"Sur la courbure des variétés non holonomes," C. R. Acad. Sci. Paris **187** (1928), 1273-1276.

On the curvature of non-holonomic manifolds

Note by **Z. HORÁK**

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When one studies non-holonomic manifolds, one confirms that the usual method for arriving at curvature is no longer applicable to a X_n^m, due to the impossibility of constructing an infinitely-small, closed parallelogram ([1]). In the present Note, I propose to generalize the usual method for a X_n^m by replacing the closed parallelogram with an infinitely-small closed cycle that is composed of an arbitrary parallelogram that is situated in X_n^m and the vector that joins its final point to its initial point, in such a way that one returns to the starting point.

Consider a X_n^m that is embedded in a X_n with a local $(n - m)$-direction and let x^λ (λ, μ, $\nu = 1, 2, \ldots, n$) be the holonomic parameters of X_n. If one introduces the non-holonomic parameters x^k ($i, j, k = 1, 2, \ldots, n$) ([2]):

$$dx^k = A_\lambda^k \, dx^\lambda , \qquad dx^\lambda = A_k^\lambda \, dx^k ,$$

and if one denotes:

$$\underset{12}{\Delta} = \underset{2}{d}\,\underset{1}{d} - \underset{1}{d}\,\underset{2}{d}$$

then one will have:

$$\underset{12}{\Delta} x^k = \Pi_{ij}^k \, \underset{1}{dx^i} \, \underset{2}{dx^j} + A_{i\lambda}^k \, \underset{12}{\Delta} x^\lambda , \tag{1}$$

in which:

$$\Pi_{ij}^k = 2 A_\lambda^k \, \partial_{[i} A_{j]}^\lambda \qquad \left(\partial_i = A_i^\lambda \frac{\partial}{\partial x^\lambda} \right),$$

in which the choice of the parameters x^k and the operation $\underset{12}{\Delta}$ is completely arbitrary. Now, one can choose the dx^k in such a manner ([3]) that the displacements in the X_n^m are coupled by the $n - m$ conditions:

([1]) J.-A. SCHOUTEN, "On non holonomic connections," Proc. Kon. Akad. v. Wet. Amsterdam **31** (1928). See also G. VRANCEANU, "Sur quelques tenseurs dans les variétés non holonomes," C. R. Acad. Sci. Paris **186** (1928), 995-996.

([2]) J.-A. SCHOUTEN, *loc. cit.*

([3]) See my paper: "Ueber die Formeln für die allgemeine lineare Uebertragung…," Nieuw Achif v. Wisk. **15** (1927), 193-201.

$$dx^r = 0 \qquad (r = m + 1, \dots, n),$$

and that the other m differentials dx^a ($a, b, c, d, e, f = 1, 2, \dots, m$) are situated in X_n^m. Having said that, we shall suppose that the displacements $\underset{1}{d}$, $\underset{2}{d}$ that intervene in the operation are placed in X_n^m in such a way that:

(2) $$\underset{1}{d}\, x^r = 0, \qquad \underset{2}{d}\, x^r = 0, \qquad \underset{12}{\Delta}\, x^r = 0.$$

Hence, the two displacements $\underset{1}{d}$, $\underset{2}{d}$ generate a parallelogram in X_n^m whose final point can agree with its initial point by displacing along the vector:

$$-\underset{12}{\Delta}\, x^\lambda = \underset{21}{\Delta}\, x^\lambda .$$

It is precisely the vector that completes the parallelogram considered to a closed cycle to which we shall appeal for our definition of curvature. Finally, if we write:

$$-A_\lambda^k \underset{12}{\Delta}\, x^\lambda = -\underset{12}{\Delta}^k = \underset{21}{\Delta}^k$$

then, by virtue of (2), the equations (1) will become:

(3) $$\underset{12}{\Delta}\, x^\lambda = \Pi_{bd}^c\, \underset{1}{d}\, x^b\, \underset{1}{d}\, x^d - \underset{21}{\Delta}^c, \qquad \underset{21}{\Delta}^r = \Pi_{bd}^r\, \underset{1}{d}\, x^b\, \underset{1}{d}\, x^d .$$

Now suppose that one introduces an arbitrary linear connection in X_n that is defined by the coefficients $\Gamma_{\lambda\mu}^\nu$, $\Gamma_{\lambda\mu}^{\prime\nu}$. If one denotes the coefficients of that connection by Λ_{ij}^k, $\Lambda_{ij}^{\prime k}$, when they expressed by means of the non-holonomic parameters x^k ([1]) – i.e., if one sets:

(4) $$\nabla_i\, v_k = \partial_i\, v_k + \Lambda_{ij}^k\, v^k dx^j, \qquad \nabla_i\, w_k = \partial_i\, w_k + \Lambda_{ij}^{\prime k}\, w_k\, dx^j,$$

then the non-holonomic connection that is induced ([2]) in the X_n^m by the given connection will be defined the coefficients Λ_{ab}^c, $\Lambda_{ab}^{\prime c}$ ([3]).

Having said that, we propose to calculate the change $\underset{12}{\mathcal{D}}\, v^c$ that a vector v^c ($v^r = 0$) that is situated in X_n^m experiences during a circulation around our closed cycle. If the symbol

([1]) HORÁK, *loc. cit.*

([2]) The covariant derivative of an affinor in X_n^m is equal to the X_n^m–component of the covariant derivative in X_n; see Schouten (*loc. cit.*).

([3]) See my Czech paper: "On a generalization of the notion of manifold," Publication of the Science Faculty at the University of Masaryk, Brno, no. 86, 1927, pp. 2.

$\underset{12}{D} v^c$ denotes the change that the same vector experiences during displacement along the parallelogram that is generated by $\underset{1}{d} x^b$, $\underset{2}{d} x^b$ then one obtains the total change $\underset{12}{\mathcal{D}} v^c$ by adding to $\underset{12}{D} v^c$ the change that the vector v^c experiences upon returning from the final point of the parallelogram to its initial point along the vector $\underset{12}{\Delta}^k$. One will obviously have:

$$\underset{12}{\mathcal{D}} v^c = \underset{12}{D} v^c + \underset{21}{\nabla_k} v^c \Delta^k$$

then, from which one can easily infer by calculation and taking (3) and (4) into account:

$$(5)\qquad \begin{cases} \underset{12}{\mathcal{D}} v^c = R_{bda}^{\cdots c}\, v^a\, \underset{1}{d} x^b\, \underset{1}{d} x^d, \\ R_{bda}^{\cdots c} = 2\,\partial_{[d} \Lambda_{|a|b]}^{c} + 2\Lambda_{f[d}^{c} \Lambda_{|a|b]}^{f} + \Lambda_{ak}^{c}\, \Pi_{bd}^{k}. \end{cases}$$

We have then arrived at a definition of the affinor (5) as the *quantity of curvature* in X_n^m. By the same reasoning, when applied to a covariant vector w_a, we will arrive at the equation:

$$\underset{12}{D} w_a = - R_{bda}'^{\cdots c}\, w_c\, \underset{1}{d} x^b\, \underset{1}{d} x^d,$$

in which the quantity $R_{bda}'^{\cdots c}$ results from (5) by replacing the Λ_{ab}^{c} with $\Lambda_{ab}'^{c}$.

If X_n becomes an A_n then (5) will reduce to the affinor that Schouten ([1]) defined to be the curvature of the A_n^m, and for a holonomic manifold, the $R_{bda}^{\cdots c}$, $R_{bda}'^{\cdots c}$ will take the form that was given by the author ([2]).

([1]) *Loc. cit.*
([2]) *Loc. cit.*