,(\gamma, \delta, \ldots, \lambda) + (\alpha, \gamma)\,(\delta, \ldots, \lambda, \beta) + \ldots + (\alpha, \lambda)\,(\beta, \gamma, \ldots, \kappa)\,,$$

combined with the relation:

$$(\alpha, \beta) = A_{\alpha\beta}\,.$$

If one is given two permutations $(\alpha, \beta, \ldots, \lambda)$ and $(\alpha', \beta', \gamma', \ldots, \lambda')$ that differ by only the order of the indices then one will have:

$$(\alpha, \beta, \ldots, \lambda) = \pm(\alpha', \beta', \gamma', \ldots, \lambda')\,,$$

in which the sign + pertains to the case in which the two permutations have the same class, and the – sign pertains to the contrary case.

Having said that, let $n = 2p + 1$, and suppose that all of the first-order minors of the determinant Δ are non-zero. One then infers an equivalent system from the relation (34):

$$(36) \qquad \frac{dx_1}{(2,3,4,\ldots,2p+1)} = \frac{dx_2}{(3,4,\ldots,2p+1,1)} = \ldots = \frac{dx_1}{(1,2,3,4,\ldots,2p)} .$$

The system (36) is no different from the system (15). However, *it admits the multiplier* $M = 1$. In order to show that, it would suffice to verify that one has indeed:

$$(37) \qquad \frac{\partial(2,3,4,\ldots,2p+1)}{\partial x_1} + \frac{\partial(3,4,\ldots,2p+1,1)}{\partial x_2} + \frac{\partial(4,5,\ldots,2p+1,1,2)}{\partial x_3} + \cdots = 0 .$$

An arbitrary term on the left-hand side of that relation has the form:

$$\frac{\partial A_{ik}}{\partial x_i}(\alpha, \beta, \gamma, \ldots, \lambda) ,$$

in which $(\alpha, \beta, \gamma, \ldots, \lambda)$ is a permutation of the $(2p - 2)$ whole numbers that remain after suppressing the three indices i, k, l. Now, it is easy to see that the three derivatives $\frac{\partial A_{ik}}{\partial x_i}$, $\frac{\partial A_{kl}}{\partial x_i}$, $\frac{\partial A_{li}}{\partial x_k}$ have the same multiplier. For example, one has the sum:

$$\left(\frac{\partial A_{ik}}{\partial x_i} + \frac{\partial A_{kl}}{\partial x_i} + \frac{\partial A_{li}}{\partial x_k}\right) \qquad (4, 5, \ldots, 2p + 1),$$

and all of the other terms can be grouped in an analogous fashion. The relation (37) will then be a consequence of the relations (32).

12. – Let I_1 be an arbitrary absolute integral invariant of the system (15):

$$I_1 = \int A_1\, dx_1 + A_2\, dx_2 + \cdots + A_n\, dx_n .$$

When the operation (E) is applied to that absolute invariant, it will lead to a first integral:

$$A_1 X_1 + A_2 X_2 + \ldots + A_n X_n = \text{const.},$$

which is a theorem that is due to Poincaré. The result might be illusory when $A_1 X_1 + A_2 X_2 + \ldots + A_n X_n$ reduces to a constant. We shall examine the most-general case in which one knows a relative invariant of the system (15):

$$J_1 = \int A_1\, dx_1 + \cdots + A_n\, dx_n \quad . \tag{38}$$

One deduces an invariant I_2^d from that invariant J_1 by means of the operation (D):

$$I_2^d = \iint \sum \left(\frac{\partial A_i}{\partial x_k} - \frac{\partial A_k}{\partial x_i} \right) dx_i\, dx_k\,, \tag{39}$$

and one then deduces an invariant $I_1^{(d,e)}$ from the invariant I_2^d by means of the operation (E):

$$I_1^{(d,e)} = \int \mu_1\, dx_1 + \mu_2\, dx_1 + \cdots + \mu_n\, dx_n\,, \tag{40}$$

in which one has set:

$$\left\{ \begin{aligned} \mu_i &= a_{i1}\, X_1 + a_{i2}\, X_2 + \cdots + a_{in}\, X_n\,, \\ a_{ik} &= \frac{\partial A_i}{\partial x_k} - \frac{\partial A_k}{\partial x_i} \end{aligned} \right. \qquad (i, k = 1, 2, \ldots, n). \tag{41}$$

If the μ_i are not all zero then *one will obtain a first-integral of the system* (15) *by quadratures:*

$$U(x_1, x_2, \ldots, x_n) = \int \mu_1\, dx_1 + \mu_2\, dx_1 + \cdots + \mu_n\, dx_n = \text{const.}$$

The proposition also applies to an absolute integral invariant I_1, provided that all of the coefficients μ_i are non-zero. However, the first integral to which one is led will be nothing but the first integral that is given by Poincaré's theorem.

It suffices to verify the equalities:

$$\frac{\partial}{\partial x_i}(A_1 X_1 + A_2 X_2 + \ldots + A_n X_n) + X_1\, a_{i1} + \ldots + X_n\, a_{in} = 0 \qquad (i = 1, 2, \ldots, n),$$

which will become:

$$A_1 \frac{\partial X_1}{\partial x_i} + \cdots + A_n \frac{\partial X_n}{\partial x_i} + X(A_i) = 0 \qquad (i = 1, 2, \ldots, n)$$

upon replacing the a_{ik} with their expressions.

One will recover the relations that express the idea that I_1 is an absolute integral invariant precisely. However, if one starts from a relative integral invariant then some quadratures will generally be necessary if one is to obtain the first integral U.

13. – The proposition does not apply when the invariant $I_1^{(d,e)}$ that is represented by formula (40) is identically zero. The second-order invariant (39) is then an invariant $I_2^{(d,e)}$. If that invariant $I_2^{(d,e)}$ is not itself identically zero then one saw before how knowing that invariant will permit one to simplify the problem of integration. The invariant $I_2^{(d,e)}$ can be identically zero only if it was deduced from an absolute invariant I_1^d by the operation (D).

If that invariant is I_1^d then one has seen how it will give a first integral by quadratures (no. **11**). The only case in which the method will seem to give no simplification is the case of an absolute invariant I_1^d that is not, at the same time, $I_1^{(d,e)}$. Let:

$$I_1^d = \int A_1\,dx_1 + A_2\,dx_2 + \cdots + A_n\,dx_n$$

be that invariant. The expression $A_1\,dx_1 + \ldots + A_n\,dx_n$ is an exact differential dU. On the other hand, the expression:

$$A_1\,X_1 + \ldots + A_n\,X_n$$

cannot be zero unless I_1^d is $I_1^{(d,e)}$. Moreover, since the coefficients μ_i are all zero then that expression will reduce to a *non-zero* constant K. One then deduces from equations (15) that:

$$\frac{A_1\,dx_1 + \cdots + A_n\,dx_n}{K} = dt\,,$$

and one will get a first integral that contains t by quadratures:

$$\int A_1\,dx_1 + \cdots + A_n\,dx_n = K\,t + C\,.$$

14. – The general result in no. **12** established a link between the search for integrable combinations of the system (15) and the first-order relative invariants of that system. It is easy to exhibit that link directly.

Finding an integrable combination of equations (15) amounts to finding a system of n functions $\mu_1, \mu_2, \ldots, \mu_n$ such that $\mu_1\,dx_1 + \mu_2\,dx_2 + \ldots + \mu_n\,dx_n$ is an exact differential and one will have, at the same time:

$$\mu_1\,X_1 + \mu_2\,X_2 + \ldots + \mu_n\,X_n = 0\,. \tag{42}$$

One can satisfy the latter relation by setting:

$$\mu_i = \lambda_{i1}\,X_1 + \lambda_{i2}\,X_2 + \ldots + \lambda_{in}\,X_n\,, \tag{43}$$

in which λ_{ik} are new functions of the variables $x_1, x_2, \ldots, x_n$ that satisfy the conditions:

$$\lambda_{ii} = 0 , \qquad \lambda_{ik} + \lambda_{ki} = 0 .$$

The integrability condition $\frac{\partial \mu_i}{\partial x_k} = \frac{\partial \mu_k}{\partial x_i}$ will then be written as:

$$X(\lambda_{ik}) - \sum_{h=1}^{n} X_k \rho_{ihk} + \sum_{h=1}^{n} \left(\lambda_{hk} \frac{\partial X_h}{\partial x_i} + \lambda_{ih} \frac{\partial X_h}{\partial x_k} \right) = 0 , \tag{44}$$

when one sets:

$$\rho_{ihk} = \frac{\partial \lambda_{ih}}{\partial x_k} + \frac{\partial \lambda_{hk}}{\partial x_i} + \frac{\partial \lambda_{ki}}{\partial x_h} .$$

If we compare those conditions (44) to the conditions:

$$X(a_{ik}) + \sum_{h=1}^{n} \left(a_{hk} \frac{\partial X_h}{\partial x_i} + a_{ih} \frac{\partial X_h}{\partial x_k} \right) = 0 , \tag{45}$$

which express the idea that:

$$I_2 = \iint \sum a_{ik} \, dx_i \, dx_k$$

is a second-order integral invariant, then we will see that they will become identical upon replacing λ_{ik} with a_{ik}, provided that one has:

$$\frac{\partial a_{ih}}{\partial x_k} + \frac{\partial a_{hk}}{\partial x_i} + \frac{\partial a_{ki}}{\partial x_h} = 0 ,$$

i.e., whenever the invariant I_2 is I_2^d, one will have deduced an invariant J_1 or I_1 by the operation (D).

15. – The combination of calculations that led to that theorem can be justified *a priori* by a remark that was the starting point for this work and which I will develop in only the case of three variables, for simplicity.

Consider a system of three first-order differential equations that I will write:

$$\frac{dx}{X} = \frac{dy}{Y} = \frac{dz}{Z} = dt , \tag{46}$$

with the usual notations, in which X, Y, Z does not depend upon t, and let:

$$J_1 = \int a\,dx + b\,dy + c\,dz$$

be a relative integral invariant of that system, which we can replace with a second-order integral invariant:

$$I_2^d = \iint A\,dx\,dy + B\,dy\,dz + C\,dz\,dx\,,$$

in which:

$$A = \frac{\partial a}{\partial y} - \frac{\partial b}{\partial x}, \quad B = \frac{\partial b}{\partial z} - \frac{\partial c}{\partial y}, \quad C = \frac{\partial c}{\partial x} - \frac{\partial a}{\partial z},$$

and the expression under the $\iint$ sign is an exact differential.

Let C_0 be an arbitrary closed curve that is not tangent to the characteristic of equations (46) that issues from any point. Take a surface Σ_0 that is bounded by Γ_0 and is such that the characteristic that issues from any of its points is not tangent to the surface.

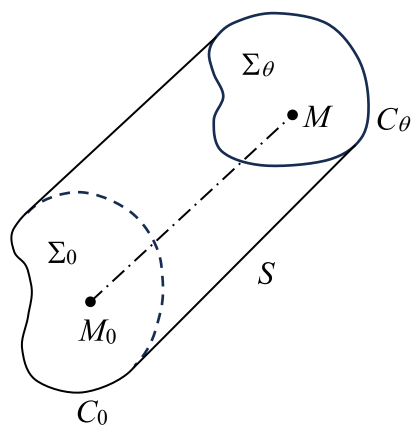

Figure 2.

Let M_0 be an arbitrary point of Σ_0 whose coordinates are x_0, y_0, z_0. If we take the initial value of t to be zero and the initial values of x, y, z to be x_0, y_0, z_0 in equations (46) then the point whose coordinates (x, y, z) will describe a segment of the characteristic M_0M when t varies from zero to θ. If θ is sufficiently small then the locus of those characteristics will be a volume that is analogous to a cylinder that is bounded by segments of the characteristics that issue from the various point of C_0 when t varies from zero to θ.

The integral I_2^d extends over the entire outer surface that bounds that volume is zero. On the other hand, since I_2^d is an integral invariant, the integral that is taken along the outer edge of Σ_0 will be equal to the integral that is taken along the internal edge of Σ_0. Consequently, when the integral I_2^d is extended over all of the surface S, it will be zero. If we consider that integral to be a function $F(\theta)$ of θ then we can $F'(\theta) = 0$. In order to evaluate that derivative, suppose that the coordinates of a point of C_0 are expressed as functions of a variable parameter u in such a fashion that we will obtain all of the points on that curve by varying u from zero to U. The coordinates of a point on the surface S will then be functions of two variables u and t:

(47) $$x = f_1(t, u), \quad y = f_2(t, u), \quad z = f_3(t, u),$$

and one will get all of the points on that surface by varying u from zero to U and t from 0 to θ. The function $F(\theta)$ will then have the expression:

$$F(\theta) = \iint \left[A \frac{D(x,y)}{D(t,u)} + B \frac{D(y,z)}{D(t,u)} + C \frac{D(z,x)}{D(t,u)} \right] dt\, du,$$

in which the double integral extends over the domain that was just defined, and x, y, z are replaced with their expressions (47) in terms of A, B, C. Upon taking the differential equations (46) themselves into account, one can further write that formula as:

$$F(\theta) = \int_0^\theta dt \int_0^U \left[A\left(X\frac{\partial y}{\partial u} - Y\frac{\partial x}{\partial u} \right) + B\left(Y\frac{\partial z}{\partial u} - Z\frac{\partial y}{\partial u} \right) + C\left(Z\frac{\partial x}{\partial u} - X\frac{\partial z}{\partial u} \right) \right] du.$$

For $\theta = 0$, the derivative $F'(\theta)$ will reduce to:

$$\int_0^U \left[(CZ - AY)\frac{\partial x}{\partial u} + (AX - BZ)\frac{\partial y}{\partial u} + (BY - CX)\frac{\partial z}{\partial u} \right] du,$$

i.e., to the curvilinear integral:

$$\int_{(C_0)} (CZ - AY)\, dx + (AX - BZ)\, dy + (BY - CX)\, dz$$

that is taken along C_0. Since that integral is zero for any closed curve C_0, the expression:

(48) $$(CZ - AY)\, dx + (AX - BZ)\, dy + (BY - CX)\, dz$$

will then be an exact differential.

Moreover, one has:

$$X(CZ - AY) + Y(AX - BZ) + Z(BY - CX) = 0,$$

and as a result, the expression (48) will be an integrable combination of equations (46).

If one has, at the same time:

$$CZ - AY = 0, \qquad AX - BZ = 0, \qquad BY - CX = 0,$$

then one can deduce that:

$$\frac{X}{B} = \frac{Y}{C} = \frac{Z}{A},$$

and the system (46) will be equivalent to the system:

$$\frac{dx}{B} = \frac{dy}{C} = \frac{dz}{A}. \tag{46′}$$

That new system will admit unity for a multiplier because one can infer the relation:

$$\frac{\partial B}{\partial x} + \frac{\partial C}{\partial y} + \frac{\partial A}{\partial z} = 0$$

from the expressions for A, B, C.

16. – To conclude, we shall once more apply the general theorem to the invariants of order n and $n - 1$. Let I_n be an invariant of order n :

$$I_n = \iint \cdots \int M\, dx_1\, dx_2 \cdots dx_n\,. \tag{49}$$

Any multiple integral of order n can be replaced with a multiple integral of order $n - 1$ that is extended over a closed multiplicity, so one can consider I_n to be an invariant I_n^d. When the operation (E) is applied to that invariant, it will lead to an invariant $I_{n-1}^{(d,e)}$.

Suppose that n is odd. We take:

$$A_{12\ldots n} = A_{23\ldots n1} = \ldots = M\,,$$

and the invariant $I_{n-1}^{(d,e)}$ will be expressed by:

$$I_{n-1}^{(d,e)} = \iint \cdots \int M\,[X_n\, dx_1\, dx_2 \cdots dx_{n-1} + X_1\, dx_2 \cdots dx_{n-1} + \cdots]\,. \tag{50}$$

The expression under the integration signs must be an exact differential. Since $n - 1$ is even, by hypothesis, one will then have the relation:

$$\frac{\partial (M\, X_1)}{\partial x_1} + \frac{\partial (M\, X_2)}{\partial x_2} + \cdots + \frac{\partial (M\, X_n)}{\partial x_n} = 0\,, \tag{51}$$

which shows that M is a multiplier, and one will recover a theorem of Poincaré. The system of differential equations (27) that is associated with the invariant $I_{n-1}^{(d,e)}$ is identical to the system (15) itself in the present case.

The conclusion will be the same when n is even. We must take:

$$A_{12\dots n} = - A_{23\dots n1} = A_{34\dots n12} = \dots = M ,$$

and the invariant $I_{n-1}^{(d,e)}$ will be expressed by:

$$I_{n-1}^{(d,e)} = \iint\cdots\int M\,[X_n\,dx_1\,dx_2\cdots dx_{n-1} - X_1\,dx_2\cdots dx_{n-1} + \cdots]. \tag{50$'$}$$

However, since $n - 1$ is odd, the condition (51) will not change.

Finally, suppose that we know an invariant I_{n-1}. There are several cases to distinguish between according to the hypotheses that one can make regarding that invariant. If one has an invariant $I_{n-1}^{(d,e)}$ then it will have the form (50) or (50′) according to the parity of u, and the relation (51) will again be verified in such a way that M will be a multiplier.

An invariant I_{n-1} that is not I_{n-1}^d will give an invariant I_n^d under the operation (D), and as a result a multiplier.

However, if one applies the operation (E) to an invariant I_{n-1}^0 then one will get an invariant I_{n-2}^e, and it would seem that the operation (D) will be necessary if one is to finally arrive at an invariant $I_{n-1}^{(d,e)}$, i.e., a multiplier. However, there is a simplification that can be made in this case as a result of a theorem by Kœnigs ([1]). The system of differential equations (27) that is associated with the invariant I_{n-2}^e is completely integrable, and one will then obtain an equation:

$$A(f) = \sum_i \mu_i \frac{\partial f}{\partial x_i} = 0 ,$$

which will define a completely-integrable system when it is combined with the equation $X(f) = 0$.

It is easy to see the reason for that simplification, and at the same time, to see that it is not possible in the general case. Suppose that one has reduced the system (15) to the form:

$$\frac{dx_1}{0} = \frac{dx_2}{0} = \dots = \frac{dx_{n-1}}{0} = \frac{dx_n}{1} = dt \tag{52}$$

by a change of variables, and let I_{n-1} be an integral invariant of order $n - 1$:

$$I_{n-1} = \iint\cdots\int \mu_1\,dx_2\cdots dx_n + \mu_2\,dx_3\cdots dx_n\,dx_1 + \cdots,$$

in which the coefficients $\mu_1, \mu_2, \dots, \mu_n$ depend upon only the variables $x_1, x_2, \dots, x_{n-1}$.

([1]) "Sur les invariants intégraux," C. R. Acad. Sci. Paris **122** (1896), 25-27.

When the operation (E) is applied to that invariant I_{n-1}, it will lead to an invariant I_{n-2}^e *in which neither x_n nor dx_n appear:*

$$I_{n-2}^e = \iint \cdots \int \sum C_{\alpha_1 \alpha_2 \cdots \alpha_{n-2}} dx_{\alpha_1} dx_{\alpha_2} \cdots dx_{\alpha_{n-2}} .$$

The system of differential equations (27) that is associated with that invariant I_{n-2}^e has the form:

$$\frac{dx_1}{\lambda_1} = \frac{dx_2}{\lambda_2} = \ldots = \frac{dx_{n-1}}{\lambda_{n-1}} ,$$

in which $\lambda_1, \lambda_2, \ldots, \lambda_{n-1}$ do not depend upon x_n, and the two equations:

$$\sum_{i=2}^{n-2} \lambda_i \frac{\partial f}{\partial x_i} = 0 , \qquad \frac{\partial f}{\partial x_n} = 0$$

will indeed define a complete system.

On the contrary, take an integral invariant of the system (52) of order less than $n - 1$, for example, an invariant I_2 :

$$I_2 = \iint \sum A_{ik} dx_i \, dx_k .$$

The coefficients A_{ik} are independent of x_n, and the invariant I_1^e that one deduces by means of the operation (E) will have the form:

$$I_1^e = \int C_1 \, dx_1 + C_2 \, dx_2 + \cdots + C_{n-1} \, dx_{n-1} ,$$

in which $C_1, C_2, \ldots, C_{n-1}$ are functions of $x_1, x_2, \ldots, x_{n-1}$ that can be arbitrary. The system (27) that is associated with that invariant I_1^e will reduce to a single equation here:

$$C_1 \, dx_1 + C_2 \, dx_2 + \cdots + C_{n-1} \, dx_{n-1} = 0 ,$$

and it is clear that this equation is not completely-integrable if n is greater than 3, in general.

____