"Ueber eine charakterische Eigenschaft der Differentialgleichungen der Variationsrechnung," Math. Ann. **2** (1897), 49-72.

On the characteristic property of the differential equations of the calculus of variations

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§ 1.

As is known from the investigations of **Jacobi** (*), the differential equations to which the determination of the maxima and minima of simple integrals leads possess the remarkable property that they can be transformed into a system of first-order differential equations whose integration is equivalent to the solution of a first-order nonlinear partial differential equation. One obtains the integral equations of the problem of the calculus of variations when one differentiates a complete integral of the partial differential equation that is connected with it with respect to its integration constants and sets those differential quotients equal to new arbitrary constants.

That interesting peculiarity of the aforementioned differential equations might seem to justify the question whose resolution will define the subject of the following efforts:

What property characterizes a differential equation:

$$F(x, y, y', y'', \ldots) = 0$$

as the equation of a problem in the calculus of variations?

The known deductions of the calculus of variations, which we would next like to recall, will likewise make a special property of the differential equations that we speak of emerge that will prove to be essentially characteristic from now on: Let $f(x, y, y', y'', \ldots, y^{(n)})$ be a given function of the argument x, an unknown function y of x, and its derivatives y', y'', …, $y^{(n)}$, and treat the problem of determining y as a function of x in such a way that the integral:

(*) "De aequationum differentialium isoperimetricarum transformationibus earumque reduction ad aequationem differentialem partialem primi ordinis non linearem," *Gesammelte Werke*, Bd. V, and *Vorlesungen über Dynamik*, Lect. 19.

$$J = \int_{x_0}^{x_1} f(x, y, y', \dots, y^{(n)})\, dx$$

will become a maximum or a minimum, in which x_0 and x_1 are unvarying quantities for which the functions y, y', y'', …, $y^{(n-1)}$ shall assume prescribed values. From the principles of the calculus of variations, in order for an extremum to occur, it is necessary that the desired function:

1. should make the first variation of the integral J :

$$\delta J = \int_{x_0}^{x_1} \delta f \cdot dx = \int_{x_0}^{x_1} \sum_{k=0}^{n} \frac{\partial f}{\partial y^{(k)}}\, u^{(k)} \cdot dx$$

equal to zero, and that:

2. it should give an unvarying sign to the second variation:

$$\delta^2 J = \int_{x_0}^{x_1} \delta^2 f \cdot dx = \int_{x_0}^{x_1} \sum_{i,k=0}^{n} \frac{\partial^2 f}{\partial y^{(i)}\, \partial y^{(k)}}\, u^{(i)}\, u^{(k)} \cdot dx\,.$$

The u in that means a function of x that vanishes at the limits of the integral, along with its first (n – 1) derivatives but is otherwise arbitrary.

We shall appeal to the abbreviated notation:

$$\Phi(x, y, y', \dots) \sim \Psi(x, y, y', \dots)$$

in order to express the idea that the difference between the functions Φ and Ψ for an undetermined y can be represented by an exact differential quotient:

$$\frac{dX}{dx}(x, y, y', \dots)\ .$$

One will then get:

$$\delta f \equiv \sum_{k=0}^{n} f_k \cdot u^{(k)} \sim u \cdot F(x, y, y', \dots, y^{(2n)})$$

by repeated partial conversion, in which f_k replaces $\partial f / \partial y^{(k)}$, and F represents the differential expression:

$$F(x, y, y', \dots, y^{(2n)}) = V(f) \equiv \sum_{k=0}^{n} (-1)^k \frac{d^k}{dx^k} f_k\ , \tag{1}$$

which has order 2*n* or lower, but has even order in any event, as one knows (*). When one recalls that the $u,\ u',\ \ldots,\ u^{(n-1)}$ vanish at the limits, it will then follow that:

$$\delta J = \int_{x_0}^{x_1} \delta f \cdot dx = \int_{x_0}^{x_1} u \cdot F \cdot dx\,,$$

from which one further concludes that the desired function *y* must satisfy the differential equation:

(2) $$F\,(x, y, y', \ldots, y^{(2n)}) = 0$$

in order to make δJ zero.

Since the operations *d* and δ commute, when one applies the δ process to:

$$\delta f \sim u \cdot F,$$

that will give:

$$\delta^2 f \sim u \cdot \delta F,$$

such that one can put the second variation of the integral into the form:

$$\delta^2 f = \int_{x_0}^{x_1} \delta^2 f \cdot dx = \int_{x_0}^{x_1} u \cdot \delta F \cdot dx\,,$$

Now, **Jacobi** (**) based the further transformation of that expression for the purpose of determining its sign upon the remarkable fact that the differential expression:

$$\delta F = \sum F_k \cdot u^{(k)}\,,$$

which is linear and homogeneous in the derivatives of *u*, is self-adjoint. The adjoint to the linear differential expression:

(3) $$P\,(u) \equiv \sum_{k=0}^{n} p_k(x)\, u^{(k)}$$

is known to be:

(4) $$P'(u) \equiv \sum_{k=0}^{n} (-1)^k \frac{d^k}{dx^k}\{p_k(x) \cdot u\}$$

and as such, it is characterized completely by the property (***) that:

(*) Cf., **Frobenius**, "Ueber adjungirte lineare Differentialausdrücke," Crelle's Journal, Bd. 85, pp. 206.
(**) "Zur Theorie der Variationsrechnung und der Differentialgleichungen," *Ges. Werke,* Bd. IV.
(***) Cf., **Frobenius**, *loc. cit.*, pp. 188.

(5) $$v \cdot P(u) \sim u \cdot P'(v) \ .$$

As a result of **Jacobi**'s theorem, one will then have:

(6) $$v \cdot \delta_u F \sim u \cdot \delta_v F \ ,$$

which can be proved as follows:

By definition, one has:

$$\delta_u f \sim u \cdot F \ ,$$

so

$$\delta_v (\delta_u f) \sim u \cdot \delta_v F \ ,$$

and similarly:

$$\delta_u (\delta_v f) \sim v \cdot \delta_u F \ ,$$

so since:

$$\delta_v (\delta_u f) = \delta_u (\delta_v f) \ ,$$

one will have:

$$u \cdot \delta_v F \sim v \cdot \delta_u F$$

Q.E.D. (*).

Conversely, the extent to which the property of the expression F that is characterized in that way implies its formal structure will emerge from the following theorems, whose proofs will be what are mainly addressed here:

I. *If the function* $F(x, y, y', \ldots, y^{(2n)})$ *of even order* $2n$ *has the property that the linear differential expression* $\delta F = \sum_{k=0}^{2n} F_k \cdot u^{(k)}$ *that is derived from it is self-adjoint then a function* $f(x, y, y', \ldots, y^{(n)})$ *can be determined by quadratures in such a way that F can be represented in terms of f in the form:*

$$F = V(f) \equiv \sum_{k=0}^{n} (-1)^k \frac{d^k}{dx^k}\left(\frac{\partial f}{\partial y^k}\right) .$$

Solving the differential equation $F = 0$ *is then equivalent to the problem in the calculus of variations of making the integral:*

$$J = \int_{x_0}^{x_1} f(x, y, y', \ldots, y^{(n)})\, dx$$

and extremum.

An immediate consequence of that theorem is the following one:

(*) Cf., **Frobenius**, *loc. cit.*, pp. 205.

II. *When the problem of seeking an extremum to the integral:*

$$\int_{x_0}^{x_1} \varphi(x, y, y', \ldots, y^{(m)})\, dx$$

leads to a differential equation that degenerates to the extent that its order is $2n < 2m$*, a function* $f(x, y, y', \ldots, y^{(n)})$ *of order n can be determined by quadratures in such a way that corresponding problem for the integral:*

$$\int_{x_0}^{x_1} f(x, y, y', \ldots, y^{(n)})\, dx$$

will imply the same differential equation.

There is no difficulty involved with proving that theorem directly.

Whereas the function F that was used as a basis up to now necessarily has even order, an expression F of odd order that one can associate with an analogous property has an essentially different character. Namely, as one can deduce by extension, one has the theorem:

III. *If the function* $F(x, y, y', \ldots, y^{(2n+1)})$ *of odd order has the property that the linear differential expression* $\delta F = \sum F_k \cdot u^{(k)}$ *that is derived from it is equal and opposite to its adjoint then F will necessarily be a function that is linear in y and its derivatives.*

The problem statement that was suggested is closely related to the problem of converting it into multiple integrals and partial differential equations. If $x_1, x_2, \ldots, x_n$ are n independent arguments, and y is an unknown function of them, and if one sets:

$$y_{\nu_1\nu_2\cdots\nu_n} = \frac{\partial^{\nu_1+\nu_2+\cdots+\nu_n} y}{\partial x_1^{\nu_1}\, \partial x_2^{\nu_2} \cdots \partial x_n^{\nu_n}},$$

then a function y that makes the n-fold integral:

$$\int_{(n)} f(x_1, x_2, \ldots, x_n, y, \ldots, y_{\nu_1\nu_2\cdots\nu_n}, \ldots)\, dx_1\, dx_2 \cdots dx_n$$

an extremum must fulfill the partial differential equation:

$$(7) \qquad F = V(f) \equiv \sum (-1)^{\nu_1+\nu_2+\cdots+\nu_n} \frac{d^{\nu_1+\nu_2+\cdots+\nu_n}}{dx_1^{\nu_1}\, dx_2^{\nu_2} \cdots dx_n^{\nu_n}} \left(\frac{\partial f}{\partial y_{\nu_1\nu_2\cdots\nu_n}} \right) = 0,$$

whose left-hand side is implied by the relation:

(8) $$\delta f \equiv \sum \frac{\partial f}{\partial y_{\nu_1\nu_2\cdots\nu_n}} u_{\nu_1\nu_2\cdots\nu_n} \sim u \cdot V(f)\,,$$

when that notation is now understood to mean that the difference between the two sides can be represented by an aggregate of n exact differential quotients that are taken with respect to the individual arguments $x_1, x_2, \ldots, x_n$. In complete analogy to the above, (8) will imply the relation:

(6) $$v \cdot \delta_u F \sim u \cdot \delta_v F,$$

and since equation (5) is also characteristic of the adjoints of linear partial differential equations (*), one can conclude from this that:

The expression $F = V(f)$ that is given in (7) *possesses the property that the linear differential expression δF that is derived from it is self-adjoint.*

Although it now seems likely that concluding the aforementioned property from the structure of the function F that is given in (7) is also allowable in this more general case, it nonetheless seems that proving the validity of that hypothesis will encounter grave difficulties. We will then restrict ourselves to the examination of second-order partial differential expression with two (three, resp.) arguments, which will, in fact, imply the theorem:

IV. *If an expression F has the property that its δF is self-adjoint then a function f can be determined by quadratures that is itself of second order and by means of which F can be represented in the form $V(f)$. The differential equation $F = 0$ is then equivalent to the problem of the calculus of variations:*

$$\delta \int_{(n)} f \cdot dx_1 \cdots dx_n = 0 \qquad\qquad n = 2, 3.$$

§ 2.

We begin with the exhibition and discussion of the conditions for the linear differential expression of even order $2n$:

(3) $$P(u) = \sum_{k=0}^{2n} p_k(x) u^k$$

to be equal to its adjoint expression:

(4) $$P'(u) = \sum_{k=0}^{2n} (-1)^k \frac{d^k}{dx^k}(p_k \cdot u)\ .$$

(*) Cf., **Frobenius**, *loc. cit.*, pp. 207.

If one develops $P'(u)$ in the derivatives of u and compares that to $P(u)$ then that will give the relations:

(9) $$\sum_{k=0}^{\lambda} (-1)^k (2n-k)_{\lambda-k}\, p_{2n-k}^{(\lambda-k)} = p_{2n-\lambda} \qquad (\lambda = 0, 1, 2, \ldots, 2n)$$

or, when written out:

(9) $$(2n)_\lambda\, p_{2n}^{(\lambda)} - (2n-1)_{\lambda-1}\, p_{2n}^{(\lambda-1)} + (2n-2)_{\lambda-2}\, p_{2n}^{(\lambda-2)} + \cdots + (-1)^\lambda\, p_{2n-\lambda} = p_{2n-\lambda}\,.$$

If one distinguishes between the odd and even values of the index λ then one will get, with the use of the Kronecker symbol:

$$\delta_{\mu\nu} = 0 \qquad \text{when} \quad \mu \neq \nu\,, \qquad \delta_{\mu\mu} = 1\,,$$

the two systems of condition equations:

(9′) $$\sum_{k=0}^{2\lambda-1} (-1)^k (2n-k)_{2\lambda-1-k}\, p_{2n-k}^{(2\lambda-1-k)} (1 + \delta_{k,2\lambda-1}) = 0 \qquad (\lambda = 1, 2, \ldots, n),$$

(9″) $$\frac{d}{dx}\left\{\sum_{k=0}^{2\lambda-1} (-1)^k (2n-k)_{2\lambda-k}\, p_{2n-k}^{(2\lambda-1-k)}\right\} = 0 \qquad (\lambda = 1, 2, \ldots, n).$$

Although it is not exactly required for our purposes, we shall however prove that the system of relations (9″) is already a consequence of the mutually-independent relations (9′) (which are obvious in their own right) since the expression in brackets in (9″) to be differentiated will vanish automatically on the basis of (9′). If follows from (9) that by $(2\nu - 2\lambda)$-fold differentiation:

(10) $$\sum_{k=0}^{2\lambda-1} (-1)^k (2n-k)_{2\lambda-1-k}\, p_{2n-k}^{(2\lambda-1-k)} (1 + \delta_{k,2\lambda-1}) = 0 \qquad (\lambda = 1, 2, \ldots, \nu),$$

and it can be shown that the equation:

$$\sum_{k=0}^{2\nu-1} (-1)^k (2n-k)_{2\lambda-k}\, p_{2n-k}^{(2\lambda-1-k)} = 0\,,$$

which corresponds to the ν^{th} equation in (9″), represents a linear combination of equations (10). To that end, we must prove that all of the determinants of degree $(\nu + 1)$ in the matrix:

$$\left\{\begin{matrix} (2n-k)_{1-k}(1+\delta_{k,1}) \\ \vdots \\ (2n-k)_{2\lambda-1-k}(1+\delta_{k,2\lambda-1}) \\ \vdots \\ (2n-k)_{2\nu-1-k}(1+\delta_{k,2\nu-1}) \\ (2n-k)_{2\nu-k} \end{matrix}\right\} \qquad \begin{matrix} (\lambda = 1, 2, \ldots, \nu) \\ (k = 0, 1, 2, \ldots, (2\nu-1)) \end{matrix}$$

vanish identically. If we multiply the rows and columns of it with suitable non-vanishing factors then we must show the same thing for the determinants of the matrix:

$$\left\{\begin{matrix} (2-1)_k(1+\delta_{k,1}) \\ \vdots \\ (2\lambda-1)_k(1+\delta_{k,2\lambda-1}) \\ \vdots \\ (2\nu-1)_k(1+\delta_{k,2\nu-1}) \\ (2\nu)_k \end{matrix}\right\} \qquad \begin{matrix} (\lambda = 1, 2, \ldots, \nu) \\ (k = 0, 1, 2, \ldots, (2\nu-1)). \end{matrix} \tag{11}$$

We now compose that system with the following one:

$$\left\{\begin{matrix} (-u_0)^k \\ (-u_1)^k \\ \vdots \\ (-u_\nu)^k \end{matrix}\right\} \qquad (k = 0,\ 1,\ 2,\ \ldots,\ (2\nu - 1))\ , \tag{12}$$

in which $u_0, u_1, \ldots, u_\nu$ are undetermined quantities, and obtain the table:

$$\left\{\begin{matrix} (1-u_k)-u_k \\ (1-u_k)^3-u_k^3 \\ (1-u_k)^5-u_k^5 \\ \vdots \\ (1-u_k)^{2\nu-1}-u_k^{2\nu-1} \\ (1-u_k)^{2\nu}-u_k^{2\nu} \end{matrix}\right\} \qquad (k = 0,\ 1,\ 2,\ \ldots,\ \nu)\ . \tag{13}$$

If one regards the determinant of the latter as a function of u_0 then it will be rational in u_0 and entire and obviously of degree at most $(2\nu - 1)$, but in that way it will vanish for the 2ν values:

$$u_0 = u_1\,, \qquad u_2\,, \qquad u_3\,, \qquad \ldots, \qquad u_\nu$$

$$u_0 = 1 - u_1\,, \quad 1 - u_2\,, \qquad 1 - u_3\,, \qquad \ldots, \qquad 1 - u_\nu\,,$$

since any two columns in the determinant will be equal or equal and opposite, and it will then be identically zero. If one then denotes the corresponding determinants of degree (ν + 1) in the composed system (11) and (12) by D_k and U_k then the extended law of multiplication for determinants will give:

$$\sum D_k \cdot U_k \equiv 0\,. \tag{14}$$

If we now regard the undetermined quantities u_0 , u_1 , …, u_ν as infinitely small then a simple argument will show that the identity (14) will have the vanishing of the individual determinants D_k as a consequence, assuming that not all sub-determinants of degree ν from the first ν columns of the system (11) vanish. The fact that one of those determinants, namely:

$$|\,(2\lambda - 1)_k \cdot (1 + \delta_{k,\,2\lambda-1})\,| \qquad \begin{pmatrix} \lambda = 1, 2, \ldots, \nu \\ k = 0, 1, \ldots, (\nu - 1) \end{pmatrix} \tag{15}$$

is, in fact, non-zero will be shown later in passing.

The relations (9′) *then represent necessary and sufficient conditions for one to have* $P'(u) = P(u)$.

§ 3.

We shall now move on to prove Theorem I, as we promised, for the case of n = 1. Should the function $F(x, y, y', y'')$ be arranged that the linear differential expression:

$$\delta F = F_0\, u + F_1\, u' + F_2\, u''$$

is self-adjoint, then from (9′), the relation must exist:

$$\frac{dF_2}{dx} - F_1 = 0\,, \tag{16}$$

which then shows that F_2 can no longer contain y'' itself, so F must then have the form:

$$F = M(x, y, y') \cdot y'' + N(x, y, y')\,.$$

If one integrates the function M over y' (while considering that quantity to be freely variable) and sets:

$$\int M(x, y, y')\, dy' = P(x, y, y')$$

then

$$\frac{dP}{dx} = M \cdot y'' + \frac{\partial P}{\partial y} y' + \frac{\partial P}{\partial x} ,$$

such that F can be put into the form:

$$F = \frac{d}{dx} P(x, y, y') + Q(x, y, y') ,$$

for which (16) will go to:

$$\frac{d}{dx}\left(\frac{\partial P}{\partial y'}\right) - \frac{\partial}{\partial y'}\left(\frac{dP}{dx}\right) - \frac{\partial Q}{\partial y'} = 0 ,$$

or since:

$$\frac{\partial}{\partial y'}\left(\frac{dP}{dx}\right) = \frac{d}{dx}\left(\frac{\partial P}{\partial y'}\right) + \frac{\partial P}{\partial y} ,$$

it will go to:

$$\frac{\partial P}{\partial y} + \frac{\partial Q}{\partial y'} = 0 .$$

As a result, one will get a function $f(x, y, y')$ by quadrature that satisfies the equations:

$$\frac{\partial f}{\partial y'} = -P , \qquad \frac{\partial f}{\partial y} = Q ,$$

and with its use, F will assume the structure:

$$F = \frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = V(f) . \qquad \text{Q.E.D.}$$

We now assume that Theorem I is correct for the order numbers 2, 4, 6, …, $(2n - 2)$, and we will show in what follows that it is also true for order $2n$ under that assumption, which will prove it in general.

§ 4.

If the linear differential expression that is derived from the function $F(x, y, y', \ldots, y^{(2n)})$:

$$\delta F = \sum_{k=0}^{2n} F_k \cdot u^{(k)}$$

is to be self-adjoint then from (9′), the following relations must be fulfilled:

$$\text{(17)} \qquad \sum_{k=0}^{2\lambda-1} (-1)^k (2n-k)_{2\lambda-1-k} (1+\delta_{k,2\lambda-1}) \frac{d^{2\lambda-1-k}}{dx^{2\lambda-1-k}} F_{2n-k} = 0 \qquad (\lambda = 1, 2, \ldots, n),$$

the first of which:

$$\text{(17′)} \qquad n \frac{d}{dx} F_{2n} - F_{2n-1} = 0 ,$$

shows that F_{2n} does not include $y^{(2n)}$, so F is linear in the highest derivative of y :

$$\text{(18)} \qquad F = M(x, y, y', \ldots, y^{(2n-1)}) \cdot y^{(2n)} + N(x, y, y', \ldots, y^{(2n-1)}) .$$

In what follows, W shall generally denote a function of the arguments that are given in each case, although it will not be important to define it more precisely. If one substitutes the expression (18) for F in (17′) then it will follow that:

$$n \frac{dM}{dx} - M_{2n-1} \cdot y^{(2n)} - N_{2n-1} = 0$$

or

$$(n-1) \frac{\partial M}{\partial y^{(2n-1)}} y^{(2n-1)} + W(x, y, y', \ldots, y^{(2n-1)}) = 0 ,$$

from which it will emerge that:

$$\frac{\partial M}{\partial y^{(2n-1)}} = 0 ,$$

so M is free of $y^{(2n-1)}$.

We shall now assume that the function M cannot include the arguments $y^{(2n-1)}$, $y^{(2n-2)}$, …, $y^{(2n-\nu+1)}$, and that will show that M is also free of $y^{(2n-\nu)}$ when $\nu < n$. On the basis of that assumption, the first ν relations of the system (17) will, in fact, take the form:

$$\begin{aligned} &(2n)_{2\lambda-1} \frac{d^{2\lambda-1}}{dx^{2\lambda-1}} + \sum_{k=1}^{\nu-1} (-1)^k (2n-k)_{2\lambda-1-k} (1+\delta_{k,2\lambda-1}) \frac{d^{2\lambda-1-k}}{dx^{2\lambda-1-k}} N_{2k-k} \\ &\qquad + (-1)^{\nu} (2n-\nu)_{2\lambda-1-\nu} (1+\delta_{\nu,2\lambda-1}) M_{2n-\nu} \cdot y^{(2n+2\lambda-1-\nu)} + W(x, y, y', \ldots, y^{(2n+2\lambda-\nu-2)}) = 0 \end{aligned}$$

$$(\lambda = 1, 2, \ldots, \nu),$$

and their $(2\mu - 2\lambda)$-fold differentiation will yield:

$$\text{(19)} \qquad M_{2n-\nu} \cdot y^{(2n+\nu-1)} \{(2n)_{2\lambda-1} + (-1)^{\nu} (2n-\nu)_{2\lambda-1-\nu} (1+\delta_{\nu,2\lambda-1})\}$$

$$+\sum_{k=1}^{\nu-1}(-1)^k(2n-k)_{2\lambda-1-k}(1+\delta_{k,2\lambda-1})\frac{d^{2\nu-1-k}}{dx^{2\nu-1-k}}N_{2n-k}$$
$$+\ W(x, y, y', \ldots, y^{(2n+\nu-2)}) = 0 \qquad\qquad (\lambda = 1, 2, \ldots, \nu).$$

We define the determinant Δ of the system of coefficients of equations (19):

(20) $$\Delta = \left| \{(2n)_{2\lambda-1} + (-1)^\nu (2n-\nu)_{2\lambda-1-\nu}(1+\delta_{\nu,2\lambda-1})\} \ \ldots \ (2n-k)_{2\lambda-1-\nu}(1+\delta_{k,2\lambda-1}) \ \ldots \right|$$

$$\begin{pmatrix} k = 1, 2, \ldots, (\nu-1) \\ \lambda = 1, 2, \ldots, \nu \end{pmatrix}$$

Now, if one can show that Δ is non-zero then the elimination of the $(\nu - 1)$ quantities $\frac{d^{2\nu-1-k}N_{2n-k}}{dx^{2\nu-1-k}}$ from the ν equations (19) will produce a relation of the form:

$$M_{2n-\nu}\cdot y^{(2n+\nu-1)} + W(x, y, y', \ldots, y^{(2n+\nu-2)}) = 0 ,$$

from which, it will emerge that one will also have:

(21) $$M_{2n-\nu} = 0 ,$$

so M will be free of $y^{(2n-\nu)}$.

We now investigate the possibility that the equation Δ = 0 is true. Upon multiplying the rows and columns of that by non-vanishing factors, it will go to:

$$\left| \{(2n)_\nu\, \nu! + (-1)^\nu (2\lambda-1)_\nu\, \nu!\,(1+\delta_{\nu,2\lambda-1})\} \ \ldots \ (2\lambda-1)_k\, k!\,(1+\delta_{k,2\lambda-1}) \ \ldots \right| = 0$$

$$\begin{pmatrix} k = 1, 2, \ldots, (\nu-1) \\ \lambda = 1, 2, \ldots, \nu \end{pmatrix},$$

or when one decomposes the aggregates that correspond to the first column of the determinant into:

(22) $$(2n)_\nu\, \nu!\, |(2\lambda-1)_{k-1}(k-1)!\,(1+\delta_{k-1,2\lambda-1})|$$
$$-\ |(2\lambda-1)_k\, k!\qquad (1+\delta_{k,2\lambda-1})| = 0 \qquad (k, \lambda = 1, 2, \ldots, \nu).$$

It is advisable to convert that equation from an arithmetic one into a functional one. To that end, we introduce ν functions:

(23) $$f_\lambda(x) = (1+x)^{2\lambda-1} + x^{2\lambda-1} \qquad (\lambda = 1, 2, \ldots, \nu)$$

and remark that:

$$f_\lambda^{(k)}(0) = (2\lambda - 1)_k \cdot k\,!\,(1 + \delta_{k,2\lambda-1}) .$$

With the use of that, one converts the equation $\Delta = 0$ into the demand that $x = 0$ must satisfy the equation:

(24) $$(2n)_\nu \cdot \nu!\,|\,f_\lambda^{(k-1)}(x)\,| - |\,f_\lambda^{(k)}(x)\,| = 0 \qquad (k, \lambda = 1, 2, \ldots, \nu).$$

The method for calculating the determinant that appears here leads us to the following remark: The ν functions $f_\lambda(x)$ fulfill a homogeneous linear differential equation of order ν:

$$\begin{vmatrix} y & y' & y'' & \cdots & y^{(\nu)} \\ f_1 & f_1' & f_1'' & \cdots & f_1^{(\nu)} \\ f_2 & f_2' & f_2'' & \cdots & f_2^{(\nu)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ f_\nu & f_\nu' & f_\nu'' & \cdots & f_\nu^{(\nu)} \end{vmatrix} = 0 ,$$

and when that is developed, it might read:

(25) $$y^{(\nu)} + p_1(x)\,y^{(\nu-1)} + p_2(x)\,y^{(\nu-2)} + \cdots + p_{\nu-1}(x)\,y' + p_\nu(x)\,y = 0 .$$

The coefficient of y : $p_\nu(x)$ in that, in particular, can be represented by means of the integral in the following way:

$$p_\nu(x) = (-1)^\nu\,|\,f_\lambda^{(k)}(x)\,|:|\,f_\lambda^{(k-1)}(x)\,| \qquad (k, \lambda = 1, 2, \ldots, \nu)$$

such that we can now write the equation $\Delta = 0$ as simply:

(26) $$p_\nu(0) = (-1)^\nu\,(2n)_\nu \cdot \nu\,! .$$

§ 5.

We shall now deal with the problem of exhibiting the linear differential equations (25) that the n functions:

(23) $$f_\lambda(x) = (1+x)^{2\lambda-1} + x^{2\lambda-1}$$

must satisfy. If we next set $x = z - 1/2$, which will make:

$$f_\lambda(x) = \left(\tfrac{1}{2} + z\right)^{2\lambda-1} - \left(\tfrac{1}{2} - z\right)^{2\lambda-1} = \varphi_\lambda(z) ,$$

then the expressions $\varphi_\lambda(z)$ will obviously be odd functions of z, so they will have the form:

$$\varphi_\lambda(z) = z \cdot \psi_\lambda(z^2) ,$$

in which $\psi_\lambda(u)$ represents a certain entire rational function of degree $(\lambda - 1)$ in u. The linear differential equation of order ν that those ν functions fulfill will then be simply:

$$\frac{d^\nu \eta}{du^\nu} = 0 , \tag{27}$$

and that is now to be transformed into one in terms of z by the relation $u = z^2$. The differential quotient $\frac{d^\nu \eta}{du^\nu}$ is represented in terms of the derivatives of η with respect to z in the form:

$$\frac{d^\nu \eta}{du^\nu} = \frac{1}{2^\lambda} \sum_{k=0}^{\lambda-1} (-1)^k A_{\lambda k}\, z^{-(\lambda+k)} \frac{d^{\lambda=k}\eta}{dz^{\lambda-k}} , \tag{28}$$

in which the quantities $A_{\lambda k}$ represent certain numerical values that satisfy the recursion formulas:

$$A_{\lambda+1,k} = A_{\lambda k} + (\lambda + k - 1)\, A_{\lambda,k-1} \qquad \begin{pmatrix} k = 0, 1, \ldots, \lambda \\ \lambda = 1, 2, 3, \ldots \end{pmatrix}, \tag{29}$$

as one sees immediately, and in which one sets:

$$A_{\lambda 0} = 1 , \qquad A_{\lambda\lambda} = 0 .$$

One obtains from (29), in succession, for $k = 1, 2, 3$:

$$A_{\lambda 1} = \lambda_2 ; \qquad A_{\lambda 1} = 3 \cdot (\lambda + 1)_4 ; \qquad A_{\lambda 3} = 3 \cdot 5 \cdot (\lambda + 6)_6 ,$$

and one easily verifies upon inferring $(k + 1)$ from k that:

$$A_{\lambda k} = 1 \cdot 3 \cdot 5 \ldots (2k - 1) \cdot (\lambda + k - 1)_{2k} = \frac{(\lambda + k - 1)!}{2^k \cdot k!(\lambda + k - 1)!} .$$

Therefore, under the substitution $u = z^2$, equation (27) will go to the following one:

(30)
$$\sum_{k=0}^{\nu-1} \frac{(-1)^k}{2^k \cdot k!} \frac{(\nu+k-1)!}{(\nu-k-1)!} z^{-k} \frac{d^{\nu-k}\eta}{dz^{\nu-k}} = 0 ,$$

in which one further substitutes $\eta = y / z$. In that way, one will then get the differential equation for the functions $y = \varphi_\lambda(z)$:

(31)
$$\sum_{\lambda=0}^{\nu} \frac{(-1)^\lambda}{(\nu-\lambda)!} C_\lambda \, z^{-\lambda} \frac{d^{\nu-\lambda}y}{dz^{\nu-\lambda}} = 0 ,$$

in which:

$$C_\lambda = \sum_{k=0}^{\lambda} \frac{(\nu+k-1)!}{2^k \cdot k!} (\nu-k) .$$

However, if one splits $(\nu - k)$ into $(\nu + k) - 2k$ here then one will have:

$$C_\lambda = \sum_{k=0}^{\lambda} \frac{(\nu+k)!}{2^k \cdot k!} - \sum_{k=1}^{\lambda} \frac{(\nu+k-1)!}{2^{k-1} \cdot (k-1)!} = \frac{(\nu+\lambda)!}{2^\lambda \cdot \lambda!} ,$$

so (31) will become:

$$\sum_{\lambda=0}^{\nu} (-1)^\lambda \nu_\lambda \frac{(\nu+\lambda)!}{2^\lambda \cdot \nu!} \, z^{-\lambda} \frac{d^{\nu-\lambda}y}{dz^{\nu-\lambda}} = 0 .$$

Finally, if one substitutes $z = \left(x + \frac{1}{2}\right)$ in that then that will give the differential equation of the functions $f_\lambda(x)$:

(32)
$$\sum_{\lambda=0}^{\nu} (-1)^\lambda \nu_\lambda \frac{(\nu+\lambda)!}{\nu!} (2x+1)^{-\lambda} \frac{d^{\nu-\lambda}y}{dx^{\nu-\lambda}} = 0 ,$$

or

(32′)
$$y^{(\nu)} - \frac{\nu \cdot (\nu+1)}{(2x+1)} y^{(\nu-1)} + \cdots + (-1)^\nu \frac{(2\nu)!}{\nu!(2x+1)^\nu} y = 0 .$$

The determinant of the ν functions $f_\lambda(x)$ is known to be:

$$\left| f_\lambda^k(x) \right| = c \, e^{-\int p_\lambda(x)\,dx} = c\,(2x+1)^{\nu(\nu+1)/2} \qquad \begin{pmatrix} k = 0, 1, \ldots, (\nu-1) \\ \lambda = 1, 2, \ldots, \nu \end{pmatrix},$$

in which c means a non-vanishing constant. If we set $x = 0$ in that then we will see that the expression:

$$| (2\lambda - 1)_k \cdot k! \, (1 + \delta_{k,2\lambda-1}) | = c \qquad \begin{pmatrix} k = 0, 1, \ldots, (\nu-1) \\ \lambda = 1, 2, \ldots, \nu \end{pmatrix},$$

and with it, the determinant (15) will be non-zero, which was promised above.

Furthermore, we infer from (32′) that:

$$p_\nu(x) = (-1)^\nu \frac{(2\nu)!}{\nu!} \frac{1}{(2x+1)^\nu},$$

which will make equation (26) go to the following one:

(33) $$(2n)_\nu = (2\nu)_\nu .$$

However, that cannot be true for $\nu < n$, one must have $\Delta \neq 0$, and therefore equation (21) is proved for $\nu < n$. By contrast, it is no longer true for $\nu = n$, because (33) is fulfilled in that case, of $\Delta = 0$.

We have then arrived at the result:

If the function $F(x, y, y', \dots, y^{(2n)})$ *is arranged such that* δF *is self-adjoint then* F *will have the form:*

(34) $$F = M(x, y, y', \dots, y^{(n)}) \cdot y^{(2n)} + N(x, y, y', \dots, y^{(2n-1)}) .$$

§ 6.

If we now set:

$$\int M(x, y, y', \dots, y^{(n)})\, dy^{(n)} = P(x, y, y', \dots, y^{(n)})$$

then that will make:

$$\frac{\partial P}{\partial y^{(n)}} = M ,$$

so

$$\frac{d^n P}{dx^n} = M \cdot y^{(2n)} + W(x, y, y', \dots, y^{(2n-1)}) ,$$

and therefore:

$$F = \frac{d^n}{dx^n} P(x, y, y', \dots, y^{(n)}) + Q(x, y, y', \dots, y^{(2n-1)}) .$$

If we further denote:

$$\int P(x, y, y', \dots, y^{(n)})\, dy^{(n)} \qquad \text{by} \qquad (-1)^n \varphi(x, y, y', \dots, y^{(n)}) ,$$

such that:

$$P = (-1)^n \frac{\partial \varphi}{\partial y^{(n)}} ,$$

then we can obviously put F into the form:

(35) $$F(x, y, y', \dots, y^{(2n)}) = \Phi(x, y, y', \dots, y^{(2n)}) + \Psi(x, y, y', \dots, y^{(2n-1)}) ,$$

in which we have set:

$$\Phi = V(\varphi) \equiv \sum_{k=0}^{n} (-1)^k \frac{d^k}{dx^k}\left(\frac{\partial\varphi}{\partial y^{(k)}}\right) . \tag{36}$$

Now, should:

$$\delta F = \delta\Phi + \delta\Psi$$

be self-adjoint, and since the adjoint expression to an aggregate is equal to the sum of the adjoint expressions of the individual summands, and since $\delta\Phi$ further has the structure of Φ in (36) as a result of the fact that it is self-adjoint, that would imply that $\delta\Psi$ must also be equal to its own adjoint. However, the relation (17′):

$$n\frac{d}{dx}\Psi_{2n} - \Psi_{2n-1} = 0$$

that Ψ must then satisfy shows that since Ψ does not include $y^{(2n)}$, one must have $\Psi_{2n-1} = 0$, so Ψ is also free of $y^{(2n)}$. Therefore, the assumption that we made at the end of § **3** must hold true for $\Psi\ (x, y, y', \ldots, y^{(2n-2)})$, and Ψ can be put into the form:

$$\Psi = V(\varphi) \equiv \sum_{k=0}^{n-1} (-1)^k \frac{d^k}{dx^k}\left(\frac{\partial\psi}{\partial y^{(k)}}\right) , \tag{37}$$

in which ψ depends upon the arguments $x, y, y', \ldots, y^{(n-1)}$. If one now sets:

$$\varphi(x, y, y', \ldots, y^{(n)}) + \psi(x, y, y', \ldots, y^{(n-1)}) = f(x, y, y', \ldots, y^{(n)})$$

then when one recalls (36) and (37), equation (35) will go to:

$$F = V(f) \equiv \sum_{k=0}^{n} (-1)^k \frac{d^k}{dx^k}\left(\frac{\partial f}{\partial y^{(k)}}\right) , \tag{38}$$

which proves Theorem I.

The function $f(x, y, y', \ldots, y^{(n)})$ is not determined uniquely by F. If a function $g(x, y, y', \ldots, y^{(n)})$ likewise implies a representation $F = V(g)$ for F then the difference:

$$f - g = h(x, y, y', \ldots, y^{(n)})$$

will satisfy the differential equation:

$$V(h) \equiv \sum_{k=0}^{n} (-1)^k \frac{d^k}{dx^k}\left(\frac{\partial h}{\partial y^{(k)}}\right) = 0 ,$$

which is known to be the necessary and sufficient condition for h to be equal to an exact differential quotient. One will then get all functions g that serve to represent F as in (38) from the relation:

$$g \sim f .$$

§ 7.

If a differential equation of order ($2n$) that can be solved for the highest derivative:

$$y^{(2n)} + \Phi(x, y, y', \ldots, y^{(2n-1)}) = 0 \tag{39}$$

belongs to a problem in the calculus of variations then, as would be clear from the remark at the end of § **5**, there must exist a multiplier function $M(x, y, y', \ldots, y^{(n)})$ such that the product:

$$F = M \cdot (y^{(2n)} + \Phi)$$

has the property that δF can be equal to its own adjoint.

In the case $n = 1$ of the differential equation $y'' + \Phi(x, y, y') = 0$, which we will treat next, $M(x, y, y')$ must satisfy the equation:

$$\frac{dM}{dx} - \frac{\partial}{\partial y'}(M \cdot y'' + M \cdot \Phi) = 0$$

or

$$\frac{\partial M}{\partial x} + \frac{\partial M}{\partial y} - \Phi \cdot \frac{\partial M}{\partial y'} - \frac{\partial \Phi}{\partial y'} \cdot M = 0 , \tag{40}$$

and since that equation always possesses an integral M, and at the same time represents the only condition in our case, we can conclude that mainly every second-order differential equation will be equivalent to a problem in the calculus of variations (*), but in general that is only true *a posteriori*. That is, in order for formulate that problem, one will need to know the integral of the given differential equation itself, because if one sets:

$$\log M = N$$

in (40) then one will get the linear, inhomogeneous, first-order partial differential equation for N:

$$\frac{\partial N}{\partial x} + \frac{\partial N}{\partial y} y' - \Phi \cdot \frac{\partial N}{\partial y'} - \frac{\partial \Phi}{\partial y'} = 0 ,$$

(*) That remark is already found in **Darboux** *Théorie Générale des surfaces*, III partie, pp. 53, *et seq.*

whose solution comes down to the integration of the system of ordinary differential equations:

$$dx : dy : dy' : dN = 1 : y' : -\Phi : \frac{\partial\Phi}{\partial y'}$$

or

(a) $$y'' + \Phi(x, y, y') = 0 ,$$

(b) $$\frac{dN}{dx} = \frac{\partial\Phi}{\partial y'}(x, y, y') .$$

In order to then determine N from (b), one must have generally integrated (a) before. There is only one case in which N can be determined *a priori*, i.e., without any prior integration of the differential equation (a), namely, *when* $\partial\Phi / \partial y'$ *is an exact differential quotient, so when* $\Phi_1 = \partial\Phi / \partial y'$ *satisfies the condition that:*

$$V(\Phi_1) \equiv \frac{\partial\Phi_1}{\partial y} - \frac{d}{dx}\left(\frac{\partial\Phi_1}{\partial y'}\right) = 0 .$$

If that is fulfilled then the given differential equation (*a*) *can be reduced to a problem on the calculus of variations a priori*.

In the case $n > 1$, the determination of M from the relation (17′) leads to the equation:

(41) $$n\frac{dM}{dx} - M\frac{\partial\Phi}{\partial y^{(2n-1)}} = 0 .$$

That will show that when if the determination of M is to be possible then $\Phi_{2n-1} = \dfrac{\partial\Phi}{\partial y^{(2n-1)}}$ *will be an exact differential quotient, so the condition must be satisfied:*

(42) $$V(\Phi_{2n-1}) \equiv \sum (-1)^k \frac{d^k}{dx^k}\left(\frac{\partial\Phi_{2n-1}}{\partial y^{(k)}}\right) = 0 .$$

When the latter is satisfied, one can determine M from (41). *If the linear differential expression:*

$$\delta F = \delta\{M \cdot (y^{(2n)} + \Phi)\}$$

is self-adjoint then the differential equation (38) *will be equivalent to a problem in the calculus of variations.*

§ 8.

The proof of Theorem III can be achieved with fewer strokes. If $F(x, y, y', \dots, y^{(2n+1)})$ is a function of odd order with the property that the differential expression that is derived from it:

$$\delta_u F = \sum_{k=0}^{2n+1} F_k \cdot u^{(k)}$$

is equal and opposite to its adjoint then as a result of (5), that situation can be expressed by the relation:

$$u \cdot \delta_v F + v \cdot \delta_u F \sim 0 , \tag{43}$$

and in particular, for $u = v$ that will imply that:

$$v \cdot \delta_v F \sim 0 . \tag{44}$$

If one applies the δ_v-process to (43) and the δ_u-process to (44) then it will follow that;

$$u \cdot \delta_v^2 F + v \cdot \delta_v(\delta_u F) \sim 0 , \tag{43'}$$

$$v \cdot \delta_u (\delta_v F) \sim 0 , \tag{44'}$$

so one will also have:

$$u \cdot \delta_v^2 F \sim 0 .$$

However, due to the arbitrariness in u, that is possible only when:

$$\delta_v^2 F \equiv \sum_{i,k=0}^{2n+1} F_{ik} \, v^{(i)} \, v^{(k)} \equiv 0 .$$

If one replaces v with $(u + v)$ in that then that will further give:

$$\sum_{i,k=0}^{2n+1} F_{ik} \, u^{(i)} \, v^{(k)} \equiv 0 ,$$

so

$$\sum_{i,k=0}^{2n+1} F_{ik} \, v^{(k)} \equiv 0 \qquad [i = 0, 1, \dots, (2n + 1)],$$

and finally:

$$F_{ik} = \frac{\partial^2 F}{\partial y^{(i)}\,\partial y^{(k)}} \equiv 0 \qquad [i = 0, 1, \ldots, (2n + 1)].$$

It will then emerge from this that F depends upon the arguments $y,\ y',\ y'',\ \ldots,\ y^{(2n+1)}$ linearly. Q.E.D.

That theorem and its proof can be adapted to partial differential expressions F immediately.

§ 9.

We shall now turn to the examination of a second-order partial differential expression $F(x, y, z, p, q, r, s, t)$ with two arguments x, y, in which z is the function that depends upon the latter, while p, r, s, t mean their first and second differential quotients in Euler's notation. The assumption that δF is self-adjoint can be expressed by the equation:

$$\delta F \equiv F_z + F_p \frac{\partial u}{\partial x} + F_q \frac{\partial u}{\partial y} + F_r \frac{\partial^2 u}{\partial x^2} + F_s \frac{\partial^2 u}{\partial x\,\partial y} + F_t \frac{\partial^2 u}{\partial y^2}$$

$$= F_z - \frac{d}{dx}(F_p \cdot u) - \frac{d}{dy}(F_q \cdot u) + \frac{d^2}{dx^2}(F_r \cdot u) + \frac{d^2}{dx\,dy}(F_s \cdot u) + \frac{d^2}{dy^2}(F_t \cdot u)$$

which splits into the two relations:

$$(45) \qquad \left\{ \begin{aligned} 2\frac{dF_r}{dx} + \frac{dF_s}{dx} &= 2F_p\,, \\ \frac{dF_s}{dx} + 2\frac{dF_t}{dx} &= 2F_q\,, \end{aligned} \right.$$

and in that way we set $F_x = \dfrac{\partial F}{\partial x}$, $F_{xy} = \dfrac{\partial^2 F}{\partial x\,\partial y}$, etc.

As far as the dependency of the function F upon r, s, t is concerned, the discussion of the conditions (45) implies the following differential equations:

$$(46) \qquad F_r = 0\,, \qquad F_{tt} = 0\,, \qquad F_{rs} = 0\,, \qquad F_{st} = 0\,, \qquad 2\,F_{rt} + F_{ss} = 0$$

whose integration for F gives the representation:

$$(47) \qquad F = M\,(r\,t - s^2) + R \cdot r + 2\,S \cdot s + T \cdot t + N\,,$$

in which M, N, R, S, T are functions of x, y, z, p, q.

If one substitutes the expression for F from (47) in (45) then one will find the following relations for the latter:

$$(48)\qquad \begin{cases} (\alpha) & M_x + M_z\, p + S_q - T_p = 0, \\ (\beta) & M_y + M_z\, q - R_q + S_p = 0, \\ (\gamma) & R_x + R_z\, p + S_y + S_z\, q - N_p = 0, \\ (\delta) & S_x + S_z\, p + T_y + T_z\, q - N_q = 0. \end{cases}$$

Now, let φ be a function of x, y, z, p, q. One will then get the following expression for the symbol $V(\varphi \cdot s)$:

$$(49)\qquad \begin{aligned} V(\varphi \cdot s) &\equiv \varphi_z \cdot s - \frac{d}{dx}(\varphi_p \cdot s) - \frac{d}{dy}(\varphi_q \cdot s) + \frac{d^2}{dx\,dy}(\varphi) \\ &= \varphi_{pq}(r\,t - s^2) + R' \cdot r + 2S' \cdot s + T' \cdot t + N', \end{aligned}$$

in which R', S', T', N' are functions of x, y, z, p, q, whose precise determination we shall not discuss. If one then determines the function φ such that:

$$\varphi_{pq} = \frac{\partial^2 \varphi}{\partial p\,\partial q} = M$$

then from (47) and (49), that can be used to represent F in the form:

$$(50)\qquad F = V(\varphi \cdot s) + R \cdot r + 2\,S \cdot s + T \cdot t + N.$$

If ψ is likewise a function of x, y, z, p, q then one will find that:

$$(51)\qquad V(\psi) = -\,\psi_{pp}\, r - 2\,\psi_{pq}\, s - \psi_{qq}\, t + N''.$$

If one then determines ψ by way of the equation:

$$\psi_{pp} = \frac{\partial^2 \psi}{\partial p^2} = -R$$

then, as a result of (50) and (51), one can put F into the form:

$$(52)\qquad F = V(\varphi \cdot s) + V(\psi) + 2\,S \cdot s + T \cdot t + N.$$

Now, since the expressions $\delta V(\varphi \cdot s)$ and $\delta V(\psi)$, by their very nature, are self-adjoint, the same thing will also be true of the expression:

$$\delta\{2\,S \cdot s + T \cdot t + N\}\,.$$

One gets the conditions that S, T, N are accordingly subject to from (48) when one sets $M = 0$, $R = 0$ in it:

$$(53) \qquad \begin{cases} (\alpha) & S_q - T_p = 0, \\ (\beta) & S_p \qquad = 0, \\ (\gamma) & S_y + S_z\, q - N_p = 0, \\ (\delta) & S_x + S_z\, p + T_y + T_z\, q - N_q = 0. \end{cases}$$

It emerges from (β) that S depends upon only x, y, z, q. Now, if χ is a function of the same arguments then one will have:

$$V(\chi \cdot p) = -\,2\chi_q \cdot s + T' \cdot t + N'\,.$$

If one then determines χ from:

$$\frac{\partial \chi}{\partial q} = -\,S$$

then one can put F into the form:

$$F = V(\varphi \cdot s) + V(\psi) + V(\chi \cdot p) + T \cdot t + N\,,$$

in which T and N now satisfy the conditions:

$$(55) \qquad \begin{cases} (\alpha) & T_p = 0, \\ (\gamma) & N_p = 0, \\ (\delta) & T_y + T_z\, q - N_q = 0. \end{cases}$$

As a result of (α), T will depend upon x, y, z, q. If χ is once more a function of the same arguments then one will have:

$$V(\chi) = -\,\chi_{qq}\, t + N'\,.$$

When one then determines χ from:

$$\frac{\partial^2 \chi}{\partial q^2} = -\,T\,,$$

replaces $T \cdot t$ with $V(\chi)$ in (54), and applies χ to ψ, one can assume that $T = 0$, such that the conditions for N that remain will be:

$$N_p = 0\,, \qquad N_q = 0\,.$$

If one then constructs a function $\chi(x, y, z)$ from:

$$\frac{\partial \chi}{\partial z} = N ,$$

then one will have $N = V(\chi)$, and when one again inserts χ into ψ, one will obtain the following result:

If the function $F(x, y, z, p, q, r, s, t)$ is arranged such that δF is self-adjoint then F can be brought into the form:

$$F = V(f)$$

by quadratures, in which:

$$f(x, y, z, p, q, r, s) = \varphi(x, y, z, p, q) \cdot s + \psi(x, y, z, p, q) ,$$

and the functions φ and ψ are subject to no restrictions.

§ 10.

Suppose that second-order differential expression is given:

$$F\{x_1, x_2, \ldots, x_n ; y ; y_1, y , \ldots, y_n ; y_{11}, y_{12}, \ldots, y_{nn}\} .$$

The $x_1, x_2, \ldots, x_n$ in it means the n arguments, y means the function that depends upon then, and one has further set:

$$y_i = \frac{\partial y}{\partial x_i} , \qquad y_{ik} = \frac{\partial^2 y}{\partial x_i \, \partial x_k} .$$

Should δF be self-adjoint then one will have:

$$\begin{aligned} \delta F &\equiv \frac{\partial F}{\partial y} \cdot u + \sum_{i=1}^{n} \frac{\partial F}{\partial y_i} u_i + \sum_{i \leq k} \frac{\partial F}{\partial y_{ik}} u_{ik} \\ &= \frac{\partial F}{\partial y} \cdot u - \sum_{i=1}^{n} \frac{d}{dx} \left(\frac{\partial F}{\partial y_i} \cdot u \right) + \sum_{i \leq k} \frac{d^2}{dx_i \, dx_k} \left(\frac{\partial F}{\partial y_{ik}} \cdot u \right) . \end{aligned}$$

The following relations emerge from that:

$$\sum_{i=1}^{n} \frac{d}{dx_i} \left(\frac{\partial F}{\partial y_{ik}} \right) (1 + \delta_{ik}) = 2 \, \frac{\partial F}{\partial y_k} \qquad (k = 1, 2, \ldots, n) , \tag{56}$$

$$(56') \qquad \sum_{i,k=1}^{n} \frac{d^2}{dx_i\, dx_k}\left(\frac{\partial F}{\partial y_{ik}}\right)\cdot(1+\delta_{ik}) = 2\sum_{k=1}^{n} \frac{d}{dx_k}\left(\frac{\partial F}{\partial y_k}\right),$$

from which, however, (56′) will be an immediate consequence of (56). If one combines the terms in the developed expression for (56) that include the third derivatives of y then that will give:

$$\sum_{i,k=1}^{n} \sum_{\mu,\nu=1}^{n} \frac{\partial^2 F}{\partial y_{ik}\,\partial y_{\mu\nu}}(1+\delta_{ik})(1+\delta_{\mu\nu})\, y_{\mu\nu i} = 0 \qquad (k = 1, 2, \ldots, n),$$

so

$$(50) \qquad \frac{\partial^2 F}{\partial y_{ik}\,\partial y_{\mu\nu}}(1+\delta_{ik})(1+\delta_{\mu\nu}) + \frac{\partial^2 F}{\partial y_{i\mu}\,\partial y_{\nu k}}(1+\delta_{i\mu})(1+\delta_{\mu k}) + \frac{\partial^2 F}{\partial y_{i\nu}\,\partial y_{k\mu}}(1+\delta_{i\nu})(1+\delta_{k\mu}) = 0$$

for

$$i \le k \le \mu \le \nu = 1, 2, \ldots, n .$$

One easily convinces oneself that this system of differential equations is satisfied by the determinant:

$$|\, y_{ik} \,| \qquad (i, k = 1, 2, \ldots, n),$$

as well as all of its subdeterminants of each order. In the case of $n = 3$, to which we shall now restrict ourselves, we likewise show with no effort that a sum of those quantities represents the general integral of equations (57). For $n = 3$, F will then take the form:

$$(58) \qquad F = A \begin{vmatrix} y_{11} & y_{12} & y_{13} \\ y_{21} & y_{22} & y_{23} \\ y_{31} & y_{32} & y_{33} \end{vmatrix}$$
$$+ B_{11}(y_{22}\, y_{33} - y_{23}^2) + B_{22}(y_{33}\, y_{11} - y_{31}^2) + B_{33}(y_{11}\, y_{22} - y_{12}^2)$$
$$+ 2\, B_{23}\, (y_{12}\, y_{13} - y_{12}\, y_{13}) + 2\, B_{31}\, (y_{23}\, y_{21} - y_{22}\, y_{31}) + 2\, B_{12}\, (y_{31}\, y_{32} - y_{33}\, y_{12})$$
$$+ C_{11}\, y_{11} + C_{22}\, y_{22} + C_{33}\, y_{33} + 2\, C_{23}\, y_{23} + 2\, C_{31}\, y_{31} + 2\, C_{12}\, y_{12} + D,$$

in which the coefficients A, B_{11}, …, C_{11}, …, D are functions of x_1, x_2, x_3, y, y_1, y_2, y_3 that must still satisfy certain conditions.

We would now like to show that we can progressively eliminate that expression for F by subtracting suitably-constructed quantities $V(f)$. If we next take $f = M\cdot(y_{22}\, y_{33} - y_{23}^2)$, in which M is independent of the second derivatives, then we will get an expression for $V(f)$ that has the same form as F in (58):

$$V\{M\cdot(y_{22}\, y_{33} - y_{23}^2)\} = -\frac{\partial^2 M}{\partial y_1^2}\begin{vmatrix} y_{11} & y_{12} & y_{13} \\ y_{21} & y_{22} & y_{23} \\ y_{31} & y_{32} & y_{33} \end{vmatrix} + \cdots$$

As a result, the initial term in F can be eliminated, and we can assume that $A = 0$. We will further find that:

$$V\{R \cdot y_{22}\} = -\frac{\partial^2 R}{\partial y_1^2}(y_{11}\,y_{22} - y_{12}^2) + B'_{11}(y_{22}\,y_{33} - y_{23}^2) + B'_{31}(y_{23}\,y_{21} - y_{22}\,y_{31}) + \sum C'_{ik}\,y_{ik} + D',$$

$$V\{S \cdot y_{23}\} = \frac{\partial^2 S}{\partial y_1^2}(y_{12}\,y_{13} - y_{11}\,y_{33}) + B''_{11}(y_{22}\,y_{33} - y_{23}^2) + B''_{31}(y_{23}\,y_{21} - y_{22}\,y_{31}) + B''_{12}(y_{31}\,y_{32} - y_{33}\,y_{12}) + \sum C''_{ik}\,y_{ik} + D'',$$

$$V\{T \cdot y_{33}\} = -\frac{\partial^2 T}{\partial y_1^2}(y_{33}\,y_{11} - y_{31}^2) + B'''_{11}(y_{22}\,y_{33} - y_{23}^2) + 2B'''_{12}(y_{31}\,y_{32} - y_{33}\,y_{12}) + \sum C'''_{ik}\,y_{ik} + D'''.$$

The respective coefficients B_{33}, B_{23}, B_{22} in (58) can be reduced to zero with the use of those expressions. If one then substitutes the expression for F in (58), thus-simplified, in equations (56) then one will get the following conditions for the coefficients B_{11}, B_{12}, B_{13} :

$$\text{(59)} \qquad \begin{cases} (\alpha) & \dfrac{\partial B_{12}}{\partial y_1} = 0, \\[2ex] (\beta) & \dfrac{\partial B_{13}}{\partial y_1} = 0, \\[2ex] (\gamma) & \dfrac{\partial B_{11}}{\partial y_1} + \dfrac{\partial B_{12}}{\partial y_2} + \dfrac{\partial B_{13}}{\partial y_3} = 0. \end{cases}$$

B_{12} and B_{13} are then independent of y_1 .

Now let S' and T' be functions of $x_1, x_2, x_3, y_1, y_2, y_3$, so they are likewise free of y_1 . One will then find that:

$$V(S' \cdot y_1 \cdot y_{23}) = -\frac{\partial S'}{\partial y_2}(y_{23}\,y_{21} - y_{22}\,y_{31}) - \frac{\partial S'}{\partial y_3}(y_{31}\,y_{32} - y_{33}\,y_{12}) + \frac{\partial^2 S'}{\partial y_2\,\partial y_3}\,y_1 \cdot (y_{22}\,y_{33} - y_{23}^2) + \cdots$$

$$V(T' \cdot y_1 \cdot y_{33}) = 2\,\frac{\partial T'}{\partial y_2}(y_{31}\,y_{32} - y_{33}\,y_{12}) - \frac{\partial^2 T'}{\partial y_2^2}\,y_1 \cdot (y_{22}\,y_{33} - y_{23}^2) + \cdots$$

One can first make B_{31} equal to zero in (58) with the help of those expressions, and then B_{12}, and (59.γ) shows that the still-remaining coefficient B_{11} will no longer include y_1 . Now when T'' is also free of y_1, one will have:

$$V(T''\cdot y_{33}) = -\frac{\partial^2 T'}{\partial y_2^2} y_1\cdot(y_{22}\,y_{33} - y_{23}^2) + \cdots,$$

with which, one can also reduce B_{11} to zero.

Furthermore, one has:

$$V(N) = -\frac{\partial^2 N}{\partial y_1^2} y_{11} + \cdots,$$

which will make C_{11} go to zero. The expression for F in (58) then reduced to the following one:

(60) $$F = 2\,C_{12}\,y_{12} + 2\,C_{13}\,y_{13} + C_{22}\,y_{22} + 2\,C_{23}\,y_{23} + C_{33}\,y_{33} + D\,.$$

Upon substituting that in (56), one will get the following relations for the coefficients C :

(61) $$\begin{cases} (\alpha) & \dfrac{\partial C_{12}}{\partial y_1} = 0, \quad \dfrac{\partial C_{13}}{\partial y_1} = 0, \\ (\beta) & \dfrac{\partial C_{23}}{\partial y_1} = \dfrac{\partial C_{31}}{\partial y_2} = \dfrac{\partial C_{12}}{\partial y_3}, \\ (\gamma) & \dfrac{\partial C_{22}}{\partial y_1} = \dfrac{\partial C_{12}}{\partial y_2}, \quad \dfrac{\partial C_{33}}{\partial y_1} = \dfrac{\partial C_{13}}{\partial y_3}, \\ (\delta) & \dfrac{\partial C_{22}}{\partial y_3} = \dfrac{\partial C_{23}}{\partial y_2}, \quad \dfrac{\partial C_{33}}{\partial y_2} = \dfrac{\partial C_{23}}{\partial y_3}. \end{cases}$$

As a result of (α) and (β), one can set:

$$C_{12} = -\frac{\partial N'(y_2, y_3)}{\partial y_2}, \quad C_{13} = -\frac{\partial N'(y_2, y_3)}{\partial y_3},$$

and since:

$$V(N'\cdot y_1) = -2\frac{\partial N'}{\partial y_2} y_{12} - 2\frac{\partial N'}{\partial y_3} y_{13} + \cdots,$$

C_{12} and C_{13} can be eliminated, and therefore C_{22}, C_{23}, C_{33} will be free of y_1, from (β) and (γ). Since the same thing is now true for D, as well, based upon (56), y_1 will no longer enter into F at all, while x_1 will act as a constant, and we will find ourselves in the case of two arguments that was resolved in § **9**. We have then arrived at the result:

If the expression for F in (58) *is arranged such that* δF *is self-adjoint then F can be brought into the form:*

$$F = V(f)$$

by quadratures, in which:

$$f = M \cdot (y_{22}\, y_{33} - y_{23}^2) + R \cdot y_{22} + 2S \cdot y_{23} + T \cdot y_{23} + N\,,$$

and the coefficients M, N, R, S, T represent arbitrary functions of x_1, x_2, x_3, y, y_1, y_2, y_3 .

Zurich, May 1896.