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On a partial differential equation

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1. – The second-order partial differential equation:

$$s^2 - 4\lambda(x, y)\,p\,q = 0, \tag{1}$$

in which $\lambda(x, y)$ is an arbitrary function of x and y, can be converted into a linear equation that combines some equations with equal invariants. Set:

$$p = u^2, \qquad q = v^2.$$

Equation (1) gives us:

$$\left\{\begin{aligned} \frac{\partial u}{\partial y} &= \sqrt{\lambda}\,v, \\ \frac{\partial v}{\partial x} &= \sqrt{\lambda}\,u, \end{aligned}\right. \tag{2}$$

and the elimination of v from those two relations will lead to the linear equation:

$$\frac{\partial^2 u}{\partial x\,\partial y} - \frac{1}{2}\frac{\partial \log \lambda}{\partial x}\frac{\partial u}{\partial y} - \lambda u = 0. \tag{3}$$

If u is an integral of equation (3) then one will deduce an integral of equation (1) by a quadrature:

$$z = \int u^2\,dx + \frac{1}{\lambda}\left(\frac{\partial u}{\partial y}\right)^2 dy,$$

and one will then obtain all of the integrals of that equation. When the function $\lambda(x, y)$ is real, any real solution of the equation in u will correspond to a real solution of the equation in z. If u has the form $i\,f(x, y)$, where $f(x, y)$ is a real function then z will once more be real. Moreover, any real integral of the equation in z will correspond to a function u that is either real or has the form $i\,f(x, y)$. Upon noting that when one changes u into $i\,u$, z will change into $-z$, one concludes that in order to have all of the real integrals of equation (1), it will suffice to know all of the real integrals of equation (3).

The invariants of equation (3) have the values:

$$h = \lambda , \qquad k = -\frac{1}{2}\frac{\partial^2 \log \lambda}{\partial x\,\partial y} + \lambda .$$

The invariants of the equation (E_1), which is obtained by applying the Laplace transformation $u_1 = \partial u / \partial y$ to equation (E), have the values:

$$h_1 = 2h - k - \frac{\partial^2 \log \lambda}{\partial x\,\partial y} = k , \qquad k_1 = h .$$

They are then equal to those of the equation (E) but taken in the opposite order. In other words, *the adjoint to the equation* (E) *has the same invariants as the equation* (E_1), *which one deduces from* (E) *by applying one of the Laplace transformations.*

If the Laplace sequence relative to the equation (E) terminates in one direction – on the side of positives indices, for example – after n transformations then one knows that the sequence relative to the adjoint equation or to equation (E_1) must likewise terminate on the side of negative indices after n transformations. Since one will recover (E) after a first application of the transformation to (E_1), one can conclude that the Laplace sequence relative to (E) will terminate on the side of negative indices after $n - 1$ transformations.

Any linear equation (E) that is such that the equation (E_1) that one deduces by the first Laplace transformation has the same invariants as the adjoint equation to (E) can be converted into an equation of the form (3). Indeed, upon changing u into $K(x, y)\,u$, one can always convert the linear equation into the form:

$$\frac{\partial^2 u}{\partial x\,\partial y} + b\frac{\partial u}{\partial y} + c\,u = 0 .$$

The invariants have the following values:

$$h = -c , \qquad k = \frac{\partial b}{\partial y} - c ,$$

whereas the invariants of the transformed (E_1) are:

$$h_1 = -c - \frac{\partial b}{\partial y} - \frac{\partial^2 \log c}{\partial x\,\partial y} , \qquad k_1 = -c .$$

In order for the invariants h_1 and k_1 of that equation to be identical to the invariants k and h of the adjoint, it will suffice that one should have $h_1 = k$, or:

$$2\,\frac{\partial b}{\partial y} = -\,\frac{\partial^2 \log c}{\partial x\,\partial y}\,.$$

The equation must then have the form:

$$\frac{\partial^2 u}{\partial x\,\partial y} - \left(\frac{1}{2}\frac{\partial \log c}{\partial x} + X\right)\frac{\partial u}{\partial y} + c\,u = 0\,,$$

and it will suffice that one can replace u with $u \exp \int_{x_0}^{x} X\,dx$ and c with $-\lambda$ in it in order to recover equation (3).

2. – Any equation of the form (3) that is integrable by the Laplace method will then correspond to an equation (1) that can be integrated by quadratures. From the preceding, the determination of the values of λ for which that is true amounts to the following problem: *Find all Laplace sequences that terminate in both directions and are composed of an even number* $2n$ *of equations such that two equations at an equal distance from the extremes will have the same invariants but arranged in the opposite order.* One can obtain the solution to that problem by considerations that are analogous to the ones that Darboux appealed to in order to obtain all of the equations with equal invariants for which the Laplace sequence does not contain a finite number of equations. One can appeal to linear differential equations with only one independent variable and of *even order* that are equivalent to their adjoint instead of the equations of odd order that occurred in the problem that Darboux treated.

When equation (3) is integrable by the Laplace method, the general integral of equation (1) will belong to the first Ampère class, which will result from the following general theorem:

If the expression:

$$(4)\qquad \varphi(x, y, X, X', \ldots, X^{(p)}, Y, Y', \ldots, Y^{(p)})\,dx + \psi(x, y, X, X', \ldots, X^{(p)}, Y, Y', \ldots, Y^{(p)})\,dy\,,$$

in which X is an arbitrary function of x, Y is an arbitrary function of y, and φ and ψ include X and Y and their derivatives up to a well-defined order, is an exact differential for all possible forms of the functions X and Y then one will have:

$$\int \varphi\,dx + \psi\,dy$$

$$= \int F(x, X, X', \ldots, X^{(p)})\,dx + \int F_1(y, Y, Y', \ldots, Y^{(p)})\,dy + F_2(x, y, X, \ldots, X^{(p-1)}.Y, Y', \ldots, Y^{(p-1)})\,,$$

in which F *includes only* x, X, X', …, $X^{(p)}$, F_1 *includes only* y, Y, Y', …, $Y^{(p)}$, *and* F_2 *is a well-defined function of the variables that appear in* y ([1]).

Suppose, to fix ideas, that the highest-order derivatives of the functions X, Y that appear in φ and ψ are the ones of order p, in such a way that one of the derivatives $X^{(p)}$, $Y^{(p)}$ enters into at least one of the functions φ and ψ, but that no derivative of order higher than p will enter in. By hypothesis, one must have:

$$\frac{d\varphi}{dy} = \frac{d\psi}{dx}$$

for all possible forms of the functions X and Y when one sets:

$$\frac{d\varphi}{dy} = \frac{\partial\varphi}{\partial y} + \frac{\partial\varphi}{\partial Y} Y' + \cdots + \frac{\partial\varphi}{\partial Y^{(p-1)}} Y^{(p)} + \frac{\partial\varphi}{\partial Y^{(p)}} Y^{(p+1)},$$

$$\frac{d\psi}{dx} = \frac{\partial\psi}{\partial x} + \frac{\partial\psi}{\partial X} X' + \cdots + \frac{\partial\psi}{\partial X^{(p-1)}} X^{(p)} + \frac{\partial\psi}{\partial X^{(p)}} X^{(p+1)}.$$

One sees that $d\psi / dx$ does not contain $Y^{(p+1)}$ and that $d\varphi / dy$ does not contain $X^{(p+1)}$. It is then necessary that one must have:

$$\frac{\partial\varphi}{\partial Y^{(p)}} = 0, \qquad \frac{\partial\psi}{\partial X^{(p)}} = 0,$$

i.e., that φ must contain the derivatives of the function Y only up to order at most $p - 1$, and similarly, that ψ contains the derivatives of X only up to order at most $p - 1$. $d\psi / dx$, and as a result $d\varphi / dy$, must then be linear functions of $X^{(p)}$. One must then have:

$$\frac{\partial^2 \left(\dfrac{d\varphi}{dy}\right)}{[\partial X^{(p)}]^2} = 0,$$

or what amounts to the same thing:

$$\frac{d}{dy}\left[\frac{\partial^2\varphi}{(\partial X^{(p)})^2}\right] = 0,$$

([1]) This proposition extends with no difficulty to the expressions:

$$\varphi_1\, dx_1 + \varphi_2\, dx_2 + \ldots + \varphi_n\, dx_n,$$

in which $\varphi_1, \varphi_2, \ldots, \varphi_n$ include n arbitrary functions $X_1, X_2, \ldots, X_n$ of $x_1, x_2, \ldots, x_n$, respectively, and a finite number of their derivatives, which are exact total differentials for all possible forms of the arbitrary functions.

which shows that $\frac{\partial^2\varphi}{(\partial X^{(p)})^2}$ is independent of $y, Y, Y', \ldots, Y^{(p-1)}$:

$$\frac{\partial^2\varphi}{(\partial X^{(p)})^2} = v(x, X, X', X'', \ldots, Y^{(p)}) .$$

One then deduces that φ has the form:

$$\varphi =$$

$$\Phi(x, X, X', \ldots, X^{(p)}) + \Phi_1(x, y, X, X', \ldots, X^{(p-1)}, Y, Y', \ldots, Y^{(p-1)})X^{(p)} + \Phi_2(x, y, X, X', \ldots, X^{(p-1)}, Y, \ldots, Y^{(p-1)}),$$

and one proves in the same fashion that ψ must have the form:

$$\psi =$$

$$\Psi(y, Y, Y', \ldots, Y^{(p)}) + \Psi_1(x, y, X, X', \ldots, X^{(p-1)}, Y, Y', \ldots, Y^{(p-1)})Y^{(p)} + \Psi_2(x, y, X, X', \ldots, X^{(p-1)}, Y, \ldots, Y^{(p-1)}).$$

The coefficient of $X^{(p)}Y^{(p)}$ is $\frac{\partial\Phi_1}{\partial Y^{(p-1)}}$ in $\frac{d\varphi}{dy}$ and $\frac{\partial\Psi_1}{\partial X^{(p-1)}}$ in $\frac{d\psi}{dx}$. One must then have:

$$\frac{\partial\Phi_1}{\partial Y^{(p-1)}} = \frac{\partial\Psi_1}{\partial X^{(p-1)}},$$

which shows that Φ_1 and Ψ_1 are the partial derivatives with respect to $X^{(p-1)}$ and $Y^{(p-1)}$, respectively, of a function:

$$U(x, y, X, X', \ldots, X^{(p-1)}, Y, \ldots, Y^{(p-1)}) = \int \Phi_1\, dX^{(p-1)} + \Psi_1\, dY^{(p-1)} .$$

The total differential of that function U has the expression:

$$dU =$$

$$\left(\frac{\partial U}{\partial x} + \frac{\partial U}{\partial X}X' + \cdots + \frac{\partial U}{\partial X^{(p-2)}}X^{(p-1)}\right)dx + \left(\frac{\partial U}{\partial y} + \frac{\partial U}{\partial Y}Y' + \cdots + \frac{\partial U}{\partial Y^{(p-2)}}Y^{(p-1)}\right)dy + \Phi_1 X^{(p)}dx + \Psi_1 Y^{(p)}dy,$$

and one can write:

$$\int \varphi\, dx + \psi\, dy = \int \Phi\, dx + \int \Psi\, dy + \int \varphi_1\, dx + \psi_1\, dy,$$

when one sets:

$$\varphi_1 = \Phi_2 - \left(\frac{\partial U}{\partial x} + \frac{\partial U}{\partial X} X' + \cdots + \frac{\partial U}{\partial X^{(p-2)}} X^{(p-1)} \right),$$

$$\psi_1 = \Psi_2 - \left(\frac{\partial U}{\partial y} + \frac{\partial U}{\partial Y} Y' + \cdots + \frac{\partial U}{\partial Y^{(p-2)}} Y^{(p-1)} \right).$$

We now remark that $\varphi_1\, dx + \psi_1\, dy$ must be an exact differential for all possible forms of the arbitrary functions X and Y and that φ_1 and ψ_1 include only the include the derivatives of those functions only up to order at most $p - 1$. Upon repeating the same operations with $\int \varphi_1\, dx + \psi_1\, dy$ and applying the process as many times as possible, one will conclude by arriving at an integral of the form:

$$\int \varphi_p(x, y, X)\, dx + \psi_p(x, y, Y)\, dy,$$

in which the expression $\varphi_p\, dx + \psi_p\, dy$ must be an exact differential for all possible forms of X and Y. One must then have:

$$\frac{\partial \varphi_p}{\partial y} = \frac{\partial \psi_p}{\partial x}.$$

Now, φ_p does not contain Y, so the same thing must be true for $\frac{\partial \psi_p}{\partial x}$, i.e., φ_p must have the form $P(y, Y) + Q(x, y)$.

One must likewise have that φ_p has the form $M(x, X) + N(x, y)$, and the integrability condition will become $\frac{\partial N}{\partial y} = \frac{\partial Q}{\partial x}$.

The integral in question can then be written:

$$\int \varphi_p\, dx + \psi_p\, dy = \int M(x, X)\, dx + \int P(y, Y)\, dy + \int N\, dx + Q\, dy.$$

Upon combining all of the results that were obtained, one will indeed get an expression of the stated form for the integral $\int \varphi\, dx + \psi\, dy$.

Remark. – If one supposes that the functions φ and ψ are linear with respect to the functions X, Y and their derivatives then one will recover the theorem that Darboux proved ([1]) and which is greatly useful in the study of linear equations. Indeed, the functions φ and ψ will then have the form:

$$\varphi = A_0 X + A_1 X' + \cdots + A_p X^{(p)} + B_0 Y + B_1 Y' + \cdots + B_{p-1} Y^{(p-1)},$$
$$\psi = C_0 X + C_1 X' + \cdots + C_{p-1} X^{(p-1)} + D_0 Y + D_1 Y' + \cdots + D_{p-1} Y^{(p)},$$

([1]) *Leçons sur la Théorie générale des surfaces*, t. II, pp. 151.

in which A_i, B_i, C_i, D_i are well-defined functions of x and y. If one sets:

$$U = A_p\, X^{(p-1)} + D_p\, Y^{(p-1)}$$

then one can write:

$$\int \varphi\, dx + \psi\, dy = U - \int \varphi_1\, dx + \psi_1\, dy ,$$

in which φ_1 and ψ_1 are expressions of the same form as φ and ψ that contain the derivatives of X and Y only up to order at most $p - 1$. Upon continuing in that way, one will arrive at an expression:

$$a\, X\, dx + b\, Y\, dy$$

that must be an exact differential for all possible forms of the functions X and Y, which demands that a must depend upon only the variable x and b must depend upon only the variable y. If those functions are not zero then one can make all of the quadrature symbols disappear by replacing X with X_1' / a and Y with Y_1' / b, in which X_1 and Y_1 denote two new arbitrary functions of x and y, respectively.

3. – When the function $\lambda\,(x, y)$ has the form $\lambda = k / (x + y)^2$, in which k denotes a constant, equations (1) and (3) will become:

$$s^2 - \frac{4k\, p\, q}{(x+y)^2} = 0 , \tag{1′}$$

$$\frac{\partial^2 u}{\partial x\, \partial y} + \frac{\partial u / \partial y}{x+y} - \frac{k\, u}{(x+y)^2} = 0 , \tag{3′}$$

respectively. If one forms the Laplace sequence relative to equation (3′) on the side of positive indices then one will find that the successive invariants have the values:

$$\frac{k}{(x+y)^2} , \qquad \frac{k-1}{(x+y)^2} , \qquad \frac{k-4}{(x+y)^2} , \qquad \ldots, \qquad \frac{k-n^2}{(x+y)^2} , \qquad \ldots$$

From that, in order for the sequence to be bounded, it is necessary and sufficient that *k must be the square of a whole number*.

Let us examine the simplest case. If $k = 1$ then the general integral of equation (3′) will be:

$$u = X' + \frac{Y - X}{x+y} .$$

From formulas (2), one will then have:

$$v = (x+y)\frac{\partial u}{\partial y} = Y' + \frac{X-Y}{x+y},$$

and with that, the general integral of the equation:

$$s = \frac{2\sqrt{pq}}{x+y} \tag{5}$$

will be:

$$z = \int \left(X' + \frac{Y-X}{x+y} \right)^2 dx + \int \left(Y' + \frac{X-Y}{x+y} \right)^2 dy. \tag{6}$$

If one applies the general theorem that was just proved to that expression then one will find that it can be written:

$$z = -\frac{(Y-X)^2}{x+y} + \int X'^2\, dx + \int Y'^2 dy.$$

In order to make the quadrature disappear, it will suffice to introduce two new independent variables α and β by setting:

$$X' = \alpha, \qquad Y' = \beta, \qquad x = \varphi''(\alpha), \qquad y = \psi''(\beta),$$

which will give us:

$$X = \int X'\, dx = \int \alpha\, \varphi''(\alpha)\, d\alpha = \alpha\, \varphi''(\alpha) - \varphi'(\alpha),$$

$$\int X'^2\, dx = \int \alpha^2\, \varphi''(\alpha)\, d\alpha = \alpha^2\, \varphi''(\alpha) - 2\alpha\, \varphi'(\alpha) + 2\varphi(\alpha),$$

and similarly:

$$Y = \beta\, \psi''(\beta) - \psi'(\beta),$$

$$\int Y'^2\, dx = \beta^2\, \psi''(\beta) - 2\beta\, \psi'(\beta) + 2\psi(\beta).$$

The general integral of equation (5) is then represented by the formulas:

$$\left\{ \begin{aligned} & x = \varphi''(\alpha), \quad y = \psi''(\beta), \\ & z = \alpha^2\, \varphi''(\alpha) - 2\alpha\, \varphi'(\alpha) + 2\varphi(\alpha) + \beta^2\, \psi''(\beta) - 2\beta\, \psi'(\beta) + 2\psi(\beta) \\ & \qquad - \frac{[\alpha\, \psi''(\alpha) - \varphi'(\alpha) - \beta\, \psi''(\beta) + \psi'(\beta)]^2}{\varphi''(\alpha) + \psi''(\beta)}, \end{aligned} \right. \tag{7}$$

in which α and β are variable parameters, and φ and ψ are two arbitrary functions. One remarks that the general integral of that equation contains the two arbitrary functions explicitly without that equation belonging to the class that Moutard studied.

When $k = 4$, the general integral of equation (3) is:

$$u = X'' - \frac{4X'}{x+y} + \frac{6X}{(x+y)^2} + \frac{2Y'}{x+y} - \frac{6Y}{(x+y)^2} \;,$$

and the corresponding value of v is:

$$v = \frac{x+y}{2}\frac{\partial u}{\partial y} = Y'' - \frac{4Y'}{x+y} + \frac{6Y}{(x+y)^2} + \frac{2X'}{x+y} - \frac{6X}{(x+y)^2} \;.$$

The general integral of the equation:

$$s = \frac{4\sqrt{pq}}{x+y}$$

is then represented by the formula:

$$z = \int u^2\,dx + v^2\,dy \;,$$

which will become:

$$z = \int X''^2 dx + \int Y''^2 dy - 4\,\frac{X'^2+Y'^2}{x+y} + 12\,\frac{X\,X'+Y\,Y'}{(x+y)^2} + \frac{4X'\,Y'}{x+y} - 12\,\frac{X\,Y'+X'\,Y}{(x+y)^2} - 12\,\frac{(X-Y)^2}{(x+y)^3}$$

upon applying the general theorem.

As another example of an equation to which one can apply the preceding transformation, I will cite the equation $s^2 = 4pq$, which was encountered by Thomas Craig in certain problems in the theory of surfaces, and which thus reduces to the well-known equation $\dfrac{\partial^2 u}{\partial x\,\partial y} = u$.

4. – The transformation that was just studied can be attached to a more general question. From the way that we obtain equation (3) itself, if we set:

$$M = \frac{1}{\lambda}\left(\frac{\partial u}{\partial y}\right)^2, \qquad N = -\,u^2$$

then we will have the identity:

$$\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} = \frac{2}{\lambda}\,\frac{\partial u}{\partial y}\left(\frac{\partial^2 u}{\partial x\,\partial y} - \frac{1}{2}\,\frac{\partial \log\lambda}{\partial x}\,\frac{\partial u}{\partial y} - \lambda\,u\right).$$

To abbreviate, agree to say that a *multiplier* of a linear equation:

(8) $$F\,(x) = \frac{\partial^2 z}{\partial x\,\partial y} + a\,\frac{\partial z}{\partial x} + b\,\frac{\partial y}{\partial y} + c\,z \;= 0$$

is any function μ that includes the variables x, y, z, and the partial derivatives of z up to any order, and is such that the product $\mu\,F\,(z)$ has the form:

$$\mu\,F\,(\mathrm{z}) = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y},$$

in which M and N are likewise well-defined functions of x, y, z, and the partial derivatives of z with respect to x and y. We then see that $\dfrac{2}{\lambda}\,\dfrac{\partial u}{\partial y}$ is a multiplier for the linear equation (3).

If one is given an arbitrary linear equation (8) then one knows that there will always exist an infinitude of multipliers u (x, y) that include only the independent variables x and y. In order for $u(x, y)$ to be a multiplier of F (z), it is necessary and sufficient that u should verify a linear equation of the same form, which called the *adjoint equation to the first one:*

(9) $$G\,(u) = \frac{\partial^2 u}{\partial x\,\partial y} - a\,\frac{\partial u}{\partial x} - b\,\frac{\partial u}{\partial y} + \left(c - \frac{\partial a}{\partial x} - \frac{\partial b}{\partial y}\right) u \;= 0\,.$$

If the coefficients a, b, c of equation (8) are arbitrary then one will easily see that there exist no other multipliers than the functions u (x, y) that satisfy the adjoint equation. Without wanting to enter into a detailed discussion of that question, we will show how one can predict *a priori* the existence of the very extensive case in which there exist multipliers that include not only the variables x and y, but also z and its derivatives up to an order that is as high as one desires.

First recall the identity:

(10) $$u\,F\,(z) - z\,G\,(u) = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y},$$

in which one sets:

$$M = a\,u\,z + \frac{1}{2}\left(u\,\frac{\partial z}{\partial y} - z\,\frac{\partial u}{\partial y}\right),$$

$$N = b\,u\,z + \frac{1}{2}\left(u\,\frac{\partial z}{\partial x} - z\,\frac{\partial u}{\partial x}\right).$$

Suppose that the proposed equation F (z) = 0 is such that one can pass from that equation to its adjoint by one of the transformations (m, n) that were studied completely by Darboux. In other words, suppose that the general integral of equation (9) is given by the formula:

(11) $$u = A\,z + B_1\,\frac{\partial z}{\partial x} + \cdots + B_m\,\frac{\partial^m z}{\partial x^m} + C_1\,\frac{\partial z}{\partial y} + \cdots + C_n\,\frac{\partial^n z}{\partial y^n}\;,$$

in which $A_1, B_1, \ldots, B_m, C_1, \ldots, C_n$ are well-defined functions of x and y, and z is the general integral of the equation $F(z) = 0$. If one substitutes the preceding expression for u in $G(u)$ then the result will be identically zero, provided that z verifies the equation $F(z) = 0$. Upon supposing that u is replaced by the expression (11), one will then have an identity of the form:

$$G(u) = \alpha F(z) + \sum \beta_{ik} \frac{\partial^{i+k} F(z)}{\partial x^i \, \partial y^k} , \tag{12}$$

in which the coefficients α and β_{ik} depend upon only x and y. If one makes the same substitution in the equation (10) then one will arrive at a new relation of the form:

$$\alpha' F(z) + \sum \beta'_{ik} \frac{\partial^{i+k} F(z)}{\partial x^i \, \partial y^k} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} , \tag{13}$$

in which P, Q, α', β'_{ij} include x, y, and z, along with the partial derivatives of z. Now, a series of integrations by parts will permit one to keep only those terms in the left-hand side that are divisible by $F(z)$. For example, if i is positive then one can write:

$$\beta'_{ik} \frac{\partial^{i+k} F(z)}{\partial x^i \, \partial y^k} = \frac{\partial}{\partial x}\left[\beta'_{ik} \frac{\partial^{i+k-1} F(z)}{\partial x^{i-1} \, \partial y^k} \right] - \frac{\partial \beta'_{ik}}{\partial x} \frac{\partial^{i+k-1} F(z)}{\partial x^{i-1} \, \partial y^k} ,$$

and one will have an analogous formula for reducing the index k by one unit. Upon continuing in that way and moving all of the derivatives to the right-hand side, one will conclude by arriving at an identity of the form:

$$\mu F(z) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} ,$$

in which m contain the partial derivatives of z up to arbitrary order, provided that m and n are very large. From the final identity, that function μ will be a multiplier for the equation $F(z) = 0$.

One can recover the transformation that was considered at the beginning of this article by supposing that the transform (E_1) of equation (8) has the same invariants as the adjoint of that equation. One can then pass from equation (8) to its adjoint by a transformation of the form:

$$u = A \frac{\partial z}{\partial y} + B z .$$

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