# Infinitesimal maps in optics 

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The construction of geometrical optics on a broader mathematical foundation that goes beyond the restricted goals of practical and computational optics has come about only recently. Once ABBE $\left.{ }^{1}{ }^{1}\right)$ had worked out the role of line geometry in optical theorems more rigorously than his predecessors, BRUNS $\left({ }^{2}\right)$ gave an extremely consistent and comprehensive presentation in which he summarized all of the ray maps that obeyed MALUS's theorem. That theorem, according to which surface-normal bundles remain surface-normal under any refraction (and reflection), is true for all optical maps in isotropic media (but not for the double refraction in crystals). BRUNS posed the converse question of whether a map that satisfies MALUS's theorem can be realized by optical devices, but he did not answer it. The following article would like to make a contribution towards that objective, and indeed by the use of an analytical procedure whose impetus was contained in BRUNS's treatise itself. Namely, a ray map that corresponds to MALUS's theorem is nothing but a contact transformation in five variables of a special form. The principal part of "eikonal" of the map is identical to the "generating function" of the contact transformation. By the way, it should be mentioned that this connection between optics and LIE's fertile theory is not the only one. The motion of the light ray in an elastic medium also leads to a certain class of contact transformations by means of HUYGENS's principle $\left({ }^{3}\right)$. Now, the great advantage that calculating with infinitesimal contact transformations affords is closely related to the idea of also examining infinitely-small ray maps in optics, i.e., ones that take every ray in the object space to an infinitelyclose ray in the image space. The result of a test of that hypothesis does not prove to be unfavorable. Among other things, one can resolve the important question of the existence of an ideal telescope objective in that way, if not also completely. One can foresee in advance what the advantage of introducing infinitesimal maps would be in comparison to finite ones: Certain relationships can then be written out explicitly, instead of implicitly, and in the infinitesimal domain, mere superposition will enter in place of successively performing two operations. For example, the eikonal for a double refraction requires an elimination of variables that is mostly impracticable. By contrast, the corresponding characteristic function for infinitely-small refractions is composed additively from two individual components (as long as one restricts oneself to the terms of lowest

[^0]order). However, the weakness in the method is also obvious: The infinitesimal operations relate to the finite ones in the same way that the first term in a series development relates to the total series, and in the same way that the closest neighborhood of a point relates to an entire curve or surface. Results that relate to infinitesimal operations cannot be adapted to finite ones with no further analysis but will give at most a heuristic rule of thumb. In addition (and this is an actual restriction for our own purposes), the assumption of the existence of infinitesimal operations is already a specialization of the existence of the finite ones, because it assumes the existence of suitably-chosen arbitrary parameters. When a relation:
$$
X=f(x, a)
$$
exists between the variables $x$ and $X$ that includes the arbitrary parameter $a$ then we will speak of a family of transformations between $x$ and $X$. Every well-defined value of $a$ corresponds to a transformation of the family. Now, if there is a certain value - say $a=0$ - of the parameter for which $f(x, a)$ reduces to $x$ identically then that transformation:
$$
X=f(x, 0)=x
$$
will be called the identity of the family, and everything infinitely-close to it will be an infinitesimal transformation. The equation of such a thing, when developed in powers of $a$, is:
$$
X=x+a x+\frac{a^{2}}{2} x_{2}+\ldots
$$
where $x_{1}, x_{2}, \ldots$ are functions of $x$. By contrast, one cannot speak of the infinitesimal transformation for an individual transformation $X=f(x)$ that does not contain arbitrary parameters. Families of transformations that do not contain the identity are likewise excluded. The same argument is true for transformations between several variables with several parameters. When we then assume the existence of infinitesimal ray maps in what follows, we will restrict ourselves to those finite maps that contain at least one arbitrary element. Isolated cases, such as maps with well-defined values of the refraction exponent and a well-defined arrangement of refracting surfaces, will remain excluded from out consideration. An even more far-reaching restriction that we can first establish in the course of the investigation shall be expressed at this point. If one imagines that a finite map has been realized optically by a system of refracting surfaces $F_{12}, F_{23}, \ldots, F_{\alpha-1, \alpha}$ that separate the media $M_{1}, M_{2}, M_{3}, \ldots, M_{\alpha}$ with indices of refraction $n_{1}, n_{2}, \ldots, n_{\alpha}$, resp., from each other then one can derive an infinitesimal map from that by an individual or combined application of the following two procedures:

Either one lets all $n_{\alpha}$ be equal to each other, up to small quantities of first order, or one chooses all refracting surfaces to be infinitely close and the first refraction exponent to be equal to the last one.

The first operation assumes that the ratios of the $n_{\alpha}$ (as long as they are different from unity) are actually available, while the refracting surfaces are likewise available or can be given from the
outset. Conversely, the $n_{\alpha}$ can be given for the second procedure, while the necessary type and number of available elements must enter into $F$. However, in the future, we will restrict ourselves to infinitesimal maps of the first kind.

For the sake of considerations that are more optical than purely-mathematical, the things that are used from the realm of contact transformations in space will be derived in particular in what follows. Moreover, one should refer to the detailed theory in the second volume of LIE's Transformationsgruppen.

## 2.

We write the equations of a ray $\sigma$ in the form:

$$
\begin{equation*}
\frac{x-0}{m}=\frac{y-h}{p}=\frac{z-k}{p}, \quad m^{2}+p^{2}+q^{2}=1, \tag{1}
\end{equation*}
$$

in which $m, p, q$ are the direction cosines of $\sigma$ with the coordinate axes, $0, h, k$ are the coordinates of a point of intersection with the $y z$-plane (viz., the base plane), and we choose $h, k, p, q$ to be the four determining data of $\sigma$ (viz., the ray coordinates). The equations:

$$
\begin{equation*}
h=y-x \frac{p}{m}, \quad k=z-x \frac{q}{m}, \tag{2}
\end{equation*}
$$

which follow from (1), give $h, k$ when the other two coordinates $p, q$, and an arbitrary point $x, y, z$ of the line is given. Along with those quantities, which refer to object space, we shall introduce the corresponding quantities for image space and denote them with uppercase characters. The "map" of the one space to the other consists of the association of a ray $\Sigma$ with each ray $\sigma$, i.e., of four equations that define the ray coordinates $H, K, P, Q$ as functions of $h, k, p, q$, and conversely.

A manifold of $\infty^{2}$ rays is called a ray congruence. One obtains such a thing when one regards $h, k, p, q$ as functions of two parameters or also regards $p, q$ as functions of $h, k$ (conversely, resp.). A surface-normal congruence is defined by the demand that:

$$
p d h+q d k
$$

is a complete differential. (Cf., Eikonal, pp. 333) If $\xi=\xi(\eta, \zeta)$ is the equation of a surface (viz., the wave surface), and if the normal that goes through the point $\xi, \eta, \zeta$ has the ray coordinates $h$, $k, p, q$ then $\left[\xi_{\eta}=\frac{\partial \xi}{\partial \eta}, \xi_{\zeta}=\frac{\partial \xi}{\partial \zeta}\right]$ :

$$
\frac{m}{-1}=\frac{p}{\xi_{\eta}}, \quad h=\eta-\xi \frac{p}{m}, \quad k=\zeta-\xi \frac{q}{m}
$$

in which $h, k, p, q$ are determined as functions of $\eta, \zeta$. It then follows that:

$$
\begin{aligned}
p d h+q d k & =p\left(d \eta-\frac{p}{m} d \xi-\xi \frac{m d p-p d m}{m^{2}}\right)+q\left(d \zeta-\frac{q}{m} d \xi-\xi \frac{m d q-q d m}{m^{2}}\right) \\
& =p d h+q d \zeta-\frac{1-m^{2}}{m} d \xi+\xi \frac{d m}{m^{2}} \\
& =-m\left(\xi_{\eta} d \eta+\xi_{\zeta} d \zeta\right)+m d \zeta-d\left(\frac{\xi}{m}\right) \\
& =-d\left(\frac{\xi}{m}\right)=d l .
\end{aligned}
$$

Thus, when the congruence is surface-normal:

$$
\begin{equation*}
d l=p d h+q d k \tag{3}
\end{equation*}
$$

will be a complete differential, and the equations:

$$
\begin{equation*}
\xi=-m l, \quad \eta=h-p l, \quad \zeta=k-q l \tag{4}
\end{equation*}
$$

will represent a wave surface as long as one sets $l=l(h, k), p=l_{h}, q=l_{k}\left({ }^{1}\right)$.
The argument presented here leads one to add a fifth variable $l$ to the four ray coordinates $h, k$, $p, q$, in such a way that the PFAFF equation (3), or:

$$
\begin{equation*}
d l-p d h-q d k=0 \tag{5}
\end{equation*}
$$

gives the condition for surface-normality. All optical maps (which are now represented by five equations between the $h, k, l, p, q$ and the $H, K, L, P, Q)$ have in common the validity of MALUS's theorem, or the conservation of surface-normality, which is a condition that expresses the idea that the PFAFF equation:

$$
\begin{equation*}
d L-P d H-Q d K=0 \tag{6}
\end{equation*}
$$

is a consequence of (5). Since the left-hand side of (6) will become linear in the differentials of the $h, k, l, p, q$ upon introducing the functions $H, K, L, P, Q$, an identity of the form:

$$
\begin{equation*}
d L-P d H-Q d K=\rho(d l-p d h-q d k) \tag{7}
\end{equation*}
$$

will then exist.
On the other hand, imagine that the five mapping equations have been posed and the four quantities $P, Q, p, q$ have been eliminated from them. In general, that will then yield an equation:
( ${ }^{1}$ ) We shall denote differential quotients in that unambiguous way throughout.

$$
\Omega(h, k, l, H, K, L)=0
$$

between the $h, k, l, H, K, L$. We assume that only that one equation is implied ( ${ }^{1}$ ). The equation $d \Omega$ $=0$, which is formed from the same differentials as in (7), will be identical to it, since otherwise more than one relation would exist between the $h, k, l, H, K, L$. One will then have:

$$
\frac{\Omega_{L}}{1}=-\frac{\Omega_{H}}{P}=-\frac{\Omega_{K}}{Q}=-\frac{\Omega_{l}}{\rho}=\frac{\Omega_{h}}{\rho p}=\frac{\Omega_{k}}{\rho q} ;
$$

in other words, the system of mapping equations is given by:

$$
\text { (8) } \quad \Omega=0, \quad p=-\frac{\Omega_{h}}{\Omega_{l}}, \quad q=-\frac{\Omega_{k}}{\Omega_{l}}, \quad P=-\frac{\Omega_{H}}{\Omega_{L}}, \quad Q=-\frac{\Omega_{K}}{\Omega_{L}} \text {, }
$$

and the factor $\rho$ in the identity (7) by:

$$
\begin{equation*}
\rho=-\frac{\Omega_{l}}{\Omega_{L}} \tag{9}
\end{equation*}
$$

The "generating function" $\Omega$ can be chosen arbitrarily as a function of its six arguments, except that it must contain all six variables, and the system of equations (8) must be soluble for the lowercase, as well as the uppercase, symbols.

If we interpret $h, k, l$ as rectangular coordinates of a point in three-fold space and $p, q$ as the direction coordinates of a surface element that go through it then we will have Lie's concept of a contact transformation of ordinary space. If $l$ is a function $h, k$ then equation (3) expresses the idea that the surface element $h, k, l, p, q$ contacts the surface $l=l(h, k)$ at the point $h, k, l$. The $\infty^{2}$ surface elements that contact a surface go to ones with the same property under a contact transformation. The three equations (4) by themselves define a contact transformation between the variables:

$$
\begin{aligned}
& l, h, k, p, q \\
& \xi, \eta, \zeta, \pi, \kappa
\end{aligned}
$$

with the generating function:

$$
\xi^{2}+(\eta-h)^{2}+(\zeta-k)^{2}-l^{2} .
$$

One does, in fact, get an identity of the form (7) from the still-missing equations $p=-\pi / m, \kappa=$ $-q / m$, namely:

$$
d \xi-\pi d \eta-\kappa d \zeta=-\frac{1}{m}(d l-p d h-q d k) .
$$

[^1]If equations (8) are to represent a ray map then the equations that are obtained by solving for $H$, $K, P, Q$ can no longer contain the adjoint variables $l, L$ any longer, so each of the last four equations in (8) must be free of $l, L$ when one eliminates one of those variables with the help of $\Omega=0$. The following equations then exist:

$$
\frac{\partial p}{\partial l}: \frac{\partial p}{\partial L}=\frac{\partial q}{\partial l}: \frac{\partial q}{\partial L}=\frac{\partial P}{\partial l}: \frac{\partial P}{\partial L}=\frac{\partial Q}{\partial l}: \frac{\partial Q}{\partial L}=\frac{\partial \Omega}{\partial l}: \frac{\partial \Omega}{\partial L},
$$

the first of which reads:

$$
\Omega_{l} p_{L}=\Omega_{L} p_{l}, \quad \Omega_{l}\left(\Omega_{l} \Omega_{h L}-\Omega_{h} \Omega_{l L}\right)=\Omega_{L}\left(\Omega_{l} \Omega_{h l}-\Omega_{h} \Omega_{l l}\right),
$$

or when one introduces the expression (9):

$$
\Omega_{l} \rho_{h}=\Omega_{h} \rho_{l} .
$$

That equation, which must be fulfilled for the variable-pairs:

$$
l h, h k, k l, \quad L H, H K, K L
$$

will be satisfied when $\rho$ is a function of $\Omega$, or $\Omega$ is a solution of the partial differential equation:

$$
\frac{\Omega_{l}}{\Omega_{L}}=-\rho(\Omega)=-\rho,
$$

whose general solution $\Omega$ will be found when one solves an arbitrary equation between the six arguments:

$$
\Omega, L-\rho l, h, k, H, K
$$

[where $\rho=\rho(\Omega)$ ] for $\Omega$. If one imagines that this equation has been solved for $L-\rho l$ and writes:

$$
\begin{equation*}
L-\rho l=\Psi(h, k, H, K, Q) \tag{10}
\end{equation*}
$$

then it will follow by differentiation, when $\rho^{\prime}=d \rho / d \Omega$, and $u$ means one of the four quantities $h$, $k, H, K$, that:

$$
1-\rho^{\prime} l \Omega_{L}=\Psi_{\Omega} \Omega_{L}, \quad-\rho-\rho^{\prime} l \Omega_{L}=\Psi_{\Omega} \Omega_{l}, \quad-\rho^{\prime} l \Omega_{u}=\Psi_{u}+\Psi_{\Omega} \Omega_{l}
$$

so:

$$
\Omega_{L}=\frac{1}{\Psi_{\Omega}+\rho^{\prime} l}, \quad \Omega_{l}=-\frac{\rho}{\Psi_{\Omega}+\rho^{\prime} l}, \quad \Omega_{u}=-\frac{\Psi_{u}}{\Psi_{\Omega}+\rho^{\prime} l},
$$

and from (8):

$$
\begin{equation*}
p=-\frac{\Psi_{h}}{\rho}, \quad q=-\frac{\Psi_{k}}{\rho}, \quad P=\Psi_{H}, \quad Q=\Psi_{K} \tag{11}
\end{equation*}
$$

One then sets $\Omega=0$ in those equations, which will make $\rho$ become a constant, and $\Psi$ will become a function of just $h, k, H, K$. The form of equations (11) shows that one makes that substitution right from the start, so the generating equation can be put into the form:

$$
L-\rho l-\Psi(h, k, H, K)=0, \quad \rho=\text { constant } .
$$

If one chooses:

$$
\begin{equation*}
\Omega=N L-n l+E(h, k, H, K), \tag{12}
\end{equation*}
$$

in order to get to the case of optics, then that will make $\rho=n / N$, and the mapping equations (8) give:

$$
\begin{equation*}
n p=E_{h}, \quad n q=E_{k}, \quad-N P=E_{H}, \quad-N Q=E_{K} . \tag{13}
\end{equation*}
$$

Here, $n$ and $N$ are the indices of refraction of object and image space, and $E$ is the eikonal of the map. An eikonal of the kind that one will be find explained in BRUNS (loc. cit., pp. 356) in the general case 16 . Here, we would only like to employ the four forms in which direction and segment quantities appear pair-wise, i.e., the four eikonal forms:

$$
\begin{aligned}
& E_{1}=E(p, q, P, Q), \\
& E_{2}=E(p, q, H, K), \\
& E_{3}=E(h, k, H, K), \\
& E_{4}=E(h, k, P, Q) .
\end{aligned}
$$

The system of mapping equation is contained in the combined formulas:

$$
\left\{\begin{array}{l}
d E_{1}=-n(h d p+k d q)+N(H d P+K d Q)  \tag{14}\\
d E_{2}=-n(h d p+k d q)-N(P d H+Q d K) \\
d E_{3}=+n(p d h+q d k)-N(P d H+Q d K) \\
d E_{4}=+n(p d h+q d k)+N(H d P+K d Q)
\end{array}\right.
$$

which are to be read in the way that equations (13) indicate in the case of $E_{3}$. If all four eikonals belong to one and the same map then the following relations will exist between them:

$$
\left\{\begin{array}{l}
E_{1}-E_{2}=E_{4}-E_{3}=N(P H+Q K),  \tag{15}\\
E_{3}-E_{2}=E_{4}-E_{1}=n(p h+q k) .
\end{array}\right.
$$

## 3.

In the case of an infinitesimal transformation, the mapping equations have the form:

$$
\left\{\begin{align*}
L & =l+v l^{\prime}+\frac{v^{2}}{2} l^{\prime}+\cdots \\
H & =h+v h^{\prime}+\frac{v^{2}}{2} h^{\prime \prime}+\cdots \\
K & =k+v k^{\prime}+\frac{v^{2}}{2} k+\cdots  \tag{16}\\
P & =p+v p^{\prime}+\frac{v^{2}}{2} p^{\prime \prime}+\cdots \\
Q & =q+v q^{\prime}+\frac{v^{2}}{2} q^{\prime \prime}+\cdots
\end{align*}\right.
$$

in which $v$ is an infinitely-small constant quantity that otherwise does not occur in the equations. The quantities $l^{\prime}, h^{\prime}, k^{\prime}, p^{\prime}, q^{\prime}$, etc., are functions of $l, h, k, p, q$. If equations (16) represent an infinitesimal contact transformation then they must fulfill (7), in which we likewise think of $\rho$ as having been developed in a power series in $v$, whose first term is obviously equal to 1 . We then introduce equations (16) and:

$$
\begin{equation*}
\rho=1+v \rho^{\prime}+\frac{v^{2}}{2} \rho^{\prime \prime}+\ldots \tag{17}
\end{equation*}
$$

into (7) and separate the individual powers of $v$, and the following identities will arise:

$$
\begin{align*}
& d l^{\prime}-p d h^{\prime}-q d k^{\prime}-p^{\prime} d h-q^{\prime} d k=\rho^{\prime}(d l-p d h-q d k),  \tag{18}\\
& d l^{\prime \prime}-p d h^{\prime \prime}-q d k^{\prime \prime}-2\left(p^{\prime} d h^{\prime}+q^{\prime} d k^{\prime}\right)-p^{\prime \prime} d h-q^{\prime \prime} d k=\rho^{\prime \prime}(d l-p d h-q d k) \tag{19}
\end{align*}
$$

etc. In order to satisfy (18), we set:

$$
\begin{equation*}
W=p h^{\prime}+q k^{\prime}-l^{\prime} \tag{20}
\end{equation*}
$$

and obtain:

$$
d W=h^{\prime} d p+k^{\prime} d p-p^{\prime} d h-q^{\prime} d k-\rho^{\prime}(d l-p d h-q d k) .
$$

Therefore, if $W=W(l, h, k, p, q)$ is an arbitrary function of the five original coordinates then the last equation will split into the five following ones:

$$
\begin{equation*}
-\rho^{\prime}=W_{l}, \quad-p^{\prime}+\rho^{\prime} p=W_{h}, \quad-q^{\prime}+\rho^{\prime} q=W_{k}, \quad h^{\prime}=W_{p}, \quad k^{\prime}=W_{q} \tag{21}
\end{equation*}
$$

which, along with (20), express the five quantities $l^{\prime}, h^{\prime}, k^{\prime}, p^{\prime}, q^{\prime}$, and $\rho^{\prime}$ explicitly in terms of $W$ and its derivatives. $W$ is the generating function or "characteristic function" of the contact function, or more precisely, its first-order term.

Similarly, with:

$$
\begin{gathered}
W^{\prime}=p h^{\prime \prime}+q k^{\prime \prime}-l^{\prime \prime}, \\
d W^{\prime}=h^{\prime \prime} d p+k^{\prime \prime} d p-p^{\prime \prime} d h-q^{\prime \prime} d k-2\left(p^{\prime} d h^{\prime}+q^{\prime} d k^{\prime}\right)-\rho^{\prime \prime}(d l-p d h-q d k),
\end{gathered}
$$

the decomposition of that equation will yield the five new quantities $l^{\prime \prime}, h^{\prime \prime}, k^{\prime \prime}, p^{\prime \prime}, q^{\prime \prime}$, and $\rho^{\prime \prime}$, as expressed in terms of differential operations on the arbitrary function $W^{\prime}$ (the characteristic function of the second-order terms) and the already-known ones $h^{\prime}$ and $k^{\prime}$. One proceeds in the same way with the terms of higher order.

Should the mapping equations be free of $l$ for the four actual ray coordinates $h, k, p, q$, then the individual $h^{\prime}, k^{\prime}, p^{\prime}, q^{\prime}, h^{\prime \prime}, k^{\prime \prime}, p^{\prime \prime}, q^{\prime \prime}, \ldots$ will be free of $l$. When the four expressions:

$$
h^{\prime}=W_{p}, \quad k^{\prime}=W_{q}, \quad p^{\prime}=-W_{p}-p W_{l}, \quad q^{\prime}=-W_{k}-q W_{l}
$$

are differentiated with respect to $l$, they will give:

$$
0=W_{p l}=W_{q l}, \quad 0=W_{h l}+p W_{l l}=W_{k l}+q W_{l l},
$$

and when one differentiates the last two equations with respect to $p$ ( $q$, resp.) and considers the first one:

$$
0=W_{l l}=W_{l h}=W_{l k}=W_{l p}=W_{l q},
$$

$W_{l}=-\rho^{\prime}$ will be a constant. The predicted result that from (17), $\rho=n / N$ will be constant implies the following Ansatz for the characteristic functions:

$$
\begin{equation*}
W=-\rho^{\prime} l+V(h, k, p, q), \tag{22}
\end{equation*}
$$

$$
W^{\prime}=-\rho^{\prime \prime} l+V^{\prime}(h, k, p, q)
$$

etc. The mapping equations (21) will then go to:

$$
\begin{equation*}
h^{\prime}=V_{p}, \quad k^{\prime}=V_{q}, \quad p^{\prime}=\rho^{\prime} p-V_{h}, \quad q^{\prime}=\rho^{\prime} q-V_{k}, \tag{23}
\end{equation*}
$$

in which $\rho^{\prime}$ is a constant, $V$ is an arbitrary function of the quantities $h, k, p, q$, which we would like to refer to as the generating or characteristic function of the infinitesimal ray map. The constant $\rho^{\prime}$ can be equal to zero when the former and latter refraction exponents are equal to each other. If it is non-zero then it can be replaced with a certain value (say, $\rho^{\prime}=-1$ ) with no loss of generality, as long as one changes the notation for the infinitely-small increment $v$.

If an infinitesimal contact transformation is known completely (i.e., all five of its variables $l$, $h, k, p, q$ ) then one will find the characteristic function $W$ simply by applying equation (20). By contrast, if only an infinitesimal map in the four variables $h, k, p, q$ is present then the search for the generating function $V$ will be achieved by a quadrature. Namely, from (23), one has to choose the constant $\rho^{\prime}$ in such a way that:

$$
\begin{equation*}
d V=h^{\prime} d p+k^{\prime} d q+\left(\rho^{\prime} p-p^{\prime}\right) d h+\left(\rho^{\prime} q-q^{\prime}\right) d k \tag{24}
\end{equation*}
$$

is a complete differential whose integration yields the function $V$. Moreover, in what follows, the additive constant that one can add to $V$ will always be dropped.

If the map exists in finite form then one can also derive $V$ from its eikonal. If the map possesses, e.g., the eikonal $E_{1}=E(p, q, P, Q)$, then:

$$
d E_{1}=-n(h d p+k d q)+N(H d P+K d Q),
$$

and:

$$
\begin{equation*}
E_{1}-h(N P-n p)-k(N Q-n q)=F, \tag{25}
\end{equation*}
$$

with

$$
d F=N(H-h) d P+N(K-k) d Q-(N P-n p) d h-(N Q-n q) d k
$$

If we substitute the expressions (16) and (17) then that will make:

$$
d F=N v d V+\ldots
$$

One then expresses $F$ as a function of $h, k, p, q$ with the help of the mapping equations, goes over to the infinitesimal map in a manner that is prescribed on a case-by-case basis, and divides by $N$ $v$. In the event that $E_{1}$ does not exist, the connection between $F$ and one of the other eikonals is mediated by equations (15) and (25).

The composition of arbitrarily many infinitesimal mappings comes about by simple superposition in the first-order terms, whereby the sequence of components is also irrelevant. The mappings with the characteristic quantities $\rho_{1}^{\prime} V_{1}, \rho_{2}^{\prime} V_{2}$, etc., then compose into a map with the characteristic quantities:

$$
\rho^{\prime}=\rho_{1}^{\prime}+\rho_{2}^{\prime}+\ldots, \quad V=V_{1}+V_{2}+\ldots
$$

## 4.

As a first example, we shall look for the characteristic function for those infinitesimal maps that correspond to just a change in coordinate system and linear scale, or as one says, as a rule: The generating function for infinitesimal motions and similarity transformations. It is obvious that the map will satisfy MALUS's theorem.

The aforementioned transformation takes each point $\xi, \eta, \zeta$ in space to a neighboring one $\xi+$ $v \xi^{\prime}, \eta+v \eta^{\prime}, \zeta+v \zeta^{\prime}$, where:

$$
\begin{aligned}
& \xi^{\prime}=a+\beta \zeta-\gamma \eta+\delta \xi, \\
& \eta^{\prime}=b+\gamma \xi-\alpha \zeta+\delta \eta, \\
& \zeta^{\prime}=c+\alpha \eta-\beta \xi+\delta \zeta,
\end{aligned}
$$

and it can be decomposed into an infinitesimal translation along the segment $v a, v b, v c$, an infinitesimal rotation through the angle $v$ around the axis with the direction cosines $\alpha, \beta, \gamma$, and finally, a constant dilatation of all spatial dimensions with a scale ratio of $1:(1+\delta v)$.

In order to find the change in ray coordinates from that, I shall transform two points with the coordinates $\boldsymbol{\xi}, \eta, \zeta$, and:

$$
\xi_{1}=\xi+x, \quad \eta_{1}=\eta+y, \quad \zeta_{1}=\zeta+z
$$

The line that goes through them has the ray coordinates $h, k, p, q$, where:

$$
\frac{m}{x}=\frac{p}{y}=\frac{q}{z}=\frac{1}{r}, \quad r^{2}=x^{2}+y^{2}+z^{2}, \quad h=\eta-\xi \frac{p}{m}, \quad k=\zeta-\xi \frac{q}{m} .
$$

The changes in $x, y, z$ then prove to be:

$$
\begin{aligned}
& x^{\prime}=\beta z-\gamma y+\delta x, \\
& y^{\prime}=\gamma x-\alpha z+\delta y, \\
& z^{\prime}=\alpha y-\beta x+\delta z, \\
& r^{\prime}=\quad+\delta r,
\end{aligned}
$$

from which:

$$
m^{\prime}=\left(\frac{x}{r}\right)^{\prime}=\frac{r x^{\prime}-x r^{\prime}}{r^{2}}=\frac{\beta z-\gamma y}{r},
$$

or

$$
\begin{equation*}
m^{\prime}=\beta q-\gamma p, \quad p^{\prime}=\gamma m-\alpha q, \quad q^{\prime}=\alpha p-\beta m \tag{26}
\end{equation*}
$$

For the changes in $h$ and $k$, an intermediate calculation will yield:

$$
\left\{\begin{array}{l}
h^{\prime}=b-a \frac{p}{m}+\delta h-\alpha k+\frac{p}{m}(\gamma h-\beta k),  \tag{27}\\
k^{\prime}=c-a \frac{q}{m}+\delta k+\alpha h+\frac{q}{m}(\gamma h-\beta k) .
\end{array}\right.
$$

If one then takes the differential (24) then it will follow that:

$$
d V=d[a m+b p+c q-h(\gamma m-\alpha q)-k(\alpha p-\beta m)]+\delta(h d p+k d q)+\rho^{\prime}(p d h+q d k) .
$$

The constant $\rho^{\prime}$ is then equal to $\delta$, and the characteristic function:

$$
\begin{equation*}
V=a m+b p+c q+\alpha(q h-p k)+\beta m k-\gamma m h+\delta(p h+q k) . \tag{28}
\end{equation*}
$$

A second example leads us to the domain of our present investigation: viz., the characteristic function for refraction at a surface.

If $n$ and $N$ are the indices of refraction of the media that are separated by the surface then the equation of the surface, which we will think of as being written in the form $x=x(y, z)$, for the time being, can also include the ratio $n / N$, so when it is developed in powers of $v$, it will assume the form:

$$
x=\varphi(y, z)+v \psi(y, z)+\ldots
$$

One immediately sees that for terms of first order in the infinitesimal map, only the first terms in that series:

$$
\begin{equation*}
x=\varphi(y, z) \tag{29}
\end{equation*}
$$

will come under consideration. If we then briefly refer to equation (29) as the equation of the refracting surface then that will really mean that the refracting surface assumes the form (29) when $\lim n / N=1$.

If lowercase symbols refer to the incident ray, while uppercase ones refer to the refracted one, then one will have the equations:

$$
\left\{\begin{align*}
h=y-x \frac{p}{m}, & k=z-x \frac{q}{m}  \tag{30}\\
H=y-x \frac{p}{m}, & K=z-x \frac{Q}{m}
\end{align*}\right.
$$

which say that both rays intersect the surface at the point $x, y, z$. In addition, the law of refraction or the principle of shortest light-time will imply the two relations:

$$
\begin{equation*}
\frac{N M-n m}{-1}=\frac{N P-n p}{\varphi_{y}}=\frac{N Q-n q}{\varphi_{z}} \tag{31}
\end{equation*}
$$

The six equations that are obtained by eliminating the parameters $y, z$ represent the desired ray map. If one goes to the case of infinitesimal maps then one will have:

$$
\begin{aligned}
N M-n m & =N\left(m+v m^{\prime}+\ldots\right)-m N\left(1+v \rho^{\prime}+\ldots\right) \\
& =N v\left(m-\rho^{\prime} m\right)+\ldots,
\end{aligned}
$$

and equations (30) and (31) will become:

$$
\left\{\begin{array}{c}
h^{\prime}=-x\left(\frac{p}{m}\right)^{\prime}=x \frac{p m^{\prime}-m p^{\prime}}{m^{2}}, \quad k^{\prime}=x \frac{q m^{\prime}-m q^{\prime}}{m^{2}}  \tag{32}\\
\frac{m^{\prime}-\rho^{\prime} m}{-1}=\frac{p^{\prime}-\rho^{\prime} p}{\varphi_{y}}=\frac{q^{\prime}-\rho^{\prime} q}{\varphi_{z}}
\end{array}\right.
$$

into which the first two of equations (30) or equations (2) enter. Since:

$$
m m^{\prime}+p p^{\prime}+q q^{\prime}=0
$$

it will follow that:

$$
\begin{aligned}
m^{\prime}-\rho^{\prime} m & =\frac{\rho^{\prime}}{m-p \varphi_{y}-q \varphi_{z}}, \\
h^{\prime} d p+k^{\prime} d q & =-\frac{x}{m}\left(m^{\prime} d m+p^{\prime} d p+q^{\prime} d q\right) \\
& =-\frac{x}{m}\left(m^{\prime}-\rho^{\prime} m\right)\left(d m-\varphi_{y} d p-\varphi_{z} d q\right)
\end{aligned}
$$

If we then form the differential (24) that we will have:

$$
\begin{aligned}
d V & =-\frac{x}{m}\left(m^{\prime}-\rho^{\prime} m\right)\left(d m-\varphi_{y} d p-\varphi_{z} d q\right)+\left(m^{\prime}-\rho^{\prime} m\right)\left(\varphi_{y} d p+\varphi_{z} d q\right), \\
\frac{d V}{m^{\prime}-\rho^{\prime} m} & =\varphi_{y}\left(d h+\frac{x}{m} d p\right)+\varphi_{z}\left(d k+\frac{x}{m} d q\right)-\frac{x}{m} d m
\end{aligned}
$$

If we introduce the differentials of $y$ and $z$, instead of those of $h$ and $k$, using (2), so we set:

$$
\begin{aligned}
& d h=d y-\frac{x}{m} d p-p d\left(\frac{x}{m}\right), \\
& d k=d z-\frac{x}{m} d q-q d\left(\frac{x}{m}\right),
\end{aligned}
$$

then:
so

$$
\begin{aligned}
\frac{d V}{m^{\prime}-\rho^{\prime} m} & =\varphi_{y}\left[d y-p d\left(\frac{x}{m}\right)\right]+\varphi_{z}\left[d z-q d\left(\frac{x}{m}\right)\right]-\frac{x}{m} d m \\
& =d x-\frac{x}{m} d m-p \varphi_{y} d\left(\frac{x}{m}\right)+q \varphi_{z} d\left(\frac{x}{m}\right) \\
& =\left(m-p \varphi_{y}-q \varphi_{z}\right) d\left(\frac{x}{m}\right)
\end{aligned}
$$

$$
d V=-\rho^{\prime} d\left(\frac{x}{m}\right)
$$

and therefore:

$$
\begin{equation*}
V=-\rho^{\prime} v, \quad v=\frac{x}{m} . \tag{33}
\end{equation*}
$$

The generating function is then rather simple, but initially seems to be expressed in terms of $p, q$, $y, z$. They are replaced with $p, q, h, k$ with the help of equations (2). If one writes:

$$
\begin{equation*}
x=m v, \quad y=h+p v, \quad z=k+q v \tag{34}
\end{equation*}
$$

then one will know the geometric meaning of $v . v$ is the line segment of the ray $(h, k, p, q)$ that is contained between the base plane and the refracting surface. If one further reverts to the previouslydefined form of equation (29) and writes the refracting surface in the form:

$$
\begin{equation*}
\Phi(x, y, z)=0 \tag{35}
\end{equation*}
$$

then one will get $v$ as a function of $h, k, p, q$ by solving the equation:

$$
\begin{equation*}
\Phi(m v, h+p v, k+q v)=0 \tag{36}
\end{equation*}
$$

for $v$. The mapping equations (2) will be:

$$
\begin{equation*}
h^{\prime}=-\rho^{\prime} v_{p}, \quad p^{\prime}=-\rho^{\prime} v_{q}, \quad p^{\prime}=\rho^{\prime}\left(p+v_{p}\right), \quad q^{\prime}=\rho^{\prime}\left(q+v_{p}\right) . \tag{37}
\end{equation*}
$$

For a composed refraction, one will likewise get:

$$
\begin{gather*}
h^{\prime}=-\sum_{\alpha} \rho_{\alpha}^{\prime} v_{\alpha p}, \quad k^{\prime}=-\sum_{\alpha} \rho_{\alpha}^{\prime} v_{\alpha q},  \tag{38}\\
p^{\prime}=\sum_{\alpha} \rho_{\alpha}^{\prime}\left(p+v_{\alpha h}\right), \quad q^{\prime}=\sum_{\alpha} \rho_{\alpha}^{\prime}\left(q+v_{\alpha q}\right),
\end{gather*}
$$

which are equations that can once more be written in the form (23) when one sets:

$$
\begin{equation*}
\rho^{\prime}=\sum_{\alpha} \rho_{\alpha}^{\prime}, \quad V=-\sum_{\alpha} \rho_{\alpha}^{\prime} v_{\alpha} . \tag{39}
\end{equation*}
$$

In that way, the object space is not indexed with $n, n_{\alpha}$ is the index of the $\alpha^{\text {th }}$ medium, $\Phi_{\alpha}(x, y, z)$ $=0$ is the equation of the $\alpha^{\text {th }}$ refracting surfaces, $v_{\alpha}$ is the solution of the equation:

$$
\Phi_{\alpha}\left(m v_{\alpha}, h+p v_{\alpha}, k+q v_{\alpha}\right)=0
$$

for $v_{\alpha}$, and:

$$
\frac{n_{\alpha-1}}{n_{\alpha}}=1+\rho_{\alpha}^{\prime} v+\cdots,
$$

in which $n$ means an infinitely-small constant that is common to all refractions. The sequence of the refractions is obviously irrelevant.

Some simple examples might clarify the use of these formulas. For a single refraction, one can set $\rho^{\prime}=1$ in equations (37).

For refraction at the plane:

$$
\alpha x+\beta y+\gamma z=\delta,
$$

one will have:

$$
\begin{equation*}
v=\frac{\delta-\beta h-\gamma k}{\alpha m+\beta p+\gamma q} . \tag{40}
\end{equation*}
$$

From (39), the characteristic function for a system of refracting planes (system of prisms) will be:

$$
\begin{equation*}
V=B h+C k-D, \tag{41}
\end{equation*}
$$

where:

$$
B=\sum_{\lambda} \frac{\rho_{\lambda}^{\prime} \beta_{\lambda}}{\alpha_{\lambda} m+\beta_{\lambda} p+\gamma_{\lambda} q}, \quad C=\sum_{\lambda} \frac{\rho_{\lambda}^{\prime} \gamma_{\lambda}}{\alpha_{\lambda} m+\beta_{\lambda} p+\gamma_{\lambda} q}, \quad D=\sum_{\lambda} \frac{\rho_{\lambda}^{\prime} \delta_{\lambda}}{\alpha_{\lambda} m+\beta_{\lambda} p+\gamma_{\lambda} q},
$$

so $V$ is linear in $h$ and $k$.
In order to obtain the characteristic function for refraction at the sphere:

$$
\begin{equation*}
(x-A)^{2}+(y-B)^{2}+(z-C)^{2}=D^{2}, \tag{42}
\end{equation*}
$$

one must solve the equation:

$$
(m v-A)^{2}+(p v+h-B)^{2}+(q v+k-C)^{2}=D^{2}
$$

or

$$
v^{2}-2 v(A m+B p+C q-p h-q k)=D^{2}-A^{2}-(h-B)^{2}-(k-C)^{2}
$$

for $v$. With:

$$
\begin{equation*}
u^{2}=D^{2}-A^{2}-(h-B)^{2}-(k-C)^{2}+(A m+B p+C q-p h-q k)^{2} \tag{43}
\end{equation*}
$$

one will have:

$$
v=A m+B p+C q-p h-q k \pm u
$$

The double-valuedness relates to the two points where the ray $(h, k, p, q)$ cuts the sphere. If we distinguish both values by $v_{1}$ and $v_{2}$ and assign the coefficients:

$$
\rho_{1}^{\prime}=\rho_{2}^{\prime}=\frac{\delta}{2}
$$

to both refractions then, from (39):

$$
\begin{aligned}
V & =\delta(p h+q k-A m-B p-C q) \\
& =\delta(p h+q k)+A m+b p+c q,
\end{aligned}
$$

when:

$$
\begin{equation*}
A=-\frac{a}{\delta}, \quad B=-\frac{b}{\delta}, \quad C=-\frac{c}{\delta} . \tag{44}
\end{equation*}
$$

From (28), the $V$ that is obtained will generate a translation and a dilatation. Those two infinitesimal transformations can then be replaced with a two-fold refraction at a spherical surface. The fact that this theorem cannot be adapted to finite refractions explains what was said about infinitesimal maps in the introduction.

A translation without dilatation ( $\delta=0$ ) can be generated by a four-fold refraction at two spherical surfaces. If $A, B, C$ and $A, B, C$ are coordinates of the centers of two spheres then one can combine the two characteristic functions:

$$
V=\delta(p h+q k-A m-B p-C q)
$$

and

$$
V^{\prime}=\delta\left(p h+q k-A^{\prime} m-B^{\prime} p-C^{\prime} q\right)
$$

with the factors +1 and -1 into:

$$
\begin{aligned}
V-V^{\prime} & =\delta\left(A^{\prime}-A\right) m+\delta\left(B^{\prime}-B\right) p+\delta\left(C^{\prime}-C\right) q \\
& =a m+b p+c q .
\end{aligned}
$$

If $a, b, c$ are given then one choose $\delta(\neq 0), A^{\prime}, B^{\prime}, C^{\prime}$ arbitrarily and find $A, B, C$ from that. Equations (44) will again be true with $A^{\prime}=B^{\prime}=C^{\prime}=0$.

## 5.

The derivation of an infinitesimal composition of refractions that was spoken of up to now assumes that the refraction exponents differed by only infinitely little, so one would be free to specify them. That method will no longer be practicable when some of the $n_{\alpha}$ possess definite numerical values. An infinitesimal map can come about in that way only when one can place the available surfaces infinitely close to each other. All cases of that type can be superposed in the simplest way, which we formulate thus:

Two infinitely-close surfaces enclose a medium whose index of refraction is $N$, while the external space possesses the index $n=1$. The ray will be refracted twice while going through the infinitely-thin layer, such that its third (i.e., final) position will differ infinitely little from its first (i.e., initial) position. We wish to express that infinitesimal map in ray coordinates and seek its characteristic function.

The two refracting surfaces are called $S, T$. Let their equations be:
(S) $x=\varphi(y, z) \quad$ and $\quad$ (T) $\quad x=\varphi(y, z)+v \psi(y, z)$.

The light-path $A B C D$ cuts the surface $S$ at $B$ and the surface $T$ at $C$. Let the coordinates of $B$ be $x$, $y, z$, and let those of $C$ be $x+v x^{\prime}, y+v y^{\prime}, z+v z^{\prime}$, so one will obviously have:

$$
x+v x^{\prime}=\varphi\left(y+v y^{\prime}, z+v z^{\prime}\right)+v \psi(y, z),
$$

$$
\begin{equation*}
x^{\prime}=\varphi_{y} y^{\prime}+\varphi_{z} z^{\prime}+\psi . \tag{45}
\end{equation*}
$$

The ray assumes three positions, of which, 1 and 3 differ from each other only by quantities of order $v$, but they differ from 2 by a finite amount. With a slight change of the previous notation, let the ray coordinates be:
$(A B): \quad h, \quad k, \quad p, \quad q$,
$(B C): \quad H, \quad K, \quad P, \quad Q$,
$(C D): \quad h+v h^{\prime}, \quad k+v k^{\prime}, \quad p+v p^{\prime}, \quad q+v q^{\prime}$.
The first refraction yields equations (2) and (31), which can be written as:

$$
\begin{equation*}
N M-m=-\lambda, \quad N P-p=-\lambda \varphi_{y}, \quad N Q-q=-\lambda \varphi_{z}, \tag{46}
\end{equation*}
$$

with a proportionality factor $\lambda$. The ray $B C$ yields the two systems of equations:
and

$$
\begin{array}{ll}
H=y-x \frac{P}{M}, & K=z-x \frac{Q}{M},  \tag{47}\\
0=y^{\prime}-x^{\prime} \frac{P}{M}, & 0=z^{\prime}-x^{\prime} \frac{Q}{M},
\end{array}
$$

from which (45) will imply:

$$
\left\{\begin{array}{c}
x^{\prime}=r M, \quad y^{\prime}=r P, \quad z^{\prime}=r R,  \tag{48}\\
r=\frac{\psi}{M-P \varphi_{y}-Q \varphi_{z}} .
\end{array}\right.
$$

The direction cosines of the normal to the surface $T$ at the point $C$ are proportional to:

$$
-1, \quad\left[\varphi_{y}+v \psi_{y}\right], \quad\left[\varphi_{z}+v \psi_{z}\right],
$$

in which $y$ and $z$ are replaced with $y+v y^{\prime}, z+v z^{\prime}$. The second refraction at the point $C$ then yields the equations:

$$
\begin{aligned}
m+v m^{\prime}-N M & =\lambda+v \mu, \\
p+v p^{\prime}-N P & =-(\lambda+v \mu)\left[\varphi_{y}+v\left(\varphi_{y y} y^{\prime}+\varphi_{y z} z^{\prime}+\psi_{y}\right)\right], \\
q+v q^{\prime}-N Q & =-(\lambda+v \mu)\left[\varphi_{z}+v\left(\varphi_{z y} y^{\prime}+\varphi_{z z} z^{\prime}+\psi_{z}\right)\right],
\end{aligned}
$$

which are analogous to (46), in which $\lambda+v \mu$ means a proportionality factor. Combining those equations with (46) gives:

$$
\begin{array}{ll}
m^{\prime}=\mu, & p^{\prime}=-\mu \varphi_{y}-\lambda\left(\varphi_{y y} y^{\prime}+\varphi_{y z} z^{\prime}+\psi_{y}\right), \\
& q^{\prime}=-\mu \varphi_{z}-\lambda\left(\varphi_{z y} y^{\prime}+\varphi_{z z} z^{\prime}+\psi_{z}\right) .
\end{array}
$$

In order to express those equations somewhat more clearly, If shall differentiate the formula:

$$
\begin{equation*}
N^{2}-1=\lambda^{2}\left(1+\varphi_{y}^{2}+\varphi_{z}^{2}\right)-2 \lambda\left(m-p \varphi_{y}-q \varphi_{z}\right) \tag{49}
\end{equation*}
$$

which follows from (46), whereby $\lambda$ will be expressed as a function of $p, q, y, z$, and then obtain for the total differential of $\lambda$ that:

$$
\begin{gathered}
0=d \lambda\left[\lambda\left(1+\varphi_{y}^{2}+\varphi_{z}^{2}\right)-\left(m-p \varphi_{y}-q \varphi_{z}\right)\right] \\
+\lambda^{2}\left(\varphi_{y} \varphi_{y y}+\varphi_{z} \varphi_{z y}\right) d y+\lambda^{2}\left(\varphi_{y} \varphi_{y z}+\varphi_{z} \varphi_{z z}\right) d z \\
+\lambda\left(p \varphi_{y y}+q \varphi_{z y}\right) d y+\lambda\left(p \varphi_{y z}+q \varphi_{z z}\right) d z \\
-\lambda\left(d m-\varphi_{y} d p-\varphi_{z} d q\right),
\end{gathered}
$$

and since:

$$
\lambda\left(1+\varphi_{y}^{2}+\varphi_{z}^{2}\right)-\left(m-p \varphi_{y}-q \varphi_{z}\right)=-N\left(M-P \varphi_{y}-Q \varphi_{z}\right)=-\frac{N \psi}{r}
$$

the former equation can be written more simply as:

$$
\begin{equation*}
\frac{N \psi}{r} d \lambda=N \lambda\left(P \varphi_{y y}+Q \varphi_{z y}\right) d y+N \lambda\left(P \varphi_{y z}+Q \varphi_{z z}\right) d z-\lambda\left(d m-\varphi_{y} d p-\varphi_{z} d q\right) \tag{50}
\end{equation*}
$$

It then follows from this that:

$$
\begin{aligned}
& \psi \lambda_{y}=\lambda r\left(P \varphi_{y y}+Q \varphi_{z y}\right)=\lambda\left(\varphi_{y y} y^{\prime}+\varphi_{z y} z^{\prime}\right), \\
& \psi \lambda_{z}=\lambda r\left(P \varphi_{y z}+Q \varphi_{z z}\right)=\lambda\left(\varphi_{z y} y^{\prime}+\varphi_{z z} z^{\prime}\right),
\end{aligned}
$$

with which the expressions for $m^{\prime}, p^{\prime}, q^{\prime}$ that were obtained above will go to:

$$
\begin{equation*}
m^{\prime}=\mu, \quad p^{\prime}=-\mu \varphi_{y}-(\lambda \psi)_{y}, \quad q^{\prime}=-\mu \varphi_{z}-(\lambda \psi)_{z} \tag{51}
\end{equation*}
$$

Since $m m^{\prime}+p p^{\prime}+q q^{\prime}=0$, one obtains $\mu$ in the form:

$$
\begin{equation*}
\mu=\frac{p(\lambda \psi)_{y}+q(\lambda \psi)_{z}}{m-p \varphi_{y}-q \varphi_{z}} \tag{52}
\end{equation*}
$$

$m^{\prime}, p^{\prime}, q^{\prime}$ are found with that, as expressed in terms of $p, q, y, z$. One gets two equations for $h^{\prime}, k^{\prime}$ from the demand that the $x+v x^{\prime}, \ldots$ lie on the ray $h+v h^{\prime}, \ldots$, namely:

$$
\begin{equation*}
h^{\prime}=y^{\prime}-x^{\prime} \frac{p}{m}-x\left(\frac{p}{m}\right)^{\prime}, \quad k^{\prime}=z^{\prime}-x^{\prime} \frac{q}{m}-x\left(\frac{q}{m}\right)^{\prime} . \tag{53}
\end{equation*}
$$

In order to find the generating function of that map, we set:

$$
d V=h^{\prime} d p+k^{\prime} d q-p^{\prime} d h-q^{\prime} d k
$$

since the constant $\rho^{\prime}$ will vanish due to the equality of the indices of refraction in object and image space. In place of the differentials of $h$ and $k$, we once more introduce the differentials of $y, z$ using (2) and the get:

$$
\begin{aligned}
h^{\prime} d p+k^{\prime} d q & =x^{\prime} d m+y^{\prime} d p+z^{\prime} d q-\frac{x}{m}\left(m^{\prime} d m+p^{\prime} d p+q^{\prime} d q\right) \\
p^{\prime} d h+q^{\prime} d k & =p^{\prime} d y+q^{\prime} d z-\frac{d x}{m}\left(p p^{\prime}+q q^{\prime}\right)-x\left[p^{\prime} d\left(\frac{p}{m}\right)+q^{\prime} d\left(\frac{q}{m}\right)\right] \\
& =m^{\prime} d x+p^{\prime} d y+q^{\prime} d z-\frac{x}{m}\left(m^{\prime} d m+p^{\prime} d p+q^{\prime} d q\right)
\end{aligned}
$$

so

$$
d V=x^{\prime} d m+y^{\prime} d p+z^{\prime} d q-m^{\prime} d x-p^{\prime} d y-q^{\prime} d z
$$

and furthermore, from (48), (46), (50):

$$
\begin{aligned}
x^{\prime} d m+y^{\prime} d p+z^{\prime} d q & =r(M d m+P d p+Q d q) \\
& =\frac{r \lambda}{N}\left(-d m+\varphi_{y} d p+\varphi_{z} d q\right) \\
& =\psi\left(\lambda_{p} d p+\lambda_{q} d q\right)
\end{aligned}
$$

and from (51), we have:

$$
\begin{aligned}
m^{\prime} d x+p^{\prime} d y+q^{\prime} d z & =\mu d x-\mu\left(\varphi_{y} d y+\varphi_{z} d z\right)-(\lambda \psi)_{y} d y-(\lambda \psi)_{z} d z \\
& =-(\lambda \psi)_{y} d y-(\lambda \psi)_{z} d z
\end{aligned}
$$

Accordingly, we will have:

$$
\left.d V=\psi\left(\lambda_{p} d p+\lambda_{q} d q\right)+(\lambda \psi)_{y} d y+(\lambda \psi)_{z} d z\right)=d(\lambda \psi),
$$

since $\lambda$ depends upon $y, z, p, q$, but $\psi$ depends upon only $y, z$.
The generating function of our infinitesimal transformation is then:

$$
\begin{equation*}
V=\lambda \psi, \tag{54}
\end{equation*}
$$

in which $\varphi(y, z)$ and $\psi(y, z)$ are given functions that define the two infinitely-close surfaces:

$$
x=\varphi \quad \text { and } \quad x=\varphi+\nu \psi .
$$

$N$ is the index of refraction of the medium that is contained between them with respect to the external space, and $\lambda$ is the function of $y, z, p, q$ that is defined by (49). $V$ is then known as a function of $y, z, p, q$, and with the help of the equations:

$$
\begin{equation*}
h=y-\varphi \frac{p}{m}, \quad k=z-\varphi \frac{q}{m} \tag{55}
\end{equation*}
$$

as a function of $h, k, p, q$.
In the simplest case that was spoken of (viz., two neighboring surfaces that enclose an infinitely-thin lens), the map with the characteristic function:

$$
V=\sum_{\alpha} \lambda_{\alpha} \psi_{\alpha}
$$

will arise by superposition, in which the functions $\varphi$ and $\psi$ are provided with the index $\alpha$, as well as the index of refraction $N$ and the variables $y, z$. The quantities $y_{\alpha}, z_{\alpha}$ are to be eliminated from each expression $\lambda_{\alpha} \psi_{\alpha}$ with the help of the equations:

$$
h=y_{\alpha}-\varphi_{\alpha} \frac{p}{m}, \quad k=z_{\alpha}-\varphi_{\alpha} \frac{q}{m} .
$$

A special case of this is that the second surface of the foregoing lens is the first surface of the following one, in such a way that we then have a system of infinitely-close surfaces:

$$
x=\varphi, \quad x=\varphi+v \psi_{1}, \quad x=\varphi+v\left(\psi_{1}+\psi_{2}\right), \ldots
$$

before us. All $\varphi_{\alpha}$ then reduce to $\varphi$ (up to terms of higher order), just as $y_{\alpha}, z_{\alpha}$ will reduce to $y, z$, resp. The various functions $\lambda_{\alpha}$ differ by only the index of refraction $N_{\alpha}$, and in addition, the elimination of the quantities $y, z$ is performed on each component $\lambda_{\alpha} \psi_{\alpha}$ using the same equations (55), so it can also be carried out on the total sum.

The case of reflection between two neighboring surfaces deserves especial interest. If we set $N=-1$ in (49) then we will have:

$$
\begin{equation*}
\lambda=2 \frac{m-p \varphi_{y}-q \varphi_{z}}{1+\varphi_{y}^{2}+\varphi_{z}^{2}} \tag{56}
\end{equation*}
$$

As an example of that, we might address the problem of finding the characteristic function for reflection between two plane that define an angle of $v / 2$ with each other and go through the coordinate origin.

The equations of the two planes can be written in the form:

$$
\begin{gathered}
A x+B y+C z=0 \\
\left(A+\frac{v}{2} A^{\prime}\right) x+\left(B+\frac{v}{2} B^{\prime}\right) y+\left(C+\frac{v}{2} C^{\prime}\right) z=0
\end{gathered}
$$

when we establish that:

$$
\begin{gathered}
A^{2}+B^{2}+C^{2}=A^{\prime 2}+B^{\prime 2}+C^{\prime 2}=1, \\
A A^{\prime}+B B^{\prime}+C C^{\prime}=0 .
\end{gathered}
$$

The line of intersection of the two planes has the following direction cosines in this case:

$$
\begin{equation*}
\alpha=B C^{\prime}-C B^{\prime}, \quad \beta=C A^{\prime}-A C^{\prime}, \quad \gamma=A B^{\prime}-B A^{\prime} . \tag{57}
\end{equation*}
$$

For the functions $\varphi$ and $\psi$, one gets:

$$
\begin{aligned}
\varphi & =-\frac{B y+C z}{A}, \\
\varphi+v \psi & =-\frac{B y+C z+\frac{v}{2}\left(B^{\prime} y+C^{\prime} z\right)}{A+\frac{v}{2} A^{\prime}}, \\
2 \psi & =-\frac{B^{\prime} y+C^{\prime} z}{A}+\frac{A^{\prime}}{A^{2}}(B y+C z),
\end{aligned}
$$

and for $\lambda$, (56) gives:

$$
\lambda=2 A(A m+B p+C q) .
$$

In order to eliminate $y$ and $z$, one appeals to the equations:

$$
h=y+\frac{p}{m} \frac{B y+C z}{A}, \quad k=z+\frac{q}{m} \frac{B y+C z}{A},
$$

from which, it follows that:

$$
\begin{aligned}
B y+C z & =A m \cdot \frac{B h+C k}{A m+B p+C q}, \\
B^{\prime} y+C^{\prime} z & =B^{\prime} h+C^{\prime} k-\left(B^{\prime} q+C^{\prime} q\right) \frac{B h+C k}{A m+B p+C q}, \\
2 \psi & =-\frac{B^{\prime} h+C^{\prime} k}{A}+\frac{B h+C k}{A} \frac{A^{\prime} m+B^{\prime} p+C^{\prime} q}{A m+B p+C q} .
\end{aligned}
$$

Therefore, one will have:

$$
V=(B h+C k)\left(A^{\prime} m+B^{\prime} p+C^{\prime} q\right)-\left(B^{\prime} h+C^{\prime} k\right)(A m+B p+C q),
$$

and from (57):

$$
\begin{equation*}
V=\alpha(q h-p k)+\beta m k-\gamma m h . \tag{58}
\end{equation*}
$$

From (28), that is the characteristic function of a rotation around the axis $\alpha, \beta, \gamma$ (viz., the axis of intersection of the two reflecting planes) and through an angle of $v$ (viz., twice the angle of reflection). Here, we are dealing with a case in which the infinitesimal result can actually be adapted to finite maps.

## 6.

The characteristic function $V$ that was presented in the previous section does not generally possess a simple form when one actually performs the elimination of $y$ and $z$, or at least imagines performing it. From the way that it came about, it must satisfy certain higher-order partial differential equations as a function of $h, k, p, q$ whose form will probably be quite complicated. We therefore return to the first process, and thus assume that a finite map that is realized by optical means is an infinitesimal one that is not derived from infinitely-close positions of the (refracting or reflecting) surfaces, but from infinitely-close values of the refraction exponents and call such an infinitesimal map optically-producible in the strict sense. As we have already remarked several times before, that conception of the infinitesimal maps assumes that the values of the indices of refraction enter into the equations of the finite map as undetermined constants, as long as they are different; in that way, one excludes, e.g., reflections. The characteristic function of an opticallyproducible map is:

$$
\begin{equation*}
V=\sum_{\alpha} \sigma_{\alpha} v_{\alpha} \tag{59}
\end{equation*}
$$

and the mapping equations are:

$$
\begin{align*}
h^{\prime}=V_{p}, \quad k^{\prime}=V_{q}, \quad p^{\prime} & =-\sigma p-V_{h}, \quad q^{\prime}=-\sigma q-V_{k},  \tag{60}\\
\sigma & =\sum_{\alpha} \sigma_{\alpha}, \tag{61}
\end{align*}
$$

in which the notation has been changed slightly ( $\sigma=-\rho$ ). The $\sigma_{\alpha}$ are constants the depend upon the ratio of the refraction exponents at the $\alpha^{\text {th }}$ refracting surface and can be written:

$$
\begin{equation*}
v \sigma_{\alpha}=1-\frac{n_{\alpha-1}}{n_{\alpha}}=\delta \log n_{\alpha} \tag{62}
\end{equation*}
$$

If the equation of the $\alpha^{\text {th }}$ refracting surface is:

$$
\Phi_{\alpha}(x, y, z)=0
$$

then we understand $v_{\alpha}$ to mean the solution of the equation:

$$
\Phi_{\alpha}\left(m v_{\alpha}, h+p v_{\alpha}, k+q v_{\alpha}\right)=0
$$

for $v_{\alpha}$.
As solutions of an equation:

$$
\begin{equation*}
\Phi(x, y, z)=0 \tag{35}
\end{equation*}
$$

with the substitutions:

$$
\begin{equation*}
x=m v, \quad y=h+p v, \quad z=k+q v \tag{34}
\end{equation*}
$$

each of the $v$ will satisfy two particular first-order partial differential equations that one can find as follows: A complete differentiation of (35) will yield:

$$
\Phi_{x}(m d v+v d m)+\Phi_{y}(d h+p d v+v d p)+\Phi_{z}(d k+q d v+v d q)=0
$$

or, when one sets $m \Phi_{x}+p \Phi_{y}+q \Phi_{y}=\Phi^{\prime}$, for the moment:

$$
-m \Phi^{\prime} d v=m \Phi_{y}(d h+v d p)+m \Phi_{z}(d k+v d q)-v \Phi_{x}(p d p+q d q)
$$

SO:

$$
\begin{array}{ll}
v_{h}=-\frac{\Phi_{y}}{\Phi^{\prime}}, & v_{k}=-\frac{\Phi_{z}}{\Phi^{\prime}} \\
v_{p}=\frac{v}{m \Phi^{\prime}}\left(p \Phi_{x}-m \Phi_{y}\right), & v_{q}=\frac{v}{m \Phi^{\prime}}\left(q \Phi_{x}-m \Phi_{z}\right)
\end{array}
$$

If one eliminates the ratios $\Phi_{x}: \Phi_{y}: \Phi_{z}$ from these four equations then the two differential equations will follow:
or

$$
\begin{align*}
& \left\{\begin{array}{l}
\frac{v_{p}}{v}=v_{h}+\frac{p}{m^{2}}\left(1+p v_{h}+q v_{k}\right) \\
\frac{v_{q}}{v}=v_{k}+\frac{q}{m^{2}}\left(1+p v_{h}+q v_{k}\right)
\end{array}\right.  \tag{63}\\
& \left\{\begin{array}{l}
v v_{h}=v_{p}-p\left(v+p v_{p}+q v_{q}\right) \\
v v_{k}=v_{q}-q\left(v+p v_{p}+q v_{q}\right)
\end{array}\right. \tag{64}
\end{align*}
$$

The last two imply the integrability condition:

$$
0=\frac{\partial}{\partial k}\left[v_{p}-p\left(v+p v_{p}+q v_{q}\right)\right]-\frac{\partial}{\partial h}\left[v_{q}-q\left(v+p v_{p}+q v_{q}\right)\right]
$$

or

$$
\begin{equation*}
0=\Delta(v)=v_{p k}-p\left(v_{k}+p v_{p k}+q v_{q k}\right)-v_{q h}+q\left(v_{h}+p v_{p h}+q v_{q h}\right) . \tag{65}
\end{equation*}
$$

That last second-order equation is linear and homogeneous, so it will be fulfilled by not only each individual $v_{\alpha}$, but also by:

$$
V=\sum_{\alpha} \sigma_{\alpha} v_{\alpha}
$$

The characteristic function V of any optically-producible map satisfies the second-order linear differential equation:

$$
\Delta(V)=0 .
$$

With that, the question of whether every ray map that satisfies MALUS's theorem can be realized by optical means has been resolved in the negative sense.

If we apply the necessary (but not also sufficient) criterion for a map to be optically-producible to the characteristic function (58) of an infinitely-small rotation then it will follow that:

$$
\Delta(q h-p k)=-2 m^{2}, \quad \Delta(m k)=-2 m p, \quad \Delta(-m h)=-2 m q,
$$

so

$$
\begin{equation*}
\Delta(V)=-2 m(\alpha m+\beta p+\gamma q), \tag{66}
\end{equation*}
$$

i.e., an infinitesimal rotation is not optically producible (in the manner that was described here).

Let us add a remark in passing: Up to now, we have used the same rectangular system of coordinate axes as a basis for both object space and image space. Nothing prevents us from doing that when the equations of the map are derived from geometric or physical data. Conversely, if we have a given map, whether it takes the form of an eikonal or a characteristic function $V$ or as a system of four equations, then we must also consider the possibility that the image space might be referred to a different system of axes than the object space. A translation, a rotation, and possibly a dilatation will enter into the optical elements of the map, and the characteristic function $V$ will contain the terms (28), in addition to the optically-producible part. From that, one can use (66) to identify the rotational term with the help of the operation $\Delta(V)$, and thus remove it afterwards, while translation and similarity belong to the optically-producible (infinitesimal) maps.

As was pointed out, the criterion $\Delta(V)=0$ is necessary, but not sufficient. A sufficient criterion for whether a given function $V(h, k, p, q)$ can be represented as a linear combination of functions $v_{\alpha}$ that each satisfy the equations (64) might be very hard to find and would also have no interest for optics itself, since the optical producibility of infinitesimal maps does not imply that of the finite ones. It is in the nature of contemporary investigations that they mainly yield negative results. The most important result of that type is the impossibility of an ideal telescope objective, whose proof we shall now go on to (with the restriction that we envisioned).
7.

A congruence of rays (i.e., a two-fold manifold of rays) is called centric when all rays go through the same point, namely, the vertex of the congruence. A centric congruence in object space that is, in turn, transformed into a centric congruence in image space together define a dicentric congruence of light paths. The vertices of the two ray congruences define a pair of conjugate dicentric points. A finite or infinite number of dicentric points can be present. In the last case, they will define dicentric lines, surfaces, or bodies in some situations. The appearance of dicentric surfaces with dicentric bodies is the most important case in optics and shall be referred to as aplanarity. The map, the surfaces, and the points of the surfaces are called aplanatic $\left({ }^{1}\right)$.

A special case of the centric congruence is the parallel congruence. In that case, the vertex is a point at infinity. If any parallel congruence in object space is taken to a centric congruence in image space then the aplanatic surface in object space will be the plane at infinity. That map can be referred to as an aplanatic map of the plane at infinity and its optical realization will be the ideal telescope objective. The case in which the aplanatic image surface is also at infinity, so the parallel congruence goes to a parallel congruence, is realized by any set of prisms. If the aplanatic object surface lies at finite points and the image surface lies at infinity then we have the ideal collimator, which is the inverse of the ideal telescope. Finally, if both aplanatic surfaces lie at finite points then we have the ideal microscope, and for a coarse classification of them, it is inessential to know what further demands (e.g., planarity of the image, correct picture, etc.) have been placed upon the optical instrument. For the sake of brevity, the different types of aplanarity shall then be referred to as microscopic (viz., object and image at finite points), telescopic (viz., object at infinity, image at finite points), and prismatic aplanarity (viz., object and image at infinity).

We now seek the characteristic functions of the aplanatic infinitesimal maps, and indeed first the microscopic ones. The equations:

$$
\begin{equation*}
h=y-x \frac{p}{m}, \quad k=z-x \frac{q}{m}, \tag{2}
\end{equation*}
$$

exist between the ray coordinates of a centric congruence with vertex $x, y, z$. The image of that congruence has the ray coordinates $h+v h^{\prime}, \ldots$, and let it be likewise centric. Its vertex will possess the coordinates $x+v x^{\prime}, y+v y^{\prime}, z+v z^{\prime}$, which differ from $x, y, z$ only infinitely little, and we will have the equation:

$$
\begin{equation*}
h^{\prime}=y^{\prime}-x^{\prime} \frac{p}{m}-x\left(\frac{p}{m}\right)^{\prime} \quad k^{\prime}=z^{\prime}-x^{\prime} \frac{q}{m}-x\left(\frac{q}{m}\right)^{\prime} . \tag{53}
\end{equation*}
$$

Should the points $x, y, z$ and $x+v x^{\prime}, y+v y^{\prime}, z+v z^{\prime}$ define a surface then one could think of $x, y, z$ and $x^{\prime}, y^{\prime}, z^{\prime}$ as functions of two parameters $\beta, \gamma$ that we can introduce in place of $h, k$ as ray

[^2]coordinates with the help of equations (2). From (60), if $\sigma$ is an arbitrary constant then the differential of the characteristic function will be:
$$
d V=h^{\prime} d p+k^{\prime} d q-p^{\prime} d h-q^{\prime} d k-\sigma(p d h+q d k),
$$
and furthermore:
\[

$$
\begin{aligned}
& h^{\prime} d p+k^{\prime} d q=x^{\prime} d m+y^{\prime} d p+z^{\prime} d q-\frac{x}{m}\left(m^{\prime} d m+p^{\prime} d p+q^{\prime} d q\right), \\
& p^{\prime} d h+q^{\prime} d k=m^{\prime} d x+p^{\prime} d y+q^{\prime} d z-\frac{x}{m}\left(m^{\prime} d m+p^{\prime} d p+q^{\prime} d q\right), \\
& p d h+q d k=m d x+p d y+q d z-d\left(\frac{x}{m}\right),
\end{aligned}
$$
\]

and accordingly:

$$
d V=x^{\prime} d m+y^{\prime} d p+z^{\prime} d q-m^{\prime} d x-p^{\prime} d y-q^{\prime} d z+\sigma d\left(\frac{x}{m}\right)-\sigma(m d x+p d y+q d z)
$$

If we then set:

$$
\begin{equation*}
V=\sigma \frac{x}{m}+x^{\prime} m+y^{\prime} p+z^{\prime} q+\varphi \tag{67}
\end{equation*}
$$

then:

$$
-d \varphi=m d x^{\prime}+p d y^{\prime}+q d z^{\prime}+m^{\prime} d x+p^{\prime} d y+q^{\prime} d z+\sigma(m d x+p d y+q d z) .
$$

If we imagine replacing $x, y, z, x^{\prime}, y^{\prime}, z^{\prime}$ with their values in terms of $\beta, \gamma$ then $d \varphi$ will become a linear combination of $d \beta$ and $d \gamma$, so $\varphi$ will be merely a function of $\beta$ and $\gamma$. One then obtains the characteristic function of the present infinitesimal microscopic-aplanatic map when one eliminates the parameters $\beta, \gamma$ from (67), in which $\varphi$ is an arbitrary function of $\beta, \gamma$, and equations (2). Due to the arbitrariness of $\varphi$, one can then generate the aplanatic map in an infinitude of ways, even when the aplanatic surface and the point-relationship between the conjugate vertices of the congruences is prescribed.

For the telescopic-aplanatic maps, the Ansatz is inadmissible in that form, since the $x, y, z$ will become infinite. The characteristic function of such a thing shall be derived in the next $\S$.

First, we ask which microscopic-aplanatic maps are optically producible. For that, the aplanatic object surface shall be taken in the form:

$$
x=x(y, z),
$$

so $y, z$ are chosen to be the independent parameters $\beta, \gamma$, resp.; $x, \varphi, x^{\prime}, y^{\prime}, z^{\prime}$ will then be functions of $y, z$. In the case of an optically-producible map, equations (59), (61), (64) are true, which we
would like to convert by introducing $y, z, p, q$ in place of $h, k, p, q$. If we set the function $v_{\alpha}$, which is deduced from the equation:

$$
\Phi_{\alpha}\left(m v_{\alpha}, h+p v_{\alpha}, k+q v_{\alpha}\right)=0
$$

equal to $x / m+w_{\alpha}$ then we will have:

$$
\begin{equation*}
v_{\alpha}=\frac{x}{m}+w_{\alpha}, \quad V=\sigma \frac{x}{m}+W, \quad \sigma=\sum_{\alpha} \sigma_{\alpha}, \quad W=\sum_{\alpha} \sigma_{\alpha} w_{\alpha}, \tag{68}
\end{equation*}
$$

and $w_{\alpha}$ is obtained as a function of $p, q, y, z$ directly from the equation:

$$
\Phi_{\alpha}\left(x+m w_{\alpha}, y+p w_{\alpha}, z+q w_{\alpha}\right)=0 .
$$

However, for the function $w\left({ }^{1}\right)$ that is defined by an equation:

$$
\begin{equation*}
\Phi(x+m w, y+p w, z+q w)=0 \tag{69}
\end{equation*}
$$

there exist two first-order differential equations that are derived in a manner that is similar to (63) or (64) when one differentiates (69) and eliminates the ratios of the derivatives of $\Phi$. The result will be:

$$
\left\{\begin{array}{l}
w w_{y}=w_{p}-\left(m x_{y}+p\right)\left(w+p w_{p}+q w_{q}\right),  \tag{70}\\
w w_{z}=w_{q}-\left(m x_{z}+p\right)\left(w+p w_{p}+q w_{q}\right) .
\end{array}\right.
$$

The integrability condition that follows from this is:

$$
E(w)=w_{p z}-\left(m x_{y}+p\right)\left(w_{z}+p w_{p z}+q w_{q z}\right)-w_{q y}-\left(m x_{z}+p\right)\left(w_{y}+p w_{p z}+q w_{q y}\right)=0,
$$

which is a homogeneous linear differential equation that is satisfied by $w_{\alpha}$, along with:

$$
W=\sum_{\alpha} \sigma_{\alpha} w_{\alpha}
$$

On the other hand, from (67) and (68):

$$
W=m x^{\prime}+p y^{\prime}+q z^{\prime}+\varphi,
$$

where $x^{\prime}, y^{\prime}, z^{\prime}, \varphi$ are functions of $y, z$. The function of $y, z$ that is defined in that way must satisfy the equation $E(W)=0$ identically. One now finds:

[^3]\[

$$
\begin{gathered}
W_{p}=y^{\prime}-x^{\prime} \frac{p}{m}, \quad W_{q}=z^{\prime}-x^{\prime} \frac{q}{m}, \\
W+p W_{p}+q W_{q}=2\left(m x^{\prime}+p y^{\prime}+q z^{\prime}\right)+\varphi-\frac{x}{m}, \\
W_{p}-\left(p+m x_{y}\right)\left(W+p W_{p}+q W_{q}\right)=y^{\prime}+x^{\prime} x_{y}-\varphi\left(m x_{y}+p\right)-2\left(m x_{z}+q\right)\left(m x^{\prime}+p y^{\prime}+q z^{\prime}\right), \\
W_{q}-\left(p+m x_{z}\right)\left(W+p W_{p}+q W_{q}\right)=z^{\prime}+x^{\prime} x_{z}-\varphi\left(m x_{z}+p\right)-2\left(m x_{z}+q\right)\left(m x^{\prime}+p y^{\prime}+q z^{\prime}\right) . \\
y_{z}^{\prime}+x_{z}^{\prime} x_{y}-\varphi_{z}\left(m x_{y}+p\right)-2\left(m x_{y}+p\right)\left(m x_{z}^{\prime}+p y_{z}^{\prime}+q z_{z}^{\prime}\right) \\
=z_{z}^{\prime}+x_{y}^{\prime} x_{z}-\varphi_{y}\left(m x_{z}+q\right)-2\left(m x_{z}+q\right)\left(m x_{y}^{\prime}+p y_{y}^{\prime}+q z_{y}^{\prime}\right) .
\end{gathered}
$$
\]

However, an identity of the form:

$$
\begin{equation*}
\alpha m+\beta p+\gamma q+a m^{2}+b p^{2}+c q^{2}+A p q+B q m+C m p=d \tag{72}
\end{equation*}
$$

decomposes into the individual equations:

$$
\alpha=\beta=\gamma=0, \quad a=b=c=d, \quad A=B=C=0 .
$$

We then obtain the following equations from (71) whose origin gives the powers of $m, p, q$ that are enclosed in parentheses:

$$
(p) \quad \varphi_{z}=0, \quad(q) \quad \varphi_{y}=0, \quad \text { so } \quad \varphi=0,
$$

in which nothing takes on an additive constant, and:

$$
(p q) \quad z_{z}^{\prime}=y_{y}^{\prime}, \quad(q m) \quad x_{y} z_{z}^{\prime}=x_{z} z_{y}^{\prime}+x_{y}^{\prime}, \quad(m p) \quad x_{y} y_{z}^{\prime}+x_{z}^{\prime}=x_{z} y_{y}^{\prime}
$$

from which, if $\psi$ means a function of $y, z$ then:

$$
\begin{equation*}
y^{\prime}=\psi_{z}, \quad z^{\prime}=\psi_{y}, \quad x_{y}^{\prime}=x_{y} \psi_{y z}-x_{z} \psi_{y y}, \quad x_{z}^{\prime}=x_{z} \psi_{y z}-x_{y} \psi_{z z} . \tag{73}
\end{equation*}
$$

Finally:

$$
\left(1, m^{2}, p^{2}, q^{2}\right) \quad y_{z}^{\prime}+x_{y} x_{z}^{\prime}-z_{y}^{\prime}-x_{z} x_{y}^{\prime}=+2\left(x_{y} x_{z}^{\prime}-x_{z} x_{y}^{\prime}\right)=2 y_{z}^{\prime}=-2 z_{y}^{\prime}
$$

which are three equations that reduce to:

$$
y_{z}^{\prime}=z_{y}^{\prime}=x_{y} x_{z}^{\prime}-x_{z} x_{y}^{\prime}=0,
$$

and from (73), to:

$$
\psi_{y y}=\psi_{z z}=0 .
$$

With the constants $\beta, \gamma, \delta$, the function $\psi$ will then have the form:

$$
\psi=\gamma y+\beta z+\delta y z,
$$

from which, it will follow from (73) that:

$$
y^{\prime}=\beta+\delta y, \quad z^{\prime}=\gamma+\delta z, \quad x_{y}^{\prime}=\delta x_{y}, \quad x_{z}^{\prime}=\delta x_{z},
$$

and with a new constant $\alpha$ :

$$
x^{\prime}=\alpha+\delta x .
$$

If we write:

$$
\alpha=-\delta a, \quad \beta=-\delta b, \quad \gamma=-\delta c
$$

then we will have the relations:

$$
\begin{equation*}
x^{\prime}=\delta(x-a), \quad y^{\prime}=\delta(y-b), \quad z^{\prime}=\delta(z-c) \tag{74}
\end{equation*}
$$

between the conjugate aplanatic points, which expresses a constant expansion of all radius vectors from the point $a, b, c$ with a ratio of $1:(1+v \delta)$. This case, which is irrelevant in optical practice, is realized by concentric aplanatic spheres for which the refracting surfaces are likewise concentric spheres with certain radii that depend upon the refraction exponents. In the case of one refraction, let $\rho$ be the radius of the refracting sphere, so one will get the aplanatic surfaces in the form of concentric spheres with the radii:

$$
r=\rho \frac{N}{n} \quad \text { and } \quad R=\rho \frac{n}{N} .
$$

If the refraction is infinitesimal then we set:

$$
\frac{n}{N}=1-\sigma v+\cdots, \quad \frac{n^{2}}{N^{2}}=1-2 \sigma v+\cdots
$$

The equation of the aplanatic object surface here is:

$$
(x-a)^{2}+(y-b)^{2}+(z-c)^{2}-r^{2}=0,
$$

when $a, b, c$ are the coordinates of the center of the sphere. The equation of the refracting surface is:

$$
\Phi(\xi, \eta, \zeta)=(\xi-a)^{2}+(\eta-b)^{2}+(\zeta-c)^{2}-r^{2}(1-2 \sigma v+\ldots)=0,
$$

and since we need only the terms of order zero in the equation of the refracting surface in order to get the first-order terms in the infinitesimal map:

$$
\Phi(\xi, \eta, \zeta)=(\xi-a)^{2}+(\eta-b)^{2}+(\zeta-c)^{2}-r^{2}=0 .
$$

One gets $W=s w$ for the characteristic function, when $w$ is determined from the equation:
$0=\Phi(x+m w)+(y+p w, z+q w)=(m w+x-a)^{2}+(p w+y-b)^{2}+(q w+z-c)^{2}-r^{2}$
or

$$
w^{2}+2 w[m(x-a)+p(y-b)+q(z-c)]=0,
$$

whose two roots are $w=0$ and:

$$
w=-2[m(x-a)+p(y-b)+q(z-c)] .
$$

Since, on the other hand:

$$
W=\delta[m(x-a)+p(y-b)+q(z-c)],
$$

it follows that $\delta=-2 \sigma$, which is a relation that agrees with (74) and the equation:

$$
R=r+v r^{\prime}=r\left(\frac{n}{N}\right)^{2}=r(1-2 \sigma v), \quad r=-2 \sigma r
$$

For a single refraction, the case of aplanatic spheres is, moreover, the only case of microscopic aplanarity, as one sees by substitution in the formula:

$$
w=\frac{W}{\sigma}=\frac{\delta}{\sigma}[m(x-a)+p(y-b)+q(z-c)]
$$

in equations (70), while for a sequence of refractions, only the relation (74) will emerge directly from our formulas.

## 8.

If we introduce the quantities:

$$
\alpha=\frac{1}{x}, \quad \delta=\frac{y}{x}, \quad \gamma=\frac{z}{x}
$$

in place of the coordinates $x, y, z$ of the vertex of the congruence then equations (2) and (53) will go to:

$$
\left\{\begin{array}{rlrl}
\frac{p}{m} & =\beta-\alpha h, & \frac{q}{m} & =\gamma-\alpha k  \tag{75}\\
\left(\frac{p}{m}\right)^{\prime} & =\beta^{\prime}-\alpha^{\prime} h-\alpha h^{\prime}, \quad\left(\frac{q}{m}\right)^{\prime} & =\gamma^{\prime}-\alpha^{\prime} k-\alpha k^{\prime}
\end{array}\right.
$$

in which $\alpha, \beta, \gamma$ and $\alpha+v \alpha^{\prime}, \beta+v \beta^{\prime}, \gamma+v \gamma^{\prime}$ are the determining data of conjugate congruence vertices. If aplanarity exists then $\alpha, \alpha^{\prime}, \beta^{\prime}, \gamma^{\prime}$ can be regarded as functions of $\beta, \gamma$. That Ansatz is also appropriate for telescopic aplanarity, for which one simply sets $\alpha=0$, while $\alpha^{\prime}$ does not vanish identically. If one also has that $\alpha^{\prime}=0$ then a parallel congruence will go to a parallel congruence (i.e., prismatic aplanarity). For telescopic aplanarity, one then gets:

$$
\left\{\begin{array}{rlrl}
\frac{p}{m} & =\beta, & \frac{q}{m} & =\gamma,  \tag{76}\\
\left(\frac{p}{m}\right)^{\prime} & =\beta^{\prime}-\alpha^{\prime} h,\left(\frac{q}{m}\right)^{\prime} & =\gamma^{\prime}-\alpha^{\prime} k,
\end{array}\right.
$$

and one regards $\alpha^{\prime}, \beta^{\prime}, \gamma^{\prime}$ in them as arbitrary functions of $p, q$ (instead of $\beta, \gamma$ ). When one recalls that $m m^{\prime}+p p^{\prime}+q q^{\prime}=0$, equations (76) will imply the following values for $p^{\prime}, q^{\prime}$ :

$$
\begin{aligned}
& \frac{p^{\prime}}{m}=\beta^{\prime}-\alpha^{\prime} h-p\left[\left(\beta^{\prime}-\alpha^{\prime} h\right) p+\left(\gamma^{\prime}-\alpha^{\prime} k\right) q\right], \\
& \frac{q^{\prime}}{m}=\gamma^{\prime}-\alpha^{\prime} k-q\left[\left(\beta^{\prime}-\alpha^{\prime} h\right) p+\left(\gamma^{\prime}-\alpha^{\prime} k\right) q\right] .
\end{aligned}
$$

If one seeks the characteristic function $V$ and observes that:

$$
p^{\prime}=-\sigma p-V_{h}, \quad q^{\prime}=-\sigma q-V_{k}
$$

then it will follow that:

$$
\left\{\begin{array}{l}
V_{h}=A[h-p(p h+q k)]+B,  \tag{77}\\
V_{k}=A[k-q(p h+q k)]+C,
\end{array}\right.
$$

in which the three quantities:

$$
\begin{equation*}
A=\alpha^{\prime} m, \quad B=-\sigma p-\beta^{\prime} m+p m\left(\beta^{\prime} p+\gamma^{\prime} q\right), C=-\sigma q-\gamma^{\prime} m+q m\left(\beta^{\prime} p+\gamma^{\prime} q\right) \tag{78}
\end{equation*}
$$

are functions of only $p, q$. It follows upon integrating (77) that:

$$
\begin{equation*}
V=\frac{A}{2}\left[h^{2}+k^{2}-(p h+q k)^{2}\right]+B h+C k+D \tag{79}
\end{equation*}
$$

in which $D$ is a fourth, arbitrary function of $p, q$. This form, which is quadratic in $h, k$, is the characteristic function of an infinitesimal telescopic-aplanatic map.

As a control, and in order to perform that type of derivation in an example, the characteristic function (79) shall be developed from the eikonal of the finite map. Let $X, Y, Z, F$ be arbitrary functions of $p, q$. We begin with the eikonal $E(p, q, P, Q)$ [so $E_{1}$, from the schema (14)]:

$$
\begin{equation*}
E=N(M X+P Y+Q Z+F) \tag{80}
\end{equation*}
$$

with the mapping equations:

$$
\left\{\begin{array} { l } 
{ - \frac { n } { N } h = M X _ { p } + P Y _ { p } + Q Z _ { p } + F _ { p } , } \\
{ - \frac { n } { N } k = M X _ { q } + P Y _ { q } + Q Z _ { q } + F _ { q } , }
\end{array} \quad \left\{\begin{array}{l}
H=Y-X \frac{Q}{M}  \tag{82}\\
K=Z-X \frac{Q}{M}
\end{array}\right.\right.
$$

which implies that the image ray goes through the point $X, Y, Z$, so each parallel object congruence with the direction cosine $m, p, q$ is associated with the centric image congruence with the vertex $X, Y, Z$, i.e., it will be mapped aplanatically to the plane at infinity. Should the map be an infinitesimal one then the point $X, Y, Z$ will lie at infinity, up to quantities of order $v$, i.e., we can set:

$$
\left\{\begin{align*}
v X & =\xi+v \xi^{\prime}+\frac{v^{2}}{2} \xi^{\prime \prime}+\cdots \\
v Y & =\eta+v \eta^{\prime}+\frac{v^{2}}{2} \eta^{\prime \prime}+\cdots  \tag{83}\\
v Z & =\zeta+v \zeta+\frac{v^{2}}{2} \zeta^{\prime \prime}+\cdots \\
v F & =\varphi+v \varphi^{\prime}+\frac{v^{2}}{2} \varphi^{\prime \prime}+\cdots
\end{align*}\right.
$$

in which the $\xi, \eta, \zeta, \varphi, \xi^{\prime}, \eta^{\prime}, \zeta^{\prime}, \varphi^{\prime}, \ldots$ are certain partially arbitrary, partially to-be-determined, functions of $p, q$. If we include those equations, as well as the developments (16) and (17), in the eikonal (80) and denote a sum over the three coordinate axes by $S$, to abbreviate, then that will give:

$$
\begin{equation*}
\frac{v E}{N}=S\left(m+v m^{\prime}+\frac{v^{2}}{2} m^{\prime \prime}+\cdots\right)\left(\xi+v \xi^{\prime}+\frac{v^{2}}{2} \xi^{\prime \prime}+\cdots\right)+\varphi+v \varphi^{\prime}+\frac{v^{2}}{2} \varphi^{\prime \prime}+\cdots \tag{84}
\end{equation*}
$$

However, from (25), one will have:

$$
\begin{equation*}
E-h(N P-n p)-k(N Q-n q)=N v V+\ldots, \tag{85}
\end{equation*}
$$

from which it emerges that $E$ must be divisible by $v$. The development on the right-hand side of (84) must then begin with $v^{2}$, and one will have the identities:

$$
\begin{gather*}
S m \xi+\varphi=0, \quad S\left(m^{\prime} \xi+m \xi^{\prime}\right)+\varphi^{\prime}=0,  \tag{86}\\
\frac{2 E}{N v}=S\left(m^{\prime \prime} \xi+2 m^{\prime} x^{\prime}+m \xi^{\prime \prime}\right)+\varphi^{\prime \prime} \tag{87}
\end{gather*}
$$

Furthermore, upon substituting the series (16), (17), (83) in the first of the mapping equations (81), one will get:

$$
-v h\left(1+\rho^{\prime} v+\cdots\right)=\varphi_{p}+v \varphi_{p}^{\prime}+\frac{v^{2}}{2} \varphi_{p}^{\prime \prime}+S\left(m+v m^{\prime}+\frac{v^{2}}{2} m^{\prime \prime}+\cdots\right)\left(\xi+v \xi^{\prime}+\frac{v^{2}}{2} \xi^{\prime \prime}+\cdots\right)
$$

and a corresponding expression from the second equation. Separating the powers of $v$ gives:

$$
\begin{gather*}
0=S m \xi_{p}+\varphi_{p}, \quad 0=S m \xi_{q}+\varphi_{q},  \tag{88}\\
-h=S\left(m^{\prime} \xi_{p}+m \xi_{p}^{\prime}\right)+\varphi_{p}^{\prime}, \quad-k=S\left(m^{\prime} \xi_{q}+m \xi_{q}^{\prime}\right)+\varphi_{q}^{\prime},  \tag{89}\\
\left\{\begin{array}{c}
-2 \rho^{\prime} h=S\left(m^{\prime \prime} \xi_{p}+2 m^{\prime} \xi_{p}^{\prime}+m \xi_{p}^{\prime \prime}\right)+\varphi_{p}^{\prime \prime}, \\
-2 \rho^{\prime} k=S\left(m^{\prime \prime} \xi_{q}+2 m^{\prime} \xi_{q}^{\prime}+m \xi_{q}^{\prime \prime}\right)+\varphi_{q}^{\prime \prime},
\end{array}\right. \tag{90}
\end{gather*}
$$

With that, equations (88) determine $\varphi$, equations (89) determine $m^{\prime}, p^{\prime}, q^{\prime}$, and equations (90) determine $m^{\prime \prime}, p^{\prime \prime}, q^{\prime \prime}$, when one considers the relations:

$$
S m m^{\prime}=0, \quad S m m^{\prime \prime}=-S m^{\prime 2} .
$$

The values obtained must then satisfy equations (86). Finally, the second pair of mapping equations (82) will yield the relations:

$$
v\left(h+v h^{\prime}+\cdots\right)=\eta+v \eta^{\prime}+\frac{v^{2}}{2} \eta^{\prime \prime}+\cdots-\left(\xi+v \xi^{\prime}+\frac{v^{2}}{2} \xi^{\prime \prime}+\cdots\right)\left(\frac{p}{m}+v\left(\frac{p}{n}\right)^{\prime}+\frac{v^{2}}{2}\left(\frac{p}{n}\right)^{\prime \prime}+\cdots\right)
$$

which split into:

$$
\begin{gather*}
0=\eta-\xi \frac{p}{m}, \quad 0=\zeta-\xi \frac{q}{m}  \tag{91}\\
h=\eta^{\prime}-\xi^{\prime} \frac{p}{m}-\xi\left(\frac{p}{m}\right)^{\prime}, \quad k=\zeta^{\prime}-\xi^{\prime} \frac{q}{m}-\xi\left(\frac{q}{m}\right)^{\prime}, \tag{92}
\end{gather*}
$$

$$
\left\{\begin{array}{l}
2 h^{\prime}=\eta^{\prime \prime}-\xi^{\prime \prime} \frac{p}{m}-2 \xi^{\prime}\left(\frac{p}{m}\right)^{\prime}-\xi\left(\frac{p}{m}\right)^{\prime \prime},  \tag{93}\\
2 k^{\prime}=\zeta^{\prime \prime}-\xi^{\prime \prime} \frac{q}{m}-2 \xi^{\prime}\left(\frac{q}{m}\right)^{\prime}-\xi\left(\frac{q}{m}\right)^{\prime \prime},
\end{array}\right.
$$

and some further relations will yield the values of $h^{\prime}, k^{\prime}, \ldots$, in addition.
Equations (86) and (91) together will give:

$$
\begin{equation*}
\xi=-m \varphi, \quad \eta=-p \varphi, \quad \zeta=-q \varphi, \tag{94}
\end{equation*}
$$

with which (88) are likewise fulfilled. It follows further from (92) and (86) that:

$$
p h+q k=m \xi^{\prime}+p \eta^{\prime}+q \zeta^{\prime}-\frac{\xi^{\prime}}{m}+\frac{\xi m^{\prime}}{m^{2}}=-\varphi^{\prime}-\frac{\xi^{\prime}}{m}+\frac{\varphi m^{\prime}}{m^{2}},
$$

so one further has:

$$
\left\{\begin{array}{l}
m^{\prime} \varphi=-m\left(p h+q k+\varphi^{\prime}\right)-\xi^{\prime},  \tag{95}\\
p^{\prime} \varphi=-p\left(p h+q k+\varphi^{\prime}\right)-\eta^{\prime}+h, \\
q^{\prime} \varphi=-q\left(p h+q k+\varphi^{\prime}\right)-\zeta^{\prime}+k,
\end{array}\right.
$$

which are equations that likewise fulfill (89). Equations (90) and (93) still remain unused, but they would yield the values of $m^{\prime \prime}, p^{\prime \prime}, q^{\prime \prime}, h^{\prime}, k^{\prime}$, which will not be used here.

It now follows from (87) and (94) that the eikonal is:

$$
\begin{aligned}
\frac{2 E}{N V} & =-S m m^{\prime \prime} \varphi+2 S m^{\prime} \xi^{\prime}+S m \xi^{\prime \prime}+\varphi^{\prime \prime} \\
& =\varphi S m^{\prime 2}+2 S m^{\prime} \xi^{\prime}+S m \xi^{\prime \prime}+\varphi^{\prime \prime}
\end{aligned}
$$

From (95), one has:

$$
\begin{aligned}
\varphi^{2} S m^{\prime 2}= & \left(p h+q k+\varphi^{\prime}\right)^{2}+h^{2}+k^{2}+\xi^{\prime 2}+\eta^{\prime 2}+\zeta^{\prime 2} \\
& -2 \varphi^{\prime}\left(p h+q k+\varphi^{\prime}\right)-2(p h+q k)\left(p h+q k+\varphi^{\prime}\right)-2\left(\eta^{\prime} h+\zeta^{\prime} k\right) \\
= & -\left(p h+q k+\varphi^{\prime}\right)^{2}+h^{2}+k^{2}-2\left(\eta^{\prime} h+\zeta^{\prime} k\right)+\xi^{\prime 2}+\eta^{\prime 2}+\zeta^{\prime 2}, \\
\varphi S m^{\prime} \xi^{\prime}= & \varphi^{\prime}\left(p h+q k+\varphi^{\prime}\right)-\xi^{\prime 2}-\eta^{\prime 2}-\zeta^{\prime 2}+\eta^{\prime} h+\zeta^{\prime} k, \\
\frac{2 E \varphi}{N v}= & \varphi^{\prime 2}-(p h+q k)^{2}+h^{2}+k^{2}-\xi^{\prime 2}-\eta^{\prime 2}-\zeta^{\prime 2}+\left(m \xi^{\prime \prime}+p \eta^{\prime \prime}+q \zeta^{\prime \prime}+\varphi^{\prime \prime}\right) \varphi,
\end{aligned}
$$

or

$$
\frac{E}{N \nu}=\frac{1}{2 \varphi}\left[h^{2}+k^{2}-(p h+q k)^{2}\right]+D,
$$

$$
\begin{aligned}
2 \varphi D & =\varphi\left(\varphi^{\prime \prime}+m \xi^{\prime \prime}+p \eta^{\prime \prime}+q \zeta^{\prime \prime}\right)+\varphi^{\prime 2}-\xi^{\prime 2}-\eta^{\prime 2}-\zeta^{\prime 2} \\
& =\varphi \varphi^{\prime \prime}+\varphi^{\prime 2}-\left(\xi \xi^{\prime \prime}+\eta \eta^{\prime \prime}+\zeta \zeta^{\prime \prime}\right)-\left(\xi^{\prime 2}+\eta^{2}+\zeta^{\prime 2}\right)
\end{aligned}
$$

so $D$ is an arbitrary function of $p, q$.
Furthermore, from (85):

$$
\begin{align*}
& V=\frac{E}{N v}-h\left(p^{\prime}+s p\right)-k\left(q^{\prime}+s q\right) \\
& =D+\frac{1}{2 \varphi}\left[h^{2}+k^{2}-(p h+q k)^{2}\right]+\frac{p h+q k}{\varphi}\left(p h+q k+\varphi^{\prime}\right)+\frac{\left(\eta^{\prime}-h\right) h+\left(\zeta^{\prime}-k\right) k}{\varphi} \\
& \quad-\sigma\left(p h+q k+\varphi^{\prime}\right) \\
& = \\
& \frac{A}{2}\left[h^{2}+k^{2}-(p h+q k)^{2}\right]+B h+C k+D,  \tag{96}\\
& \qquad A=-\frac{1}{\varphi}, \quad B=\frac{\eta^{\prime}+p \varphi^{\prime}}{\varphi}-\sigma p, \quad C=\frac{\zeta^{\prime}+q \varphi^{\prime}}{\varphi}-\sigma q,
\end{align*}
$$

will coincide with (78) and (79) when one observes that the quantities $\alpha^{\prime}, \beta^{\prime}, \gamma^{\prime}$ in the former are connected with the ones in the latter by the equations:

$$
v \alpha^{\prime}=\frac{1}{X}-\left(\frac{1}{X}\right)_{0}, \quad v \beta^{\prime}=\frac{Y}{X}-\left(\frac{Y}{X}\right)_{0}, \quad v \gamma^{\prime}=\frac{Z}{X}-\left(\frac{Z}{X}\right)_{0},
$$

in which the index 0 means the substitution $v=0$. In fact, one will then have:

$$
\begin{gathered}
\alpha^{\prime}=\frac{1}{\xi}, \quad \beta^{\prime}=\frac{\eta^{\prime}}{\xi}-\frac{\eta \xi^{\prime}}{\xi^{2}}, \quad \gamma^{\prime}=\frac{\zeta^{\prime}}{\xi}-\frac{\zeta \xi^{\prime}}{\xi^{2}}, \\
p \beta^{\prime}+q \gamma^{\prime}=-\frac{\varphi+m \xi^{\prime}}{\xi}+(\varphi+m \xi) \frac{\xi^{\prime}}{\xi^{2}}=\frac{\xi^{\prime} \varphi}{\xi^{2}}-\frac{\varphi^{\prime}}{\xi},
\end{gathered}
$$

from which, the identity of the values (78) and (96) can be read off easily.
If we pose the question of the optical producibility of the telescopic-aplanatic map then upon applying the criterion $\Delta(V)=0$ to the function (79), we will first get an identity that is linear with respect to $h, k$, and which decomposes into the three equations:

$$
\begin{equation*}
A_{p}=0, \quad A_{q}=0, \quad B_{q}-q\left(B+p B_{p}+q B_{q}\right)=C_{q}-p\left(C+p C_{p}+q C_{q}\right) \tag{97}
\end{equation*}
$$

after some reductions. Thus, $A$ would be constant, $\varphi$ would be constant, and from (94):

$$
\xi^{2}+\eta^{2}+\zeta^{2}=\varphi^{2}
$$

which is the anastigmatic image surface of a sphere with the radius $\varphi / v$. However, a more precise consideration shows that a term like the one that is multiplied by $A$ cannot appear in an opticallyproducible $V$ at all. $V$ is a linear combination of functions $v$ that each satisfy the partial differential equations (64). In regard to the manner by which $v$ includes the variables $h$ and $k$, it follows that $v$ cannot become infinite in the limit of $h=\infty$ to order higher than $h^{1}$. Assuming that, if $\lim v=\infty$ to the order of a function $\psi(h)$ such that for large values of $h$ there is a development of the form:

$$
v=v_{0} \psi(h)+\ldots, \quad v_{0}=v_{0}(k, p, q),
$$

where the ratio of the remainder to $\psi(h)$ drops below any limit with increasing $h$, then one will have:

$$
\frac{v}{h}=v_{0} \frac{\psi(h)}{h}+\cdots,
$$

on the one hand, and:

$$
\lim \frac{v}{h}=\lim v_{h}=\lim \frac{v_{p}-p\left(v+p v_{p}+q v_{q}\right)}{v}=\lim \frac{v_{0 p}-p\left(v_{0}+p v_{0 p}+q v_{0 q}\right)+\cdots}{v_{0}+\cdots},
$$

on the other, such that $\lim (v / h)$ can remain finite, as well as vanish, but not become infinite.
For sufficiently large values of $h$ and $k$, each individual $v$, and therefore $V$, as well, have dimensions at most that of $h^{1}, k^{1}$, resp., which excludes the terms in (79) that is multiplied by $A$. Accordingly, telescopic aplanarity is not optically producible, and the ideal telescope objective cannot be realized by any choice of refracting surfaces with values of the refraction exponents that remain undetermined.

If one sets $A=0$ then $V$ will be the generating function for prismatic aplanarity, which will map parallel congruences to other parallel ones; the condition (97) still remains to be fulfilled. From (78), if we set:

$$
\beta=m \beta^{\prime}, \quad \gamma=m \gamma^{\prime},
$$

for the moment, then:

$$
\left\{\begin{array}{l}
B=p(\beta p+\gamma q-\sigma)-\beta  \tag{98}\\
C=q(\beta p+\gamma q-\sigma)-\gamma .
\end{array}\right.
$$

The requirement (97) is written:

$$
\begin{aligned}
C_{p}-B_{q}= & p\left(C+p C_{p}-q B_{p}\right)+q\left(-B+p C_{q}-q B_{q}\right) \\
& =p(p C-q B)_{p}+q(p C-q B)_{q},
\end{aligned}
$$

and when one substitutes (98), one will get:

$$
m^{2}\left(\beta_{q}-\gamma_{p}\right)=0,
$$

i.e., $\beta d p+\gamma d q$ is a complete differential, or with an arbitrary function $s$ :

$$
\beta=s_{p}, \quad \gamma=s_{q} .
$$

For example, the assumption:

$$
s=\sigma \log (a m+b p+c q)
$$

corresponds to a refraction in the plane:

$$
a x+b y+c z=d
$$

Let me point out once more that I by no means fail to appreciate the incompleteness in the proof that was given for the non-existence of an ideal telescope objective. If it is also unlikely that an optical system should possess essential properties like aplanarity for well-defined numerical values of the refraction exponents that it does not possess for generally-available values of them then an adaptation of the principle of the proof to finite maps would still be desirable in order to gain a deeper insight into the nature of the contradiction between aplanarity and optical producibility as it manifests itself in the purely-formal structure of a function $V(h, k, p, q)$. Perhaps the shortcoming that takes the form of the exclusion of catoptric systems is even more palpable. In order to rectify that, the formulas of $\S \mathbf{5}$ will be discussed in more detail than they were here, but in which, it would seem, the meaning of the infinitesimal maps will fade noticeably. That is because, when considered in a purely-mathematical way, they can be composed of the characteristic functions of reflections in neighboring surfaces that also include the term that is critical for telescopic aplanarity:

$$
\begin{equation*}
h^{2}+k^{2}-(p h+q k)^{2} . \tag{99}
\end{equation*}
$$

If we set, e.g.:

$$
x^{2}+y^{2}+z^{2}=r^{2}, \quad \varphi=\sqrt{r^{2}-y^{2}-z^{2}}, \quad \psi=1
$$

in (54), (55), (56), such that reflection takes place from two spheres of radius $r$ whose centers lie along the $X$-axis at a distance of $v$ from each other, then the characteristic function will be:

$$
\begin{aligned}
V & =2 \frac{m-p \varphi_{y}-q \varphi_{z}}{1+\varphi_{y}^{2}+\varphi_{z}^{2}}=\frac{2 x}{r^{2}}(m x+p y+q z) \\
& =\frac{2 m v}{r^{2}}(v+p h+q k),
\end{aligned}
$$

in which $v$ is determined from the equation:

$$
(m v)^{2}+(h+p v)^{2}+(k+q v)^{2}=r^{2} .
$$

With:

$$
u^{2}=r^{2}-h^{2}-k^{2}+(p h+q k)^{2},
$$

it will then follow that:

$$
\begin{gathered}
v=-p h-q k \pm u \\
V=\frac{2 m}{r^{2}}\left[u^{2} \mp u(p h+q k)\right],
\end{gathered}
$$

and when one composes a generating function $V=V_{1}+V_{2}$ from:

$$
V_{1}=\frac{2 m}{r^{2}}\left[u^{2}+u(p h+q k)\right] \quad \text { and } \quad V_{2}=\frac{2 m}{r^{2}}\left[u^{2}-u(p h+q k)\right],
$$

one will get:

$$
V=\frac{4 m}{r^{2}} u^{2}=4 m-\frac{4 m}{r^{2}}\left[h^{2}+k^{2}-(p h+q k)^{2}\right] .
$$

Thus, telescopic aplanarity would be achieved by four-fold reflection from two neighboring spheres, which is an infinitesimal property for which one knows no finite analogue. At any rate, the impossibility of the ideal telescope objective can initially be asserted for only the purelydioptric systems.

While reserving the imagined extensions, one must also renounce telescopic aplanarity for arbitrary congruence apertures theoretically and restrict oneself to the examination of elementary congruences. That is, anastigmatic structures enter in place of the dicentric ones, and series developments up to a certain order must be appealed to for the mapping functions (equations, resp.), which are not series developments in the parameters (e.g., refraction exponents) of the map, as was done here, but in the variables (e.g., ray coordinates). The fact that it is precisely in the case of the ideal telescope objective that optical practice has instinctively hit upon the right idea by seeking to approximate it with refracting spherical surfaces emerges almost immediately from the form (99) of the critical term. For refraction at a sphere of radius $r$ about a coordinate origin, there exists the characteristic function $V=\sigma v$, with:

$$
v=-p h-q k+\sqrt{r^{2}-h^{2}-k^{2}+(p h+q k)^{2}} .
$$

For sufficiently-small values of $h, k$, i.e., for rays that pass sufficiently close to the center of the refracting surface, one then has the series development:

$$
v=r-p h-q k-\frac{1}{2 r}\left[h^{2}+k^{2}-(p h+q k)^{2}\right]+\ldots
$$

from which the aplanarity of paraxial rays will follow for a centric lens system. In addition to the engineering considerations, the theoretical fact that it is precisely just the combination (99) of the terms that are quadratic in $h$ and $k$ that enter into the associated characteristic function also speaks in favor of preserving the use of spherical surfaces.
9.

As an appendix, the case of double refraction in uniaxial and biaxial crystals will be treated. In that case, MALUS's theorem is no longer valid, and relationship to contact transformations breaks down. Nonetheless, the infinitesimal maps of the type still possess a self-evident form and allow the question of telescopic aplanarity to be answered, and in the negative here, as well.

Following the known laws of theoretical optics, we must first present the relationship between rays and wave normals in anisotropic media, for which we shall choose the coordinate planes to be the optical symmetry planes of the medium (viz., the principal planes of the elasticity ellipsoid). The equation of the wave surface, when referred to its center, has the form:

$$
\left\{\begin{array}{c}
\frac{a^{2} x^{2}}{a^{2}-u^{2}}+\frac{b^{2} y^{2}}{b^{2}-u^{2}}+\frac{c^{2} z^{2}}{c^{2}-u^{2}}=0  \tag{100}\\
\text { or } \\
\frac{x^{2}}{a^{2}-u^{2}}+\frac{y^{2}}{b^{2}-u^{2}}+\frac{z^{2}}{c^{2}-u^{2}}+1=0 \\
u^{2}=x^{2}+y^{2}+z^{2}
\end{array}\right.
$$

where $a, b, c$ are the reciprocal axes of FRESNEL's elasticity ellipsoid. If $\mu, \pi, \kappa$ are the direction cosines of the ray, so:

$$
x=u \mu, \quad y=u \pi, \quad z=u \kappa,
$$

then the associated radius vector $u$ of the wave surface will follow from one of the two equations:

$$
\left\{\begin{array}{c}
\frac{a^{2} \mu^{2}}{a^{2}-u^{2}}+\frac{b^{2} \pi^{2}}{b^{2}-u^{2}}+\frac{c^{2} \kappa^{2}}{c^{2}-u^{2}}=0  \tag{101}\\
\text { or } \quad \\
\frac{\mu^{2}}{a^{2}-u^{2}}+\frac{\pi^{2}}{b^{2}-u^{2}}+\frac{\kappa^{2}}{c^{2}-u^{2}}+\frac{1}{u^{2}}=0
\end{array}\right.
$$

which can be developed quadratically in $u^{2}$, which corresponds to the two sheets of the wave surface. The elasticity ellipsoid and the wave surface might refer to the time unit.

We place the tangent plane to the wave surface at the point $x, y, z$ and drop an altitude from the center to that plane. Let the length of the altitude be $w$ and let its direction cosines be $\rho, \sigma, \tau$, such that:

$$
\begin{equation*}
w=x \rho+y \sigma+z \tau=u(\mu \rho+\pi \sigma+\kappa \tau) . \tag{102}
\end{equation*}
$$

The quantities $\rho, \sigma, \tau$ are proportional to the derivatives of the equation (100) with respect to $x, y$, $z$. We choose the second form, so we can assume that:

$$
\left\{\begin{array}{c}
\rho=\lambda x\left(\frac{1}{a^{2}-u^{2}}+\frac{1}{\vartheta^{2}}\right), \\
\sigma=\lambda y\left(\frac{1}{b^{2}-u^{2}}+\frac{1}{\vartheta^{2}}\right), \\
\tau=\lambda z\left(\frac{1}{c^{2}-u^{2}}+\frac{1}{\vartheta^{2}}\right), \\
\frac{1}{\vartheta^{2}}=\left(\frac{x}{a^{2}-u^{2}}\right)^{2}+\left(\frac{y}{b^{2}-u^{2}}\right)^{2}+\left(\frac{z}{c^{2}-u^{2}}\right)^{2} . \tag{104}
\end{array}\right.
$$

Upon squaring and adding (103), it will follow that:

$$
1=\lambda^{2}\left(\frac{1}{\vartheta^{2}}-\frac{2}{\vartheta^{2}}+\frac{u^{2}}{\vartheta^{4}}\right)=\lambda^{2} \frac{u^{2}-\vartheta^{2}}{\vartheta^{4}}
$$

and on the other hand, it follows from (102) that:

$$
w=\lambda\left(1-\frac{1}{\vartheta^{2}}+\frac{u^{2}}{\vartheta^{2}}\right)=\lambda \frac{u^{2}-\vartheta^{2}}{\vartheta^{2}},
$$

and therefore:

$$
w \lambda=\vartheta^{2}=u^{2}-w^{2} .
$$

It will then follow from (103) that:

$$
\left\{\begin{align*}
\frac{\rho w}{a^{2}-w^{2}} & =\frac{\mu u}{a^{2}-u^{2}}, \\
\frac{\sigma w}{b^{2}-w^{2}} & =\frac{\pi u}{a^{2}-u^{2}},  \tag{105}\\
\frac{\tau w}{c^{2}-w^{2}} & =\frac{\kappa u}{a^{2}-u^{2}} .
\end{align*}\right.
$$

The quantities $\mu, \pi, \kappa, u$ and $\rho, \sigma, \tau, w$ appear symmetrically here. In order to find an equation between just $\rho, \sigma, \tau, w$, from (105), we envision writing:

$$
\begin{equation*}
\mu \frac{u}{w}=\rho \frac{a^{2}-u^{2}}{a^{2}-w^{2}}=\rho\left(1-\frac{\vartheta^{2}}{a^{2}-w^{2}}\right) \tag{106}
\end{equation*}
$$

and the two corresponding equations. It then follows from (102) that:

$$
1=\frac{u}{w}(\mu \rho+\pi \kappa+\sigma \tau)=1-\vartheta^{2}\left(\frac{\rho^{2}}{a^{2}-w^{2}}+\frac{\sigma^{2}}{b^{2}-w^{2}}+\frac{\tau^{2}}{c^{2}-w^{2}}\right),
$$

or

$$
\frac{\rho^{2}}{a^{2}-u^{2}}+\frac{\sigma^{2}}{b^{2}-u^{2}}+\frac{\tau^{2}}{c^{2}-u^{2}}=0 .
$$

If we combine the successive corresponding formulas then we will get the equations:

$$
\begin{align*}
& \left\{\begin{array}{c}
\frac{a^{2} \mu^{2}}{a^{2}-u^{2}}+\frac{b^{2} \pi^{2}}{b^{2}-u^{2}}+\frac{c^{2} \kappa^{2}}{c^{2}-u^{2}}=0, \\
\frac{\rho^{2}}{a^{2}-u^{2}}+\frac{\sigma^{2}}{b^{2}-u^{2}}+\frac{\tau^{2}}{c^{2}-u^{2}}=0,
\end{array}\right.  \tag{107}\\
& \left\{\begin{array}{l}
\frac{a^{2} \mu^{2}}{a^{2}-u^{2}}+\frac{b^{2} \pi^{2}}{b^{2}-u^{2}}+\frac{c^{2} \kappa^{2}}{c^{2}-u^{2}}=0, \\
\frac{\rho^{2}}{a^{2}-u^{2}}+\frac{\sigma^{2}}{b^{2}-u^{2}}+\frac{\tau^{2}}{c^{2}-u^{2}}=0,
\end{array}\right. \\
& \left\{\begin{array}{l}
\left(\frac{\mu}{a^{2}-u^{2}}\right)^{2}+\left(\frac{\pi}{b^{2}-u^{2}}\right)^{2}+\left(\frac{\kappa}{c^{2}-u^{2}}\right)^{2}=\frac{1}{u^{2} \vartheta^{2}}, \\
\left(\frac{\rho}{a^{2}-u^{2}}\right)^{2}+\left(\frac{\sigma}{b^{2}-u^{2}}\right)^{2}+\left(\frac{\tau}{c^{2}-u^{2}}\right)^{2}=\frac{1}{w^{2} \vartheta^{2}},
\end{array}\right.  \tag{108}\\
& u^{2}=w^{2}+\vartheta^{2}, \tag{109}
\end{align*}
$$

in addition to (105).
Thus, if $\mu, \pi, \kappa$ is given then one $u$ from (107) $)_{1} \vartheta$ from (108) $)_{1}, w$ from (109), and $\rho, \sigma, \tau$ from (105). Conversely, if $\rho, \sigma, \tau$ is given then one finds $w$ from (107) $)_{2}, \vartheta$ from (108) $)_{2}, u$ from (109), and $\mu, \pi, \kappa$ from (105). The equations for $u$ and $w$ are quadratic. For each ray $\mu, \pi, \kappa$ there are two wave normals $\rho, \sigma, \tau$, and conversely, for any wave normal there are $\rho, \sigma, \tau$ there are two rays $\mu$, $\pi, \kappa$. Moreover, it is unnecessary in what follows to treat the two systems of values (viz., the ordinary and extraordinary) separately.

Differentiating (107)2, in which $w$ is regarded as a function of $\rho, \sigma, \tau$, will yield:

$$
\frac{\rho d \vartheta}{a^{2}-w^{2}}+\frac{\sigma d \sigma}{b^{2}-w^{2}}+\frac{\tau d \tau}{c^{2}-w^{2}}+\frac{d w}{w \vartheta^{2}}=0 .
$$

It then follows from (106) that:

$$
\frac{u}{w}(\mu d \rho+\pi d \sigma+\kappa d \tau)=-\vartheta^{2}\left(\frac{\rho d \vartheta}{a^{2}-w^{2}}+\frac{\sigma d \sigma}{b^{2}-w^{2}}+\frac{\tau d \tau}{c^{2}-w^{2}}\right)=\frac{d w}{w}
$$

Those formulas, which represent the known enveloping relation between wave planes and wave surfaces, are true with no changes for an arbitrary choice of coordinate axes. Therefore, if $m, p, q$ are the direction cosines of a ray, when referred to a new fixed system of axes, and $r, s, t$ are those of the wave normal then one will also have:

$$
\left\{\begin{align*}
m d r+p d s+q d t & =\frac{1}{u} d w \\
r d m+s d p+t d q & =w d\left(\frac{1}{u}\right),  \tag{111}\\
r m+s p+t q & =\frac{w}{u}
\end{align*}\right.
$$

in which $u$ is the radius vector of the wave surface that is associated with $m, p, q$, and $w$ is the altitude that is dropped to the wave plane and, at the same time, the propagation velocity of the wave.

For refraction at the boundary between two anisotropic media, the wave normals follow the usual law of refraction, but with the variable propagation velocities $w$ (reciprocal indices of refraction). One then goes from the incident ray to the incident wave normal using the formulas above, and from there, the law of refraction will give the refracted wave normal, and that will give the refracted ray. Since two of those individual relations are double-valued, the total relation is four-valued. In isotropic media, the ray coincides with its wave normal. The transition from isotropic media to crystalline media and back again is double-valued, while the transition from isotropic ones to isotropic ones is single-valued. Obviously, it will suffice to consider the transition between isotropic and crystalline media, since one can think of an isotropic medium as being inserted between two crystalline ones that bound each other directly.

We denote the direction cosines of the ray $U$ in isotropic media, when referred to arbitrary axes, by $M, P, Q$, those of the ray $u$ in the crystalline medium by $m, p, q$, and those of the wave normal $w$ by $r, s, t$. Furthermore, as we have done up to now, we let the direction cosines of the ray $u$ and wave normal $w$, when referred to the symmetry planes of the crystals, be $\mu, \pi$, $\kappa$ and $\rho$, $\sigma, \tau$, resp., and let $u, w, \vartheta$ be the quantities that were defined above. The following orthogonal relations with constant coefficients $\xi, \eta, \zeta, \ldots$ exist between the direction cosines in two systems of axes, which we can write in the form of tables:

|  | $\mu$ | $\pi$ | $\kappa$ |
| :---: | :---: | :---: | :---: |
| $m$ | $\xi$ | $\xi^{\prime}$ | $\xi^{\prime \prime}$ |
| $p$ | $\eta$ | $\eta^{\prime}$ | $\eta^{\prime \prime}$ |
| $q$ | $\zeta$ | $\zeta^{\prime}$ | $\zeta^{\prime \prime}$ |


|  | $\rho$ | $\sigma$ | $\tau$ |
| :---: | :---: | :---: | :---: |
| $r$ | $\xi$ | $\xi^{\prime}$ | $\xi^{\prime \prime}$ |
| $s$ | $\eta$ | $\eta^{\prime}$ | $\eta^{\prime \prime}$ |
| $t$ | $\zeta$ | $\zeta^{\prime}$ | $\zeta^{\prime \prime}$ |

The $\xi, \eta, \zeta, \ldots$ are the direction cosines of the optical symmetry axes with respect to the coordinate axes. In addition to the relations between the ray $U$ and the wave normal $w$ that were derived above, one also has an ordinary refraction with the indices $N=1$ and $n=1 / w$, so the differences:

$$
M-\frac{r}{w}, \quad P-\frac{s}{w}, \quad Q-\frac{t}{w}
$$

will be proportional to the direction cosines of the incident altitude, so if $x=x(y, z)$ is the equation of the refracting surface then:

$$
\begin{equation*}
M=\frac{r}{w}+f, \quad P=\frac{s}{w}-f x_{y}, \quad Q=\frac{t}{w}-f x_{z} . \tag{112}
\end{equation*}
$$

Upon exiting the crystal, $m, p, q$ are given. From them, one finds $\mu, \pi, \kappa$, and one further finds $r$, $s, t$ from $u, \vartheta, w, \rho, \sigma, \tau$, and one gets the unknowns $f, M, P, Q$ with the help of equations (112) and the relation:

$$
M^{2}+P^{2}+Q^{2}=1 .
$$

The entrance into crystalline media is more complicated. $M, P, Q$ are given in that case, and the unknowns $r, s, t, f$ are determined from (112), but in which $w$ depends upon $\rho, \sigma, \tau$ or $r, s, t$ according to (107) $)_{2}$, and when one appends $r^{2}+s^{2}+t^{2}=1$, with which the further calculation for the inverse transition proceeds as before.

We now go on to infinitesimal maps in the case of exiting by setting $a, b, c$ equal to unity, up to quantities of the same order as the infinitely-small constant $v$, so:

$$
a=1+\alpha v, \quad b=1+\beta v, \quad c=1+\gamma v .
$$

Since $u$ and $w$ lie between the largest and smallest of the values $a, b, c$, they will also be equal to 1 , up to quantities of order $v$, so:

$$
u=1+v \varphi+\ldots, \quad w=1+v \psi+\ldots
$$

When only the lowest-order powers of $v$ are retained, as always, equation (107) ${ }_{1}$ gives:

$$
\begin{equation*}
\frac{\mu^{2}}{\alpha-\varphi}+\frac{\pi^{2}}{\beta-\varphi}+\frac{\kappa^{2}}{\gamma-\varphi}=0 \tag{113}
\end{equation*}
$$

and equation (108) $)_{1}$ gives:

$$
\vartheta=2 v \varepsilon,
$$

when

$$
\frac{1}{\varepsilon^{2}}=\left(\frac{\mu}{\alpha-\varphi}\right)^{2}+\left(\frac{\pi}{\beta-\varphi}\right)^{2}+\left(\frac{\kappa}{\gamma-\varphi}\right)^{2} .
$$

It then follows from (109) that:

$$
w^{2}=u^{2}-\vartheta^{2}=1+2 v \varphi+\ldots, \quad \psi=\varphi,
$$

such that $u$ and $w$ first differ in their terms in $v^{2}$. Finally, (105) implies that:

$$
\rho=\frac{u}{w} \mu\left(1+\frac{\vartheta^{2}}{a^{2}-u^{2}}\right)=\mu\left(1+2 v \frac{\varepsilon^{2}}{\alpha-\varphi}\right)
$$

or

$$
\rho-\mu=2 \nu \mu \frac{\varepsilon^{2}}{\alpha-\varphi}, \quad \sigma-\pi=2 v \pi \frac{\varepsilon^{2}}{\beta-\varphi}, \quad \tau-\kappa=2 \nu \kappa \frac{\varepsilon^{2}}{\gamma-\varphi} .
$$

The ray and its associated wave normal then differ by only terms of order $v$, and one also has formulas of the form:

$$
r=m-v m_{1}, \quad s=p-v p_{1}, \quad t=q-v q_{1}
$$

accordingly, in which $m_{1}, p_{1}, q_{1}$ denote certain functions of $m, p, q$ for which (111) will imply two relations. On the one hand, from $(111)_{3}$ will give:

$$
\frac{w}{u}=1-v\left(m m_{1}+p p_{1}+q q_{1}\right)
$$

so since $w / u-1$ has order $v^{2}$, one will have:

$$
\begin{equation*}
0=m m_{1}+p p_{1}+q q_{1} . \tag{114}
\end{equation*}
$$

On the other hand, it follows from $(111)_{2}$ that:

$$
w d\left(\frac{1}{r}\right)=-v\left(m_{1} d m+p_{1} d p+q_{1} d q\right)=-v d \varphi+\ldots
$$

so

$$
\begin{equation*}
d \varphi=m_{1} d m+p_{1} d p+q_{1} d q \tag{115}
\end{equation*}
$$

such that $m_{1}, p_{1}, q_{1}$ can be expressed in a simple way in terms of the derivatives of the single function $\varphi=\varphi(p, q)$ with respect to $p$ and $q$. From (112), one again has:

$$
\begin{aligned}
M & =m(1-v \varphi)-v m_{1}+f, \\
P & =p(1-v \varphi)-v p_{1}+f x_{y}, \\
Q & =q(1-v \varphi)-v q_{1}+f x_{z},
\end{aligned}
$$

and since:

$$
M m+P p+Q q=1-v \varphi+f\left(m-p x_{y}-q x_{z}\right)
$$

is equal to unity, up to terms of order $v^{2}$, one will have:

$$
f=\frac{v \varphi}{m-p x_{y}-q x_{z}} .
$$

Finally, if we once more set:

$$
M=m+v m^{\prime}, \quad P=p+v p^{\prime}, \quad Q=q+v q^{\prime},
$$

as before, then it will follow that:

$$
\left\{\begin{array}{l}
m^{\prime}=-m \varphi-m_{1}+\frac{\varphi}{m-p x_{y}-q x_{z}},  \tag{116}\\
p^{\prime}=-p \varphi-p_{1}-\frac{\varphi x_{y}}{m-p x_{y}-q x_{z}}, \\
q^{\prime}=-q \varphi-q_{1}-\frac{\varphi x_{z}}{m-p x_{y}-q x_{z}},
\end{array}\right.
$$

The entrance into crystalline media also leads to exactly the same formulas when $M, P, Q$ means the ray in the isotropic medium, and $m, p, q$ means the ray in the crystalline medium, as always. By contrast, should $m, p, q$ mean the ray in the foregoing medium and $M, P, Q$, the ray in the following one ("foregoing" and "following" are used in the sense of the motion of light), which would be consistent with our original usage of the symbols, then the signs in (116) would have to be inverted in the case of entrance. If the ray goes from an anisotropic medium 1 into an anisotropic medium 2 then the total change $v m^{\prime}$ in $m$ will be the combination:

$$
m^{\prime}=m_{1}^{\prime}+m_{2}^{\prime},
$$

with:

$$
\begin{aligned}
& m_{1}^{\prime}=-\varphi_{1}\left(m-\frac{1}{m-p x_{y}-q x_{z}}\right)-m_{1}, \\
& m_{2}^{\prime}=+\varphi_{2}\left(m-\frac{1}{m-p x_{y}-q x_{z}}\right)+m_{2},
\end{aligned}
$$

in which each individual $\varphi$ is defined by an equation of the form (113) or:

$$
\frac{(\xi m+\eta p+\zeta q)^{2}}{\alpha-\varphi}+\frac{\left(\xi^{\prime} m+\eta^{\prime} p+\zeta^{\prime} q\right)^{2}}{\beta-\varphi}+\frac{\left(\xi^{\prime \prime} m+\eta^{\prime \prime} p+\zeta^{\prime \prime} q\right)^{2}}{\gamma-\varphi}=0 .
$$

The constants $\alpha, \beta, \gamma, \xi, \eta, \zeta, \ldots$ are provided with indices 1,2 ; (114) and (115) are true for both indices.

One sees from this that in every case of a single refraction between anisotropic or isotropic media, the mapping equations have the form:

$$
\left\{\begin{array}{l}
m^{\prime}=\varphi(m-\lambda)+\xi  \tag{117}\\
p^{\prime}=\varphi\left(p+\lambda x_{y}\right)+\eta \\
q^{\prime}=\varphi\left(q+\lambda x_{z}\right)+\zeta \\
\lambda\left(m-p x_{y}-q x_{z}\right)=1
\end{array}\right.
$$

in which $\varphi, \xi, \eta, \zeta$ represent certain functions of $p, q$ between which the following relations exist:

$$
\left\{\begin{align*}
\xi m+\eta p+\zeta q & =0  \tag{118}\\
\xi d m+\eta d p+\zeta d q & =d \varphi
\end{align*}\right.
$$

For a refraction between two isotropic media, $\varphi$ is a constant, while $\xi, \eta, \zeta$ vanishes. For the stillmissing ray coordinates $h, k$ and their increments $h^{\prime}, k^{\prime}$, one again has the equations:

$$
\begin{array}{ll}
h=y-x \frac{p}{m} & k=z-x \frac{q}{m} \\
h^{\prime}=\frac{x}{m^{2}}\left(p m^{\prime}-m p^{\prime}\right), & k^{\prime}=\frac{x}{m^{2}}\left(q m^{\prime}-m q^{\prime}\right),
\end{array}
$$

which define the point $x, y, z$ as the point of intersection of the refracted and the incident ray. As before, the first pair facilitates the transition from the coordinates $p, q, y, z$ to the actual ray coordinates $p, q, h, k$. In that sense, one has:

$$
\begin{gathered}
p^{\prime} d h+q^{\prime} d k=m^{\prime} d x+p^{\prime} d y+q^{\prime} d z-\frac{x}{m}\left(m^{\prime} d m+p^{\prime} d p+q^{\prime} d q\right) \\
h^{\prime} d p+k^{\prime} d q=-\frac{x}{m}\left(m^{\prime} d m+p^{\prime} d p+q^{\prime} d q\right)
\end{gathered}
$$

If one defines the differential expression:

$$
\begin{equation*}
D=p^{\prime} d h+q^{\prime} d k-h^{\prime} d p-k^{\prime} d q=m^{\prime} d x+p^{\prime} d y+q^{\prime} d z \tag{119}
\end{equation*}
$$

then the term in (117) that is multiplied by $\lambda$ will drop out, and what will remain is:

$$
D=\varphi(m d x+p d y+q d z)+\xi d x+\eta d y+\zeta d z
$$

In general, one has:

$$
\xi d x+\eta d y+\zeta d z=\eta d h+\zeta d k+(\xi m+\eta p+\zeta q) d\left(\frac{x}{m}\right)+\frac{x}{m}(\xi d m+\eta d p+\zeta d q)
$$

so, from (118):

$$
\xi d x+\eta d y+\zeta d z=\frac{x}{m} d \varphi+\eta d h+\zeta d k
$$

and furthermore:

$$
m d x+p d y+q d z=d\left(\frac{x}{m}\right)+p d h+q d k
$$

so:

$$
\begin{equation*}
D=d\left(\varphi \frac{x}{m}\right)+\varphi(p d h+q d k)+\eta d h+\zeta d k=d(\varphi v)+\varphi(p d h+q d k)+\eta d h+\zeta d k \tag{120}
\end{equation*}
$$

in which $v$, in turn, is the solution of the equation:

$$
\Phi(m v, h+p v, k+q v)=0
$$

and $\Phi(x, y, z)=0$ is the equation of the refracting surface. From (119) and (120), the mapping equations then have the form:

$$
\begin{aligned}
p^{\prime} & =(\varphi v)_{h}+\varphi p+\eta, & & h^{\prime}=-(\varphi v)_{p}, \\
q^{\prime} & =(\varphi v)_{k}+\varphi q+\zeta, & & k^{\prime}=-(\varphi v)_{q},
\end{aligned}
$$

which can then be defined by differential operations on two characteristic functions $\varphi$ and $v$, since $\eta, \zeta$ are also expressible in terms of $\varphi_{p}$ and $\varphi_{q}$ using (118). Equations exist for a sum of refractions whose individual characteristic quantities are $\varphi$ and $V=\varphi v$ that are analogous to (121) with the characteristic quantities:

$$
\varphi=\sum_{\alpha} \varphi_{\alpha}, \quad V=\sum_{\alpha} \varphi_{\alpha} \nu_{\alpha}
$$

The previously-established form for $v$ also resolves the question here of the possibility of telescopic aplanarity, since any individual $v_{\alpha}$ has a dimension of at most $h^{1}, k^{1}$ for large values of $h, k$, and each $\varphi_{\alpha}$ depends upon only $p, q$, so $V$ has the same form as $v$ with respect to the quantities $h, k$, so $p^{\prime}, q^{\prime}$ will have dimensions at most $h^{0}, k^{0}$. In the case of telescopic aplanarity, however, $p^{\prime}$, $q^{\prime}$ must include the quantities $h, k$ in the combinations:

$$
A p(h p+k q)-A h, \quad A q(h p+k q)-A k
$$

which contradicts the result that was just obtained for $A \neq 0$. One will then be led to $A=0$, so to the case of prismatic aplanarity, and the ideal telescope objective is itself an impossibility when one appeals to anisotropic media.


[^0]:    ( ${ }^{1}$ ) CZAPSKI, Theorie der optischen Instrumente nach ABBE, Breslau, 1893.
    $\left(^{2}\right)$ Das Eikonal, Abhandlungen der K. Sächs. Gesellschaft der Wissenschaften, 1895.
    $\left(^{3}\right)$ LIE-SCHEFFER, Geometrie der Berührungstransformationen I., Leipzig, 1896.

[^1]:    $\left({ }^{1}\right)$ Maps that lead to two or three such equations play no special in optics.

[^2]:    $\left({ }^{1}\right)$ That term, which BRUNS used throughout his lectures on optics, agrees in principle with that of ABBE, but deviates somewhat from the one that was employed in "Eikonal." Now, an element of a congruence (viz., an elementary congruence) is called anastigmatic when it is centric on both sides, and a congruence of arbitrary aperture is dicentric. In the treatise on the eikonal, the latter term subsumed both dicentric and anastigmatic structures.

[^3]:    $\left({ }^{1}\right)$ When considered geometrically, $w$ is the part of the ray $h, k, p, q$ that is contained between the refracting and aplanatic surfaces.

