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On a Monge problem with several independent variables

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The link between the integration of a first-order partial differential equation in two independent variables and the determination in explicit form of all solutions to a Monge equation in two unknown functions $F\left(x, y, z, \frac{dy}{dx}, \frac{dz}{dx}\right) = 0$ is classical. When one passes from a partial differential equation in two independent variables to a first-order equation in $(n + 1)$ independent variables, the Monge equation will itself be replaced by a relation between n independent variables, two unknown functions of those n variables, and their first-order partial derivatives.

1. – Let:

$$F(x_1, x_2, \ldots, x_{n+1}, x_{n+2}\,;\, P_1, P_2, \ldots, P_{n+1}) = 0 \tag{1}$$

be a first-order partial differential equation, in which $x_1, x_2, \ldots, x_{n+1}$ are independent variables, x_{n+2} is the unknown function, and $P_1, P_2, \ldots, P_{n+1}$ are its partial derivatives. An integral multiplicity M_{n+1} is generally determined when it is subject to contain a point-like multiplicity M_n in $(n + 2)$-dimensional space. Let:

$$x_{n+1} = f_1(x_1, x_2, \ldots, x_n)\,, \qquad x_{n+2} = f_2(x_1, x_2, \ldots, x_n) \tag{2}$$

be the equations of M_n . We set:

$$p_i = \frac{\partial f_1}{\partial x_i}\,, \qquad q_i = \frac{\partial f_2}{\partial x_i} \qquad (i = 1, 2, \ldots, n)\,.$$

If M_n belongs to an integral multiplicity M_{n+1} of (1) then the values of $P_1, \ldots, P_{n+1}$ at a point of M_n must verify equation (1), as well as the n relations:

$$q_i = P_i + P_{n+1}\, p_i \qquad (i = 1, 2, \ldots, n), \tag{3}$$

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which are deduced from the general relation:

$$dx_{n+1} = P_1\, dx_1 + \ldots + P_n\, dx_n + P_{n+1}\, dx_{n+1} .$$

The value of P_{n+1} at a point of M_n is then given by the equation:

$$F(x_1, x_2, \ldots, x_{n+1}, x_{n+2} ; q_1 - p_1 P_{n+1}, \ldots, q_n - p_n P_{n+1}, P_{n+1}) = 0 . \tag{4}$$

From a general theorem of Cauchy, any root of that equation that is holomorphic in the neighborhood of a point (x_i^0) of M_n will give an integral M_{n+1} that is holomorphic in the same neighborhood. The conclusion breaks down for a root P_{n+1} of equation (4) that satisfies, at the same time, the condition that:

$$\frac{\partial F}{\partial P_1} p_1 + \cdots + \frac{\partial F}{\partial P_n} p_n - \frac{\partial F}{\partial P_{n+1}} = 0 . \tag{5}$$

The elimination of P_{n+1} from equations (4) and (5) leads to a relation of the form:

$$\Phi(x_1, \ldots, x_{n+1}, x_{n+2}, p_1, \ldots, p_n, q_1, \ldots, q_n) = 0 , \tag{6}$$

which the differential equation of the *singular multiplicity* M_n of equation (1). Those multiplicities are obviously entirely analogous to the *integral curves* for an equation in three variables.

2. – One can further find the explicit equations that define those multiplicities when one knows the general integral of equation (1). Let:

$$V(x_1, \ldots, x_{n+1} ; a_1, a_2, \ldots, a_{n+1}) = 0 \tag{7}$$

be a complete integral. The singular multiplicities M_n are represented by the equations:

$$V = 0 , \qquad \frac{dV}{da_1} = 0 , \qquad \ldots, \qquad \frac{dV}{da_n} = 0 , \tag{8}$$

$$H = \begin{vmatrix} \dfrac{d^2V}{da_1^2} & \dfrac{d^2V}{da_1\, da_2} & \cdots & \dfrac{d^2V}{da_1\, da_n} \\ \cdots & \cdots & \cdots & \cdots \\ \dfrac{d^2V}{da_n\, da_1} & \cdots & \cdots & \dfrac{d^2V}{da_n^2} \end{vmatrix} = 0 , \tag{9}$$

in which one has replaced a_{n+1} with an arbitrary function $f(a_1, a_2, \ldots, a_{n+1})$, and in which $\frac{d}{da_i}$, $\frac{d^2}{da_i\, da_k}$ have the usual significance.

One can easily deduce some *necessary* conditions for an equation of the form (6) to define the singular multiplicities of a first-order partial differential equation from one or the other of those definitions. *The ratio* $\frac{\partial \Phi}{\partial p_i} : \frac{\partial \Phi}{\partial q_i}$ *must be independent of* i, while taking into account the equation itself.

Those conditions are not sufficient in all cases, but if they are fulfilled then one can always integrate equation (6) by expressing the variables and the two unknown functions explicitly in terms of the n auxiliary parameters, an arbitrary function of those parameters, and its derivatives up to second order. I shall rapidly indicate the proof of that: Suppose that equation (6) has been solved for one of the derivatives; q_n, for example, so $q_n = f(x_i, p_1, \ldots, p_n ; q_1, \ldots, q_{n-1})$.That function f must satisfy the conditions:

$$\frac{\partial f}{\partial p_k} + \frac{\partial f}{\partial q_k} \frac{\partial f}{\partial p_n} = 0 \qquad (k = 1, 2, \ldots, n-1) , \tag{10}$$

and the integration of equation (6) is equivalent to the search for n-dimensional integrals of the system of two Pfaff equations in $3n + 1$ variables:

$$\left\{ \begin{aligned} \omega_1 &= dx_{n+1} - p_1\, dx_1 - \cdots - p_n\, dx_n = 0, \\ \omega_2 &= dx_{n+2} - q_1\, dx_1 - \cdots - q_{n-1}\, dx_{n-1} - f\, dx_n = 0. \end{aligned} \right. \tag{11}$$

Now, the conditions (10) express the idea that the equation $\Omega = \omega_2 - \frac{\partial f}{\partial p_n} \omega_1 = 0$ is a *singular equation* ([1]) of that system for which there exists a family of ∞^n linear integral elements in involution with all of the linear integral elements of the system relative to the equation $\Omega = 0$. In general, that equation has class $2n + 1$, and for a change of variables that amounts to putting Ω into its canonical form, so one can convert the equations of the system (11) into the form:

$$\left\{ \begin{aligned} & dZ - P_1\, dX_1 - \cdots - P_n\, dX_n = 0, \\ & A_1\, dX_1 + \cdots + A_n\, dX_n + B_1\, dP_1 + \cdots + B_n\, dP_n = 0, \end{aligned} \right. \tag{11′}$$

in which the coefficients A_i, B_k are functions of the new variables X_i, P_k, Z, and n other variables Q_i that are independent of the former. One satisfies the equations of the new system by taking Z to be an arbitrary functions $F(X_1, \ldots, X_n)$, while the variables P_i, Q_i are determined from the relations:

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$$P_i = \frac{\partial F}{\partial X_i}, \qquad A_i + B_1 \frac{\partial^2 F}{\partial X_i \, \partial X_1} + \cdots + B_n \frac{\partial^2 F}{\partial X_i \, \partial X_n} = 0 \qquad (i = 1, 2, \ldots, n) .$$

Those results can be further generalized by replacing equation (1) with a system in involution of first-order partial differential equations for one unknown function.

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