"Über den Zusammenhang zwischen den Theorien der zweiten Variation und der Weierstrass'schen Theorie der Variationsrechnung," Rend. Circ. matem. Palermo (1909), 49-78.

# On the connection between the theories of the second variation and Weierstrass's theory of the calculus of variations.

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In the following, it will be shown how the transformation of the second variation that was first performed by A. CLEBSCH ([1]) and later in a simpler form by G. von ESCHERICH ([2]) can be derived from the so-called WEIERSTRASS formula that represented the different between two integrals by means of $E$-function. Whereas for the simplest problem in the calculus of variations, the integral:

$$\int [f(x, y, p) - p\, f_p(x, y, p)]\, dx + f_p(x, y, p)\, dy$$

will become independent of the path when one replaces the $p$ in it with the slope function $p\ (x, y)$ of any extremal field, for the problem with several unknown functions, that theorem is not true for the analogous integral:

$$(A) \qquad \int [f(x, y_1, \ldots, y_n; p_1, \ldots, p_n) - \sum_{i=1}^{n} p_i\, f_{p_i}(x, y_1, \ldots, y_n; p_1, \ldots, p_n)]\, dx$$
$$+ \sum_{i=1}^{n} f_{p_i}(x, y_1, \ldots, y_n; p_1, \ldots, p_n)\, dy_i\ .$$

However, as HILBERT ([3]) remarked, that integral will be independent on any surface that is defined by a one-parameter family of extremals that is selected from a field. A proof of that will now be communicated that perhaps displays certain advantages over the original Hilbert proof.

([1]) "Ueber die Reduction der zweiten Variation auf ihre einfachste Form," J. reine angew. Math. **55** (1858), 254-273.

([2]) "Die zweite Variation der einfachen Integrale," Sitzber. math.-naturw. Classe der Kaiser. Akad. Wiss. (Wien), Abt. II.a **107** (1898), 1191-1250, 1267-1326, 1381-1428; *ibid.* **108** (1899), 1269-1340; *ibid.* **110** (1901), 1355-1421.

([3]) "Zur Variationsrechnung," Nachr. v. d. Kgl. Ges. d. Wiss. Göttingen (1905), 159-180.

Some consequences will then be inferred from it, and in particular, one can easily prove in that way the theorem that a certain bilinear expression that ESCHERICH denoted by $\psi(z, r; u, \rho)$ can reduce to a constant identically when one substitutes solutions of the so-called *accessory linear system of differential equations* for the $z, r$ in it, on the one hand, and the $u, \rho$, on the other. That same bilinear expression allows one to characterize those extremal fields for which the integral ($A$) is independent of the path, which will be shown in § **2**. The result overlaps completely with the one that A. MAYER found ([4]). The aforementioned fields (they will be referred to briefly as MAYER fields) make it possible to expression the integral difference in terms of the $E$-function by means of the WEIERSTRASS formula, from which one will arrive at the CLEBSCH transformation of the second variation by a suitable series development and restricting to second-order terms. Certain systems of $n$ linearly-independent solutions of the accessory system of differential equations play an essential role in that transformation, which ESCHERICH refers to as *conjugate* systems, and which are closely related to the MAYER families of extremals as will be shown in what follows. CLEBSCH and ESCHERICH have given all conjugate systems whose determinants do not vanish at a given location. In a previous treatise ([5]), I showed how one would also get those conjugate systems whose determinants vanished at the location in question. I shall take up that question again here in § **4**, while § **5** then offers the geometric interpretation of the results that were found in § **4**.

## § 1.

One treats the following problem in the calculus of variations: Make the integral:

$$\int f(x, y_1, \ldots, y_n; y_1', \ldots, y_n')\, dx \tag{1}$$

an extremum under the auxiliary conditions ($m < n$):

$$\varphi_k(x, y_1, \ldots, y_n; y_1', \ldots, y_n') = 0 \qquad (k = 1, 2, \ldots, m). \tag{2}$$

For simplicity, we assume that all of the functions that appear are regularly analytic. $y_i'$ is understood to mean the derivative of $y_i$ with respect to $x$. The partial derivatives of a function with respect one of its variables will be denoted by appending the variable in question as an index to the function symbol. Finally, for brevity, we will frequently not write out all variables in full in $f$, $\varphi_k$, and other functions, but only suggest them in the following way: $f(x, y, y')$.

We set:

([4]) "Über den HILBERTschen Unabhängigkeitssatz in der Theorie des Maximums und Minimums der einfachen Integrale (II. Mitteilung)," Ber. Verh. Kgl. Sächsisch. Ges. Wiss. Leipzig, Math.-Phys. Klasse **57** (1905), 49-67; *b*) "Nachträgliche Bemerkung zu meiner II. Mitteilung über den HILBERTschen Unabhängigkeitssatz," *ibid.*, pp. 313-314.

([5]) "Zur Theorie der zweiten Variation einfacher Integrale," Monatsh. Math. Phys. **14** (1903), 3-57.

(3) $$F(x, y, y', \lambda) = f + \lambda_1\,\varphi_1 + \lambda_2\,\varphi_2 + \ldots + \lambda_m\,\varphi_m\,,$$

in a known way, in which $\lambda_1, \lambda_2, \ldots, \lambda_m$ mean functions of $x$. If the $n + m$ functions:

(4) $$y_1(x), \ldots, y_n(x)\,; \qquad \lambda_1(x), \ldots, \lambda_m(x)$$

satisfy not only equations (2), but also the equations:

(5) $$F_{y_i} - \frac{d}{dx} F_{y'_i} = 0 \qquad (i - 1, 2, \ldots, n),$$

then the curve $y_i = y_i(x)$ $(i = 1, 2, \ldots, n)$ will be referred to as an *extremal*, and the functions $\lambda_r(x)$ $(r = 1, 2, \ldots, m)$ will be referred to as the associated *multipliers*. If the determinant:

(6) $$\begin{vmatrix} F_{y'_i y'_k} & \varphi_{r\,y'_k} \\ \varphi_{s\,y'_i} & 0 \end{vmatrix} \qquad \begin{pmatrix} i = 1, 2, \ldots, n; & r = 1, 2, \ldots, m \\ k = 1, 2, \ldots, n; & s = 1, 2, \ldots, m \end{pmatrix}$$

proves to be nonzero for all $x$ that satisfy the inequality $x_0 \le x \le x_1$ when one substitutes the functions (4) then the arc $(x_0, x_1)$ of the extremal in question will be called *regular*.

Let an $n$-parameter family of extremals (together with the associated multipliers) be given by:

(7) $$y_i(x\,; a_1, \ldots, a_n)\,; \qquad \lambda_r(x\,; a_1, \ldots, a_n) \qquad (i = 1, 2, \ldots, n\,; r = 1, 2, \ldots, m).$$

Let $\mathfrak{R}$ denote a closed, simply-connected, $(n + 1)$-dimensional continuum in the space of $(n + 1)$ variables $x, y_1, \ldots, y_n$ in which the extremals of (7) are regular, and which simply covers those extremals. Furthermore, let the functional determinants be non-zero:

$$\left| y_{i a_k}(x, a_1, \ldots, a_n) \right| \qquad (i, k = 1, 2, \ldots, n)$$

in $\mathfrak{R}$. One then says: The extremal family (7) defines a *field* in $\mathfrak{R}$.

By solving the equations:

(8) $$y_i = y_i(x\,; a_1, \ldots, a_n) \qquad (i = 1, 2, \ldots, n),$$

one will obtain:

(9) $$a_i = a_i(x\,; y_1, \ldots, y_n) \qquad (i = 1, 2, \ldots, n).$$

The functions $a_i$ are then single-valued in $\mathfrak{R}$ and regularly analytic. If one substitutes the expressions (9) in $y_{ix}(x, a_1, \ldots, a_n)$ then one will get:

(10) $$y_{ix}(x, a_1, \dots, a_n) = p_i(x, a_1, \dots, a_n) \qquad (i = 1, 2, \dots, n) .$$

Those functions are also single-valued and regularly analytic in $\mathfrak{R}$, and they are called the *slope functions* of the field, because they are nothing but the direction coefficients of the extremal of the field at the point $(x, y_1, \dots, y_n)$ that goes through that point. If one substitutes the expression (9) in $\lambda_r(x, a_1, \dots, a_n)$ then one will get:

(11) $$\lambda_r(x, a_1, \dots, a_n) = l_r(x, y_1, \dots, y_n) \qquad (r = 1, 2, \dots, m).$$

We refer to those functions, which are likewise single-valued and regularly analytic in $\mathfrak{R}$, as the *multiplier functions* of the field.

Now let:

(12) $$y_i = y_i(x, c) \qquad (i = 1, 2, \dots, n)$$

be a one-parameter family of extremals ([6]) that is selected from (7). The associated multipliers are:

(13) $$\lambda_i = \lambda_i(x, c) \qquad (r = 1, 2, \dots, m).$$

The family (12) fills up a two-dimensional surface $f$ in $\mathfrak{R}$, and we can consider $x$ and $c$ to be its parameters. An equation $c = c(x)$ will then produce a curve that lies in $f$. In particular, the extremals (12) will be given by $c$ = const.

We assert: The family of curves $c$ = const. is a family of extremals of the following variational problem with only one unknown function: Determine $c$ as a function of $x$ such that the integral:

(14) $$\left\{ \begin{aligned} &\int F[x, y_1(x,c), \dots, y_n(x,c); y_{1x}(x,c) + y_{1c}(x,c)\,c', \dots, y_{nx}(x,c) + y_{nc}(x,c)\,c'; \\ &\qquad\qquad\qquad\qquad\qquad\qquad \lambda_1(x,c), \dots, \lambda_n(x,c)]\,dx, \end{aligned} \right.$$

in which $F$ is the expression (3), is an extremum. In fact, the LAGRANGE equation for that variational problem reads:

(15) $$\sum_{i=1}^{n} \left[ F_{y_i}\, y_{ic} + F_{y_i'}(y_{ixc} + y_{icc}\,c') - \frac{d}{dx}(F_{y_i'}\, y_{ic}) \right] + \sum_{r=1}^{m} \lambda_{rc}\,\varphi_r = 0 ,$$

in which the arguments of $F_{y_i}$, $F_{y_i'}$, $\varphi_r$ are: $y_i(x, c)$, $y_{ix}(x, c) + y_{ic}(x,c)\,c'$. If one now sets $c$ = const., so $c' = 0$, then one will also have $\varphi_r = 0$, since the extremals (12) satisfy the condition equations (2), and equation (15) will reduce to:

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([6]) In this, it is assumed that different extremals of (7) also correspond to different values of the parameter $c$.

$$\sum_{i=1}^{n}\left\{F_{y_i}\left[y(x,c),y_x(x,c),\lambda(x,c)\right]-\frac{d}{dx}F_{y_i'}\left[y(x,c),y_x(x,c),\lambda(x,c)\right]\right\}y_{ic}(x,c)=0\,.$$

However, the latter equation is fulfilled because (12) and (13) satisfy equations (5).

If the independence theorem is applied to the integral (14) and the extremal field $c$ = const. then that will imply: The integral:

$$\int F_{y_i}\left[y(x,c),y_x(x,c),\lambda(x,c)\right]dx+\sum_{i=1}^{n}F_{y_i'}\left[y(x,c),y_x(x,c),\lambda(x,c)\right]y_{ic}(x,c)\,dc \tag{16}$$

has the same value for two curves in the $(x, c)$-plane with the same starting points and endpoints that lie completely in the field.

Now, let any curve that lies on the surface $f$ be given, along which *the parameter c varies continuously*. On that curve, one has:

$$dy_i = y_{ix}(x, c)\,dx + y_{ic}(x, c)\,dc \qquad (i = 1, 2, \ldots, n)\,.$$

If we then extend the integral:

$$\left\{\begin{aligned}&\int\left\{F[x,y,p(x,y),l(x,y)]-\sum_{i=1}^{n}p_i(x,y)F_{y_i'}[x,y,p(x,y),l(x,y)]\right\}dx\\&\qquad+\sum_{i=1}^{n}F_{y_i'}[x,y,p(x,y),l(x,y)]\,dy_i\end{aligned}\right. \tag{17}$$

[where $p(x, y)$ and $l(x, y)$ mean the functions (10) and (11)] over such a curve then it will reduce to precisely the integral (15), and we will have the theorem:

*The integral* (17) *has the same value for any two curves that lie on the surface f and have equal starting points and endpoints and along which the parameter c varies continuously.*

Now let a closed curve $C$ that does not intersect itself be given that cuts no extremal of the field more often than once. The extremals of the field that go through its points define a one-parameter family, and the two-dimensional surface that they fill up has the type of a cylindrical surface. We take a point $P_0$ on the curve $C$. We can choose the parameter $c$ of the aforementioned family of extremals to be the arc-length that is measured along them. If $S$ is the length of the curve $C$ then the cylindrical surface $f$ will be spanned when $c$ satisfies the inequality $0 \le c < S$. One observes that the parameter value $c$ will not be distributed continuously over $f$, but jump by $S$ along the extremal that goes through $P_0$. In particular, one cannot, say, conclude that the integral (17) must have the value zero when it is extended over a closed curve that lies on $f$, such as the curve $C$. However, the following theorem is true:

*The integral* (17) *will always have the same value when it is extended over an arbitrary closed curve that lies on f and encircles the opening of the cylindrical surface f once.*

For simplicity, we would like to restrict ourselves to proving the somewhat-more-specialized theorem: The integral (17) will always have the same value when it is extended over any curve that lies on *f* and is cut by each of the extremals that generate *f* at one and only one point.

Let $C'$ be one such curve. It will be cut by the extremal that goes through the point $P_0$ of $C$ at the point $P_0'$. We denote the value that the integral (17) assigns to the extremal arc $P_0 P_0'$ by $I_{P_0 P_0'}$. We take a second point $P_1$ on $C$. The extremal that goes through it meets $C'$ at $P_1'$. We let $I_{P_1 P_1'}$ denote the value that the integral (17) assigns to the extremal arc $P_1 P_1'$. $C$ will be divided into two arcs by $P_0$ and $P_1$ and $C'$ will be divided by $P_0'$ and $P_1'$; let the values that the integral (17) assigns to those arcs be: $I_{P_0 P_1}^{(1)}$, $I_{P_0 P_1}^{(2)}$, $I_{P_0' P_1'}^{(1)}$, $I_{P_0' P_0'}^{(2)}$. Indeed, let the upper indices be chosen such that integrals that are denoted with the same upper indices belong to arcs of $C$ and $C'$ that are cut by the same extremals. It will follow from the independence theorem that it is valid for (17) with no further discussion that:

$$I_{P_0 P_1}^{(1)} + I_{P_1 P_1'} = I_{P_0 P_0'} + I_{P_0' P_1'}^{(1)} ,$$
$$I_{P_0 P_1}^{(2)} + I_{P_1 P_1'} = I_{P_0 P_0'} + I_{P_0' P_1'}^{(2)} ,$$

from which, it will follow by subtraction that:

$$I_{P_0 P_1}^{(1)} - I_{P_0 P_1}^{(2)} = I_{P_0' P_1'}^{(1)} - I_{P_0' P_1'}^{(2)} ,$$

which is, however, what we asserted.

Now let:

$$y_i = y_i\,(x, c_1, c_2)\,, \qquad \lambda_i = \lambda_i\,(x, c_1, c_2) \tag{18}$$

be a two-parameter family of extremals that selected from (7), along with the associated multipliers. The one-parameter family that is selected by:

$$\begin{array}{llll} c_1^0 \leq c_1 \leq c_1^0 + \Delta c_1\,, & c_2 = c_2^0\,, & c_1 = c_1^0 + \Delta c_1\,, & c_2^0 \leq c_2 \leq c_2^0 + \Delta c_2\,, \\ c_1^0 + \Delta c_1 \geq c_1 \geq c_1^0\,, & c_2 = c_2^0 + \Delta c_2\,, & c_1 = c_1^0\,, & c_2^0 + \Delta c_2 \geq c_2 \geq c_2^0 \end{array}$$

fills up a surface of the type that was just considered. If we intersect it with an arbitrary plane $x = x_0$ and extend the integral (17) over the closed intersection curve that arises then its value will be independent of $x_0$. However, that value is:

$$
(19)\quad
\left\{
\begin{aligned}
&\int_{c_1^0}^{c_1^0+\Delta c_1}\sum_{i=1}^{n}\{F_{y_i'}[x_0,y(x_0,c_1,c_2^0),y_x(x_0,c_1,c_2^0),\lambda(x_0,c_1,c_2^0)]\,y_{ic_1}(x_0,c_1,c_2^0)\\
&\qquad -F_{y_i'}[x_0,y(x_0,c_1,c_2^0+\Delta c_2),y_x(x_0,c_1,c_2^0+\Delta c_2),\lambda(x_0,c_1,c_2^0+\Delta c_2)]\\
&\qquad\qquad \times y_{ic_1}(x_0,c_1,c_2^0+\Delta c_2)\}\,dc_1\\
&+\int_{c_2^0}^{c_2^0+\Delta c_2}\sum_{i=1}^{n}\{F_{y_i'}[x_0,y(x_0,c_1+\Delta c_1,c_2^0),y_x(x_0,c_1+\Delta c_1,c_2^0),\lambda(x_0,c_1+\Delta c_1,c_2^0)]\\
&\qquad\qquad \times y_{ic_1}(x_0,c_1+\Delta c_1,c_2^0)\\
&\qquad -F_{y_i'}[x_0,y(x_0,c_1^0,c_2),y_x(x_0,c_1^0,c_2),\lambda(x_0,c_1^0,c_2)]\,y_{ic_2}(x_0,c_1^0,c_2)\}\,dc_2\,.
\end{aligned}
\right.
$$

If one divides that expression by $\Delta c_1 \cdot \Delta c_2$ and lets $\Delta c_1$ and $\Delta c_2$ go to zero, and further writes $x$ in place of the arbitrary $x_0$ and $c_1$, $c_2$ in place of the general $c_1^0$, $c_2^0$ then one will get:

$$
\begin{aligned}
&\sum_{i,k=1}^{n}\{F_{y_i' y_k}[x,y(x,c_1,c_2),y_x(x,c_1,c_2),\lambda(x,c_1,c_2)][y_{ic_2}(x,c_1,c_2)\,y_{kc_1}(x,c_1,c_2)\\
&-[y_{ic_1}(x,c_1,c_2)\,y_{kc_2}(x,c_1,c_2)]+F_{y_i' y_k'}[x,y(x,c_1,c_2),y_x(x,c_1,c_2),\lambda(x,c_1,c_2)]\\
&\qquad\times[y_{ic_2}(x,c_1,c_2)\,y_{kxc_2}(x,c_1,c_2)-y_{ic_1}(x,c_1,c_2)\,y_{kxc_1}(x,c_1,c_2)\}\\
&\qquad\qquad+\sum_{i=1}^{n}\sum_{r=1}^{m}\varphi_{r y_1'}[x,y(x,c_1,c_2),y_x(x,c_1,c_2)]\\
&\qquad\times[y_{ic_2}(x,c_1,c_2)\,\lambda_{rc_1}(x,c_1,c_2)-y_{ic_1}(x,c_1,c_2)\,\lambda_{rc_2}(x,c_1,c_2)]\ .
\end{aligned}
$$

We then have the theorem: *Along each individual extremal of the family* (18), *the expression:*

$$
(20)\quad
\left\{
\begin{aligned}
&\sum_{i,k=1}^{n}\{F_{y_i' y_k}(x,y,y',\lambda)(y_{ic_2}\,y_{kc_1}-y_{ic_1}\,y_{kc_2})+F_{y_i' y_k'}(x,y,y',\lambda)(y_{ic_2}\,y_{kxc_1}-y_{ic_1}\,y_{kxc_2})\}\\
&\qquad\qquad+\sum_{i=1}^{n}\sum_{r=1}^{m}\varphi_{r y_1'}(x,y,y')(y_{ic_2}\,\lambda_{rc_1}-y_{ic_1}\,\lambda_{kc_2})
\end{aligned}
\right.
$$

*remains constant.* That theorem was proved in an entirely different way by A. CLEBSCH ([7]) and G. v. ESCHERICH ([8]).

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([7]) *Loc. cit.* ([1]), pp. 260.

([8]) *Loc. cit.* ([2]), **107** (1898), pp. 1245. G. Herglotz was kind enough to communicate to me that the line of reasoning that was presented above had many points of contact with the arguments that H. POINCARÉ presented in his *Méthode Nouvelles de la Mécanique céleste* in the chapter on integral invariants (added by the editor).

## § 2.

We are now in a position to give those extremal fields for which the integral (17) will be path-independent, with no further analysis.

If we choose $c_1$ and $c_2$ in (18) to be any two of the parameters that appear in (7) – say, $a_\mu$ and $a_\nu$ – then if (17) is to be independent of path, the expression (19), and therefore (20), as well, must always have the value zero. That will imply the following $n$ ($n$ – 1) / 2 conditions as the necessary condition for the integral (17) to be independent of path for the extremal field that is defined by (7):

$$(21)\quad \left\{\begin{aligned} &\sum_{i,k=1}^{n}\{F_{y_i' y_k}(x,y,y',\lambda)(y_{i a_\nu}\, y_{k a_\mu} - y_{i a_\mu}\, y_{k a_\nu}) + F_{y_i' y_k'}(x,y,y',\lambda)(y_{i a_\nu}\, y_{k x a_\mu} - y_{i a_\mu}\, y_{k x a_\nu})\} \\ &\qquad + \sum_{i=1}^{n}\sum_{r=1}^{m}\varphi_{r y_1'}(x,y,y')(y_{i a_\nu}\,\lambda_{r a_\mu} - y_{i a_\mu}\,\lambda_{k a_\nu}) = 0 \quad (\mu,\nu = 1,2,\ldots,n). \end{aligned}\right.$$

We easily convince ourselves that the existence of those conditions is also sufficient. To that end, we first remark that (21) can also be written in the form:

$$(22)\qquad \sum_{i=1}^{n}\left(\frac{\partial y_i}{\partial a_\nu}\frac{\partial F_{y_i'}(x,y,y',\lambda)}{\partial a_\mu} - \frac{\partial y_i}{\partial a_\mu}\frac{\partial F_{y_i'}(x,y,y',\lambda)}{\partial a_\nu}\right) = 0 .$$

We let $x_0$ denote an arbitrary value of $x$ and introduce the initial coordinates of our extremal for $x = x_0$ in place of the $a$ as new parameters, which will happen by way of:

$$a_i = a_i(x_0, y_1^0, \ldots, y_n^0) \qquad (i = 1, 2, \ldots, n)$$

by means of equations (9). If we now define the expression:

$$(23)\qquad \sum_{i=1}^{n}\left(\frac{\partial y_i}{\partial y_\nu^0}\frac{\partial F_{y_i'}(x,y,y',\lambda)}{\partial y_\mu^0} - \frac{\partial y_i}{\partial y_\mu^0}\frac{\partial F_{y_i'}(x,y,y',\lambda)}{\partial y_\nu^0}\right)$$

then due to the fact that:

$$\frac{\partial}{\partial y_\nu^0} = \sum_{\mu=0}^{m}\frac{\partial a_\rho}{\partial y_\nu^0}\frac{\partial}{\partial a_\rho},$$

it will be equal to:

$$\sum_{\mu=0}^{m}\frac{\partial a_\rho}{\partial y_\nu^0}\frac{\partial a_\sigma}{\partial y_\mu^0}\sum_{i=1}^{n}\left(\frac{\partial y_i}{\partial a_\rho}\frac{\partial F_{y_i'}}{\partial a_\sigma} - \frac{\partial y_i}{\partial a_\sigma}\frac{\partial F_{y_i'}}{\partial a_\rho}\right),$$

but that is equal to zero, due to (22). If one sets $x = x_0$ in (23), in particular, and considers that $\partial y_i / \partial y_\nu^0$ will then be equal to 1 or 0 according to whether $\nu$ is or is not equal to $i$, respectively, then one will get the $n (n - 1) / 2$ equations:

$$\left[\frac{\partial F_{y'_\nu}(x, y, y', \lambda)}{\partial y_\mu^0}\right]_{x=x_0} = \left[\frac{\partial F_{y'_\mu}(x, y, y', \lambda)}{\partial y_\nu^0}\right]_{x=x_0},$$

which say the following: There exists a function $\Phi(y_1^0, \ldots, y_n^0)$ whose partial derivative with respect to $y_i^0$ is equal to the expression $F_{y'_i}(x, y, y', \lambda)$ when one replaces the $x$, $y_k$, $y'_k$, $\lambda_r$ in it with $x_0$, $y_k^0$, $p_k(x, y_1^0, \ldots, y_n^0)$, $l_k(x, y_1^0, \ldots, y_n^0)$, respectively. That has the consequence that the integral (17) will be path-independent for curves that lie completely in the plane $x = x_0$. However, that will suffice to make it possible to prove the path-independence of the integral (17) in general ([9]). We will then have the theorem:

*In order for the integral* (17) *to be path-independent for the extremal field that is defined by* (7), *it is necessary and sufficient that equations* (21) *or* (22) *must be satisfied.*

Indeed, it will suffice when those equations are true for any special value of $x$, since they will then be true in general, from § **1**.

A. MAYER ([10]) was probably the first to point out the extremal fields that we speak of, which is why we, with O. BOLZA ([11]), will refer to them as MAYER fields.

We shall now prove that there is always an $n$-parameter family of extremals in a sufficiently-small neighborhood of a sufficiently-short regular extremal arc that includes the extremal arc in question and define a MAYER field in that neighborhood.

Let: $x_0$, $y_1^{00}, \ldots, y_n^{00}$ ; $y_1^{00\prime}, \ldots, y_n^{00\prime}$ be a line element of the extremal arc, and let $\lambda_1^{00}, \ldots, \lambda_n^{00}$ be the values of the associated multipliers for $x = x_0$ ; we set:

$$w_i^{00} = F_{y'_i}(x_0, y^{00}, y^{00\prime}, \lambda^{00}) \qquad (i = 1, 2, \ldots, n).$$

We shall now consider the $n + m$ equations:

$$\text{(24)} \qquad \begin{cases} F_{y'_i}(x_0, y^0, y^{0\prime}, \lambda^0) = 0 & (i = 1, 2, \ldots, n), \\ \varphi_r(x_0, y^0, y^{0\prime}) = 0 & (r = 1, 2, \ldots, m). \end{cases}$$

([9]) *Loc. cit.* ([3]), pp. 166.
([10]) *Loc. cit.* ([4]). *a.*
([11]) "WEIERSTRASS's theorem and KNESER's theorem on transversals for the most general case of an extremum of a simple definite integral," Trans. Am. Math. Soc. **7** (1906), 459-488, pp. 481.

Since their functional determinant with respect to the $n + m$ quantities $y^{0'}$ and $\lambda^0$ is nothing but the determinant (6), and is therefore nonzero, their solution will give:

$$
(25) \qquad \begin{cases} y_i^{0'} = y_i^{0'}(y_1^0, \dots, y_n^0; w_1^0, \dots, w_n^0) & (i = 1, 2, \dots, n), \\ \lambda_i^0 = \lambda_i^0(y_1^0, \dots, y_n^0; w_1^0, \dots, w_n^0) & (r = 1, 2, \dots, m), \end{cases}
$$

and the functions on the right-hand sides take the values $\Phi_{y_i^0}$, $\lambda_r^{00}$ for $y_1^0, \dots, y_n^0$ ; $w_1^0, \dots, w_n^0$ and are regularly analytic in a neighborhood of that location.

Now let $\Phi(y_1^0, \dots, y_n^0)$ be a function whose partial derivatives $\Phi_{y_i^0}$ at the location $y_1^{00}, \dots, y_n^{00}$ have the values $w_1^{00}, \dots, w_n^{00}$, respectively. We can then replace $w_i^0$ with $\Phi_{y_i^0}(y_1^0, \dots, y_n^0)$ in equations (24). Equations (25) will imply $y_i^{0'}$ and $\lambda_r^0$ as functions of $y_1^0, \dots, y_n^0$ alone:

$$
\begin{aligned} y_i^{0'} &= \varphi_i(y_1^0, \dots, y_n^0) && (i = 1, 2, \dots, n), \\ \lambda_r^0 &= \psi_r(y_1^0, \dots, y_n^0) && (r = 1, 2, \dots, m). \end{aligned}
$$

If we now construct solutions to equations (2) and (5) from the initial values:

$$
\begin{aligned} & x_0, \; y_i^0 && (i = 1, 2, \dots, n), \\ y_i^{0'} &= \varphi_i(y_1^0, \dots, y_n^0) && (i = 1, 2, \dots, n), \\ \lambda_r^0 &= \psi_r(y_1^0, \dots, y_n^0) && (r = 1, 2, \dots, m) \end{aligned}
$$

then those solutions will define an $n$-parameter family:

$$
(26) \qquad \begin{cases} y_i = y_i(x, y_1^0, \dots, y_n^0) & (i = 1, 2, \dots, n), \\ \lambda_r = \lambda_r(x, y_1^0, \dots, y_n^0) & (r = 1, 2, \dots, m), \end{cases}
$$

and since obviously $y_{i\,y_k^0}(x_0, y_1^0, \dots, y_n^0)$ is equal to one or zero according to whether $i$ is or is not equal to $k$, resp., the determinant:

$$
\frac{\partial(y_1, y_2, \dots, y_n)}{\partial(y_1^0, y_2^0, \dots, y_n^0)}
$$

will reduce to 1 identically for $x = x_0$. As is known, one infers from this that the family (26) defines a field in the neighborhood of the location $x_0$, $y_1^{00}, \dots, y_n^{00}$. However, one will show that this field is a MAYER field with no further analysis when one sets $x = x_0$ in equations (22). They will then reduce to:

$$
\Phi_{y_i^0 y_k^0}(y_1^0, \dots, y_n^0) - \Phi_{y_k^0 y_i^0}(y_1^0, \dots, y_n^0) \equiv 0
$$

and are therefore certainly fulfilled. We have then, in fact, shown:

*Any sufficiently-short, regular extremal arc can be surrounded by a MAYER field.*

If an $n$-parameter family of extremal curves – say (7) – is given, and one sets:

$$y_{i a_k}(x, a_1, \dots, a_n) = u_i^{(k)}, \qquad \lambda_{r a_k}(x, a_1, \dots, a_n) = \rho_r^{(k)}$$

then, as is known, the $u_i^{(k)}$, $\rho_r^{(k)}$ will satisfy the system of linear differential equations ([12]):

$$(27) \quad \begin{cases} \sum_{i=1}^{n}(F_{y_i y_j} u_i + F_{y_i' y_j} u_i') - \frac{d}{dx}\sum_{i=1}^{n}(F_{y_i y_j'} u_i + F_{y_i' y_j'} u_i') \\ \qquad + \sum_{r=1}^{m} \varphi_{r y_j} \rho_r - \frac{d}{dx}\sum_{r=1}^{m} \varphi_{r y_j'} \rho_r = 0 & (j = 1, 2, \dots, n), \\ \qquad \sum_{i=1}^{n}(\varphi_{s y_j} u_i + \varphi_{s y_j'} u_i') = 0 & (s = 1, 2, \dots, m) \end{cases}$$

that ESCHERICH referred to as the *accessory system of differential equations* ([13]).

ESCHERICH called a system of $n$ linearly-independent solutions $u^{(\mu)}$, $\rho^{(\mu)}$ ($\mu = 1, 2, \dots, n$) to the accessory system of equations that satisfies the $n (n - 1) / 2$ conditions:

$$(28) \quad \begin{cases} \sum_{i,k=1}^{n}[F_{y_i' y_k}(u_i^{(\mu)} u_k^{(\nu)} - u_i^{(\nu)} u_k^{(\mu)}) + F_{y_i' y_k'}(u_i^{(\mu)} u_k^{(\nu)\prime} - u_i^{(\nu)} u_k^{(\mu)\prime})] \\ + \sum_{i=1}^{n}\sum_{r=1}^{m} \varphi_{r y_i'}(u_i^{(\mu)} \rho_r^{(\nu)} - u_i^{(\nu)} \rho_r^{(\mu)}) = \psi(u^{(\mu)}, \rho^{(\mu)}; u^{(\nu)}, \rho^{(\nu)}) = 0 \quad (\mu, \nu = 1, 2, \dots, n) \end{cases}$$

a *conjugate system* ([14]). We can then expression our results above thusly:

*In order for a field that is constructed from an n-parameter family of extremals to be a MAYER field, it is necessary and sufficient that the solutions of the accessory system of equations that emerge upon differentiating with respect to the parameters should define a conjugate system* ([15]).

*At every location $x_0$ there is a conjugate system of solutions to* (27) *whose determinant:*

$$\left| u_i^{(\mu)} \right| \qquad (\mu, i = 1, 2, \dots, n)$$

---

([12]) That system of differential equations depends upon the $n$ parameters $a_1, a_2, \dots, a_n$ .

([13]) *Loc. cit.* ([2]), **107** (1898), pp. 1236.

([14]) *Loc. cit.* ([2]), **107** (1898), pp. 1246. The fact that the expression $\psi(u, \rho; \bar{u}, \bar{\rho})$ reduces to a constant for any two solutions of (27) is nothing but the theorem that was proved at the end of § **1**.

([15]) And indeed for all values of the parameters $a_1, a_2, \dots, a_n$ .

*does not vanish there* (16).

## § 3.

Let a MAYER field be given by:

(29) $$y_i = y_i\,(x, a_1, \ldots, a_n)\,, \qquad \lambda_r = \lambda_r\,(x, a_1, \ldots, a_n)\,.$$

Let its slope functions and multipliers be:

$$p_i\,(x, y_1, \ldots, y_n)\,, \qquad l_r\,(x, y_1, \ldots, y_n)\,.$$

The arc $(A, B)$ of the extremal that is defined by $a_1^0, \ldots, a_n^0$ lies completely inside the field. We can then set, for brevity:

(30) $$y_i(x, a_1^0, \ldots, a_n^0) = y_i\,(x)\,, \qquad \lambda_r(x, a_1^0, \ldots, a_n^0) = \lambda_r\,(x)\,.$$

Since the integral (17) is path-independent, its integral will be a complete differential – say, the differential of $V\,(x, y_1, \ldots, y_n)$ – such that we will have:

(31) $$\begin{cases} V_x(x, y_1, \ldots, y_n) = F\,[x, y, p\,(x, y), l\,(x, y)] - \sum_{i=1}^{n} p_i(x, y)\, F_{y_i'}\,[x, y, p\,(x, y), l\,(x, y)]\,, \\ V_{y_i}(x, y_1, \ldots, y_n) = F_{y_i'}\,[x, y, p\,(x, y), l\,(x, y)]\,. \end{cases}$$

We further set, in the known way:

$$E\,(x, y, p, l, \bar{y}') = F\,(x, y, \bar{y}', l) - F\,(x, y, p, l) - \sum_{i=1}^{n} (\bar{y}_i' - p_i)\, F_{y_i'}(x, y, p, l)\;.$$

If an arbitrary curve that satisfies the condition equations (2) and runs inside of the field is given by:

$$y_i = \bar{y}_i\,(x) \qquad (i = 1, 2, \ldots, n),$$

and $x_0$ and $x$ belong to the interval $(A, B)$ then it will follow from the path-independence of the integral (17) with no further discussion that:

---

(16) See., ESCHERICH, *loc. cit.* (2), **107** (1898), pp. 1324.

$$(32)\quad \left\{\begin{aligned} &\int_{x_0}^{x} f[x, \bar{y}(x), \bar{y}'(x)]\,dx - \int_{x_0}^{x} f[x, y(x), y'(x)]\,dx \\ = &\int_{x_0}^{x} E\{x, \bar{y}(x), p[x, \bar{y}'(x)], l[x, \bar{y}(x)], \bar{y}'(x)\}\,dx \\ &+\{V[x, \bar{y}_1(x), \ldots, \bar{y}_n(x)] - V[x, y_1(x), \ldots, y_n(x)]\} \\ &-\{V[x_0, \bar{y}_1(x_0), \ldots, \bar{y}_n(x_0)] - V[x_0, y_1(x_0), \ldots, y_n(x_0)]\}, \end{aligned}\right.$$

and if one lets $\bar{\lambda}_r(x)$ ($r = 1, 2, \ldots, m$) denote entirely-arbitrary functions then one can also replace the left-hand side of that equation with:

$$\int_{x_0}^{x} \{F[x, \bar{y}(x), \bar{y}'(x), \bar{\lambda}(x)]\,dx - F[x, y(x), y'(x), \lambda(x)],$$

and that expression will be further equal to:

$$\begin{aligned} \int_{x_0}^{x} \sum_{i=1}^{n} \{F_{y_i}[x, y(x), y'(x), \lambda(x)] - \frac{d}{dx} F_{y_i'}[x, y(x), y'(x), \lambda(x)]\}[\bar{y}_i(x) - y_i(x)]\,dx \\ + \sum_{i=1}^{n} F_{y_i'}[x, y(x), y'(x), \lambda(x)]\,[\bar{y}_i(x) - y_i(x)] \\ - \sum_{i=1}^{n} F_{y_i'}[x_0, y(x_0), y'(x_0), \lambda(x_0)]\,[\bar{y}_i(x_0) - y_i(x_0)]\ , \end{aligned}$$

up to second-order terms in the differences $(\bar{y}_i - y_i)$ and $(\bar{y}_i' - y_i')$, and the integral in that will be equal to zero, due to the existence of equations (5). On the right-hand side of (32), one has:

$$V(x, \bar{y}_1, \ldots, \bar{y}_n) - V(x, y_1, \ldots, y_n) = \sum_{i=1}^{n} V_{y_i}(x, y_1, \ldots, y_n)(\bar{y}_i - y_i) + [\bar{y} - y]_2\ ,$$

and when one observes equations (31), one will see that the first-order terms on the left-hand side of (32) precisely cancel the first-order terms on the right-hand side outside the differences inside of the integral sign. If one writes out the second-order terms on the left-hand side, as well as the right outside of the integral sign, explicitly then one will get:

(33)
$$\left\{\begin{aligned}
&\tfrac{1}{2}\int_{x_0}^{x}\Big\{\sum_{i,k=1}^{n}[F_{y_i y_k}(x,y,y',\lambda)(\bar{y}_i - y_i)(\bar{y}_k - y_k)\\
&\qquad + 2F_{y_i' y_k}(x,y,y',\lambda)(\bar{y}_i' - y_i')(\bar{y}_k - y_k)\\
&\qquad + F_{y_i' y_k'}(x,y,y',\lambda)(\bar{y}_i' - y_i')(\bar{y}_k' - y_k')\\
&+2\sum_{i=1}^{n}\sum_{r=1}^{m}\Big[\varphi_{r y_i}(x,y,y')(\bar{y}_i - y_i) + \varphi_{r y_i'}(x,y,y')(\bar{y}_i' - y_i')\Big](\bar{\lambda}_r - \lambda_r)\Big\}\,dx\\
&\qquad = \int_{x_0}^{x} E[x, y, p(x,\bar{y}), l(x,\bar{y}), \bar{y}']\,dx\\
&\quad + \tfrac{1}{2}\sum_{i,k=1}^{n} V_{y_i y_k}(x,y)(\bar{y}_i - y_i)(\bar{y}_k - y_k)\Big]_{x_0}^{x} + [\bar{y}_i - y_i, \bar{y}_i' - y_i'] .
\end{aligned}\right.$$

We now set:

(34)
$$\left\{\begin{aligned}
\bar{y}_i(x) &= y_i(x) + \varepsilon\,\eta_i(x) + [\varepsilon]_2 \qquad (i = 1, 2, \ldots, n),\\
\bar{\lambda}_r(x) &= \lambda_r(x) + \varepsilon\,\sigma_r(x) + [\varepsilon]_2 \qquad (r = 1, 2, \ldots, m),
\end{aligned}\right.$$

in which $\varepsilon$ means a parameter.

If one considers the fact that the $\bar{y}_i(x)$ should satisfy the condition equations (2) then one will get:

(35) $$\sum_{i=1}^{n}\{\varphi_{r y_i}[x, y(x), y'(x)]\,\eta_i(x) + \varphi_{r y_i'}[x, y(x), y'(x)]\,\eta_i'(x)\} = 0 \qquad (r = 1, 2, \ldots, m) .$$

No terms of order one in $\varepsilon$ occur in (33). We would now like to set the second-order terms in $\varepsilon$ on the left and right equal to each other.

If we set:

$$\begin{aligned}
&\sum_{i,k=1}^{n}[F_{y_i y_k}(x,y,y',\lambda)\,\eta_i\,\eta_k + 2F_{y_i' y_k}(x,y,y',\lambda)\,\eta_i'\,\eta_k + F_{y_i' y_k'}(x,y,y',\lambda)\,\eta_i'\,\eta_k']\\
&\qquad + 2\sum_{i=1}^{n}\sum_{r=1}^{m}[\varphi_{r y_i}(x,y,y')\,\eta_i + \varphi_{r y_i'}(x,y,y')\,\eta_i']\,\sigma_r = 2\Omega(\eta,\eta',\sigma) ,
\end{aligned}$$

to abbreviate, then when we consider (35), we will get:

$$\int_{x_0}^{x}\sum_{i=1}^{n}[\Omega_{\eta_i}(\eta,\eta',\sigma)\,\eta_i + \Omega_{\eta_i'}(\eta,\eta',\sigma)\,\eta_i']\,dx$$

as the coefficient of $\varepsilon^2/2$ on the left. By partial integration, in the known way, we will bring that into the form:

$$\int_{x_0}^{x}\sum_{i=1}^{n}\left[\Omega_{\eta_i}(\eta,\eta',\sigma)\,\eta_i - \frac{d}{dx}\Omega_{\eta_i'}(\eta,\eta',\sigma)\right]\eta_i\,dx + \sum_{i=1}^{n}\Omega_{\eta_i'}(\eta,\eta',\sigma)\,\eta_i\Big]_{x_0}^{x} .$$

That expression is nothing but the so-called *second variation*.

We then go on to calculate the coefficient of $\varepsilon^2 / 2$ in the difference that stands outside of the integral sign on the right-hand side of (33). If we set (in order to be consistent with the CLEBSCH notation):

$$V_{y_i y_k}[x, y_1(x), \dots, y_n(x)] = \beta_{ik}(x) \qquad (i, k = 1, 2, \dots, n)$$

then that coefficient will read:

$$\sum_{i,k=1}^{n} \beta_{ik}(x)\,\eta_i(x)\,\eta_k(x) .$$

Now, when one considers (31), one will have:

$$V_{y_i y_k}(x, y_1, \dots, y_n) = \frac{\partial}{\partial y_k} F_{y_i'}[x, y, p(x, y), l(x, y)] ,$$

or when one observes the meanings of $p(x, y)$ and $l(x, y)$ according to their definitions (10) and (11), resp.:

$$V_{y_i y_k}(x, y_1, \dots, y_n) = \frac{\partial}{\partial y_k} F_{y_i'}[x, y(x, a), y'(x, a), \lambda(x, a)] ,$$

in which the $a_i$ are replaced with the functions (9): $a_i(x, y_1, \dots, y_n)$. However, one will then have:

$$\frac{\partial}{\partial y_k} = \sum_{\nu=1}^{n} a_{\nu y_k}(x, y_1, \dots, y_n)\frac{\partial}{\partial a_\nu} .$$

The expressions $a_{\nu y_k}(x, y_1, \dots, y_n)$ are easily determined in the following way: One partially differentiates the identities:

$$y_i = y_i[x, a_1(x, y_1, \dots, y_n), \dots, a_n(x, y_1, \dots, y_n)] \qquad (i = 1, 2, \dots, n)$$

with respect to $y_k$ . That will give:

$$\sum_{\nu=1}^{n} y_{i a_\nu}(x, a_1, \dots, a_n)\, a_{\nu y_k}(x, y_1, \dots, y_n) = \varepsilon_{ik} ,$$

in which $\varepsilon_{ik} = 0$ or 1 according to whether $i \neq k$ or $i = k$, resp. If one sets ([17]):

([17]) That notation does not overlap completely with the one that was employed at the end of § **2**, insofar as the expressions that are denoted by $u^{(\nu)}$, $\rho^{(\nu)}$ here will first arise from the ones that were denoted that way there by specializing the parameters $a_1, \dots, a_n$ to $\overset{0}{a}_1, \dots, \overset{0}{a}_n$ .

(36) $$y_{i a_\nu}(x, a_1^0, \dots, a_n^0) = u_i^{(\nu)}(x), \quad \lambda_{r a_\nu}(x, a_1^0, \dots, a_n^0) = \rho_r^{(\nu)}(x),$$

and one further sets:

$$\Delta = |u_i^{(\nu)}(x)| \qquad (i, \nu = 1, 2, \dots, n),$$

and one denotes the subdeterminant of the determinant that belongs to the element $u_i^{(\nu)}(x)$ by $\Delta_i^{(\nu)}$ then one will have:

(37) $$\begin{cases} \beta_{ik}(x) & = \displaystyle\sum_{\nu=1}^{n} \frac{\partial}{\partial a_\nu} F_{y_i'}[x, y(x, a^0), y_x(x, a^0), \lambda(x, a^0)] \frac{\Delta_k^{(\nu)}}{\Delta} \\ & = \displaystyle\sum_{\nu=1}^{n} \left\{ \sum_{\mu=1}^{n} \{F_{y_i' y_\mu}[x, y(x), y'(x), \lambda(x)] u_\mu^{(\nu)}(x) \right. \\ & \left. + F_{y_i' y_\mu'}[x, y(x), y'(x), \lambda(x)] u_\mu^{(\nu)\prime}(x)\} + \displaystyle\sum_{r=1}^{m} \varphi_{r y_i'}[x, y(x), y'(x), \lambda(x)] \rho_r^{(\nu)}(x) \right\} \frac{\Delta_k^{(\nu)}}{\Delta} \end{cases}$$

The coefficients of $\varepsilon^2 / 2$ in the integral that appears on the right in (33) still remain to be determined. It is known that:

$$E(x, y, p, l, \bar{y}') = \tfrac{1}{2} \sum_{i,k=1}^{n} F_{y_i' y_k'}(x, y, p, l)(\bar{y}_i' - p_i)(\bar{y}_k' - p_k) + [\bar{y}' - p]_3 .$$

We set:

(38) $$p_i[x, \bar{y}_1(x), \dots, \bar{y}_n(x)] = a_i(x, \varepsilon) \qquad (i = 1, 2, \dots, n),$$

in which the $a_i$ are the expressions (9). That gives:

$$p_i[x, \bar{y}_1(x), \dots, \bar{y}_n(x)] = y_{i\,x}[x, a_1(x, \varepsilon), \dots, a_n(x, \varepsilon)] \qquad (i = 1, 2, \dots, n),$$

and when one develops the right-hand side in $\varepsilon$ and recalls (36):

$$p_i[x, \bar{y}_1(x), \dots, \bar{y}_n(x)] = y_i'(x) + \varepsilon \sum_{k=1}^{n} u_i^{(k)\prime}(x)\, a_{k\varepsilon}(x, 0) + [\varepsilon]_2 .$$

If we introduce the abbreviated notation for $a_{k\varepsilon}(x, 0)$:

$$a_{k\varepsilon}(x, 0) = \delta_k(x) \qquad (k = 1, 2, \dots, n)$$

then we will have:

(39) $$\bar{y}_i'(x) - p_i[x, \bar{y}_1(x), \ldots, \bar{y}_n(x)] = \varepsilon\left[\eta_i'(x) - \sum_{k=1}^{n} u_i^{(k)'}(x)\,\delta_k(x)\right] + [\varepsilon]_2 \qquad (i = 1, 2, \ldots, n).$$

There is a much simpler expression for the $\delta_k(x)$: (38) implies the identity:

$$\bar{y}_i'(x) = y_i[x, a_1(x, \varepsilon), \ldots, a_n(x, \varepsilon)] \qquad (i = 1, 2, \ldots, n),$$

and it will follow upon differentiation with respect to $\varepsilon$ for $\varepsilon = 0$ that:

(40) $$\eta_i(x) = \sum_{k=1}^{n} u_i^{(k)}(x)\,a_{k\varepsilon}(x, 0) = \sum_{k=1}^{n} u_i^{(k)}(x)\,\delta_k(x).$$

One then has simply:

(41) $$\delta_k(x) = \frac{1}{\Delta}\sum_{\nu=1}^{n} \eta_\nu(x)\,\Delta_\nu^{(k)}.$$

Finally, if we, with ESCHERICH, set:

(42) $$\zeta_i(x) = \eta_i'(x) - \sum_{k=1}^{n} u_i^{(k)'}(x)\,\delta_k(x) \qquad (i = 1, 2, \ldots, n)$$

then the desired coefficient of $\varepsilon^2 / 2$ in the first term on the right-hand side of (33) will be:

$$\sum_{i,k=1}^{n} F_{y_i' y_k'}[x, y(x), y'(x), \lambda(x)]\,\zeta_i(x)\,\zeta_k(x).$$

If one observes that this satisfies the condition equations (2) then, due to (39) and (42), one will have:

$$\varphi_r\{x, \bar{y}(x), p[x, \bar{y}(x)] + \varepsilon\,\zeta(x) + [\varepsilon]_2\} = 0 \qquad (r = 1, 2, \ldots, m)$$

or

$$\varphi_r\{x, \bar{y}(x), p[x, \bar{y}(x)]\} + \varepsilon\sum_{k=1}^{n} \varphi_{r\,y_k'}\{x, \bar{y}(x), p[x, \bar{y}(x)]\}\,\zeta_k(x) + [\varepsilon]_2\} = 0.$$

Now, since:

$$\varphi_r\{x, \bar{y}(x), p[x, \bar{y}(x)]\} = 0,$$

after dividing by $\varepsilon$ and setting $\varepsilon = 0$, one will get:

(44) $$\sum_{k=1}^{n} \varphi_{r\,y_k'}[x, y(x), y'(x)]\,\zeta_k(x) = 0 \qquad (r = 1, 2, \ldots, m).$$

If we combine all of the foregoing then we will get the following result:

*One has the following relation for arbitrary functions* $\eta(x)$ *that satisfy the condition equations* (35):

$$(45)\qquad \begin{cases} \int_{x_0}^{x}\sum_{i=1}^{n}\left[\Omega_{\eta_i}(\eta,\eta',\sigma)\,\eta_i - \frac{d}{dx}\Omega_{\eta'_i}(\eta,\eta',\sigma)\right]\eta_i\,dx + \sum_{i=1}^{n}\left[\Omega_{\eta'_i}(\eta,\eta',\sigma)\,\eta_i\right]_{x_0}^{x} \\ = \int_{x_0}^{x}\sum_{i=1}^{n} F_{y'_i y'_k}[x, y(x), y'(x), \lambda(x)]\,\zeta_i\,\zeta_k\,dx + \sum_{i,k=1}^{n}\beta_{ik}\,\eta_i\,\eta_k\Big]_{x_0}^{x} \end{cases}$$

The $\zeta_i$ in that are defined by (43) and satisfy equations (44), and the $\beta_{ik}$ are defined by equations (37). The expressions $u^{(k)}$, $\rho^{(k)}$ ($k$ = 1, 2, …, $n$) that appear in the definitions of the $\zeta_i$ and $\beta_{ik}$ define a conjugate system of solutions to the accessory system of differential equations that correspond to the extremal (30) ([18])[which arise from (27) for the values $a_1^0$, …, $a_n^0$ of the parameters $a_1$, …, $a_n$]. The formula (45) is nothing but the *transformation of the second variation* that was given by CLEBSCH ([19]).

If we consider:

$$\Omega_{\eta'_i}(\eta,\eta',\sigma) = \sum_{k=1}^{n}\{F_{y'_i y_k}[x, y(x), y'(x), \lambda(x)]\,\eta_k + F_{y'_i y'_k}[x, y(x), y'(x), \lambda(x)]\,\eta'_k\}$$
$$+\sum_{r=1}^{m}\varphi_{r\,y'_i}[x, y(x), y'(x)]\,\sigma_r\ ,$$

as well as formula (40), then we will have:

$$(46)\qquad \begin{cases} \sum_{i=1}^{n}\Omega_{\eta'_i}(\eta,\eta',\sigma)\,\eta_k \\ = \sum_{\nu=1}^{n}\delta_\nu\left[\sum_{i,k=1}^{n}(F_{y'_i y_k}u_i^{(\nu)}\,\eta_k + F_{y'_i y'_k}u_i^{(\nu)}\,\eta'_k) + \sum_{i=1}^{n}\sum_{r=1}^{m}\varphi_{r\,y'_i}\,u_i^{(\nu)}\,\sigma_k\right]. \end{cases}$$

On the other hand, upon substituting the values (37) for $\beta_{ik}$ and considering (41), we will get:

$$(47)\qquad \sum_{i,k=1}^{n}\beta_{ik}\,\eta_i\,\eta_k = \sum_{\nu=1}^{n}\delta_\nu\left[\sum_{i,k=1}^{n}(F_{y'_i y_k}\eta_i\,u_k^{(\nu)} + F_{y'_i y'_k}\,\eta_i\,u_k^{(\nu)\prime}) + \sum_{i=1}^{n}\sum_{r=1}^{m}\varphi_{r\,y'_i}\eta_k\,\rho_i^{(\nu)}\right].$$

If we differentiate (45) with respect to $x$ and observe (46) and (47) then we will recognize that following equation ([20]), which is valid for all $\eta$ that satisfy the conditions (35):

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([18]) The fact that any arbitrary conjugate system of solutions to the stated equations can be chosen for the $u^{(k)}$, $\rho^{(k)}$ emerges from the arguments of § **5**.

([19]) *Loc. cit.* ([1]).

([20]) See ESCHERICH, *loc. cit.* ([2]), **107** (1898), pp. 1249.

(48)
$$\left\{\begin{aligned} &\sum_{i=1}^{n}\left[\Omega_{\eta_i}(\eta,\eta',\sigma)-\frac{d}{dx}\Omega_{\eta_i'}(\eta,\eta',\sigma)\right]\eta_i(x) \\ = \;&\sum_{i,k=1}^{n}F_{y_i' y_k'}[x,y(x),y'(x),\lambda(x)]\zeta_i(x)\zeta_k(x) \\ &-\frac{d}{dx}\sum_{\nu=1}^{n}\delta_\nu(x)\psi[u^{(\nu)},\rho^{(\nu)}(x);\eta(x),\sigma(x)], \end{aligned}\right.$$

in which we set:

$$\begin{aligned}\psi(u,\rho;\,\eta,\sigma) = &\sum_{i,k=1}^{n}\{F_{y_i' y_k'}[x,y(x),y'(x),\lambda(x)](u_i\,\eta_k-\eta_i\,u_k) \\ &\quad + F_{y_i' y_k'}[x,y(x),y'(x),\lambda(x)](u_i\,\eta_k'-\eta_i\,u_k')\} \\ &+\sum_{i=1}^{n}\sum_{r=1}^{m}\varphi_{r\,y_k'}[x,y(x),y'(x),\lambda(x)](u_i\,\sigma_r-\eta_i\,\rho_k)\},\end{aligned}$$

to abbreviate. Using formula (48), it can now be proved, with no further difficulties ([21]), that when we are not dealing with the so-called exceptional case, the extremals that start from a fixed point of the extremal (30) will always define a field that surrounds the extremal (30), assuming that the quadratic form:

$$\sum_{i,k=1}^{n}F_{y_i' y_k'}[x,y(x),y'(x),\lambda(x)]\,\zeta_i\,\zeta_k$$

is definite under the auxiliary conditions:

$$\sum_{i=1}^{n}\varphi_{r\,y_i'}[x,y(x),y'(x),\lambda(x)]\,\zeta_i = 0 \qquad (r = 1, 2, \ldots, m).$$

That formula (again, except for the so-called exceptional case) also allows us to exhibit the non-existence of an extremum for extremal arcs that reach from the starting point to a conjugate point, and without exception ([22]).

## § 4.

If a regular extremal is given by:

(49)
$$\left\{\begin{aligned} y_i &= y_i(x) &\qquad& (i = 1, 2 \ldots, n), \\ \lambda_r &= \lambda_r(x) && (r = 1, 2, \ldots, m) \end{aligned}\right.$$

for $x = x_0$, and one sets:

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([21]) ESCHERICH, *loc. cit.* ([2]), **110** (1901), pp. 1404.
([22]) ESCHERICH, *loc. cit.* ([2]), **110** (1901), pp. 1411.

$$y_i^{00} = y_i(x_0), \quad y_i^{00\prime} = y_i'(x_0) \qquad (i = 1, 2 \dots, n),$$
$$\lambda_i^{00} = \lambda_r(x_0) \qquad (r = 1, 2, \dots, m)$$

for its initial values, then every system of values:

$$y_i^0, \quad y_i^{0\prime} \qquad (i = 1,2 \dots, n),$$
$$\lambda_r^0 \qquad (r = 1, 2, \dots, m)$$

that are sufficiently close to those values will yield one and only one regular extremal as long as the equations:

$$\varphi_r(x_0, y_0, y_0') = 0 \qquad (r = 1, 2, \dots, m)$$

are satisfied with the initial values for $x = x_0$. Indeed, one will obtain all of the extremals in a neighborhood of the extremal (49) in that way.

We would now like to refer to the expressions $F_{y_i'}(x_0, y^0, y^{0\prime}, \lambda^0)$ as the *momenta with respect to the coordinate directions* and set:

$$w_i^0 = F_{y_i'}(x_0, y^0, y^{0\prime}, \lambda^0) \qquad (i = 1, 2, \dots, n),$$
$$w_i^{00} = F_{y_i'}(x_0, y^{00}, y^{00\prime}, \lambda^{00}) \qquad (i = 1, 2, \dots, n),$$

to abbreviate. An extremal that is regular for $x = x_0$ is then determined uniquely by the initial values of its coordinates: $y_i^0$ $(i = 1, 2, \dots, n)$ and its momenta with respect to the coordinate directions: $w_i^0$ $(i = 1, 2, \dots, n)$, and indeed one will get all extremals in a neighborhood of the extremal (49) when one lets the quantities $y_i^0$, $w_i^0$ $(i = 1, 2, \dots, n)$ run through all of the values in a neighborhood of the quantities $y_i^{00}$, $w_i^{00}$. The proof of that is obtained from the fact that is a system of quantities $y_i^0$, $w_i^0$ is given then the associated $y_i^{0\prime}$, $\lambda_0$ will be determined from the equations:

$$\text{(50)} \qquad \begin{cases} F_{y_i'}(x_0, y^0, y^{0\prime}, \lambda^0) = w_i^0 & (i = 1, 2, \dots, n) \\ \varphi_r(x_0, y^0, y^{0\prime}) = 0 & (r = 1, 2, \dots, m), \end{cases}$$

whose functional determinant with respect to the $y^{0\prime}$ and $\lambda_0$ is nothing but the determinant (6), and it is therefore nonzero at the location $y_i^{00}$, $y_i^{00\prime}$, $\lambda_r^{00}$, $w_i^{00}$. Hence, the equations (50) are uniquely soluble for the $y_i^{0\prime}$, $\lambda_r^0$ in a neighborhood of $y_i^{00}$, $y_i^{00\prime}$, $\lambda_r^{00}$, $w_i^{00}$ ([23]).

Let an orthogonal transformation of the $y$ be given by:

([23]) A precise presentation of that proof is in O. BOLZA, *loc. cit.* ([11]).

(51) $$Y_i = \sum_{k=1}^{n} a_k^{(i)} y_k\,, \qquad y_i = \sum_{k=1}^{n} a_k^{(i)} Y_k \qquad (i = 1, 2, \ldots, n).$$

One will have:

$$F(x, y, y', \lambda) = G(x, Y, Y', \lambda)\,.$$

One gets from this upon differentiating with respect to $Y_i'$:

$$G_{Y_i'}(x, Y, Y', \lambda) = \sum_{k=1}^{n} a_k^{(i)} F_{y_k'}(x, y, y', \lambda)\,.$$

We will then refer to the expression:

$$\sum_{k=1}^{n} a_k^{(i)} F_{y_k'}(x, y, y', \lambda)$$

as the *momentum with respect to the direction of the* $Y_i$*-axis*. We shall denote the initial values of that momentum for $x = x_0$ by $W^0$ ($i = 1, 2, \ldots, n$), and the initial values of the new coordinates $Y_i$ by $Y_i^0$. Obviously, one can also employ the $y_i^0$ and $w_i^0$ as initial values, as above, as well as the $Y_i^0$ and $W_i^0$.

As was mentioned at the conclusion of § **2**, every extremal belongs to an accessory system of differential equations. The accessory system that belongs to the extremal (49) reads:

(52) $$\begin{cases} \displaystyle\sum_{k=1}^{n} \left[ F_{y_i y_k} z_k + F_{y_i y_k'} z_k' - \frac{d}{dx}(F_{y_i' y_k} z_k + F_{y_i' y_k'} z_k') \right] \\ \qquad\qquad \displaystyle + \sum_{r=1}^{m} \left[ \varphi_{r y_i} \sigma_r - \frac{d}{dx}(\varphi_{r y_i'} \sigma_r) \right] = 0 \qquad (i = 1, 2, \ldots, n), \\ \qquad\qquad \displaystyle\sum_{k=1}^{n} (\varphi_{r y_k} z_k + \varphi_{r y_k'} z_k') = 0 \qquad (r = 1, 2, \ldots, m), \end{cases}$$

in which, as will always be the case in what follows, the $y$, $y'$, $\lambda$ in $F_{y_i y_k}$, $F_{y_i' y_k}$, $F_{y_i y_k'}$, $\varphi_{r y_k}$, $\varphi_{r y_k'}$ have been replaced with the functions (49): $y(x)$, $y'(x)$, $\lambda(x)$.

That system of equations possesses $2n$ linearly-independent solutions, in terms of which all of the remaining ones can be expressed linearly with constant coefficients. One will get such a system of fundamental solutions in the following way:

Let:

(53) $$\begin{cases} y_i = y_i(x, y_i^0, \ldots, y_n^0, w_1^0, \ldots, w_n^0) & (i = 1, 2, \ldots, n) \\ \lambda_r = \lambda_r(x, y_i^0, \ldots, y_n^0, w_1^0, \ldots, w_n^0) & (r = 1, 2, \ldots, m) \end{cases}$$

be the family of extremals, when expressed in terms of the initial values $y_i^0$ and $w_i^0$. The expressions:

(54) $$\begin{cases} z_i^{(k)}(x) = y_{i\,y_k^0}(x, y_i^{00}, \ldots, y_n^{00}, w_1^{00}, \ldots, w_n^{00}) & (i = 1, 2, \ldots, n) \\ \sigma_i^{(k)}(x) = \lambda_{r\,y_k^0}(x, y_i^{00}, \ldots, y_n^{00}, w_1^{00}, \ldots, w_n^{00}) & (r = 1, 2, \ldots, m) \end{cases}$$

for $k = 1, 2, \ldots, n$, together with the expressions:

(54′) $$\begin{cases} z_i^{(n+k)}(x) = y_{i\,w_k^0}(x, y_i^{00}, \ldots, y_n^{00}, w_1^{00}, \ldots, w_n^{00}) & (i = 1, 2, \ldots, n) \\ \sigma_i^{(n+k)}(x) = \lambda_{r\,w_k^0}(x, y_i^{00}, \ldots, y_n^{00}, w_1^{00}, \ldots, w_n^{00}) & (r = 1, 2, \ldots, m) \end{cases}$$

for $k = 1, 2, \ldots, n$, define a fundamental system. One gets that easily by the following argument:

A solution of (52) is determined completely by its initial values $z_i^0$, $z_i^{0'}$, $\sigma_r^0$ for $x = x_0$. On the other hand, if one prescribes those initial values arbitrarily such that they satisfy the $m$ conditions:

$$\sum_{k=1}^{n} [\varphi_{r\,y_k}(x_0, y^{00}, y^{00'})\, z_k^0 + \varphi_{r\,y_k'}(x_0, y^{00}, y^{00'})\, z_k^{0'}] = 0 \qquad (r = 1, 2, \ldots, m)$$

then one will get all solutions of (52). If one sets:

(55) $$v_i = \sum_{k=1}^{n} (F_{y_i' y_k}\, z_k + F_{y_i' y_k'}\, z_k') + \sum_{r=1}^{m} \varphi_{r\,y_i'}\, \sigma_r$$

and one denotes the values of the $v_i$ for $x = x_0$ by $v_i^0$ then one can obviously now give the $z_i^0$, $v_i^0$ ($i = 1, 2, \ldots, n$) in place of the $z_i^0$, $z_i^{0'}$, $\lambda_r^0$, but they are completely arbitrary. One and only one solution to (52) is determined by every system of values $z_i^0$, $v_i^0$. If one further sets:

$$w_i(x, y^0, w^0) = F_{y_i'}[x, y(x, y^0, w^0), y_x(x, y^0, w^0), \lambda(x, y^0, w^0)] \qquad (i = 1, 2, \ldots, n)$$

and

$$v_i^{(k)}(x) = \sum_{\nu=1}^{n} (F_{y_i' y_\nu}\, z_\nu^{(k)} + F_{y_i' y_\nu'}\, z_\nu^{(k)'}) + \sum_{r=1}^{m} \varphi_{r\,y_i'}\, \sigma_r^{(k)}$$

then one will have precisely:

(56) $$\begin{cases} v_i^{(k)}(x) = w_{i\,y_k^0}(x, y^{00}, w^{00}) & (k = 1, 2, \ldots, n), \\ v_i^{(n+k)}(x) = w_{i\,w_k^0}(x, y^{00}, w^{00}) & (k = 1, 2, \ldots, n). \end{cases}$$

We now infer the initial values of the $z^{(k)}(x)$ and $v^{(k)}(x)$ from (54), (54′), (56):

$$(57)\begin{cases} z_k^{(k)}(x_0)=1 & (k\ \ =1,2,\dots,n), & v_k^{(n+k)}(x_0)=1 & (k\ \ =1,2,\dots,n), \\ z_k^{(i)}(x_0)=0 & (i,k=1,2,\dots,n; i\neq k), & v_k^{(n+i)}(x_0)=0 & (i,k=1,2,\dots,n; i\neq k), \\ z_k^{(n+i)}(x_0)=0 & (i,k=1,2,\dots,n), & v_k^{(i)}(x_0)=0 & (i,k=1,2,\dots,n). \end{cases}$$

We are now in a position to put an arbitrary solution $z_i(x)$, $\sigma_r(x)$ of (52) in the form:

$$(58)\qquad \begin{cases} z_i(x)=\sum_{k=1}^{2n} c_k\, z_i^{(k)}(x) & (i=1,2,\dots,n) \\ \sigma_i(x)=\sum_{k=1}^{2n} c_k\, \sigma_i^{(k)}(x) & (r=1,2,\dots,m) \end{cases}$$

with no further analysis. We must only observe that one must also have:

$$v_i(x)=\sum_{k=1}^{2n} c_k\, z_i^{(k)}(x) \qquad (i=1,2,\dots,n)$$

then, such that when we consider (57), we will immediately have:

$$c_k = z_k(x_0)\,, \qquad c_{n+k} = v_k(x_0) \qquad (k = 1, 2, \dots, n).$$

Any solution of (52) can then, in fact, be expressed in terms of our $2n$ special solutions.

If we introduce new coordinates by the orthogonal transformation (51) then we can, as we said, also write the family of extremals in the form:

$$(59)\qquad \begin{cases} y_i = y_i^0(x, Y_1^0,\dots,Y_n^0; W_1^0,\dots,W_n^0) & (i\ =1,2,\dots,n), \\ \lambda_r = \lambda_r^0(x, Y_1^0,\dots,Y_n^0; W_1^0,\dots,W_n^0) & (r=1,2,\dots,m), \end{cases}$$

in which the $Y_i^0$ mean the initial values of the new coordinates for $x = x_0$, and the $W^0$ mean the initial values of the momenta with respect to the new coordinate directions. The extremal (49) might correspond to the initial values $Y^{00}$, $W^{00}$. The expressions:

$$(60)\qquad \begin{cases} z_i^{\bullet(k)}(x) = y^{\bullet}_{iY_k^0}(x, Y_1^{00},\dots,Y_n^{00}; W_1^{00},\dots,W_n^{00}) & (i\ =1,2,\dots,n), \\ \sigma_r^{\bullet(k)}(x) = \lambda^{\bullet}_{rY_k^0}(x, Y_1^{00},\dots,Y_n^{00}; W_1^{00},\dots,W_n^{00}) & (r=1,2,\dots,m), \end{cases}$$

for $k = 1, 2, \dots, n$, together with the expressions:

$$(60') \qquad \begin{cases} z_i^{\bullet(n+k)}(x) = y^{\bullet}_{iW_k^0}(x, Y_1^{00}, \dots, Y_n^{00}; W_1^{00}, \dots, W_n^{00}) & (i = 1, 2, \dots, n), \\ \sigma_r^{\bullet(n+k)}(x) = \lambda^{\bullet}_{rW_k^0}(x, Y_1^{00}, \dots, Y_n^{00}; W_1^{00}, \dots, W_n^{00}) & (r = 1, 2, \dots, m), \end{cases}$$

for $k = 1, 2, \dots, n$, likewise define a fundamental system of (52). That is because if one observes that:

$$\frac{\partial}{\partial Y_k^0} = \sum_{\nu=1}^{n} a_\nu^{(k)} \frac{\partial}{\partial y_\nu^0}, \qquad \frac{\partial}{\partial y_k^0} = \sum_{\nu=1}^{n} a_\nu^{(k)} \frac{\partial}{\partial Y_\nu^0},$$

and likewise

$$\frac{\partial}{\partial W_k^0} = \sum_{\nu=1}^{n} a_\nu^{(k)} \frac{\partial}{\partial w_\nu^0}, \qquad \frac{\partial}{\partial w_k^0} = \sum_{\nu=1}^{n} a_\nu^{(k)} \frac{\partial}{\partial W_\nu^0},$$

then one will get:

$$(61) \qquad \begin{cases} z_i^{\bullet(k)}(x) = \sum_{\nu=1}^{n} a_\nu^{(k)} z_i^{(\nu)}(x), & \sigma_r^{\bullet(k)}(x) = \sum_{\nu=1}^{n} a_\nu^{(k)} \sigma_r^{(\nu)}(x) \qquad (k = 1, 2, \dots, n), \\ z_i^{\bullet(n+k)}(x) = \sum_{\nu=1}^{n} a_\nu^{(k)} z_i^{(n+\nu)}(x), & \sigma_r^{\bullet(n+k)}(x) = \sum_{\nu=1}^{n} a_\nu^{(k)} \sigma_r^{(n+\nu)}(x) \qquad (k = 1, 2, \dots, n), \end{cases}$$

and

$$z_i^{(k)}(x) = \sum_{\nu=1}^{n} a_\nu^{(k)} z_i^{\bullet(\nu)}(x), \qquad \sigma_r^{(k)}(x) = \sum_{\nu=1}^{n} a_\nu^{(k)} \sigma_r^{\bullet(\nu)}(x) \qquad (k = 1, 2, \dots, n),$$

$$z_i^{(n+k)}(x) = \sum_{\nu=1}^{n} a_\nu^{(k)} z_i^{\bullet(n+\nu)}(x), \qquad \sigma_r^{(n+k)}(x) = \sum_{\nu=1}^{n} a_\nu^{(k)} \sigma_r^{\bullet(n+\nu)}(x) \qquad (k = 1, 2, \dots, n).$$

Any solution $z_i$, $\sigma_r$ of (52) can also be written in the form:

$$z_i = \sum_{k=1}^{2n} C_k \, z_i^{\bullet(k)}, \qquad \sigma_r = \sum_{k=1}^{2n} C_k \, \sigma_r^{\bullet(k)}$$

then, and the following relations will exist between the constants $C_k$ and the $c_k$ in (58):

$$(62) \qquad \begin{cases} C_k = \sum_{\nu=1}^{n} c_\nu \, a_\nu^{(k)}, \quad C_{n+k} = \sum_{\nu=1}^{n} c_{n+\nu} \, a_\nu^{(k)}, & (k = 1, 2, \dots, n), \\ c_k = \sum_{\nu=1}^{n} C_\nu \, a_k^{(\nu)}, \quad c_{n+k} = \sum_{\nu=1}^{n} C_{n+\nu} \, a_k^{(\nu)}, & (k = 1, 2, \dots, n). \end{cases}$$

With ESCHERICH ([24]), we set:

([24]) *Loc. cit.* ([2]), **107** (1898), pp. 1244.

$$\sum_{i,k=1}^{n}[F_{y_i' y_k}(z_i\,\bar z_k - \bar z_i\,z_k) + F_{y_i' y_k'}(z_i\,\bar z_k' - \bar z_i\,z_k')] + \sum_{i=1}^{n}\sum_{r=1}^{m}\varphi_{r\,y_i'}(z_i\,\bar\sigma_r - \bar z_i\,\sigma_r)$$
$$= \psi(z,\sigma;\bar z,\bar\sigma) \ ,$$

as we already did before. If one defines $v_i$ by (55) and $\bar v_i$ by:

$$\bar v_i = \sum_{k=1}^{n}(F_{y_i' y_k}\bar z_k + F_{y_i' y_k'}\,\bar z_k')] + \sum_{r=1}^{m}\varphi_{r\,y_i'}\,\bar\sigma_r$$

then one will have:

$$\psi(z,\sigma;\bar z,\bar\sigma) = \sum_{i=1}^{n}(z_i\,\bar v_i - \bar z_i\,v_i) \ .$$

As we have already mentioned, the expression $\psi(z,\sigma;\bar z,\bar\sigma)$ will reduce to a constant when $z_i$, $\sigma_r$, as well as $\bar z_i$, $\bar\sigma_r$ are solutions of (52). If it so happens that:

$$\psi(z,\sigma;\bar z,\bar\sigma) = 0$$

then those solutions will be called *conjugate* if they are linearly independent.

We infer from formulas (57), with no further discussion, that for our fundamental system: $\psi(z,\sigma;\bar z,\bar\sigma)$ *is always equal to zero, except when* $k = n + i$ ($i = n + k$, resp.), *and we will then have:*

$$\psi(z^{(i)},\sigma^{(i)};\bar z^{\,(n+i)},\bar\sigma^{\,(n+i)}) = 1 \qquad (i = 1, 2, \ldots, n).$$

Our fundamental system will then be one of the ones that ESCHERICH referred to as *involutory fundamental systems* ([25]).

However, the fundamental system of $z^{\bullet(k)}$, $\sigma^{\bullet(k)}$ is involutory, as we now easily convince ourselves: If we set:

$$v_i^{\bullet(\nu)} = \sum_{k=1}^{n}(F_{y_i' y_k} z_k^{\bullet(\nu)} + F_{y_i' y_k'} z_k^{\bullet(\nu)'}) + \sum_{r=1}^{m}\varphi_{r\,y_i'}\,\sigma_r^{\bullet(\nu)}$$

then:

$$v_i^{\bullet(\nu)} = \sum_{k=1}^{n} a_k^{(\nu)}\,v_i^{(k)} \ , \qquad v_i^{\bullet(n+\nu)} = \sum_{k=1}^{n} a_k^{(\nu)}\,v_i^{(n+k)} \ ,$$

and therefore:

$$\psi(z^{\bullet(\mu)},\sigma^{\bullet(\mu)},z^{\bullet(\nu)},\sigma^{\bullet(\nu)}) = \sum_{k,k'=1}^{n} a_k^{(\mu)}\,a_{k'}^{(\nu)}\sum_{i=1}^{n}(z_i^{(k)}\,v_i^{(k')} - z_i^{(k')}\,v_i^{(k)})$$

([25]) *Loc. cit.* (2), **107** (1898), pp. 1307.

$$= \sum_{k,k'=1}^{n} a_k^{(\mu)}\, a_{k'}^{(\nu)}\, \psi(z^{(k)}, \sigma^{(k)}; z^{(k')}, \sigma^{(k')}) = 0 \qquad (\mu,\ \nu = 1, 2, \ldots, n).$$

One likewise gets:

$$\psi(z^{\bullet(n+\mu)}, \sigma^{\bullet(n+\mu)}, z^{\bullet(n+\mu)}, \sigma^{\bullet(n+\mu)}) = 0 \qquad (\mu,\ \nu = 1, 2, \ldots, n).$$

Furthermore:

$$\psi(z^{\bullet(\mu)}, \sigma^{\bullet(\mu)}, z^{\bullet(n+\mu)}, \sigma^{\bullet(n+\mu)}) = \sum_{k,k'=1}^{n} a_k^{(\mu)}\, a_{k'}^{(\nu)} \sum_{i=1}^{n} (z_i^{(k)}\, v_i^{(n+k')} - z_i^{(n+k')}\, v_i^{(k)})$$

$$= \sum_{k,k'=1}^{n} a_k^{(\mu)}\, a_{k'}^{(\nu)}\, \psi(z^{(k)}, \sigma^{(k)}; z^{(n+k')}, \sigma^{(n+k')}) = \sum_{k=1}^{n} a_k^{(\mu)}\, a_k^{(\nu)},$$

and that expression is, equal to 0 or 1 according to whether $\mu \neq \nu$ or $\mu = \nu$, resp., due to the orthogonality of the transformation (51). The assertion is proved with that.

A system of $n$ linearly-independent solutions of (52):

$$\begin{array}{ll} u_1^{(i)}, \ldots, u_n^{(i)}; & \rho_1^{(i)}, \ldots, \rho_m^{(i)}; \\ \ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots \\ u_1^{(n)}, \ldots, u_n^{(n)}; & \rho_1^{(n)}, \ldots, \rho_m^{(n)} \end{array}$$

will be referred to as a *conjugate system* when the $n$ $(n - 1) / 2$ conditions exist:

$$\psi(u^{(i)}, \rho^{(i)}; u^{(k)}, \rho^{(k)}) = 0 \qquad (i, k = 1, 2, \ldots, n). \tag{63}$$

If one performs a linear transformation with non-vanishing determinant on a conjugate system, i.e., one sets:

$$\bar{u}_k^{(i)} = \sum_{\nu=1}^{n} b_\nu^{(i)}\, u_k^{(\nu)}, \qquad \bar{\rho}_r^{(i)} = \sum_{\nu=1}^{n} b_\nu^{(i)}\, \rho_r^{(\nu)},$$

then one will again obtain a conjugate system. We shall refer to all conjugate systems that can go to each other under such a linear transformation as a *group of conjugate systems* and pose the problem of giving one and only one representative of each such group.

We imagine that the individual solutions of our conjugate system are represented in terms of the fundamental system of the $z_i^{(k)},\ \rho_r^{(k)}$:

$$u_i^{(k)} = \sum_{\nu=1}^{2n} c_\nu^{(k)}\, z_i^{(\nu)}, \qquad \rho_r^{(k)} = \sum_{\nu=1}^{2n} c_\nu^{(k)}\, \sigma_r^{(\nu)}.$$

The systems is determined completely by the matrix:

$$\left\| c_\nu^{(k)} \right\| \qquad \begin{pmatrix} \nu = 1, 2, \ldots 2n \\ k = 1, 2, \ldots, n \end{pmatrix}.$$

We distinguish two types of conjugate systems according to whether the determinant:

$$\left| u_i^{(k)} \right| \qquad (i, k = 1, 2, \ldots, n)$$

is or is not equal to zero at the location $x_0$, resp. Since:

$$c_\nu^{(k)} = u_\nu^{(k)}(x_0) \qquad (\nu, k = 1, 2, \ldots, n),$$

that distinction is equivalent to distinguishing whether the determinant:

$$\left| c_i^{(k)} \right| \qquad (i, k = 1, 2, \ldots, n) \tag{64}$$

is not or is equal to zero. In the former case, we can always arrive at a matrix of the $c_i^{(k)}$ that assumes the form:

$$\begin{Vmatrix} 1 & 0 & \cdots & 0; & c_{n+1}^{(1)} & \cdots & c_{2n}^{(1)} \\ 0 & 1 & \cdots & 0; & c_{n+1}^{(2)} & \cdots & c_{2n}^{(2)} \\ \vdots & \vdots & \ddots & \vdots; & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 1; & c_{n+1}^{(n)} & \cdots & c_{2n}^{(n)} \end{Vmatrix} \tag{65}$$

by a linear transformation of the conjugate system. However, with a calculation that was performed before:

$$\begin{aligned} \psi\left( \sum_{k=1}^{2n} c_k^{(\mu)} z^{(k)}, \sum_{k=1}^{2n} c_k^{(\mu)} \sigma^{(k)}; \sum_{k=1}^{2n} c_k^{(\nu)} z^{(k)}, \sum_{k=1}^{2n} c_k^{(\nu)} \sigma^{(k)} \right) &= \sum_{k,k'=1}^{2n} c_k^{(\mu)} c_{k'}^{(\nu)} \psi(z^{(k)}, \sigma^{(k)}; z^{(k')}, \sigma^{(k')}) \\ &= \sum_{k=1}^{n} (c_k^{(\mu)} c_{n+k}^{(\nu)} - c_k^{(\nu)} c_{n+k}^{(\mu)}) = c_{\mu+n}^{(\nu)} - c_{\nu+n}^{(\mu)}. \end{aligned}$$

The system that the matrix (65) defines will then be a conjugate one if and only if *the determinant:*

$$\left| c_{n+k}^{(i)} \right| \qquad (i, k = 1, 2, \ldots, n)$$

*is symmetric*. One and only one representative of each group of conjugate systems of the first kind is given by that. That result was derived by ESCHERICH ([26]) in the way that was given here and overlaps completely with the result that CLEBSCH ([27]) derived in a different way.

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([26]) *Loc. cit.* (2), **107** (1898), pp. 1309.

([27]) "Ueber diejenigen Probleme der Variationsrechnung, welche nur eine unabhängige Variable enthalten," J. reine und angew. Math. **55** (1858), 335-355, pp. 343.

We would now like to also determine the conjugate systems of the second kind. The determinant (64) will then be equal to zero, and we would like to assume that all of its $(n - e + 1)$-rowed subdeterminants are zero, but at least one of the $(n - e)$-rowed ones is non-zero.

The equations:

$$\sum_{\nu=1}^{n} c_\nu^{(k)} a_\nu = 0 \qquad (k = 1, 2, \ldots, n)$$

then have $e$ linearly-independent solutions:

$$\begin{matrix} a_1^{(1)}, \ldots, a_n^{(1)} \\ \ldots\ldots\ldots\ldots \\ a_1^{(e)}, \ldots, a_n^{(e)} \end{matrix}$$

that can be chosen such that the relations exist:

$$\sum_{\nu=1}^{n} a_\nu^{(k)} a_\nu^{(k')} = 0 \qquad (k \neq k'),$$

$$\sum_{\nu=1}^{n} \left(a_\nu^{(k)}\right)^2 = 1 .$$

We can extend that system to an orthogonal one by adding a suitable system:

$$\begin{matrix} a_1^{(e+1)}, \ldots, a_n^{(e+1)} \\ \ldots\ldots\ldots\ldots\ldots \\ a_1^{(n)}, \ldots, a_n^{(n)} . \end{matrix}$$

We shall now employ the coefficients $a_k^{(i)}$ to exhibit a new fundamental system $z^{\cdot(\nu)}$, $\sigma^{\cdot(\nu)}$ ($\nu = 1, 2, \ldots, 2n$) using (61). Let the matrix of constants by which the solutions to our conjugate system is composed from that fundamental system be:

$$\left\| C_\nu^{(k)} \right\| \qquad \begin{pmatrix} \nu = 1, 2, \ldots 2n \\ k = 1, 2, \ldots, n \end{pmatrix},$$

in which the $C_\nu^{(k)}$ have the values:

$$C_\nu^{(k)} = \sum_{i=1}^{n} c_i^{(k)} a_i^{(\nu)} , \qquad C_{n+\nu}^{(k)} = \sum_{i=1}^{n} c_{n+i}^{(k)} a_i^{(\nu)} \qquad (\nu, k = 1, 2, \ldots, n).$$

The first $e$ columns of our matrix then consist of nothing buts zeroes. One can, with no loss of generality, assume that the matrix has the form:

$$\left\| \begin{array}{ccccccccccc} 0 & \cdots & 0 & 1 & 0 & \cdots & 0; & C_{n+1}^{(1)} & \cdots & C_{2n}^{(1)} \\ \vdots & \cdots & \vdots & 0 & 1 & \cdots & 0; & \vdots & \cdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & C_{n+1}^{(e)} & \cdots & C_{2n}^{(e)} \\ 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & C_{n+1}^{(e+1)} & \cdots & C_{2n}^{(e+1)} \\ \vdots & \cdots & \vdots & \vdots & \vdots & \ddots & \vdots; & \vdots & \cdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 1; & C_{n+1}^{(n)} & \cdots & C_{2n}^{(n)} \end{array} \right\|$$

by means of formulas (62), which can be achieved by a linear transformation of the conjugate system. In order for the system of the $u^{(k)}$, $\rho^{(k)}$ to be conjugate, as we saw, it is necessary and sufficient that the equations must be true:

$$\sum_{k=1}^{n} (C_k^{(\mu)} C_{n+k}^{(\nu)} - C_k^{(\nu)} C_{n+k}^{(\mu)}) \qquad (\mu, \nu = 1, 2, \ldots, n).$$

That gives:

$$C_{n+\nu}^{(\mu)} = 0 \qquad (\mu \leq e, e < \nu \leq n),$$
$$C_{n+\mu}^{(\nu)} = C_{n+\nu}^{(\mu)} \qquad (e < \mu \leq n, e < \nu \leq n).$$

Since not all $n$-rowed determinants in our matrix can vanish [otherwise the solutions $u^{(k)}$, $\rho^{(k)}$ ($k = 1, 2, \ldots, n$) would not be linearly independent], it will follow that the determinant:

$$\left| \begin{array}{ccc} C_{n+1}^{(1)} & \cdots & C_{n+e}^{(1)} \\ \vdots & \cdots & \vdots \\ C_{n+1}^{(e)} & \cdots & C_{n+e}^{(e)} \end{array} \right|$$

must be nonzero. Therefore, the matrix can be brought into the form:

$$(66) \qquad \left\| \begin{array}{cccccccccccc} 0 & \cdots & 0 & 1 & \cdots & 0; & 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \cdots & \vdots & 0 & \cdots & 0; & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 1 & \cdots & 0 & 0 & \cdots & 0 & C_{n+e+1}^{(e+1)} & \cdots & C_{2n}^{(e+1)} \\ \vdots & \cdots & \vdots & \vdots & \ddots & \vdots; & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 1; & 0 & \cdots & 0 & C_{n+e+1}^{(n)} & \cdots & C_{2n}^{(n)} \end{array} \right\|$$

by a further linear transformation, in which the subdeterminant that is composed of the $C$ is symmetric.

If one would once more like to express the constants $c_i^{(k)}$ that make the general conjugate system whose determinant for $x = x_0$ has all of its subdeterminants up to order $(n - e + 1)$ vanishing in terms of the fundamental system of the $z_i^{(k)}$, $\sigma_r^{(k)}$ then one will achieve that in the following way: We again denote the elements of the matrix (66) by $C_i^{(k)}$ $(k = 1, 2, \ldots, n\,;\ i = 1, 2, \ldots, 2n)$ and set:

$$c_\nu^{(k)} = \sum_{i=1}^{n} C_i^{(k)} a_\nu^{(i)}\,, \qquad c_{n+\nu}^{(k)} = \sum_{i=1}^{n} C_{n+i}^{(k)} a_\nu^{(i)}\,.$$

One must then set the $a_\nu^{(i)}$ equal to the coefficients of the most general orthogonal transformation, with the restriction that no two transformations $a_\nu^{(i)}$ and $\bar{a}_\nu^{(i)}$ for which:

$$\bar{a}_\nu^{(i)} = \sum_{k=1}^{e} \gamma_k^{(i)} a_\nu^{(k)} \qquad (i = 1, 2, \ldots, e\,;\ \nu = 1, 2, \ldots, n)$$

are employed.

*One will thus obtain one and only one representative from every group of conjugate systems with the stated property*. That is true for $e = 1, 3, \ldots, n - 1$. For $e = n$, so when all $u_k^{(i)}$ vanish for $x = x_0$, one will get the single group of conjugate systems, which is given by the following matrix:

$$\text{(67)} \qquad \begin{Vmatrix} 0 & \cdots & 0; & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0; & 0 & 0 & \cdots & 1 \end{Vmatrix}.$$

The problem of giving one and only one representative from each group of conjugate systems is thus solved.

## § 5.

We saw in § **2** that when a MAYER field is given by:

$$\text{(68)} \qquad y_i = y_i(x, a_1, \ldots, a_n)\,, \qquad \lambda_r = \lambda_r(x, a_1, \ldots, a_n)$$

and yields the extremal (49) for the values $a_1^0, \ldots, a_n^0$ of the parameters $a$ then the expressions:

$$\text{(69)} \qquad u_i^{(k)}(x) = y_{i a_k}(x, a_1^0, \ldots, a_n^0)\,, \qquad \rho_r^{(k)}(x) = y_{r a_k}(x, a_1^0, \ldots, a_n^0)$$

will yield a conjugate system of solutions to (52).

We would now like to call, in general, an $n$-parameter family of extremals (68) a MAYER family of extremals, with no concern for whether it does or does not define a field, when the $n$ ($n$ – 1) / 2 conditions ([28]):

$$(70) \qquad \sum_{i=1}^{n} (y_{i a_\nu} w_{i a_\mu} - y_{i a_\mu} w_{i a_\nu}) = 0 \qquad (\mu, \nu = 1, 2, \ldots, n)$$

exist identically in the $a$ for one value of $x$ (and therefore for all values of $x$), in which:

$$w_i (x, a_1, \ldots, a_n) = F_{y'_i} [x, y(x,a), y_x(x,a), \lambda(x,a)] \qquad (i = 1, 2, \ldots, n)$$

denote the momenta with respect to the coordinate directions.

If the constants $a_k^{(i)}$ mean the coefficients of an orthogonal transformation and one sets:

$$Y_i (x, a_1, \ldots, a_n) = \sum_{k=1}^{n} a_k^{(i)} y_k(x, a_1, \ldots, a_n) \qquad (i = 1, 2, \ldots, n),$$
$$W_i (x, a_1, \ldots, a_n) = \sum_{k=1}^{n} a_k^{(i)} w_k(x, a_1, \ldots, a_n) \qquad (i = 1, 2, \ldots, n)$$

then the $n$ ($n$ – 1) / 2 conditions (70) are obviously equivalent to the $n$ ($n$ – 1) / 2 conditions:

$$(71) \qquad \sum_{i=1}^{n} (Y_{i a_\nu} W_{i a_\mu} - Y_{i a_\mu} W_{i a_\nu}) = 0 \qquad (\mu, \nu = 1, 2, \ldots, n),$$

as a simple calculation that is entirely analogous to one that was performed in § **4** will show.

In order for the family (68) to actually depend upon $n$ parameters and no fewer, it is necessary and sufficient that not all $n$-rowed determinants in the matrix:

$$(72) \qquad \left\| y_{i a_k} \quad \cdots \quad y_{n a_k}; \quad \lambda_{i a_k} \quad \cdots \quad \lambda_{m\, a_k} \right\| \qquad (k = 1, 2, \ldots, n)$$

vanish identically. If we assume that even when we replace the $a_k$ with the special values $a_k^0$, not all determinants in (72) will vanish identically in $x$, then we will see that the $u_i^{(k)}$, $\rho_r^{(k)}$ that are defined by (69) will again define a conjugate system of solutions, because on the one hand, they are linearly independent, since not all $n$-rowed determinants in the matrix

$$\left\| u_1^{(k)} \quad \cdots \quad u_n^{(k)}; \quad \rho_1^{(k)} \quad \cdots \quad \rho_m^{(k)} \right\| \qquad (k = 1, 2, \ldots, n)$$

([28]) That form of the conditions is merely an abbreviated notation for (21) [(22), resp.].

will vanish, and on the other hand, due to (70), they will satisfy the conditions (63).

One now sees that, conversely, *all conjugate systems can be obtained in the given way from MAYER families of extremals by differentiation with respect to the parameters.*

That is because if a conjugate system is given:

$$u_1^{(k)}(x), \dots, u_n^{(k)}(x), \qquad \rho_1^{(k)}(x), \dots, \rho_n^{(k)}(x) \qquad (k = 1, 2, \dots, n) \tag{73}$$

and $x_0$ is any special value of $x$ then not all $n$-rowed determinants in the matrix:

$$\left\| u_1^{(k)} \quad \cdots \quad u_n^{(k)}; \quad \rho_1^{(k)} \quad \cdots \quad \rho_m^{(k)} \right\| \qquad (k = 1, 2, \dots, n) \tag{74}$$

will be zero, since otherwise the functions (73) would not be linearly independent. We again set:

$$\begin{aligned} v_i^{(k)}(x_0) = \sum_{\nu=1}^{n} \{ & F_{y_i' y_\nu}[x_0, y(x_0), y'(x_0), \lambda(x_0)] u_\nu^{(k)}(x_0) + F_{y_i' y_\nu'}[x_0, y(x_0), y'(x_0), \lambda(x_0)] u_\nu^{(k)'}(x_0) \\ & + \sum_{\nu=1}^{m} \varphi_{\nu y_i'}[x_0, y(x_0), y'(x_0)] \rho_\nu^{(k)}(x_0) \qquad (i, k = 1, 2, \dots, n), \end{aligned}$$

and furthermore:

$$\left\{ \begin{aligned} y_i^0 &= y_i(x_0) + \sum_{\nu=1}^{n} u_i^{(\nu)}(a_\nu - a_\nu^0) && (i = 1, 2, \dots, n), \\ w_i^0 &= F_{y_i'}[x_0, y(x_0), y'(x_0), \lambda(x_0)] + \sum_{\nu=1}^{n} v_i^{(\nu)}(x_0)(a_\nu - a_\nu^0) && (i = 1, 2, \dots, n). \end{aligned} \right. \tag{75}$$

We replace the $y_i^0$, $w_i^0$ in the expressions:

$$\begin{aligned} y_i &= y_i(x, y_1^0, \dots, y_n^0, w_1^0, \dots, w_n^0) && (i = 1, 2, \dots, n), \\ \lambda_r &= \lambda_r(x, y_1^0, \dots, y_n^0, w_1^0, \dots, w_n^0) && (r = 1, 2, \dots, m), \end{aligned}$$

which represent the extremals by their dependency on the initial values of the ordinates and momenta for $x = x_0$ with the expressions (75). We thus obtain an extremal family that depends upon $n$ parameters $a_1, \dots, a_n$ that includes the extremals (49) for $a_\nu = a_\nu^0$, and we can easily show that it is a MAYER family.

In fact, one has:

$$\begin{aligned} y_{i a_k}(x_0, a_1, \dots, a_n) &= u_i^{(k)}(x_0) && (i, k = 1, 2, \dots, n), \\ \frac{\partial}{\partial a_k} F_{y_i'}[x_0, y(x_0, a), y_x(x_0, a), \lambda(x_0, a)] &= v_i^{(k)}(x_0) && (i, k = 1, 2, \dots, n) \end{aligned}$$

identically in the $a$, such that equations (70) will reduce to:

$$\sum_{i=1}^{n}[u_i^{(\mu)}(x_0)\,v_i^{(\nu)}(x_0)-u_i^{(\nu)}(x_0)\,v_i^{(\mu)}(x_0)] = 0 \qquad (\mu,\ \nu = 1, 2, \ldots, n)$$

for $x = x_0$, which are certainly fulfilled in their own right, because the system $u^{(k)}$, $\rho^{(k)}$ ($k = 1, 2, \ldots, n$) is a conjugate one, by assumption. The matrix (72) reduces to the matrix (74) for $x = x_0$, such that certainly not all of its $n$-rowed determinants vanish. The family that is thus found is a MAYER family. Since the functions (73) will be obtained from it by differentiation with respect to the $a_\nu$ for $a_\nu = a_\nu^0$, our assertion is proved.

We now introduce new parameters $b$ into the MAYER system (68) instead of the parameters $a$ such that the values of the parameters $a$ and the parameters $b$ will correspond to each other in a one-to-one way by means of the relations:

$$(76) \qquad \begin{cases} a_\nu = a_\nu(b_1, b_2, \ldots, b_n) \\ b_\nu = b_\nu(b_1, b_2, \ldots, b_n) \end{cases} \qquad (\nu = 1, 2, \ldots, n),$$

and such that the functional determinant:

$$\left| a_{\nu b_\mu}(b_1^0, \ldots, b_n^0) \right| \qquad (\mu,\ \nu = 1, 2, \ldots, n),$$

in which one sets:

$$b_\nu^0 = b_\nu(a_1^0, \ldots, a_n^0) \qquad (\nu = 1, 2, \ldots, n)$$

does not vanish. If one denotes the conjugate system that arises from our MAYER family by differentiating with respect to the new parameter $b$ by $\bar{u}^{(k)}$, $\bar{\rho}^{(k)}$, then one will have:

$$\bar{u}_i^{(k)} = \sum_{\nu=1}^{n} a_{\nu b_k}(b_1^0, \ldots, b_n^0)\, u_i^{(\nu)} \qquad (i = 1, 2, \ldots, n),$$

$$\bar{\rho}_r^{(k)} = \sum_{\nu=1}^{n} a_{\nu b_k}(b_1^0, \ldots, b_n^0)\, \rho_r^{(\nu)} \qquad (r = 1, 2, \ldots, m),$$

for $k = 1, 2, \ldots, n$. The conjugate system $\bar{u}^{(k)}$, $\bar{\rho}^{(k)}$ then arises from the conjugate system $u^{(k)}$, $\rho^{(k)}$ by a linear transformation with a non-vanishing determinant.

If, conversely, an arbitrary linear transformation with a non-vanishing determinant is given by the coefficients $\alpha_i^{(k)}$ ($i, k = 1, 2, \ldots, n$) then one sets:

$$a_\nu = a_\nu^0 + \sum_{k=1}^{n} \alpha_\nu^{(k)}(b_k - b_k^0).$$

Upon differentiating our MAYER system with respect to the parameters $b$, one will then get precisely that conjugate system that arises from the system of $u^{(k)}$, $\rho^{(k)}$ by the given linear transformation. We will then have:

*If one represents the same MAYER family by different parameters then that will yield only a conjugate system in the same group. Conversely, all conjugate systems in a group can be obtained from the same MAYER family.*

Previously, we distinguished two types of conjugate systems according to whether their determinant at the given location $x_0$ was or was not equal to zero. Those of the first type arise from MAYER families that define a field in the neighborhood of the abscissa $x_0$, and conversely only one conjugate system of the first type can arise from MAYER family that defines a field in the neighborhood of $x_0$. We will then get certain representatives from all groups of conjugate systems of the first type when we differentiate the general MAYER family that defined a field in the neighborhood of $x_0$ and includes the extremal (49) with respect to its parameters. However, for those parameters, one can choose the initial values $y_1^0, \ldots, y_n^0$ of the ordinates for $x = x_0$ for an extremal family that defines a field for $x = x_0$ . As was shown in § **2**, one will get the most general MAYER family that defined a field in the neighborhood of $x_0$ when one prescribes the initial values of the momenta with respect to the coordinate directions $x = x_0$ to be the partial derivatives with respect to the corresponding coordinates of one and the same function $\Phi(y_1^0, \ldots, y_n^0)$. If one makes use of the notation (53) then the most general MAYER family that defines a field in the neighborhood of $x_0$ will take the form:

$$
(77)\qquad \begin{cases} y_i = y_i\,[x, y_1^0, \ldots, y_n^0; \Phi_{y_1^0}(y_1^0, \ldots, y_n^0), \ldots, \Phi_{y_n^0}(y_1^0, \ldots, y_n^0)] & (i = 1, 2, \ldots, n), \\ \lambda_r = \lambda_r\,[x, y_1^0, \ldots, y_n^0; \Phi_{y_1^0}(y_1^0, \ldots, y_n^0), \ldots, \Phi_{y_n^0}(y_1^0, \ldots, y_n^0)] & (r = 1, 2, \ldots, m), \end{cases}
$$

and should that family contain the extremal (49) then the function $\Phi$ would have to be subject to the conditions:

$$
\Phi_{y_i^0}(y_1^{00}, \ldots, y_n^{00}) = w_i^{00} \qquad (i = 1, 2, \ldots, n).
$$

Now, if $u^{(k)}$, $\rho^{(k)}$ is the conjugate system that arises from (77) by differentiation with respect to the parameters then when we recall (54) and (54′), we will have:

$$
u_i^{(k)} = z_i^{(k)} + \sum_{\nu=1}^{n} \Phi_{y_\nu^0 y_k^0}(y_1^{00}, \ldots, y_n^{00})\, z_i^{(n+\nu)} \qquad (i = 1, 2, \ldots, n),
$$

$$
\rho_r^{(k)} = \sigma_r^{(k)} + \sum_{\nu=1}^{n} \Phi_{y_\nu^0 y_k^0}(y_1^{00}, \ldots, y_n^{00})\, \sigma_r^{(n+\nu)} \qquad (r = 1, 2, \ldots, m).
$$

If we further set:

$$
\Phi_{y_\nu^0 y_k^0}(y_1^{00}, \ldots, y_n^{00}) = c_\nu^{(k)}
$$

then one will have:

$$c_\nu^{(k)} = c_k^{(\nu)},$$

*and we have obtained the result of CLEBSCH and ESCHERICH that was given in* § **4** *once more.* At the same time, we see that two different functions $\Phi(y_1^0, \ldots, y_n^0)$ – say, $\Phi_1$ and $\Phi_1$ – correspond to conjugate systems of the same group if and only if the second derivatives $\Phi_{1 y_\nu^0 y_k^0}(y_1^{00}, \ldots, y_n^{00})$ and $\Phi_{2 y_\nu^0 y_k^0}(y_1^{00}, \ldots, y_n^{00})$ coincide for $\nu, k = 1, 2, \ldots, n$.

It still remains for us to investigate the MAYER families from which the conjugate systems of the second type will arise.

Let the conjugate system that is constructed from the fundamental system (60), (60′) with the help of the matrix (66) be given. We get an associated MAYER family of extremals in the following way: We start from the representation (59) of the extremals of our problem and replace the $Y_1^0, \ldots, Y_e^0$ in it with $Y_1^{00}, \ldots, Y_e^{00}$, where:

$$Y_i^{00} = \sum_{k=1}^{n} a_k^{(i)} y_k^{00} \qquad (i = 1, 2, \ldots, n),$$

and $y_k^{00}$ mean the initial ordinates of the extremals (49) for $x = x_0$, as they have up to now. Moreover, we let $\Phi(Y_{e+1}^0, \ldots, Y_n^0)$ denote an arbitrary function of $Y_{e+1}^0, \ldots, Y_n^0$ that satisfies the conditions:

$$\Phi_{Y_\nu^0}(Y_{e+1}^{00}, \ldots, Y_n^{00}) \equiv W_\nu^{00} \qquad (\nu = e + 1, \ldots, n).$$

In that, one has set:

$$W_i^{00} = \sum_{k=1}^{n} a_k^{(i)} w_k^{00} \qquad (i = 1, 2, \ldots, n),$$

such that the $W_i^{00}$ are the initial values of the momenta with respect to the directions of the $Y_i$-axis for $x = x_0$ when they are defined on the extremal (49), We now replace $W_{e+1}^0, \ldots, W_n^0$ in (59) with:

$$W_\nu^0 = \Phi_{Y_\nu^0}(Y_{e+1}^0, \ldots, Y_n^0) \qquad (\nu = e + 1, \ldots, n).$$

We thus obtain the $n$-parameter family of extremals:

$$(78) \qquad \left\{ \begin{aligned} y_i &= y_i^{\bullet}[x, Y_i^{00}, \ldots, Y_e^{00}, Y_{e+1}^0, \ldots, Y_n^0; W_1^0, \ldots, W_e^0, \\ &\qquad \Phi_{Y_{e+1}^0}(Y_{e+1}^0, \ldots, Y_n^0), \ldots, \Phi_{Y_n^0}(Y_{e+1}^0, \ldots, Y_n^0)], \\ \lambda_r &= \lambda_r^{\bullet}[x, Y_i^{00}, \ldots, Y_e^{00}, Y_{e+1}^0, \ldots, Y_n^0; W_1^0, \ldots, W_e^0, \\ &\qquad \Phi_{Y_{e+1}^0}(Y_{e+1}^0, \ldots, Y_n^0), \ldots, \Phi_{Y_n^0}(Y_{e+1}^0, \ldots, Y_n^0)], \end{aligned} \right.$$

which depend upon the parameters $W_1^0, \ldots, W_e^0$ ; $Y_{e+1}^0, \ldots, Y_n^0$. One obtains the extremal (49) for the values $W_1^{00}, \ldots, W_e^{00}$ ; $Y_{e+1}^{00}, \ldots, Y_n^{00}$ of those parameters. The family is a MAYER family. In order to see that, we write it in the form:

$$y_i = \bar{y}_i(x, W_1^0, \ldots, W_e^0, Y_{e+1}^0, \ldots, Y_n^0),$$
$$\lambda_r = \bar{\lambda}_r(x, W_1^0, \ldots, W_e^0, Y_{e+1}^0, \ldots, Y_n^0),$$

denote the associated momenta with respect to the directions of the $y^i$-axes by:

$$w_i = \bar{w}_i(x, W_1^0, \ldots, W_e^0, Y_{e+1}^0, \ldots, Y_n^0) \qquad (i = 1, 2, \ldots, n),$$

and set:

$$Y_i(x, W_1^0, \ldots, W_e^0, Y_{e+1}^0, \ldots, Y_n^0) = \sum_{k=1}^{n} a_k^{(i)}\, \bar{y}_i\,(x, W_1^{\,0}, \ldots, W_e^{\,0}; Y_{e+1}^{\,0}, \ldots, Y_n^{\,0}),$$
$$W_i(x, W_1^0, \ldots, W_e^0, Y_{e+1}^0, \ldots, Y_n^0) = \sum_{k=1}^{n} a_k^{(i)}\, \bar{w}_i\,(x, W_1^{\,0}, \ldots, W_e^{\,0}; Y_{e+1}^{\,0}, \ldots, Y_n^{\,0}).$$

We remark that the relations will then exist:

$$Y_{iW_\nu^0}(x, W_1^0, \ldots, W_e^0, Y_{e+1}^0, \ldots, Y_n^0) = 0 \qquad (\nu = 1, 2, \ldots, e),$$
$$W_{iW_\nu^0}(x, W_1^0, \ldots, W_e^0, Y_{e+1}^0, \ldots, Y_n^0) = \varepsilon_{i\nu} \qquad (\nu = 1, 2, \ldots, e),$$
$$Y_{iY_\nu^0}(x, W_1^0, \ldots, W_e^0, Y_{e+1}^0, \ldots, Y_n^0) = 0 \qquad (\nu = e+1, 2, \ldots, n),$$
$$W_{iY_\nu^0}(x, W_1^0, \ldots, W_e^0, Y_{e+1}^0, \ldots, Y_n^0) = \Phi_{Y_i^0 Y_\nu^0}(Y_{e+1}^0, \ldots, Y_n^0) \qquad (\nu = e+1, 2, \ldots, n),$$

in which $\varepsilon_{i\nu} = 0$ or 1 according to whether $i \neq \nu$ or $i = \nu$, resp. The conditions (71) are therefore fulfilled for $x = x_0$ and our family is a MAYER family.

One will get the values $W_1^{00}, \ldots, W_e^{00}$, $Y_{e+1}^{00}, \ldots, Y_n^{00}$ of the parameters $W_1^{\,0}, \ldots, W_e^{\,0}$, $Y_{e+1}^{\,0}, \ldots, Y_n^{\,0}$ by differentiating (78) with respect to them:

$$u_i^{(\nu)} = z_i^{\bullet(n+\nu)}, \qquad \rho_i^{(\nu)} = \sigma_i^{\bullet(n+\nu)}, \qquad (\nu = 1, 2, \ldots, e),$$

$$\left.\begin{aligned} u_i^{(\nu)} &= z_i^{\bullet(\nu)} + \sum_{k=e+1}^{n} \Phi_{Y_k^0 Y_\nu^0}(Y_{e+1}^{00}, \ldots, Y_n^{00})\, z_i^{\bullet(n+\nu)} \\ \rho_i^{(\nu)} &= \sigma_i^{\bullet(\nu)} + \sum_{k=e+1}^{n} \Phi_{Y_k^0 Y_\nu^0}(Y_{e+1}^{00}, \ldots, Y_n^{00})\, \sigma_i^{\bullet(n+\nu)} \end{aligned}\right\} \qquad (\nu = e+1, \ldots, n),$$

such that the system that emerges from our MAYER family will be nothing but the one that emerges from the fundamental system (60), (60′) by means of the matrix (66) when we set:

$$C_{n+k}^{(\nu)} = \Phi_{Y_k^0 Y_\nu^0}(Y_{e+1}^{00}, \ldots, Y_n^{00}) \qquad (\nu, k = e+1, \ldots, n).$$

*In order to obtain all conjugate systems whose determinants have all of their subdeterminants with more than* ($n$ – $e$) *rows vanishing for* $x = x_0$, *while at least one of the* ($n$ – $e$)*-rowed subdeterminants does not vanish there, one can proceed as follows:* One takes an ($n$ – $e$)-dimensional linear manifold through the point $y_1^{00}$, …, $y_n^{00}$ in the $n$-dimensional space of $y_1^0$, …, $y_n^0$ and then chooses new rectangular coordinates $Y_1^0$, …, $Y_n^0$ in that space such the $Y_{e+1}^0$, …, $Y_n^0$ axes lie on the aforementioned manifold. One now prescribes an arbitrary function $\Phi(Y_{e+1}^0, \ldots, Y_n^0)$ on that manifold whose partial derivatives at the location $Y_{e+1}^{00}$, …, $Y_n^{00}$ coincide with the initial values of the momenta of the extremal (49) with respect to the directions of the $Y_{e+1}^0$, …, $Y_n^0$ axes. One then lays the $e$-parameter family of extremals through point of that manifold whose momenta with respect to the $Y_{e+1}^0$, …, $Y_n^0$ axes coincide with the partial derivatives of Φ at the point in question, while the momenta with respect to the $Y_1^0$, …, $Y_e^0$ axes remain arbitrary. One then obtains an $n$-parameter MAYER family of extremals that yields a conjugate system of the desired type. If one lets the aforementioned manifold assume all possible positions and prescribes the function Φ in all possible ways then one will get conjugate systems from all possible groups. Different positions of the aforementioned manifold yield conjugate systems from different groups. Two different functions Φ on the same manifold will yield conjugate system in different groups if and only if their second derivative at the location $Y_{e+1}^{00}$, …, $Y_n^{00}$ do not all coincide.

Finally, as far as a conjugate system that emerges from the fundamental system (54), (54′) by the matrix (67) is concerned, it will arise from the MAYER family that consists of all extremals that go through the point $(x_0, y_1^{00}, \ldots, y_n^{00})$.

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HANS HAHN