"Les formules de Frenet pour un espace de M. Weyl," C. R. Acad. Sci. Paris **6** (27 June 1921), 1647-1650.

The Frenet formulas for a Weyl space

By

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Blaschke recently gave (1) the Frenet formulas for a curve that is traced in a space (R_n) with a Riemannian metric in which one defines parallel displacement as Levi-Cività did (2). Upon employing the same calculation procedures that Blaschke did, we have obtained the *Frenet formulas for a curve that is traced in a space* (W_n) *with a Weyl metric.*

An n-dimensional Weyl (3) space (W_n) is an n-dimensional multiplicity in which the metric is defined by two forms (one quadratic and one linear):

$$ds^2 = \sum_{i,k=1}^{n} g_{ik}\, dx_i\, dx_k \,, \qquad d\varphi = \sum_{i=1}^{n} \varphi_i\, dx_i \,.$$

$d\varphi$ is an invariant for any continuous transformation (T) of the form:

$$x_i = \psi_i\,(y_1, \ldots, y_n) \qquad (i = 1, 2, \ldots, n)\,.$$

Moreover, if one makes a change of calibration – i.e., if one takes a unit of length that is $\sqrt{\lambda}$ times smaller (λ = continuous function of $x_1, \ldots, x_n$) – then the two forms will become:

$$ds'^2 = \sum_{i,k=1}^{n} \lambda\, g_{ik}\, dx_i\, dx_k \,, \qquad d\varphi' = d\varphi - \frac{d\lambda}{\lambda}\,.$$

The laws of geometry must be satisfied under the following two conditions:

1. They are expressed by formulas that are invariant under any transformation (T).

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(1) Math. Zeit. **6** (1919).
(2) Rendiconti del Circolo mat. di Palermo **42** (1917).
(3) See WEYL, *Raum, Zeit, Materie*, 4th edition, § 16.

2. Those formulas remain invariant if one changes g_{ik} into $\lambda\, g_{ik}$ and φ_i into $\varphi_i - \frac{1}{\lambda}\frac{\partial\lambda}{\partial x_i}$.

Weyl define parallel displacement in relation to this new concept. Let a vector with components $(\xi^1, \dots, \xi^n)$ be attached to the point $P\,(x_1, \dots, x_n)$. We say that its *measure* is:

$$m = \sum_{i,k=1}^{n} g_{ik}\, \xi^i \xi^k .$$

Upon displacing P to $P'(x_i + dx_i)$ by congruence, its components will become $\xi^i + d\xi^i$, with:

$$d\xi^i = -\tfrac{1}{2}\sum_{k,r,t=1}^{n} g^{ik}\left[\frac{\partial g_{ik}}{\partial x_r} + \frac{\partial g_{kr}}{\partial x_t} - \frac{\partial g_{rt}}{\partial x_k} + g_{rk}\varphi_t + g_{tk}\varphi_r - g_{rt}\varphi_k\right]\xi^r dx_t .$$

Let C be a curve whose parametric equations are $x_i = x_i\,(s)$. Imagine that we have fixed a vector Ξ at each point $P\,(s)$ whose components (ξ^i) are continuous functions of s according to a continuous law. Let $P\,(s)$ and $P'(s + ds)$ be two neighboring points, so they will then correspond to the two vectors Ξ and Ξ'. Displace Ξ from P to P' by congruence, so one will get a vector Ξ^* at P that is generally different from Ξ'. The difference $\Xi' - \Xi^*$ is an infinitely small vector that is attached to the arc PP'. Form:

$$\theta\,(\Xi) = \frac{\Xi' - \Xi^*}{ds} .$$

We will then get a new vector $\theta(\Xi)$ that is attached to the point $P\,(s)$ of the curve C and depends upon the field Ξ in an invariant manner ([1]). Then let $\Xi = \Xi_1$ with the components $\xi^i_{(1)} = dx_i / ds$. Then set:

$$\begin{aligned}
\theta\,(\Xi_1) &= \Xi_2 \quad \text{with components } \xi^i_{(2)} , \\
\theta\,(\Xi_2) &= \Xi_3 \quad \quad '' \quad\quad '' \quad\quad \xi^i_{(3)} , \\
&\dots\dots\dots\dots\dots\dots, \\
\theta\,(\Xi_{n-1}) &= \Xi_n \quad \text{with components } \xi^i_{(n)} .
\end{aligned}$$

The n-hedron $\Xi_1, \Xi_2, \dots, \Xi_n$ is not orthogonal, in general. Orthogonalize it using Schmidt's method ([2]) by defining an n-hedron that is composed of the n vectors:

([1]) WEYL, *loc. cit.*, pp. 103.
([2]) "Integralgleichungen, etc….," Math. Ann. **63**.

$$H_p = \begin{vmatrix} (1,1) & (1,2) & \cdots & (1,p-1) & \Xi_1 \\ (2,1) & \cdots & \cdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ (p,1) & \cdots & \cdots & (p,p-1) & \Xi_p \end{vmatrix} \qquad (p = 1, 2, \ldots, n),$$

in which one sets:

$$(\alpha, \beta) = \sum_{i,k=1}^{n} g_{ik}\, \xi^i_{(\alpha)}\, \xi^k_{(\beta)}\,, \qquad D_p = \begin{vmatrix} (1,1) & \cdots & (1,p) \\ \vdots & \ddots & \vdots \\ (p,1) & \cdots & (p,p) \end{vmatrix}, \qquad D_0 = 1.$$

The n-hedron (N), $H_1, \ldots, H_n$ is orthogonal and normalized; i.e.:

$$\sum_{i,k=1}^{n} g_{ik}\, \eta^i_{(\alpha)}\, \eta^k_{(\beta)} = \delta_{pq} = \begin{cases} 1 & \text{is} \quad p = q \\ 0 & \text{is} \quad p \neq q. \end{cases}$$

The Frenet formulas for the curve C are the formulas that give the values of s :

$$\theta(H_p) = \frac{W'_p - W_p}{ds}.$$

One finds, by some simple calculations, that:

$$(F) \qquad \begin{cases} \theta\eta^i_{(1)} = \dfrac{1}{2}\dfrac{d\varphi}{ds}\eta^i_{(1)} + \dfrac{1}{\rho_1}\eta^i_{(2)}, \\ \theta\eta^i_{(2)} = -\dfrac{1}{\rho_1}\eta^i_{(1)} + \dfrac{1}{2}\dfrac{d\varphi}{ds}\eta^i_{(2)} + \dfrac{1}{\rho_2}\eta^i_{(3)}, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \theta\eta^i_{(p)} = -\dfrac{1}{\rho_{p-1}}\eta^i_{(p-1)} + \dfrac{1}{2}\dfrac{d\varphi}{ds}\eta^i_{(p)} + \dfrac{1}{\rho_p}\eta^i_{(p+1)}, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots, \\ \theta\eta^i_{(n)} = -\dfrac{1}{\rho_{n-1}}\eta^i_{(n-1)} + \dfrac{1}{2}\dfrac{d\varphi}{ds}\eta^i_{(n)}, \end{cases} \qquad (p = 2, \ldots, n-1),$$

in which:

$$\rho_k = \frac{D_k}{\sqrt{D_{k-1}\, D_{k+1}}}.$$

The $\theta\eta^i_{(k)}$ are then homogeneous linear functions of the $\eta^i_{(q)}$. The determinant of those functions is *skew-symmetric*; the ρ_i are the $(n-1)$ *radii of curvature* of the curve. That determinant possesses a principal diagonal whose terms are all *equal to* $\frac{1}{2}\frac{d\varphi}{ds}$. For a space ($R_n$), the formulas ($F$) will be the same as the ones that we just found, except that all of the terms in the principal diagonal will be equal to zero. (The ρ_i will not have the same value, since they depend upon the φ_i.) If one regards the trihedron (N) as moving along the curve C then one can say that one passes from one of its position to the neighboring position *by displacing by congruence* and then *subjecting it to a rotation that is defined by the curvatures* $1 / \rho_i$ *of* C, and finally *deforming it with a homothety of ratio* $1 + d\varphi / 2$.

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