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On the foundations of geometry

(By Herrn *Wilhelm Killing* in Braunsberg)

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I have already presented some prior works before the present treatise, namely, a paper that appeared in the Jahresbericht des Gymnasiums zu Brilon in 1880 and a second one that appeared as a preprint in the directory of lectures at our Lyceum for the Winter of 1884/85. However, individual parts of those works were included in the present work, but not without drastic alterations. Nonetheless, most parts of it are entirely new, namely, the last sections.

The fact that geometry assumes certain theorem without proof and must then base the further theorems on the assumption of that proof has been recognized since time immemorial. However, completely analogous statements are true for the concepts. The construction of concepts in geometry comes about by way of definitions whose essence consists of the fact that there are also limits to the act of definition and that certain concepts are at the basis of all definitions without themselves being definable; they might be referred to as the "basic concepts" of geometry. In order to be able to assign that name to a system of concepts, it must satisfy three conditions: First of all, each of those concepts must be necessary for geometry. Secondly, the system cannot be reduced to a smaller number of concepts, and thirdly, it must succeed in arriving at all geometric concepts.

However, the possibility of making the definitions is by no means assured in the presentation of the basic concepts. Before one can combine several concepts into one definition, one must recognize the possibility of connecting them to begin with, and since that connection should imply a new concept, the connection might not be necessary. Furthermore, in many cases, a definition cannot be posed in full generality immediately, but one must appeal to an auxiliary concept that has a specialized character, at least apparently. In those cases, it will be necessary to verify that the new concept actually has general validity. Thus, we recognize that any definition already assumes that certain judgements have been made. Those judgements (i.e., theorems) whose proofs are not traceable to other ones then define the foundation for the definitions and further judgements; one might be permitted to refer to them as the *postulates* of geometry.

From the great variety of opinions that have prevailed in regard to the first principles of geometry, I have regarded it as necessary to base my intuition more firmly and to preclude any possible misunderstandings as much as is feasible in presenting the basic concepts, and in so doing, I would not like to refrain from confessing that I glimpse the justification for the basic concepts and postulates that I shall present mainly in the fact that a consistent construction of geometry is possible with their help. However, at the same time, I shall not hide the fact that a second problem now necessarily confronts us, namely, the intrinsic justification for those concepts and judgements. That problem, which can always be addressed at a later point, might be less mathematical in character. However, I have believed that I should stay away from the realm of philosophy completely, even though it would certainly be interesting to at least add some general thoughts.

If I also place all of the postulates directly next to each other in § **2** then I would nonetheless deem it appropriate to pursue the consequences of each of them up to the point that it becomes necessary to add more of them. However, that prevented me from having any ambition to allow the exposition to diverge beyond what was already necessary. A certain breadth is unavoidable in foundational works. I believe that I devoted a painstaking thoroughness to the derivation of the concepts. However, I believed that some proofs could be addressed with only brief suggestions in order to not go much-too-far afield.

The first nine sections are concerned with only those consequences that are implied by the first seven postulates, and arrive at the entirely remarkable fact that the realm of knowledge that is determined in that way is essentially identical to rigorously-defined analytical theory, namely, the theory of finite transitive transformation groups. Should one prefer to find an oversight in the fact that one does not immediately include the intransitive groups then one might only consider the fact that the intransitive groups are closed in themselves and assume that a larger domain goes beyond the scope of those groups. On the other hand, the theory of transitive groups leads to the intransitive groups by way of the subgroups that they include, such that the latter are not excluded by our choice of starting point either.

Just as there are unimaginably many different transitive groups, one can also exhibit entirely different closed systems that satisfy the given postulates. All of them show an essential agreement in structure since they rest upon the same foundations. They must then be referred to as a branch of a single science, and it would be suitable to give a distinctive name to it. However, it would be better to call that science a "generalized geometry" and to assign the names of "space forms in the general sense" to its branches, in contrast to the "proper space forms" that will be determined by postulate VIII and are consistent with our experience for the case of three dimensions.

Above all, it might be questionable whether the latter postulate can be placed on entirely the same level with the remaining ones. According to it, a point at rest can either assume a second arbitrary position or a structure that goes through the point should necessarily preserve the rest state of the point. Whereas the remaining postulates must be assumed to be completely general, the property that is postulated here might be assumed for only a single point and then follow for all other ones. Thus, that postulate is only the basis for certain subdomain of a larger domain.

The following fact also serves to convince me that geometry, in the narrow sense, can be regarded as a small part of a general realm of knowledge: If one pursues one of **Plücker**'s ideas and regards a straight line, a plane, or any structure at all to be a proper space form that remains unchanged by all transformations of a subgroup then one will arrive at only another treatment of the same space form at the most fundamental level. The system to which one will be led in that way will also always exhibit the properties of a more general proper space form, but only in the rarest cases. One will then be led to the more general space form by an entirely simple operation that is performed on a proper space form.

Finally, I might probably support my opinion by drawing attention to a two-dimensional space form that I will characterize more precisely in § **13**, and which is perhaps compatible with my eight postulates. The circle will be replaced with a certain spiral in it. Herr **von Helmholtz**, who described it precisely without exhibiting its equations, excluded it by the postulate that a second closed line moves through the point at rest. However, if we restrict ourselves to experiences in which only motions in a plane are employed then the impossibility of such a space form cannot be proved; for higher-dimensional space, they will generally be excluded by themselves.

From the close relationship that exists between generalized space forms and transformation groups, I deem it necessary to point out the subtle differences that exist between the two theories. One of them is based upon the fact that complex values of the variables have the same status as real ones in the group, but not in the space form, and as a result of that, two groups are considered to be essentially identical (i.e., similar) when they can be converted into each other by an imaginary transformation, while the corresponding space forms exhibit essential differences. A second difference is the fact that many space forms can belong to the same group. Namely, there are cases in which a single system of values for the variables can be assigned to a point, as well as many of them. I recall the two space forms of positive constant curvature that are possible for each number of dimensions (this journal, Bd. 83, pp. 72).

I shall give an overview of the theory of transformation groups in § **9**. In it, I must, above all, consider those parts of it that will find application in the following sections or serve to ease one's understanding of the proofs that are employed in them. I have completely avoided any references to where the individual theorems were first presented or where one might find the simplest proofs of them. Indeed, the literature on that topic is still rather limited. For § **9**, one mostly has the first volume of **Lie**'s *Transformationsgruppen*, Leipzig, 1888. Along with that, one should confer a paper by Herrn **Schur** in volume 35 of the Annalen, and finally my own papers, namely, the ones that were published in volumes 31-36 of the Annalen. If I have connected the development of the concept of transformation group in § **8** quite closely with that of **Lie** then I was led to do that by, above all, the desire to make their complete agreement all the more transparent. I have also been able to mention in § **10** that **Lie** has directed his considerations to the linear group many times already, by which the motion of a fixed point will determine infinitely-close points.

The last four sections are devoted to the proper space forms. Just as **von Helmholtz** started from certain properties of motion and first referred to the true foundations of geometry in that way, I shall also assume some properties of motion, and indeed I shall likewise consider the rotation of a body around a fixed point. My assumption then leads me to an invariant between the coordinates of two points, so to a distance function in the sense of Herrn **Weierstrass**. When I let the two points move infinitely close to each other, **Riemann**'s invariant expression for the square of the arc-length element will replace it, and I shall prove that it is linear and homogeneous of degree two in the differentials. I shall deal with it in two different ways: once in conjunction with and with the use of the investigations that were present in this journal by Herren **Christoffel** (Bd. **70**) and **Lipschitz** (Bd. **70-72**). Finally, I shall apply the theory of transformation groups, and in that way arrive at a very simple and elementary derivation of the proper space forms.

In conclusion, allow me to make the following remark: For the proper space forms, the important phenomenon appears that a body or a boundary structure can never be made to overlap one of its parts. From my postulate VIII (and likewise from **von Helmholtz**'s assumptions), that fact represents a proposition that can be proved. One might also attempt to choose that property as a foundation, except that one must then pose it for not only bodies, but also all boundary structures. I still cannot predict whether one will then succeed in doing without motion entirely. Trying to do that would likewise present great difficulties. It would also emerge that one would arrive at position relations in geometry especially, and that laws of magnitudes would always take second place.

§ 1. – Basic concepts.

We propose the following basic concepts of geometry:

Solid body, parts of a body, space, parts of space, occupying (covering) a space, time, rest, motion.

The ancients believed it to be generally known that motion should be foreign to geometry, and as a result they went to the greatest efforts to also ban the use of motion from geometric proofs completely. However, they did not by any means succeed in getting along without motion. When **Euclid** proved the theorem of the congruence of triangles and theorems about the circle, he employed motion quite obviously. Even the use of the circle in those constructions/problems that served as existence proofs mostly came down to motions. Finally, in very many cases where **Euclid** employed his laws of magnitude, the deeper basis was found in motion. However, as soon as a theorem whose proof was found by applying motion was employed to prove another theorem, the latter would also be based upon motion. One can by no means conclude from the fact that many proofs only applied the congruence theorem for triangles that motion was irrelevant in them. Rather, one would have to confess that the direct application of motion would make the proof simpler, clearer, and more natural in many cases. If one could actually do without motion in geometry then the attempts of the ancients would have quite certainly been more successful.

In recent times, one often says that one does not deal with the motion of bodies, but with the comparison of different parts of space. That thought is all the more reasonable since the wonderful advances that geometry has made in latter times have come about precisely because one has been allowed to make the comparison in various ways. However, one foresees that the various types of comparisons would require special derivations and that all of them could be derived from congruence. Moreover, geometry, in and of itself, deals with only conceptual operations, it is irrelevant which representation is connected with the concepts. One can then undoubtedly place a certain comparison of parts of space at the pinnacle of geometry, but that comparison must be identical to the one that is mediated by the motion of rigid bodies. Meanwhile, one initially arrives at the spirit of the comparison of parts of space by means of the motion of solid bodies. That will then be first mediated by the use of eyes and hands (so also essentially by motion), and when one ultimately refrains from doing that, one appeals to the given tool, at least indirectly. On those grounds already, it is most natural to preserve the solid body as the basic concept for geometry. Once one tries to express the fundamental theorems without the use of the concept of solid bodies, one will recognize how difficult that it, and how much their naturality will suffer from that.

Motion is linked with its opposite, namely, rest. However, the concept of time cannot be avoided then. The necessity of that division has been recognized since time immemorial.

However, one must then admit that some of those concepts have more of the character of auxiliary concepts. Above all, that is true of time, in regard to which geometry found that it could not do without the concepts of "simultaneously, before, and after." The solid body was not used, in and of itself, but only insofar as further concepts could be derived with its help.

We cannot address the proof that it is impossible to reduce the given concepts to a smaller number here. When we look more closely at the attempts that have been made up to now to produce definitions of the concepts that were presented, we would certainly convince ourselves that we would be dealing with only an explanation, but not a rigorous definition. For example, if a solid body is defined as one for which the size and form remain unchanged then two new basic concepts, namely, "size" and "form of a body" would appear in place of the one basic concept of "solid body," and their explanation would create even more work than the concept that was given originally.

Thus, all that is required is the proof that the concept that was proposed suffices for geometry, and that proof can be carried out only by its actual construction.

§ 2. – Postulates of geometry.

The extension and impermeability of the body must also be asserted from the outset in geometry; we combine them into a postulate:

1. *Any body occupies a space at any time. The space that a body occupies cannot be simultaneously occupied by a second body.*

Geometry requires the unlimited subdivision of a body. However, the geometric subdivision is essentially different from the mechanical one in that the latter separates the parts from each other, while the parts of the geometric subdivision can be thought of as belonging to the whole. For example, if two positions of the same body have a spatial region in common, but do not coincide in another, then a geometric subdivision of the body will be effected in that way. One distinguishes those parts of the body whose second position was occupied by the body in its first position from those parts whose second position does not belong to the first position of the body. In that way, the demand that there must be an unlimited geometric subdivision will be entirely compatible with the physical assumptions of molecules and atoms.

II. *Any space (body) can be subdivided. Any part of a space (body) is, in turn, a space (body). If A is a part of B and B is a part of C then A is also a part of C, where one understands A, B, C to mean spaces, as well as bodies.*

The independence of space from the bodies that occupy it leads to the following postulate:

III. *Any body can move. If a body at any time covers the space that a second body occupied at an earlier time then it can be made to cover any space that the second one occupied at any time.*

That postulate allows one to define congruence for spaces and bodies by making that concept independent of time, and makes spaces independent of the bodies employed, whereas bodies are independent of the space employed. We will then refer to two spaces as congruent when they can cover the same spaces. Likewise, congruent bodies can be made to cover the same space. When one recalls the mobility of the body, the space that is occupied by a body will be referred to as its *position*. In regard to that postulate, geometry completely ignores the material of the body. It is only in that sense that the occasionally-used expression that "geometry considers space to be empty" can be understood.

IV. *Any body can move in such a way that one part of it can come to overlap a part of an arbitrary space.*

Let *M* be the given space, which is perhaps determined by a body *m* that occupies it at a well-defined point in time. Let *a* be the moving body, and let *A* be the space that it covers by the motion that this postulate demands. Four cases are then possible: *A* and *M* are completely identical, *A* is a part of *M*, *M* is a part of *A*, or fourthly, *A* and *M* have a part in common, while a part of *A* does not belong to *M* and a part of *M* does not belong to *A*. If one of those four cases is true without having decided which one it is, one might speak of a *partial covering*.

Let *A* be a position of a body *a* that has no part in common with a space *M*. By contrast, *a* shall also be capable of taking a position A_1 that is completely a part of *M*. There is also a position A' for the same body *a* then that has the property that a part of A' belongs to the space *M*, but another part of A' does not. Under any motion that takes the body *a* from *A* to A_1, it will arrive at a position that was given for A'. We give the following expression to that law:

V. *If a body has no part in common with a space before its motion, but belongs to that space afterwards, then it will arrive at a position in which one a part of it belongs to the space by its motion.*

Let the space *A* be divided arbitrarily into two parts *M* and *N*. One can always determine a body *k* then and perform a motion of it that satisfies the following conditions: At the beginning of the motion, *k* shall be a part of *M*, at the end of it, *k* shall cover a part of *N*, and during the motion, *k* and any part of *k* shall always belong to the space *A*. That yields the postulate:

VI. *If a (connected) space A is divided into any two parts M and N then one can always determine a body k that can move in such a way that during the motion, no part of the body will leave the space A, and k covers a part of M at the beginning of the motion and a part of N at its conclusion.*

If a space is divided into two parts *M* and *N* then we call those two parts *connected*. Above all, we refer to two spaces as *connected* when it can be considered to be part of a single space according to the foregoing postulate. [To clarify, it might be remarked that in that way two parts of space in three-dimensional space can be considered to be connected (i.e., defining a single space) when they meet at a surface, but that this expression should not be permissible when they meet at a point or along a line.]

VII. *As soon as one part a of a solid body once more comes to a position such that every part of a arrives at a part-wise covering of its initial position, every part of the body will once more take on its initial position.*

In that way, the concept of position takes on a more precise meaning than the one that was given by postulate III. Thus, two positions of the same body are referred to as *identical* only when not only the body as a whole, but also each arbitrary part of it, covers the same space at both times.

We next express the following postulate in a somewhat less precise form:

If we subdivide a body into two parts *m* and *n* and move it in such a way that *m* partially covers its initial position then *n* cannot be made to partially cover each space, but *n* will not describe a subspace in whose interior *m* is placed.

The word "interior" requires some explanation. For that reason, we express the law in the following form:

VIII. *A body a consists of two parts m and n; m covers the space M in the body's initial position. Should m continue to cover M with its parts under a motion, then one could always determine a subspace P such that with the given motion, n cannot arrive at the same covering by its parts. However, if one moves a body k in such a way that it part-wise covers one such space P at the beginning of the motion and covers M with its parts at its conclusion then it must necessarily also arrive under the motion at a covering by its parts of a space that n can occupy under the cited motion. That is true for any body and any arbitrary subdivision of it.*

We must now verify that the proposed postulates are mutually independent. The fact that none of the later postulates is contained in the previous ones is implied by the fact the previous laws are also valid for concepts that are not compatible with the following ones. "Occupying a space" is coupled with the "existence of a material." The subdivision of a body likewise subdivides the material. However, the possibility of covering that space does not require that the material must be of all one type. Therefore, postulate III does not follow from laws I and II. In the second law, one cannot replace "subdivision" with "combination" and "define a part of" with "taken by itself." However, that concept breaks down for IV. Whereas III and IV consider only the result of a motion, V represents the evolution of it, since it ascribes continuity to it. All of those laws will remain valid when one replaces a single body with a combination of several bodies and a space with a combination of separate spaces, which is a concept that was excluded by VI. One can likewise link the first six laws with the concept of fluid and gaseous bodies; it is VII that first adds the solid coupling of the individual parts. However, the fact that postulate VIII is not also redundant, but first implies the concepts of size and form, will emerge quite clearly from the following presentation. Conversely, however, one also sees that the foregoing laws will not be made redundant by the later ones. I would not like to go into the verification of that in detail, but only point out that the latter laws almost always assume one of the previous ones, so they emerge from the content of the previous ones.

§ 3. – Independence of the investigation of the body employed.

We can initially regard the space that is covered by a single body as a single space. However, we can also regard it as all of the positions that a body arrives at during any motion collectively. One sees that in the following way: If A and M are any two positions of the same body a then one can select a sequence of positions from the ones that a arrives at under the motion that takes A to M such that each of them has a part in common with the foregoing and following one. If those positions are B, C, D, E, … then B shall have a part in common with A and C, C with B and D, and D with C and E. The newly-considered part S shall belong to each space A, B, C, … However, one ignores the possibilities that a certain part of A also belongs to B and a certain part of B also belongs to C, etc. Any space A' that a occupies once under the motion considered shall be a part of S. Above all, a spatial region R shall belong to the space S when every part of R is covered by the parts of the body a during its motion. One then sees immediately that the postulates above are also true for the space S.

However, one can continue the process that was given here without limit. One considers a second motion that takes the body a from its previous initial position A to any other one M'. Ultimately, one can refer to the set of all positions that a body can attain at all as *space*. An easy application of the postulates that were posed shows that it is entirely irrelevant which body and which position one starts from. The demands that the given process can be continued without limit and that the union of the spaces that are covered can be referred to as "space" in the absolute sense frees us from those peculiarities. When the word "space" is used in that sense, we will refer to the space that any body occupies at a certain point in time as a *spatial region*.

One can even make motion independent of the body that is employed. Let a and b be any two parts of the same solid body. Let two positions A and A' for a be determined completely in the sense that was given in postulate VII that the first position of each part of a is given, as well as the second position. Since b belongs to the same solid body, (from VII) the first position of b, as well as the second one, will also be determined completely. I shall now give any part b' of that body the position B or also let it assume only a part of B. A further part c will then likewise arrive at a well-defined position C. If I now bring b' to the position B' then c will also assume a well-defined position C'. The spaces A, B, C (and any part of them) are now the spaces A', B', C' (and their corresponding parts), which are associated in such a way that when parts of those solid bodies cover each of the spaces A, B, C in their first positions, the same parts will occupy each of A', B', C' in the second position. However, the association of the spaces C and C' that is found in that way is independent of the insertion of B into B' that is employed in the present case. One would associate C with the same spatial region C' if one had chosen any other intermediate term. If one proceeds in the same way then it must be possible to start with any arbitrary spatial region M and associate another well-defined spatial region M' with it. That type of association is completely independent of the intermediate term. If one were to start, conversely, with the association of M and M' then one would also arrive at the same association for the remaining spatial regions, and in particular, one would associate A with A'. If a part of a solid body covers a space M in its first position, moreover, then there will be a second position in which M is covered by the same parts, and every space that is covered by any other part I of that body in the first position will correspond to the space that the part I occupied in its second position under the association above.

In that way, we will be completely independent of the moving body. The two positions of the body have given us an association in which every spatial region corresponds to a well-defined second region. It is not necessary to give the details of that law of association. For the sake of simplicity, in what follows, we will also further appeal to the mediation of bodies in order to derive further laws. However, we no longer need to expressly emphasize that the choice of body has no influence on the result.

When a motion of a body *a* is given, one can associate a well-defined motion with any other body *m*. Namely, if any part of *m* covers a space in the initial position of a space that part of *a* assumed in its initial position (in which it is assumed that the motions do not result simultaneously) then one can make that part of *m* perform the same motion that the part of *a* performed. In that way, a certain motion of *m* will be given. However, when the initial position of *m* has no part in common with *a*, one appeals to certain solid bodies and determines the associated motions for them in succession. It we were to choose any other body instead of *m* then we would get a "continuous sequence" (an expression that shall not be explained in more detail here) of associations; we might refer to such a thing as a "motion of space." That expression is entirely inadmissible, by itself, since space, in contrast to bodies, must be assumed to be immobile. However, one would be hard pressed to understand it verbatim. In addition, it clearly says that the investigation is independent of the body and that the same laws would be true for the sequence of those association of any spatial region with another one that was developed in the foregoing that are true for the motion of rigid bodies. For that reason, the expression "motion of space" might be applied naively.

Indeed, no direct applications of the results of this section will be made in the next section. Nonetheless, I believe that the developments that were expounded should still be discussed at this point.

§ 4. – The boundary structure and the number of dimensions of a space form.

When the parts of a body *k* cover a space, the same thing will be true for each subdivision of *k* for at least one of the parts obtained. For a body that covers several spaces with its parts simultaneously, we shall therefore also investigate the possibility that one part of it remains covered by parts of the same spaces under an arbitrary decomposition.

If we decompose a space *A* into two parts *B* and *C* then each body *k* can be made to simultaneously cover *B* and *C* with its parts. For our purposes, it will suffice to assume that each part of *k* belongs to the space *A*. Should that not in fact be the case, then we could nonetheless succeed in making one part k' if *k* lie completely in the space *A* and, at the same time, cover *B* and *C* with its parts. In that case, we would consider only the cited part k' instead of *k*. We carry out further subdivisions with that body. Such a thing can consist of saying that we consider the part of *k* that belongs to the space *B* to be one part and the one that belongs to the space *C* to be the other part. For the time being, we shall skip over that entirely specialized decomposition. However, for any other one, we will obtain at least one part of it whose parts simultaneously cover *B* and *C*. In that case, we say that *k* lies on the *boundary* of *B* and *C*.

We separate those parts of *B* that are not connected with *C*. Thus, let *B* be subdivided into B' and B'', where B', but not B'', is connected with *C*. *k* must then be simultaneously covered by the parts of *B* and *C* in its given position. The same thing will be true when we separate a part C'' from *C* that is not connected to *B*. Naturally, we can likewise add arbitrary parts to *B* that are not connected with *C* such that we will not be dependent upon the parts of space *B* and *C* that were just selected in many respects.

We start with a space *A* and decompose it into two parts *B* and *C*. We separate arbitrary parts from *B* that are not connected to *C* and, and we separate corresponding parts of *C*. Under that separation, the remaining parts will either always define a single space or decompose into several spaces. In the former case, the boundary can consist of one boundary structure, while in the latter case, it can consist of several. (*BC*) then represents a single boundary structure when:

a) The spaces *B* and *C* define a single space, and

b) After splitting off those parts of *B* that are not connected to *C*, the remaining parts will always define a single space. If the boundary consists of several structures then one can determine each of them by itself.

Since the same decomposition that was carried out here for a space can also be performed on a body, and the definition of the boundary structure can be adapted to it, that will make it possible for we to speak of the motion of a boundary structure. The same thing will also be true for the further boundary structures that will be defined in what follows, which is why that shall not be further pointed out for them. In the following sections, we would also like to express many of the definitions and propositions for only the case in which the boundary structure can be obtained by subdividing a single space without having to give the alterations that would be necessary for us to think of the boundary structure as arising from the subdivision of a body.

One gets two cases for a boundary structure (*BC*): Either each part of *C* implies a single part that is connected to *C*, or one can decompose *C* into two parts such that each of them defined a connected space with *B*. In the first case, every body that lies on the boundary structure covers the space that any other body has occupied in an earlier position of that type with its parts. Under motion, one finds that every space that a body can arrive at under motion will have the same property. In that way, one will be led to two equally-justified possibilities. If one of them is true for a specialized spatial region then it must be true for each spatial region. The meaning of the word “space” that was defined in the previous sections in the absolute sense then admits several possibilities. We refer to each of them as *space forms* and initially distinguish between *simply-extended* and *multiply-extended* space forms, which are space forms of one and several dimensions, respectively. For a one-dimensional space form, once a spatial region *A* has been decomposed into two parts *B* and *C*, any further decomposition of *B* will yield only a single spatial region that is connected to *C*. (In that definition, one generally employs a special spatial region *A* and a special decomposition. However, it immediately shows that the same property will be true for any other spatial region and any other decomposition.)

If one can decompose *B* into two parts *D* and *E* that are both connected to *C*, in which one understands (*BC*) to mean a single boundary structure, then (*CD*), as well as (*CE*), will represent a single boundary structure. Now, one can move every body k such that it succeeds in simultaneously covering all three parts C, D, E. In that way, every part that lies in C shall be connected to the one that lies in D, as well as the one that lies in E, and the same thing shall be true for the parts that lie in D and E. At the same time, once k has assumed that position, every arbitrary part of k will necessarily yield at least one part that either once more covers C, D, E with its parts or is connected with a part that is not covered. In that position, k will lie on the boundary of the three parts of space C, D, E. One separates an arbitrary piece from C that is not simultaneously connected with D and E (such that a piece that is connected to only D can be omitted) and proceeds similarly with D and E. The remaining part, for which the given laws are again valid, is either decomposed into several spaces or it always defines a single space. In the latter case, one will obtain a single boundary structure, while in the former, one will get several of them. When the subdivision of space A into the three spaces C, D, E determines a single boundary structure, one must ask whether C can be decomposed into two parts that are both connected to D and E. If that is not possible then the space form will be doubly extended. If there is such a decomposition then one will get one boundary and boundary structures with four spatial parts. The question of whether a decomposition that always increases the number of spatial parts that are connected to each other by one can be continued for a space form without limit, in which case one must assign infinitely many dimensions to it, must be set aside for now. However, in what follows, we shall consider only the case in which that process will reach its end after a finite number of operations, and we can then determine the number of dimensions in the following way:

We subdivide any spatial region into two parts and examine whether one of them, which is not connected to the other, decomposes the region by branching that region. If that is true then we shall consider only a region in which that is not the case. We again subdivide one of the two parts in such a way that each of the two new parts are connected to the other one and observe whether a splitting of such a region in which no connection between all three parts takes place will lead to a decomposition. We continue in the same way until any further subdivision of that kind is impossible. When that happens after n decompositions, we will assign n dimensions to the space form. We can summarize that briefly in the following way:

If each of $n + 1$ *parts of space are connected to each of the others and that connection must remain valid once one has removed an arbitrary region from each part that is not connected to any other then one will refer to the largest number* n *that is possible in that regard as the number of dimensions.*

§ 5. – Definitions and theorems about boundary structures.

In order to not get too involved, I shall give the following simple definitions and propositions, not in the most natural sequence that they would appear, but in such a way that their presentation will be as brief as possible.

If n is once more the number of dimensions for the space form considered then the number of decompositions that will lead to a boundary structure will be at most n. If a boundary structure is obtained by $n - m$ decompositions then we will assign it m dimensions. An m-dimensional structure is determined by $n - m + 1$ parts of space, each of which is connected with $n - m$ other ones, and their connection will not be destroyed when one removes any parts that are not connected to all of the other ones. The zero-dimensional structure is called a *point* (or also *element*), the one-dimensional one, a *line*, the two-dimensional one, a *surface*.

If a boundary structure is obtained from $n - m$ decompositions, and those parts are M_0, M_1, …, M_{n-m}, then for $m > 0$, one can decompose the part M_0 into two parts M'_0 and M''_0 such that M'_0, as well as M''_0, are connected with all of the parts M_0, …, M_{n-m}. Namely, if such a decomposition is impossible for M_0 then one will get an $(n - m)$-dimensional space, which we have excluded. That implies the further definitions and propositions:

Two boundary structures are identical when any body that lies in one of them also lies in the other one. Such things always possess equal dimensions.

A boundary structure is part of a second one when both of them possess equal dimensions and each body that lies in the first one also lies in the second one, but a body can lie in the second one that is disjoint to the first.

The point is indivisible. By contrast, any line, surface, and multi-dimensional structure is divisible.

If an (isolated) boundary structure γ decomposes into two parts α and β then any body that lies in α, as well as β, can be considered to belong to the boundary of both structures. The same argument can again be posed for the mutual boundary of two such structures that was posed for the subdivision of space, and one will also arrive at boundary structures in that way.

If an m-dimensional structure $(0 < m < n)$ *is decomposed into two parts and they determine a single boundary structure then it will have* $m - 1$ *dimensions and can also be obtained directly by* $n - m + 1$ *decompositions of space.*

The boundary structures to which one arrives by subdividing boundary structures are not distinct from the ones that are obtained by subdividing space.

Under the motion of a body, three cases are possible for a boundary structure that is obtained by subdividing that body:

a) The parts of any bounded part of the body that lies on the boundary structure will continue to cover its initial position under the motion.

b) A part of a body that lies on the structure will leave its initial position, but it will always lie on the structure in each new position (i.e., it will cover only those parts of space that belonged to the structure at the onset of the motion).

c) A part of a body that belongs to the boundary structure at the onset of the motion will also assume positions that do not belong to the initial position under the motion.

In the first case, we say that the structure *remains fixed*. In the second, it will *move into itself*, and in the third, it will *leave its initial position*. We now have the theorem:

If any part of an m-fold extended structure leaves its initial position under a motion then it will describe an $(m + 1)$*-dimensional structure, i.e., n – m subdivisions of an n-dimensional space are necessary for determining a boundary structure. If such a structure moves in such a way that it leaves its initial position then one can always perform* $n - m - 1$ *subdivisions of space in such a way that any part of a body that belongs to the structure that is obtained from the first subdivision at the onset of the motion, while it will always lie on the latter, and conversely any spatial region that lies on the latter is at least partially covered by such a part of the moving body in such a way that this part of the body belongs to the given structure in the initial position.*

In regard to the proof of that, which would be quite involved, it might suffice to give the basic ideas in it. One takes all positions that the structure assumes before and after the motion of a well-defined structure, and the mutual boundary will be defined by an *m*-dimensional structure. However, under a subdivision into two parts, the boundary structure will have a dimension that is one less than that of each part.

§ 6. – The intermediate position between two positions of the same fixed body. Uniform motion.

The three foregoing sections mainly used only the first six postulates, and the one part (namely, a section in § **3**) that was based upon postulate VII could be skipped without compromising logical continuity, since it also found no application in the two following sections. From now on, we must, above all, infer some further consequences of the seventh postulate, and that will be closely linked with the second part of the third section.

Let *M* and *N* be two positions of the same solid body, in the sense that was given to that concept in the context of postulate VII. We must then establish the spatial region that each part of the moving body occupies in the first position, as well as the second one. Since we then associate the parts of space *M* and *N* with each other as two positions of the same solid body, we likewise associate each part of *M* with a well-defined part of *N*. Whereas the body I occupies the position *M*, a body II that is solidly coupled with I will cover the position *N*. Now, as along as I preserves the position *N*, II must assume a well-determined position *P*. Naturally, *P* must also be capable of being covered by the body I. If we consider *M* to be the initial position of the body and *P* to be the final one then we shall call *N* its *intermediate position*. We define it in the following way:

If two positions M and P of the same body I *are distinguished as its initial and final positions then the intermediate position N will have the property that a body* II *that is solidly coupled with*

I *and coincides with* N *in the initial position of* I *will arrive at the final position* P *as soon as the body* I *assumes the position* N.

We would like to use the word "congruent," not just for an isolated body or spatial region, but also for the combination of parts of space in the sense of § **3**. We can also say more briefly: N is the *intermediate position* to M and P when the combination MN is congruent to NP.

While it is self-explanatory that with an arbitrary choice of congruent parts M and N, a P that corresponds to the conditions can always be found, one must always ask whether N can always be determined by that demand with an arbitrary choice of M and P. The following considerations show that this question is answered in the affirmative:

We introduce another terminology, whereby we once more emphasize that the words "position" and "congruent" shall be give the meanings that correspond to postulate VII. Let A_1 and B_1 be any two positions of the same body, such that A_1 and B_1 are its initial final positions, respectively. Let A_ι be any spatial region that is congruent to A_1 . We imagine a solid body, one part of which covers the spatial region A_1, while a second one covers B_1, and a third covers A_ι . (From § **3**, that body can be replaced with a sequence of solid bodies without altering the following developments.) Therefore, as soon as the part k_1 that was first covered by A_1 arrives at the position A_ι , one other particular part of that body k_2 will assume the position B_1 . However, that body k_2 will assume a position B_i as soon as one returns k_1 to the position A_1 . B_ι is determined completely by A_1, B_1, and A_ι then. When we apply a concise (if imprecise) expression from § **3**, we can say: If the three congruent parts of space A_1, B_1, A_ι are given arbitrarily then there will exist a well-defined spatial region B_i that will arrive at B_1 when A_1 comes to lie at A_ι . At the same time, the solid combination $A_1\, B_i$ is congruent to $A_\iota\, B_1$. In that way, every spatial region A_ι that is congruent to A_1 will be associated with a spatial region B_ι . That association is reciprocal. Namely, when the given body is assumed to be B_1 initially and A_1 is chosen to be its second position then A_ι will arrive at A_1 as soon as B_1 assumes the position B_ι . Had we started from the two positions A_ι and B_i , and had we sought the position that is associated with A_1 , then we would have arrived at B_1 . That further implies that when A_κ and B_κ are associated with each other by the first association, they will also remain associated when one considers A_ι and B_ι to be the first mutually-associated positions. That argument yields the theorem:

All of the positions that the same solid body can assume can be reciprocally associated with each other in a one-to-one way. That association can be obtained in such a way that one associates any two positions A_1 *and* B_1 *with each other, and the position* B_ι *that is associated with an arbitrary position* A_ι *is determined by the demand that one must have* $A_1\, B_\iota \cong A_\iota\, B_1$. *Once that association has been made, it will be independent of the pair that is chosen. If* A_κ *and* B_κ *are a pair to which one arrives in the given way then one will get the same association when one starts with that pair.*

In order to determine such an association, it is irrelevant which position A_ι one starts from, since any position of the body will occur as a position A_ι , in any event. One can then fix a well-defined position A and think of it as being associated with every position B of the same body. The symbol [AB] might therefore mean the association that makes the spatial region A correspond to

B. Thus, I can denote any second association by the symbol [AB'], and I will obtain all such associations when I imagine that all of the various positions of the body can be chosen for B'. [*AB*] and [*AC*] also denote the same association when *B* and *C* are identical then.

Now let [*AB*] once more be an association. I assume that one can simultaneously cover the two spaces *A* and *B* with the same body *k* such that *A* is occupied by *a* and *B* is occupied by *b*. One now moves the body *k* arbitrarily, and in a second position *a* covers the space A', while *b* covers the space B'. Such an association will also be given by [A' B'] then. That association will be identical to [*AB*] only in special cases, and one can likewise bound *A* with a certain spatial region such that whenever A' remain in it the association [$A'B'$] will be different from [*AB*] for $A'B' \cong AB$. At the same time, as long as A' remains in a certain spatial region, different positions of *A* and A' will also determine different associations.

We then have two methods for always getting new associations from a given one [*AB*]: The first one fixes *A* and replaces *B* with all parts of space that are congruent to it. The other method lets $\overline{AB}$ be congruent to it and replaces *A* with all parts of space that are congruent to it. The first method yields all possible associations. We then come down to the task of showing that the second way, which is just as powerful as the first one, also implies all possible associations.

We assume that the associations [*AB*] and [*CD*] are identical, i.e., *B* corresponds to *A* under the association [*CD*]. If one now moves $\overline{CD}$ in such a way that its figure remains unchanged then the *B* that corresponds to the fixed *A* will also change. However, as one easily proves, the same law is true for those changes in *B* that are true for rigid motions, as we can obtain the various *B* from the motion of a rigid body. Thus, any motion of the system *CD* corresponds to a well-defined motion of *B*. As long as *C* stays inside of a certain spatial region, *B* will also remain within a well-defined spatial region. If one pursues the two-sided change further, but in such a way that one restricts each of them to the given spatial region, then one will see that every motion of *B* can actually be obtained in the given way. We can then state the following theorem:

If the positions A and B are associated with each other under any of the reciprocal associations of all positions that the same solid body can assume that were characterized above then one can obtain all other such associations in such a way that one moves a body that likewise covers the spaces A and B, and thus derive a congruence of $\overline{A'B'}$ *with* $\overline{AB}$ *and regard* A' *and* B' *as associated positions.*

If we now use the term "rigid motion of space" in the sense that was proposed above then we can say that:

If any one of the associations of all positions of the same solid body that were characterized above is given then any second such association will be obtained by a rigid motion of space.

All such associations are then essentially identical and differ only in position.

Thus, the association [*AB*] is also well-defined when the position *A* coincides with *B*, so when a position *A* corresponds to itself. Hence, from the foregoing theorem, one also obtains any other association in such a way that one associates A with itself and establishes that the new position that is obtained by the motion is associated with itself. It then follows that:

Any position is associated with itself under an association of that kind.

Therefore, that also proves the theorem:

If A and B are any two positions of the same solid body then there will always be an intermediate position for it.

It was already suggested in the proof that by suitably bounding a spatial region (R) that contains A and B, one can always succeed in having only a single intermediate position inside of (R). However, the choice of (R) can also be arranged such that the intermediate position of A and M also belongs to the same spatial region as M and B and that just one such thing occurs in it. In that way, one can find that the problem of seeking the intermediate position not only admits a single solution for the given positions, but that this property will remain valid when one poses the same problem for any two of the positions that one gradually finds.

When one applies the concept of congruence to systems of spatial regions, one can give the following expression to the theorem of the intermediate position:

If A_0 and A_1 are any two positions of the same body then there will always be a third position $A_{1/2}$ of the same body such that $A_0\,A_{1/2} \cong A_{1/2}\,A_1$.

That theorem can be extended in the following way:

If A_0 and A_1 are any two positions of the same solid body then one can always determine m – 1 positions $A_{1/m}$, $A_{2/m}$, …, $A_{(m-1)/m}$ such that in each case one has:

$$A_{(i-1)/m}\,A_{i/m} \cong A_{i/m}\,A_{(i+1)/m}\,, \quad \text{where one must have} \quad \frac{m}{m} = 1,$$

whenever $0 < i < m$. The problem of finding $A_{1/m}$ is also either intrinsically unique, or it can be easily made to be.

We do not need to go further into the details of the proof of this theorem, since we can do without the theorem quite well. That comes down to the fact that in what follows we shall restrict ourselves to fractional indices whose denominators are powers of 2.

We start from two positions A_0 and A_1 and define the position A_2 by the equation $A_0\,A_1 \cong A_1\,A_2$, and also define the position A_{i+1} more generally by $A_0\,A_1 \cong A_i\,A_{i+1}$, once we come to a position A_i for a whole number i. We likewise associate every positive and negative whole number i with a certain position A_i by the equations $A_{-1}\,A_0 \cong A_0\,A_1$ and $A_{-i-1}\,A_{-i} \cong A_0\,A_1$. [In so doing, it would be good to initially just always remain inside of the spatial region (R) that was defined above and also introduce suitable restrictions on the size of the spatial region.] The demands that $A_0\,A_{1/2} \cong A_{1/2}\,A_1$,

$A_0\,A_{1/4} \cong A_{1/4}\,A_{1/2}$, etc., make it possible to give a single well-defined position for $A_{1/m}$ when m is a power of two. If one then lets i assume the values 1, 2, 3, … successively in $A_0\,A_{1/m} \cong A_{i/m}\,A_{(i+1)/m}$, and then lets $i + 1$ assume the values 0, − 1, − 2, … in succession then one will arrive at a well-defined position $A_{i/m}$. If one would like to free m from the indicated restriction then one must apply the theorem that was stated without proof above. In any event, one has the following theorem:

If one starts with the two positions A_0 and A_1 and associates a number a / m with a position $A_{a/m}$ in the given way then the position $A_{a/m}$ will be identical to the position $A_{b/n}$ when the value of a / m is equal to that of b / n .

It still remains for us to associate all of the numbers for whose representation one requires infinitely many fractions when they result in the given way with a well-defined position. However, I believe that I should not go further into that question at this point.

I now take any set of numbers and arrange them in increasing magnitudes $\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_\nu$. They might belong to the positions $A_{\alpha_0}, A_{\alpha_1}, A_{\alpha_2}, \ldots, A_{\alpha_\nu}$. I can then move the intermediate body k from the position A_{α_0} to A_{α_1}, and from there to A_{α_2}, etc. If A_{α_0} and A_{α_1} have a certain region in common then k shall cover that region, even when throughout all of the motion from A_{α_0} to A_{α_1}. When one proceeds in the same way without limit, one will get the theorem:

One can move a body in such a way that when A and B are any two positions that it takes during the motion, the intermediate position of A and B will be simply covered by the motion.

I shall call such a motion *uniform*. It has the following properties:

1. Any line that a point describes under its motion will be displaced into itself during that motion.

2. If A, B, C, D are points on such a line, and if $AB \cong CD$ then one will also have $AC \cong BD$.

We can then express the result that was obtained above in the following form:

If two positions of a body are given then one can always take the first position to the second one by a uniform motion.

§ 7. – Defining coordinates.

I would now like to show how one can measure a piece of a displaceable line by means of another arbitrary piece of that line. Since the measurement is performed by means of uniform motion, under which the line continues to cover its initial position, but it is conceivable that the line can be displaced into itself under various uniform motions, the measurement will be valid only relative to a particular uniform motion. (The time that is employed for the measurement is largely irrelevant, since there is no means for measuring it from our starting point. It should be added that, from our definition of uniform motion, an unequal velocity is not excluded.)

It obviously suffices to assume that the line segments to be measured start from the same point. Let α_0 and α_1 be two points on the line in question. One initially looks at those positions A_0 of the solid body in which a point π of it coincides with α_0 and then considers a second position A_1 to be one for which π coincides with the point α_1 . One chooses the line $\alpha_0\ \alpha_1$ to be the unit. Now, if σ is any number then one uses it to arrange the positions A_σ of the body k that were determined at the conclusion of the previous section as belonging to σ. One can let the points on the line that π occupies in the positions A_σ correspond to the number σ. In that way, any number σ will associated with a well-determined point α_σ , and thus will a well-defined line segment $\alpha_0\ \alpha_1$, as well. One then has the following theorems:

If $\mathfrak{s}$ *and* $\mathfrak{t}$ *are two equal numbers that are also obtained in different ways then the associated points* $\alpha_{\mathfrak{s}}$ *and* $\alpha_{\mathfrak{t}}$ *will coincide.*

If $\mathfrak{s}$ *and* $\mathfrak{t}$ *are two unequal positive numbers, and* $\mathfrak{s} < \mathfrak{t}$, *then the point* $\alpha_{\mathfrak{s}}$ *will lie along the segment* $\alpha_0\ \alpha_1$.

Conversely, if a further point β on the line is given after an arbitrary choice of α_0 and α_1 has been made then a single number can be assigned to that point in the given way. Namely, one can, by assumption, find a number $\mathfrak{a}_m$ for each whole number m such that the point that is associated with $\mathfrak{a}_m / 2^m$ belongs to the segment $\alpha_0\ \beta$, while in any case the point that is associated with $(\mathfrak{a}_m + 1) / 2^m$ will no longer lie on that segment. At the same time, the $\mathfrak{a}_m$ satisfy the conditions:

$$\mathfrak{a}_{m+p} \geq \mathfrak{a}_m \cdot 2^p \qquad \text{and} \qquad \mathfrak{a}_{m+p} + 1 \leq (\mathfrak{a}_m + 1) \cdot 2^p .$$

That boundary consideration, as well as another one that belongs with it, emerges from considerations that we already had to invoke in the previous section, but passed over at the time in order to not stray too far afield.

The construction of a coordinate system offers no difficulties. One chooses an arbitrary point α_0 to be the origin and establishes that all coordinates shall have the value zero for it. One then performs a uniform motion of a body from the point α_0 that coincides with a point of the body in its initial position. All points of the line from the origin that are described in that way shall have the coordinates $x_1 = 0,\ x_2 = 0,\ \ldots,\ x_{n-1} = 0$. In order to determine the coordinate x_n of an arbitrary point β on that line, one chooses a further point α_1 on that line arbitrarily. x_n shall then be the positive or negative ratio of $\alpha_0\ \beta$ to $\alpha_0\ \alpha_1$.

One now chooses any second uniform motion by which the line that was just employed will either remain fixed or be displaced into itself. The line that the origin now defines has vanishing values for $x_1, \ldots, x_{n-2}, x_n$, and after fixing an arbitrary unit, the value of x_{n-1} will be determined in the same way that x_n was determined before. However, the line $(x_1 = 0, \ldots, x_{n-1} = 0, x_n)$ will also determine a surface by that motion, and the position of the line each time will be determined as soon as the origin is known. Thus, when the origin arrives at $(0, \dots, 0, a_{n-1}, 0)$, i.e., at the point:

$$x_1 = 0,\ x_2 = 0,\ \dots,\ x_{n-1} = x_n = 0\ , \qquad x_{n-1} = a_{n-1}\ ,$$

the point $(0, 0, \dots, 0, a_n)$ will arrive at a well-determined point, and its coordinates shall be $x_1 = \dots, = x_{n-2} = 0$, $x_{n-1} = a_{n-1}$, $x_n = a_n$.

One continues in the same way by adding a third uniform motion under which the origin arrives outside of the structure $x_1 = \dots, = x_{n-2} = 0$. For the line that the origin then describes, all coordinates except for x_{n-2} shall vanish, and the value of x_{n-2} will be determined similarly to before. The position that the point $(0, 0, \dots, 0, a_{n-1}, a_n)$ assumes when the starting point arrives at $(0, 0, \dots, 0, a_{n-1}, 0)$ shall be denoted by $(0, \dots, 0, a_{n-2}, a_{n-1}, a_n)$. One then adds new motions in succession until one arrives at n uniform motions that associate each system $(x_1, \dots, x_n)$ with one point.

One then has the theorem:

With the established association, every system of values $(x_1, \dots, x_n)$ *corresponds to a single point in space, and conversely one can delimit a region around the origin and likewise determine an n-fold extended continuum* $(x_1, \dots, x_n)$ *around the system of values* $(0, \dots, 0)$ *such that every point of the region corresponds to a system in the continuum, and indeed only one.*

The first part of the theorem requires no further explanation. The second part is obtained as follows: If one lets the x_n increase from zero for vanishing values of $x_1, \dots, x_{n-1}$ then one will either return to the starting point or not. In the latter case (for which the line by no means needs to be infinite), unequal values of x_n will also correspond to different points. In the former case, there is a first positive and a first negative value of x_n that refer to the same point. As long as x_n remains between those values, every point of the line will correspond to only a single value of x_n . Those limits on x_n will not be restricted by the fact that x_{n-1} takes on non-zero values as long as the line $(0, \dots, 0, x_n)$ contains no point that is fixed under the second motion. However, when a point on that line is fixed under that motion, the uniqueness will be true up to the first fixed point, so it will still be true up to a finite value of x_n . As a result of the later motions, one can likewise restrict the region for x_n by saying that a point $(a_1, \dots, a_n)$ will coincide with the origin for non-vanishing values of $a_1, \dots, a_n$. Similar restrictions can be applied to the other coordinates; however, each coordinate will still remain with a finite range. Nevertheless, the totality of points that are represented by all allowable systems of values fills up a certain spatial region that lies around the origin, which can be shown in the same way as the way that we arrived at a definition of the number of dimensions and proved that this number has a fixed meaning for any space form.

In that way, the determination of the coordinates is reduced to n uniform motions, or if one prefers, $n + 1$ positions of the body. Those motions must be subjected to only certain restrictions, but other than that they can be chosen arbitrarily. That arbitrariness can often be used to obtain coordinate systems whose application will lead to the simplest formulas. One can also determine the coordinates $x_1, \dots, x_n$ by way of entirely arbitrary motions and then introduce new coordinates $y_1, \dots, y_n$ as functions of the x. Those functions must be mutually independent. If they further assume the values $b_1, \dots, b_n$ for $x_1 = x_2 = \dots = x_n = 0$ then one must be able to delimit a region of

x around (0, …, 0) and a region of y around (b_1, …, b_n) in such a way that a continuous and single-valued relationship will exist for both regions.

When we fix our attention on the possibility of making such a general coordinate determination, that will imply the development of the theorem:

n coordinates and uniform motions can be chosen in any n-dimensional space form such that the following relationship exists between them: Under the first motion, all coordinates x_2, …, x_n *will remain unchanged, and all* x_1 *will simultaneously increase by the same amount. Under the second motion, the* x_3, …, x_n *will remain unchanged in the structure for which* $x_1 = 0$, *and the* x_2 *will change regularly. Corresponding statement will be true for the structure* $x_1 = x_2 = 0$, *under the third motion, etc., and finally, the coordinates* x_n *will increase by the same amount for the structure* $x_1 = \ldots = x_{n-1} = 0$ *under the final motion.*

§ 8. – Mobility of a space form. The transformation group that belongs to a space form.

If an arbitrarily-chosen body is brought from a first position to a second one by any motion then a point of the body that has the coordinates x_1, …, x_n in the first position might coincide with the point x'_1, …, x'_n in its second position. In that way, it is by no means necessary for the coordinates x_1, …, x_n to be restricted to the points that actually belong to the body in its first position. Rather, one can imagine that the body continues without limit, and in that way any point x (to the extent that the given coordinate system reaches) that belongs to the first position is associated with a well-defined point x'. Those two positions then mediate a transformation:

$$x'_i = f_i(x_1, \ldots, x_n) \qquad (i = 1, \ldots, n) . \tag{1}$$

We now imagine that the body has been brought from the second position x' to a third position x''. That might be mediated by the transformation:

$$x''_i = g_i(x'_1, \ldots, x'_n) . \tag{2}$$

The values from (1) might be substituted in the last equation, by which we will get:

$$x''_i = g_i(f_1(x_1), \ldots, f_n(x_n)) = h_i(x_1, \ldots, x_n) . \tag{3}$$

However, from the basic concepts, it is possible to also obtain the third position from the first one. As a result, the transformation (3) also represented the result of a motion. If we imagine that all transformations that represent the result of motions in space that are possible at all are combined into a system then we will see that any two transformations that are performed in succession will also yield a transformation that once more belongs to the system. Whenever the transformations (1) and (2) belong to the system, the transformation (3) must also belong to the system. Herr **Lie** called such a system a *transformation group*.

Now, it is clear that the individual transformations must not be discrete, but continuous. **Lie** classified continuous transformation groups as *finite* and *infinite*. Whether the latter obey postulates I to VII at all might remain unresolved. In any event, one cannot apply postulate VIII to such a system of motions. We shall then exclude those groups from our consideration. However, any finite transformation group can be obtained in such a way that one assigns all possible values to a finite number r of parameters $a_1, \ldots, a_r$, so n functions $f_1(x_1, \ldots, x_n, a_1, \ldots, a_r), \ldots, f_n(x_1, \ldots, x_n, a_1, \ldots, a_r)$ can always be determined by any system of values for $a_1, \ldots, a_r$, and the transformation that is associated with the system $(a_1, \ldots, a_r)$ can be represented by the equations:

$$x_i' = f_i(x_1, \ldots, x_n, a_1, \ldots, a_r) \qquad (i = 1, \ldots, n) . \tag{4}$$

A group is represented by that when it follows from the $2n$ equations:

$$x_i' = f_i(x_1, \ldots, x_n, a_1, \ldots, a_r), \quad x_i'' = f_i(x_1', \ldots, x_n', b_1, \ldots, b_r) \tag{5}$$

that with an arbitrary choice of $a_1, \ldots, a_r$; $b_1, \ldots, b_r$, one will have:

$$x_i'' = f_i(x_1, \ldots, x_n, c_1, \ldots, c_r), \tag{6}$$

in which one has:

$$c_k = \varphi_k(a_1, \ldots, a_r ; b_1, \ldots, b_r) . \tag{7}$$

If one now assumes that as long as $a_1, \ldots, a_r$, $b_1, \ldots, b_r$ remain in a certain neighborhood of a chosen well-defined system of parameters $a_1^{(0)}, \ldots\ a_r^{(0)}$, for arbitrary values of $x_1, \ldots, x_n$, the n equations:

$$f_i(x_1, \ldots, x_n, a_1, \ldots, a_r) = f_i(x_1, \ldots, x_n, b_1, \ldots, b_r)$$

can be fulfilled for only $a_1 = b_1, \ldots, a_r = b_r$ then the parameters $a_1, \ldots, a_r$ will all be essential and cannot be replaced with a smaller number of them. If all motions can be represented by a transformation group with r essential parameters then we will say that the space form has *r degrees of mobility*. We have arrived at the theory of finite continuous transformation groups upon starting from postulates I to VII. The latter theory is more general than the theory of the space forms that fall within those postulates in only one respect: From the fifth postulate, any body can succeed in covering an arbitrary spatial region with its parts. As a result, it must be possible to take any point x to any point x' using a transformation of the group, or the group must be *transitive*. We then get the following theorem:

All motions that a body can make within a space form will be determined by a finite transitive transformation group. The number of essential parameters in that group gives the degree of mobility of the space form. That number must be greater than or equal to the number of dimensions of the space form.

The results of the last two sections allow us to arrive at that result in yet a second way. A uniform motion that takes the body from the first position to the second one is determined by any two positions of the same body. However, along with the uniform motions, the infinitely-small motions are also given at the same time. That concept shall not be developed in more detail here; it suffices to state the following theorem:

The infinitely-small changes $dx_1, \ldots, dx_n$ *that the coordinates* $x_1, \ldots, x_n$ *suffer under an infinitely-small motion can be represented as products of n functions* $\xi_1(x_1, \ldots, x_n), \ldots, \xi_n(x_1, \ldots, x_n)$ *with an infinitely-small quantity* dt.

If the point $(x_1, \ldots, x_n)$ then arrives at the position $x_1 + dx_1, \ldots, x_n + dx_n$ then one can set:

(8) $$dx_1 = \xi_1\, dt\,, \quad dx_2 = \xi_2\, dt\,, \quad \ldots, \quad dx_n = \xi_n\, dt\,.$$

If a space form admits the infinitely-small motions:

(9) $$\left\{ \begin{array}{cccc} \xi_{11} & \xi_{12} & \cdots & \xi_{1n} \\ \xi_{21} & \xi_{22} & \cdots & \xi_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ \xi_{r1} & \xi_{r2} & \cdots & \xi_{rn} \end{array} \right.$$

in which the k^{th} horizontal row refers to the fact that the components of the infinitely-small displacement are proportional to $\xi_{k1} : \xi_{k2} : \ldots : \xi_{kn}$, so the infinitely-small displacement:

(10) $$\sum_k a_k\, \xi_{k1}\,, \quad \sum_k a_k\, \xi_{k2}\,, \quad \ldots, \quad \sum_k a_k\, \xi_{kn}$$

is also possible, in which the $a_1, \ldots, a_r$ are independent of the $x_1, \ldots, x_n$, but otherwise arbitrary. Infinitely-small motions commute with each other (as long as one does not bring higher-order infinitesimals under consideration). The motion (10) is composed of the motions (9), and the motions (9) are mutually independent if and only if none of them is composed of the remaining ones, so when the n expressions (10) can be made to vanish for constant values of a only by setting all of the coefficients a equal to zero. Two cases are possible now: All infinitely-small motions that a space form admits can be composed of either a finite or an infinitely-large number of infinitesimal motions. We can overlook the latter possibility and assume that the number of mutually-independent infinitesimal motions is finite. We then propose the definition:

A space form has r degrees of mobility when all of the infinitely-small motions of it that are possible can be composed from r of them, and no fewer.

We would now like to prove that this definition coincides with the one that was given above and that the representation of the group that is given by equation (4) can be derived from infinitesimal transformations.

Since the most general infinitely-small motion is determined by (10), we integrate the system of n differential equations:

$$dx'_\alpha = \sum_{\kappa=1}^{n} a_\kappa\, \xi_{\kappa\alpha}(x')\, dt \qquad (\alpha = 1, \ldots, n) \tag{11}$$

and determine their constants in such a way that we will have $x'_\iota = x_\iota$ in each case for $t = 0$. We will then get equations of the form:

$$x'_\kappa = f_\kappa(x_1, \ldots, x_n; a_1, \ldots, a_r, t)\,. \tag{12}$$

Now, equations (11) will not change when one multiplies all $a_1, \ldots, a_r$ by the same constant as long as one simultaneously divides t by the same constant. Therefore, equations (12) must also remain unchanged under the same operations, or they must depend upon only $a_1 t, a_2 t, \ldots, a_r t$, in addition to $x_1, \ldots, x_n$. One then gets the form that was assumed in (4) here.

The infinitely-small motions also allow one to see whether a group is transitive or not. In order for the latter to be true, it is necessary and sufficient that the point x is taken to each infinitely-close point by them. That comes down to saying that not all of the determinants that can be formed from any n of the r rows in (9) vanish identically.

Since we have employed only the assumptions I to VII from the foregoing study, we can say:

All motions that are determined by the first seven postulates lead to a continuous transformation group. Conversely, one can associate every finite continuous transitive group of transformations with a system that satisfies the stated assumptions.

We state that theorem in yet another form:

If one takes the concept of a space form in its most general sense such that only postulates I to VII are true for it then its theory will coincide completely with that of finite continuous transitive transformation groups.

That theorem can be restricted only for an infinitely-large number of dimensions and for an infinitely-large degree of mobility.

Except for the transitivity, the groups that we have arrived at here also seem to be more special than those of **Lie** in a second respect, in that each of the groups that appear here include the identity transformation, whereas **Lie** did not assume that expressly. However, it has already been proved in various way that no restriction is implied by that.

§ 9. – Overview of the theory of transformation groups.

At this point, it is necessary to give some of the most important theorems on transformation groups, on the grounds that each of those theorems will be likewise true for space forms when one gives the most general meaning to those words. That comes down to the fact that we must support the derivation of the special (viz., the "proper") space forms on the theorems that are to be communicated here.

If the individual coordinates x_ι suffer the variations $\xi_{\rho\iota}\ \delta t$ under an infinitely-small transformation then an entirely general function $f(x_1, \ldots, x_n)$ will be changed by $\delta t\ \sum_\iota \xi_{\rho\iota} \frac{\partial f}{\partial x_\iota}$. **Lie** then denoted an infinitesimal transformation symbolically by:

(1)
$$X_\rho\ f = \sum_\iota \xi_{\rho\iota} \frac{\partial f}{\partial x_\iota}.$$

Here, ρ, σ, τ, … might be given the values 1, 2, …, r, and the r transformations thus-obtained shall be mutually independent. We introduce the notation:

(2)
$$(X_\rho\ X_\sigma) = ((X_\rho\ (X_\sigma\ f)) - ((X_\sigma\ (X_\rho\ f)) = \sum_{\alpha,\beta}\left(\xi_{\rho\alpha} \frac{\partial \xi_{\sigma\alpha}}{\partial x_\alpha} - \xi_{\sigma\alpha} \frac{\partial \xi_{\rho\alpha}}{\partial x_\alpha}\right)\frac{\partial f}{\partial x_\beta},$$

and say that the infinitesimal transformation that is thus-represented is the *composition* of $X_\rho f$ and $X_\rho f$. It must also belong to the group then, and we get:

(3)
$$(X_\rho\ X_\sigma) = \sum_\mu c_{\rho\sigma\mu} X_\mu\ f,$$

in which the c are constants that satisfy the condition that $c_{\rho\sigma\mu} + c_{\sigma\rho\mu} = 0$.

If one introduces other coordinates in place of $x_1, \ldots, x_n$ then one will only have to make the corresponding change in the symbol $X_\rho f$ in order to obtain the same transformation in the new variables. Similarly, $(X_\rho X_\sigma)$ will go to the composition of the corresponding expressions, and the constants $c_{\rho\sigma\mu}$ will remain unchanged.

If the $r(r-1)/2$ equations (3) are satisfied then the r infinitesimal transformations $X_1 f, \ldots, X_\rho f$ will lead to a group. However, the integrability conditions demand that:

(4)
$$\sum_\mu \left| c_{\sigma\tau\mu}(X_\rho\ X_\mu) + c_{\tau\rho\mu}(X_\sigma\ X_\mu) + c_{\rho\sigma\mu}(X_\tau\ X_\mu)\right| = 0$$

or

(5)
$$\sum_\mu \left| c_{\sigma\tau\mu}\, c_{\mu\rho\alpha} + c_{\tau\rho\mu}\, c_{\mu\sigma\alpha} + c_{\rho\sigma\mu}\, c_{\mu\tau\alpha}\right| = 0,$$

in which ρ, σ, τ are three distinct indices, and α is any one of them. However, those equations again represent the complete conditions. If the $\binom{r}{1}\binom{r}{2}$ constants c are chosen such that they satisfy the given conditions then one can always define corresponding groups.

If a group is determined by the r infinitesimal transformations $X_1 f, \ldots, X_r f$ then one can form r mutually-independent linear expressions $Y_1 f, \ldots, Y_r f$ in a variety of ways, where:

$$Y_\kappa f = \sum_\rho m_{\kappa\rho} X_\rho f.$$

The same group is also determined by $Y_1 f, \ldots, Y_r f$ then. However, the coefficients c now change in such a way that equations (5) are also true for the new constants. Two groups for which the coefficients c are either equal or they can be made equal by a suitable choice of the defining transformations are called *equally-composed* or *holohedrally-isomorphic*, or they are also sometimes said to have *equal form*. In a completely corresponding way, the r-parameter group $X_1 f, \ldots, X_r f$ for which equations (3) are true is called *merohedrally-isomorphic* with the $(r - q)$-parameter group $Y_1 f, \ldots, Y_{r-q} f$ when one can linearly determine r infinitesimal transformations $\mathfrak{Y}_1 f, \ldots, \mathfrak{Y}_r f$ from the latter, $r - q$ of which are again mutually independent, such that one also has:

$$(\mathfrak{Y}_\rho\, \mathfrak{Y}_\sigma) = \sum_{\kappa=1}^{r} c_{\rho\sigma\kappa} X_\kappa f$$

in each case.

A simply-infinite sequence of finite transformations can be derived from any infinitesimal one, and we would like to say that such finite transformations belong to the given infinitesimal one. If the relation $(X_\rho\, X_\sigma) = 0$ exists between two infinitely-small transformations $X_\rho f$ and $X_\sigma f$ then all of the finite transformations that belong to $X_\rho f$ can commute with the ones that belong to $X_\sigma f$; i.e., when two finite transformations of that kind are performed in succession, the result will be independent of the sequence in which that happened.

With that, one can derive the two following theorems from the theorem that was communicated at the conclusion of § **7**:

If there exist n mutually-independent and mutually-commuting uniform motions in an n-dimensional space form, and if their composition at a point does not define a structure of less than n dimensions then one can arrange, by a suitable choice of coordinates, that the n associated infinitely-small transformations can be represented symbolically in the form $\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n}$.

If there are more than n mutually-independent and mutually-commuting uniform motions then every determinant of degree n that is formed from the components of the corresponding infinitesimal transformations must vanish identically. All of those motions can be composed with the ones that are composed from them at each point to produce a structure of dimension at most n – 1.

It then follows from this that for $n = 1$, the only transformations that commute with each other are the ones that emerge from the same infinitesimal transformation. However, only three types of groups satisfy that condition under composition:

1. One-parameter groups.

2. The two-parameter groups for which the condition $(X_1\, X_2) = X_1 f$ is satisfied for a suitable choice of $X_1 f$ and $X_1 f$.

3. The three-parameter groups in which three infinitesimal transformations X_1, X_2, X_3 can be chosen in such a way that one will have:

$$(X_1\, X_2) = X_2\,, \qquad (X_1\, X_3) = -X_3\,, \qquad (X_2\, X_3) = X_1\,.$$

Thus, there are only three groups with one variable, and in the first case one can represent the general transformation in the form $x' = x + a$, in the second, by the equation $x' = a\,x + b$, and in the third by $x' = \frac{ax+b}{cx+d}$ for $ad \neq bc$.

By contrast, there is a much larger multiplicity of them for more variables. Namely, in that case, there are always groups that cannot lead back to projective ones by any choice of variables. **Lie** has exhibited all possible groups for two variables a long time ago (Math. Ann., Bd. XVI).

However, when one once more represents $Y_\kappa f$ linearly in terms of $X_1 f, \ldots, X_r f$, only $m < r$ such transformations will come under consideration, and if one makes the assumption that all $m\,(m-1)/2$ of the $(Y_\iota\, Y_\kappa)$ can be represented in terms of $Y_1, \ldots, Y_m$ then the $Y_1 f, \ldots, Y_m f$ will determine a subgroup of the given group. **Lie** called such a thing *invariant* when every $(Y_\iota\, X_\rho)$ can also be represented in terms of $Y_1 f, \ldots, Y_m f$ for $\iota = 1, \ldots, m$, $\rho = 1, \ldots, r$. When only p ($< r$) mutually-independent transformations are present in the $r\,(r-1)/2$ transformations $\sum_\mu c_{\rho\sigma\mu}\, X_\mu\, f$, they will determine an especially important invariant subgroup that shall be referred to as the *principal subgroup*.

Any infinitesimal transformation determines a one-parameter subgroup in its own right, and it will be contained in one or more two-parameter subgroups. Above all, one has the theorem that any arbitrary transformation will belong to an m-parameter subgroup when m is one of the numbers 1, …, 6, and less than r. However, it is necessary to allow imaginary transformations in that theorem.

Among the groups that are not their own principal subgroups – so the ones for which less than r of the transformations $(X_\rho\, X_\sigma)$ are mutually-independent – the ones that I refer to as groups of *rank zero* (for reasons that I shall not discuss here) occupy a prominent place. A characteristic property of them is that all of the transformations in each of their two-parameter subgroups commute with each other. In such groups, there are invariant subgroups with each number of parameters that is less than r, so in particular, there is also at least one one-parameter subgroup that commutes with all transformations of the group.

The groups that possess no invariant subgroups are called *simple*. Next to them, one has the ones that are defined by mere composition of several simple groups. Namely, if no two of the groups $G_1, \ldots, G_h$ have a common transformation (besides the identity), but the transformations each G_α commute with each G_β for unequal values of α, β (= 1, …, k), and all of them (and no more) comprise a new group H, then one might be permitted to say that H is defined from $G_1, \ldots, G_h$ by composition. In particular, if $G_1, \ldots, G_h$ are all simple then H might be referred to as *semi-simple*, for lack of a better term. Any group that is its own principal subgroup will be defined by a simple or semi-simple group and an invariant subgroup of rank zero.

Two groups that can be taken to each other by transformations of the variables will be referred to as *similar*. The equality of composition is necessary for that. However, in most cases a further condition will enter in. For example, the equality of the composition for those subgroups of transitive groups for which a point in general position remains fixed is such thing. Whereas that condition (or a similar one) will suffice from an analytical standpoint, for the sake of geometry, one adds the requirement that the variables must also be real.

Whereas all systems of values $x_1, \ldots, x_n$ are considered in the study of groups, one must regard only the points in general position as the proper points of the corresponding space form; i.e., since only transitive groups are considered here, they will be only those points for which not all of the determinants that are formed from n of the rows in [(9), § **8**] vanish. Now, the vanishing of those determinants would imply certain *invariants* of the group, which are structures that remain unchanged under every transformation of the group.

A point in general position (at least within certain limits) will either be taken to any other point by a subgroup or it will remain on a certain boundary structure. In the second case, the structure in question will again have the properties of a general space form. Any motion of space either leaves the structure alone or it makes it coincide with a congruent structure. If the given group has r parameters and the subgroups for which the structures moves within itself have m parameters then the totality of all congruent structures will define an $(r - m)$-fold extended manifold. One can regard the individual structure as an element in that manifold, and the laws of a space form in the general sense will then be true for it. The laws of motion that are true for that space form are essentially the same, in general, as they are for the given space form. The corresponding groups are at most holohedrally isomorphic, and at least meromorphically isomorphic. In that way, given any space form, one will arrive at a sequence of further ones that are very closely connected with it.

Any intransitive group has invariants, i.e., functions of $x_1, \ldots, x_n$ that do not change under any transformation of the group. However, one can also speak of invariants for transitive groups when one considers the coordinates of several points. Namely, if an arbitrary transformation of the group is defined by:

$$y_\iota = f_\iota (x_1, \ldots, x_n , a_1, \ldots, a_r)$$

for $\iota = 1, \ldots, n$, or more concisely, by $y_\iota = f_\iota (x, a)$, then for an arbitrary choice of $\rho + 1$ points x, x', …, $x^{(\rho)}$, the equations:

$$y_\iota = f_\iota(x,a), \quad y'_\iota = f_\iota(x',a), \quad \ldots, \quad y^{(\rho)} = f_\iota(x^{(\rho)},a)$$

will collectively represent a group with r parameters and $(\rho + 1) \cdot n$ variables that has the same form as the given group. As a result, as **Lie** had already emphasized, there is always a smallest number such that invariants will exist for that number of points and any larger number of them. Therefore, if the number of invariants seems to be infinitely large then the number of mutually-independent ones will nonetheless be finite, and indeed less than or equal to n. Therefore, in many cases, all invariants can be derived from a single one. However, in order for that to be true, it is necessary (but not at all sufficient) that all n variables in the invariant are essential and cannot be reduced to a smaller number by transformation. If all invariants of that type emerge from a single one in the coordinates of two points then the group will possess $n (n + 1) / 2$ parameters. However, if an invariant is added to an invariant for which all of the coordinates of two points are essential, and the former invariant is independent of the latter, then the number of parameters will be reduced by that.

§ 10. – Consequences of the eight postulates.

Up to now, we have employed only the first seven postulates. If we add the last one to them then we will speak of the "proper" or also "special space forms," in contrast to the "space forms in the general sense" that were considered up to now. We would now like to express that postulate in a different form:

When a point is fixed in an n-dimensional space, no second point can be moved in such a way that it fills up an n-dimensional region. At the same time, no structure can go through the fixed point that is necessarily likewise fixed or displaced into itself.

The one-dimensional space forms can be excluded completely. For multi-dimensional ones, the first part of the postulate says that for a point at rest, any second one will move in a structure that is at most $(n - 1)$-dimensional. Now, the second part next says that the moving point can, in fact, arrive at any position on an $(n - 1)$-dimensional structure. namely, one can assume that the moving point can describe a structure that is at most $(n - \rho)$-dimensional, where $\rho > 1$. One then draws an arbitrary line from the fixed point, and for each of its points, one determines the structure on which it can move. The totality of them defines a structure of dimension at most $n - \rho + 1 \leq n - 1$, and it will be displaced into itself by the motion.

Furthermore, our postulate excludes those groups that **Lie** called *systatic*, which are the ones for which a line or surface or a multi-dimensional structure will remain fixed whenever a point of it remains fixed; at the very least, such a structure cannot be real. Finally, no real structure at all that is displaced into itself by any motion that is still possible can go through the fixed point.

Since the group is transitive, no invariants between the coordinates $x_1, \ldots, x_n$ of its points can exist. By contrast, a relation between the $2n$ coordinates $x'_1, \ldots, x'_n$ and must exist that remains unchanged under all transformations of the group, since in the other case, a fixed point could go to a second one with an arbitrary position. The presence of a second invariant between the coordinates of two points would restrict the possible motions of a fixed point and a second one on an $(n - 2)$-dimensional structure. Therefore, a single invariant exists between two points. Whether or not a further invariant exists for several points that is independent of it must remain undecided for the time being.

Let that invariant between the x and x' be:

(1) $$J(x_1, \ldots, x_n; x'_1, \ldots, x'_n) .$$

If one permutes x and x' in that then one will merely get a function of J. However, if one sets $x_1 = x'_1, \ldots, x_n = x'_n$ then J must be completely independent of $x_1, \ldots, x_n$, because no invariant between the coordinates of a point can exist. Therefore, $J(x_1, \ldots, x_n; x_1, \ldots, x_n)$ will either be a constant or it will be undetermined by being completely independent of the point x. In the latter case, we have to decide whether the points x' that are infinitely close to x do or do not attain the same limiting value. Namely, in general, the function must be single-valued and continuous. One might then draw an arbitrary line from the point x; x' might be infinitely close to x on that line. If $J(x, x')$ then proceeds continuously then J will approach a certain limiting value for $x = x'$. The same thing is true for a second such line. However, should J now contain another limiting value then one would observe points on the two lines that are infinitely close to the point x and connect them with a line. A breakdown of continuity would then take place on the latter. Therefore, there would be a point x on a structure along which continuity broke down. However, that structure must also go to itself since x is fixed. That is excluded by our assumption, so the function (1) must approach the same limiting value for all values of x' that lie infinitely close to x, and we can therefore always arrange a small conversion that will make:

$$J(x_1, \ldots, x_n ; x_1, \ldots, x_n) = 0 ,$$

in the sense that the function will become infinitely small for all real values of x' that are placed infinitely close to x.

Now, if the point x is fixed and $J(x, x')$ takes the value a for a second point x' then x' can occupy any point for which the equation $J(x, x') = a$ is true, since x is fixed. At the very least, one must be able to bound an $(n - 1)$-dimensional region around the point x' on that structure such that any point of the region can actually be attained by x'.

We assume that the point x' is infinitely close to x and set $x'_\iota - x_\iota = dx_\iota$. In that way, $J(x_1, \ldots, x_n ; x'_1, \ldots, x'_n)$ will go to a function $K(dx_1, \ldots, dx_n)$ in which $x_1, \ldots, x_n$ are to be regarded as constants since we are investigating only those motions for which that point will remain fixed. At the same time, we replace the differentials $dx_1, \ldots, dx_n$ with $z_1, \ldots, z_n$ and write r_ι in place of $\partial f / \partial z_\iota$ in order to be able to give the changes that the z_ι experience under an infinitely-small motion in **Lie**'s symbolic notation. The symbol of one such infinitesimal transformation will then be:

$$\sum m_{\iota\kappa}\, z_\iota\, r_\kappa\,, \tag{2}$$

in which the coefficients $m_{\iota\kappa}$ are just coordinates. Since multiplying all of the dx_ι by the same constant cannot produce a change, the $\sum m_{\iota\iota}\, z_\iota\, r_\iota$ must be excluded from the transformations, and we can assume that the sum $\sum m_{\iota\iota} = 0$ in (2). If we then consider only those changes that are experienced by the points that are infinitely close to the fixed point under the transformations of the group then their totality can be considered to be a subgroup of the $(n^2 - 1)$-parameter group:

$$z_\iota\, r_\iota - z_\kappa\, r_\kappa, \quad z_\iota\, r_\kappa \quad \text{for} \quad \iota \neq \kappa\,.$$

That subgroup must obviously be intransitive; it should leave the function $K(x_1, \ldots, x_n)$ precisely unchanged. On the other hand, one sees very easily that when a linear form $\sum c_\iota\, z_\iota$ remains unchanged by the group, under any motion that keeps the point x at rest, an infinitely-small region that lies around the point will also be displaced within itself. However, when several such linear forms exist, several such structures, and therefore their intersections, as well, must move within themselves. However, when a real linear function of the z is an invariant for the subgroup in question, an $(n - 1)$-dimensional infinitely-small region must go into itself, since the point x is fixed; that is continued further, and an $(n - 1)$-dimensional structure will move within itself. However, when an imaginary linear function remains invariant, its complex conjugate form must be connected with it. Their intersection is real for $n > 2$, and we see that an $(n - 2)$-dimensional structure will be displaced within itself. Finally, when the function $K(z_1, \dots, z_n)$ includes only the variables $z_1, \dots, z_m$, but $z_{n+1}, \dots, z_n$ do not enter into it for $m < n$, the structure $z_1 = z_2 = \dots = z_m = 0$ will be displaced within itself.

We can then pose the following conditions for $n > 2$:

The group that transforms the infinitely-small quantities $dx_1, \dots, dx_n$ must possess one, and in fact only one, invariant between the coordinates of each of its points. No linear form can enter into that invariant for $n > 2$. At the same time, all n variables must be essential in it.

We would like to first treat two-dimensional space later on. For a multi-dimensional one, the infinitely-close points might be varied by the following infinitesimal transformations:

(3) $$\sum m_{\iota\kappa}\, z_\iota\, r_\kappa\,, \quad \sum m'_{\iota\kappa}\, z_\iota\, r_\kappa\,, \quad \dots$$

The invariant K must then satisfy the equations:

(4) $$\sum m_{\iota\kappa}\, z_\iota \frac{\partial K}{\partial z_\iota} = 0\,, \quad \sum m'_{\iota\kappa}\, z_\iota \frac{\partial K}{\partial z_\iota} = 0\,, \quad \dots$$

Those expressions can be arranged to have all of their derivatives expressible in terms of a single one, because otherwise we would get several invariants. Since none of those derivatives vanishes, it must be possible to express all of them in terms of $\partial K / \partial z_\iota$, and we will have the equations:

(5) $$\frac{\partial K}{\partial z_2} = P_2 \cdot \frac{\partial K}{\partial z_1}, \quad \dots, \quad \frac{\partial K}{\partial z_n} = P_n \cdot \frac{\partial K}{\partial z_1},$$

in which the $P_2, \dots, P_n$ cannot be merely constants.

The system of equations (5) is essentially identical to the system (4) and can be obtained from the latter by mere elimination. Thus, $n - 1$ mutually-independent equations also occur in (4), and the number of mutually-independent infinitesimal transformation (3) will be greater than or equal to $n - 1$. If one applies the same elimination that led from (4) to (5) to the symbol (3) then one must arrive at the symbol $r_\kappa - P_\kappa\, r_1$ for $\kappa = 2, \dots, n$, and the symbol $Q_\kappa\, r_\kappa - P_\kappa\, Q_\kappa\, r_1$ must, in turn, represent an infinitely-small motion by a suitable determination of the Q_κ that can be composed from (3). We write it in the form:

(6) $$\varphi_{21}\, r_2 - \varphi_{12}\, r_1\,, \quad \varphi_{31}\, r_3 - \varphi_{13}\, r_1\,, \quad \dots, \quad \varphi_{n1}\, r_n - \varphi_{1n}\, r_1\,,$$

in which the $\varphi_{\iota\kappa}$ are linear functions. No two functions $\varphi_{1\kappa}$ and $\varphi_{\kappa 1}$ can be taken to each by multiplying by a constant factor.

We now combine the transformations $\varphi_{21}\, r_2 - \varphi_{12}\, r_1$ and $\varphi_{31}\, r_3 - \varphi_{13}\, r_1$ using the rule that was given in equation (2) of § **9**. The new infinitely-small transformation must be linearly representable in terms of the transformations (3), and therefore in terms of the transformations (6) and any other one that might possibly be present. Since the $r_4, \ldots, r_n$ do not enter into the combination $(\varphi_{21}\, r_2 - \varphi_{12}\, r_1, \varphi_{31}\, r_3 - \varphi_{13}\, r_1)$, they must either be linearly representable in terms of $\varphi_{21}\, r_2 - \varphi_{12}\, r_1$ and $\varphi_{31}\, r_3 - \varphi_{13}\, r_1$ alone, or one must add a transformation $M_1\, r_1 + M_2\, r_2 + M_3\, r_3$ that must be reducible to the form $\varphi_{22}\, r_2 - \varphi_{32}\, r_3$.

If we next treat the first case then we will get the equations:

$$\varphi_{21}\frac{\partial\varphi_{13}}{\partial x_2} - \varphi_{12}\frac{\partial\varphi_{13}}{\partial x_1} - \varphi_{31}\frac{\partial\varphi_{12}}{\partial x_3} + \varphi_{13}\frac{\partial\varphi_{12}}{\partial x_1} = \alpha\,\varphi_{12} + \beta\varphi_{13}\ ,$$

$$-\varphi_{31}\frac{\partial\varphi_{21}}{\partial x_3} + \varphi_{13}\frac{\partial\varphi_{21}}{\partial x_1} = \alpha\,\varphi_{21},$$

$$\varphi_{21}\frac{\partial\varphi_{21}}{\partial x_2} - \varphi_{12}\frac{\partial\varphi_{31}}{\partial x_1} = \beta\,\varphi_{31},$$

at least two of which are independent, such that we can express φ_{31} and φ_{13} linearly in terms of φ_{12} and φ_{21} . If the group were determined by the $n - 1$ infinitesimal transformations (6) then all of the $\varphi_{\kappa 1}$ and $\varphi_{1\kappa}$ would be representable in terms of φ_{12} and φ_{21} . Therefore, no motion must be possible for $\varphi_{12} = \varphi_{21} = 0$, which was excluded.

In addition to the first two transformations (6), a transformation $\varphi_{23}\, p_2 - \varphi_{32}\, p_3$ might now occur. That would then give the equations:

$$\varphi_{23} = \alpha\,\varphi_{13} + \beta\,\varphi_{31} + \gamma\,\varphi_{23}\ , \quad \varphi_{32} = \kappa\,\varphi_{12} + \lambda\,\varphi_{21} + \mu\,\varphi_{31}\ ,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

from which the simpler ones would result immediately:

$$(7)\qquad \left\{\begin{array}{r@{\;}l|r@{\;}l} a\,\varphi_{13} &= b\,\varphi_{23} + c\,\varphi_{32}, & a\,\varphi_{12} &= p\,\varphi_{23} + q\,\varphi_{32}, \\ a'\,\varphi_{13} &= b'\,\varphi_{23} + c'\,\varphi_{32}, & a'\,\varphi_{23} &= p'\,\varphi_{23} + q'\,\varphi_{32}, \\ a''\,\varphi_{32} &= b''\,\varphi_{12} + c''\,\varphi_{21}, & a''\,\varphi_{31} &= p''\,\varphi_{12} + q''\,\varphi_{21}, \end{array}\right.$$

in which the a, a', a'' do not vanish, and for that reason can be set equal to unity. If one substitutes the values in the first equations in the left-hand side of the third equation and the right-hand side of the second equation then all six functions would be represented in terms of φ_{23} and φ_{32} if c'' and p' did not vanish. That argument shows that c, c', c'' and p, p', p'' would vanish. In that case, the six functions can be represented in terms of three.

Therefore, of the six linear expressions $\varphi_{1\kappa}$, $\varphi_{1\lambda}$, $\varphi_{\kappa 1}$, $\varphi_{\lambda 1}$, $\varphi_{\kappa\lambda}$, $\varphi_{\lambda\kappa}$, at most three of them are mutually-independent, and indeed under the two conditions that:

1. $\varphi_{\kappa\lambda} r_\kappa - \varphi_{\lambda\kappa} r_\lambda$ is also a transformation that is independent of $\varphi_{1\kappa} r_1 - \varphi_{\kappa 1} r_\kappa$ and $\varphi_{1\lambda} r_\kappa - \varphi_{\lambda 1} r_\lambda$, and

2. $\varphi_{\iota\lambda}$ and $\varphi_{\kappa\lambda}$ differ by only a constant factor.

However, if two functions suffice to represent the remaining ones in just one instance then a further structure would remain fixed, along with the point, which is excluded. Hence, one already gets $n (n - 1) / 2$ different infinitesimal motions $r_\iota L_\kappa - r_\kappa L_\iota$ for infinitely-close points. Since the fixed point will give n of those $n (n - 1) / 2$, due to transitivity, and can be subjected to mutually-independent infinitesimal transformations, the group of the space that is considered here will contain $n (n + 1) / 2$ parameters. However, if we recall that when a group possesses an invariant between the $2n$ quantities $x_1, \ldots, x_n$ and $x'_1, \ldots, x'_n$, and all of those variables are essential in it, so the number of parameters will amount to $n (n + 1) / 2$ precisely, and the addition of further invariants (that are independent of them) will lower the number of parameters, then that will imply the following theorem:

When a fixed point has the property that each of the structures that goes through that point can move from its initial position, the space form will possess $n (n + 1) / 2$ *degrees of mobility. All invariants that exist between an arbitrary number of points can be reduced to a single one in which the coordinates of only two points occur. It is not possible to move a body in such a way that a point will remain fixed and all of the directions that emanate from it will be displaced into themselves.*

Namely, should the latter be possible then one could represent the components of an infinitely-small motion of that type by a power series in $(x'_1 - x_1), \ldots, (x'_n - x_n)$, in which $x_1, \ldots, x_n$ represent fixed points. The terms of null and first powers must not occur in any of the components, or there must be more than $n (n + 1) / 2$ mutually-independent infinitesimal transformations.

We can infer a simple and important consequence of the previous theorem. If we choose n points $x, x', x'', \ldots, x^{(n-1)}$ then the following invariants will exist:

$$J(x, x'), \qquad J(x, x''), \qquad J(x, x'''), \qquad \ldots, \qquad J(x, x^{(n-1)}),$$

among others. If we keep the point x' fixed then x will remain on an $(n - 1)$-dimensional structure. Since x' and x'' are fixed, x must remain on the two structures $J(x, x') = a$, $J(x, x'') = a'$. However, since they have no $(n - 1)$-dimensional structure in common when x and x' are in arbitrary positions, the motion will be restricted to an $(n - 2)$-dimensional structure. When we adapt that argument to more points, we will get the theorem:

For the proper n-dimensional space forms, if one point is fixed then any other one will remain on an $(n - 1)$*-dimensional structure, if two points are fixed then it will be an* $(n - 2)$*-dimensional structure, if three points are fixed, it will be* $(n - 3)$*-dimensional, etc., and if n points are fixed then no motion will be possible unless the mutual positions of the points satisfy some special conditions*.

We once more return to the infinitesimal transformations (3) [(5), resp.]. We saw that we could give them the same symbolic form in which the $L_1, \ldots, L_n$ are homogeneous linear functions of $z_1, \ldots, z_n$. It follows from the equation:

$$(r_\iota L_\kappa - r_\kappa L_\iota , r_\iota L_\lambda - r_\lambda L_\iota) = k (r_\kappa L_\lambda - r_\lambda L_\kappa) + k'(r_\lambda L_\iota - r_\iota L_\lambda) + k''(r_\iota L_\kappa - r_\kappa L_\iota)$$

that:

$$\frac{\partial L_\kappa}{\partial z_\lambda} = \frac{\partial L_\lambda}{\partial z_\kappa},$$

or the $L_1, \ldots, L_n$ are the differential quotients of a quadratic form in $z_1, \ldots, z_n$, and since one has:

$$L_\iota \frac{\partial K}{\partial z_\kappa} - L_\kappa \frac{\partial K}{\partial z_\iota} = 0 ,$$

K must be a function of that form, and K can be set equal to that function immediately. That implies the theorem:

If a homogeneous linear group of n variables does not leave any planar structure fixed or displace it within itself then either it is transitive or it will divide space into quadratic structures, each of which moves within itself.

If one represents the quadratic form $K(z_1, \ldots, z_n)$ in the form of n squares then all of those squares must have the same signs in the present case; that gives:

The invariant $J(x_1, \ldots, x_n, x'_1, \ldots, x'_n)$ *that exists for the coordinates of two points under the conditions that were imposed goes over to a continually-positive form in* $dx_1, \ldots, dx_n$ *under* $x'_\iota = x_\iota + dx_\iota$ *in which all n differentials are essential, and its coefficients can still include the* $x_1, \ldots, x_n$.

The $n(n-1)/2$ infinitesimal transformation that determined the group, which gave the motion near the point at rest x, can be chosen in such a way that one can set:

$$X_{\iota\kappa} f = (x'_\iota - x_\iota)\, p_\kappa - (x'_\kappa - x_\kappa)\, p_\iota + \cdots$$

for $\iota, \kappa = 1, \ldots, n$, to which even higher powers of $x'_1 - x_1, \ldots, x'_n - x_n$ are added. If one forms their compositions then the transformations that are represented in that way must be expressible in terms of the ones that are present in a homogeneous and linear way. However, those expressions are already obtained from the linear terms, so it will follow that:

For all proper n-dimensional space forms, the subgroups by which all possible motions near a fixed point can be represented will always have the same composition. One can choose $n(n-1)/2$ *infinitesimal transformations* $X_{\iota\kappa} f$ *with* $X_{\iota\kappa} + X_{\kappa\iota} = 0$ *that will determine them such that for unequal indices* $\iota, \kappa, \lambda, \mu$, *one has the relations:*

$$(X_{\iota\kappa}\, X_{\kappa\lambda}) = X_{\iota\lambda}\,, \qquad (X_{\iota\kappa}\, X_{\lambda\mu}) = 0\,.$$

§ 11. – The second-degree differential expression.

If the point x is fixed and a motion takes place around it then a quadratic form $\sum a_{\iota\kappa}\, dx_\iota\, dx_\kappa$ will remain unchanged, in which the $a_{\iota\kappa}$ are functions of $x_1, \ldots, x_n$. However, if the point x arrives at a point x' under any rigid motion of space then the form $\sum a_{\iota\kappa}\, dx_\iota\, dx_\kappa$ must go to the corresponding form $\sum a'_{\iota\kappa}\, dx'_\iota\, dx'_\kappa$. That property leads to a better understanding of the differential expression. We once more employ an infinitely-small motion and choose its symbol to be:

$$X\, f = \sum \xi_\rho \frac{\partial f}{\partial x_\rho}\,.$$

In order for the quadratic form to not be changed in that way, the following $n\,(n + 1) / 2$ equations must be satisfied:

$$\sum_\rho \left(\frac{\partial a_{\iota\kappa}}{\partial x_\rho} \xi_\rho + a_{\iota\rho} \frac{\partial \xi_\rho}{\partial x_\kappa} + a_{\kappa\rho} \frac{\partial \xi_\rho}{\partial x_\iota} \right) = 0\,, \tag{1}$$

in which, as will always be true in what follows, the indices employed refer to the numbers 1, …, n, and a summation shall always be extended over those numbers.

The further investigation presents a great similarity with those of **Christoffel** and **Lipschitz** on differential expressions of degree two. Namely, the necessity of using the abbreviations that those two scholars employed will soon emerge here; we shall use **Christoffel**'s notation.

Equation (1) is differentiated with respect to x_λ , and that will then yield:

$$\sum_\rho \left(\frac{\partial^2 a_{\iota\kappa}}{\partial x_\lambda\, \partial x_\rho} \xi_\rho + \frac{\partial a_{\iota\kappa}}{\partial x_\rho} \frac{\partial \xi_\rho}{\partial x_\lambda} + \frac{\partial a_{\iota\rho}}{\partial x_\lambda} \frac{\partial \xi_\rho}{\partial x_\kappa} + \frac{\partial a_{\kappa\rho}}{\partial x_\lambda} \frac{\partial \xi_\rho}{\partial x_\iota} + a_{\iota\rho} \frac{\partial^2 \xi_\rho}{\partial x_\kappa\, \partial x_\lambda} + a_{\kappa\rho} \frac{\partial^2 \xi_\rho}{\partial x_\iota\, \partial x_\lambda} \right) = 0.$$

Since the partial differential quotients of ξ_ρ include x_κ and x_λ , as well as x_ι and x_λ , here, we first form equation (1) for ι and λ and differentiate with respect to x_κ and then form it for the indices κ and λ and differentiate with respect to x_ι . When we add the last two of those equations and subtract the first one, we will get an equation in which the only the second derivatives of ξ_ρ with respect to x_ι and x_κ will occur. In order to be able to write it conveniently, we shall employ the abbreviations:

$$2\begin{bmatrix} \iota\,\kappa \\ \lambda \end{bmatrix} = \frac{\partial a_{\kappa\lambda}}{\partial x_\iota} + \frac{\partial a_{\iota\lambda}}{\partial x_\kappa} - \frac{\partial a_{\iota\kappa}}{\partial x_\lambda}\,, \tag{2}$$

from which it further follows that:

(3) $$\frac{\partial a_{\iota\kappa}}{\partial x_\lambda} = \begin{bmatrix} \lambda\,\iota \\ \kappa \end{bmatrix} + \begin{bmatrix} \lambda\,\kappa \\ \iota \end{bmatrix}, \qquad \begin{bmatrix} \iota\,\kappa \\ \lambda \end{bmatrix} = \begin{bmatrix} \kappa\,\iota \\ \lambda \end{bmatrix}.$$

We now get the following equation:

(4) $$\sum_\rho \left(\xi_\rho \frac{\partial}{\partial x_\rho} \begin{bmatrix} \iota\,\kappa \\ \lambda \end{bmatrix} + \frac{\partial \xi_\rho}{\partial x_\iota} \begin{bmatrix} \rho\,\kappa \\ \lambda \end{bmatrix} + \frac{\partial \xi_\rho}{\partial x_\kappa} \begin{bmatrix} \rho\,\iota \\ \lambda \end{bmatrix} + \frac{\partial \xi_\rho}{\partial x_\lambda} \begin{bmatrix} \iota\,\kappa \\ \rho \end{bmatrix} + a_{\lambda\rho} \frac{\partial^2 \xi_\rho}{\partial x_\iota\,\partial x_\kappa} \right) = 0 .$$

We recall that all of the n differentials are necessary in the quadratic form $\sum a_{\iota\kappa}\,dx_\iota\,dx_\kappa$, so the determinant $|a_{\iota\kappa}| = A$. We denote the coefficients of $a_{\iota\kappa}$ in that determinant by $A_{\iota\kappa}$, so the following equations will be valid:

$$\sum_\rho a_{\iota\rho}\,A_{\iota\rho} = A, \quad \sum_\rho a_{\iota\rho}\,A_{\kappa\rho} = 0, \quad \text{for} \quad \iota \neq \kappa.$$

We can then use (4) to express the values of each second differential quotient of ξ_ρ in terms of the ξ_ρ and their first differential quotients. With **Christoffel**, we further introduce the notation:

(5) $$\sum_\rho \begin{bmatrix} \iota\,\kappa \\ \rho \end{bmatrix} \frac{A_{\rho\lambda}}{A} = \begin{Bmatrix} \iota\,\kappa \\ \lambda \end{Bmatrix},$$

replace the symbol λ with σ in (4), multiply by $A_{\sigma\alpha} / A$, and also sum over σ; it will then follow that:

(6) $$\frac{\partial^2 \xi_\alpha}{\partial x_\iota\,\partial x_\kappa} = -\sum_{\rho,\sigma} \left(\xi_\rho \frac{A_{\sigma\alpha}}{A} \frac{\partial}{\partial x_\rho} \begin{bmatrix} \iota\,\kappa \\ \sigma \end{bmatrix} + \frac{\partial \xi_\rho}{\partial x_\iota} \begin{Bmatrix} \rho\,\kappa \\ \alpha \end{Bmatrix} + \frac{\partial \xi_\rho}{\partial x_\kappa} \begin{Bmatrix} \rho\,\iota \\ \alpha \end{Bmatrix} + \frac{\partial \xi_\rho}{\partial x_\sigma} \begin{bmatrix} \iota\,\kappa \\ \rho \end{bmatrix} \frac{A_{\sigma\alpha}}{A} \right).$$

I differentiate (4) with respect to x_μ and obtain an equation in which the first and second derivatives of ξ_ρ occur:

(7) $$\sum_\rho a_{\lambda\rho} \frac{\partial^3 \xi_\rho}{\partial x_\iota\,\partial x_\kappa\,\partial x_\mu}$$

in addition to the ξ_ρ. I now switch the indices κ and μ in (4) and then differentiate with respect to x_κ. The third derivatives occur in the new equation only in the combination (7). If the equations that are thus defined are subtracted from each other then the result will assume the following form when one applies equations (3):

$$\sum_\rho \xi_\rho \frac{\partial}{\partial x_\rho} \left(\frac{\partial}{\partial x_\mu} \begin{bmatrix} \iota\,\kappa \\ \lambda \end{bmatrix} - \frac{\partial}{\partial x_\kappa} \begin{bmatrix} \iota\,\mu \\ \lambda \end{bmatrix} \right) + \frac{\partial \xi_\rho}{\partial x_\iota} \left(\frac{\partial}{\partial x_\mu} \begin{bmatrix} \rho\,\kappa \\ \lambda \end{bmatrix} - \frac{\partial}{\partial x_\kappa} \begin{bmatrix} \rho\,\mu \\ \lambda \end{bmatrix} \right)$$

$$+\frac{\partial\xi_\rho}{\partial x_\lambda}\left(\frac{\partial}{\partial x_\mu}\begin{bmatrix}\iota\kappa\\ \lambda\end{bmatrix}-\frac{\partial}{\partial x_\kappa}\begin{bmatrix}\iota\mu\\ \lambda\end{bmatrix}\right)+\frac{\partial\xi_\rho}{\partial x_\kappa}\left(\frac{\partial}{\partial x_\mu}\begin{bmatrix}\iota\rho\\ \lambda\end{bmatrix}-\frac{\partial}{\partial x_\rho}\begin{bmatrix}\iota\mu\\ \lambda\end{bmatrix}\right)+\frac{\partial\xi_\rho}{\partial x_\mu}\left(\frac{\partial}{\partial x_\rho}\begin{bmatrix}\iota\kappa\\ \lambda\end{bmatrix}-\frac{\partial}{\partial x_\kappa}\begin{bmatrix}\iota\rho\\ \lambda\end{bmatrix}\right)$$

$$+\sum_\tau\left(\frac{\partial^2\xi_\tau}{\partial x_\iota\,\partial x_\kappa}\begin{bmatrix}\lambda\kappa\\ \tau\end{bmatrix}-\frac{\partial^2\xi_\tau}{\partial x_\iota\,\partial x_\kappa}\begin{bmatrix}\kappa\mu\\ \tau\end{bmatrix}-\frac{\partial^2\xi_\tau}{\partial x_\kappa\,\partial x_\lambda}\begin{bmatrix}\iota\mu\\ \tau\end{bmatrix}+\frac{\partial^2\xi_\tau}{\partial x_\lambda\,\partial x_\mu}\begin{bmatrix}\iota\kappa\\ \tau\end{bmatrix}\right)=0\,,$$

where the summation symbol ρ has been replaced with τ in the last four sums. That exchange is appropriate in order to be able to replace the values for the second differential quotients that follow from (6) more simply. If that has happened then we will obtain expressions for the $\frac{\partial\xi_\rho}{\partial x_\iota}$, $\frac{\partial\xi_\rho}{\partial x_\lambda}$, $\frac{\partial\xi_\rho}{\partial x_\kappa}$, $\frac{\partial\xi_\rho}{\partial x_\mu}$ that are formed in an entirely analogous way. In that way, we will be led to the known abbreviation:

(8) $$(\iota\lambda\kappa\mu)=\frac{\partial}{\partial x_\mu}\begin{bmatrix}\iota\kappa\\ \lambda\end{bmatrix}-\frac{\partial}{\partial x_\kappa}\begin{bmatrix}\iota\mu\\ \lambda\end{bmatrix}+\sum_{\sigma,\tau}\frac{A_{\sigma\tau}}{A}\left(\begin{bmatrix}\iota\mu\\ \sigma\end{bmatrix}\begin{bmatrix}\kappa\lambda\\ \tau\end{bmatrix}-\begin{bmatrix}\iota\kappa\\ \sigma\end{bmatrix}\begin{bmatrix}\lambda\mu\\ \tau\end{bmatrix}\right).$$

In addition, we set, for the moment:

$$P_\rho=\frac{\partial}{\partial x_\rho}\left(\frac{\partial}{\partial x_\mu}\begin{bmatrix}\iota\kappa\\ \lambda\end{bmatrix}-\frac{\partial}{\partial x_\kappa}\begin{bmatrix}\iota\mu\\ \lambda\end{bmatrix}\right)+\sum_{\sigma,\tau}\frac{A_{\sigma\tau}}{A}\frac{\partial}{\partial x_\rho}\left(\begin{bmatrix}\iota\mu\\ \sigma\end{bmatrix}\begin{bmatrix}\kappa\lambda\\ \tau\end{bmatrix}-\begin{bmatrix}\iota\kappa\\ \sigma\end{bmatrix}\begin{bmatrix}\lambda\mu\\ \tau\end{bmatrix}\right)$$

and obtain:

(9) $$\left\{\begin{aligned}&\sum_\rho\left(\xi_\rho P_\rho+\frac{\partial\xi_\rho}{\partial x_\iota}(\rho\lambda\kappa\mu)+\frac{\partial\xi_\rho}{\partial x_\lambda}(\iota\rho\kappa\mu)+\frac{\partial\xi_\rho}{\partial x_\kappa}(\iota\lambda\rho\mu)+\frac{\partial\xi_\rho}{\partial x_\mu}(\iota\lambda\kappa\rho)\right)\\ &\quad-\sum_{\rho,\sigma,\tau}\frac{\partial\xi_\rho}{\partial x_\sigma}\frac{A_{\sigma\tau}}{A}\left(\begin{bmatrix}\iota\kappa\\ \rho\end{bmatrix}\begin{bmatrix}\lambda\mu\\ \tau\end{bmatrix}-\begin{bmatrix}\kappa\lambda\\ \rho\end{bmatrix}\begin{bmatrix}\iota\mu\\ \tau\end{bmatrix}+\begin{bmatrix}\lambda\mu\\ \rho\end{bmatrix}\begin{bmatrix}\iota\kappa\\ \tau\end{bmatrix}\right)=0\,.\end{aligned}\right.$$

That equation can be simplified significantly in the following way:

Since $\sum_\tau a_{\iota\kappa}\,A_{\kappa\tau}/A$ = 1 or 0, the differential quotient with respect to each x_ρ will be equal to zero; one then has the equation:

$$\sum_{\tau,\rho}\left(a_{\iota\tau}\frac{\partial(A_{\kappa\tau}/A)}{\partial x_\rho}\xi_\rho+\frac{A_{\kappa\tau}}{A}\frac{\partial a_{\iota\tau}}{\partial x_\rho}\xi_\rho\right)=0\,.$$

When I replace the second sum with its value in (1), it will follow that:

$$\sum_{\tau,\rho}\left(a_{\iota\tau}\frac{\partial(A_{\kappa\tau}/A)}{\partial x_\rho}\xi_\rho-\frac{A_{\kappa\tau}}{A}a_{\iota\rho}\frac{\partial\xi_\rho}{\partial x_\tau}-\frac{\partial\xi_\kappa}{\partial x_\iota}\right)=0\,.$$

Here, I replace the index ι with σ, multiply by $A_{\sigma\lambda} / A$, and sum over σ. When I then replace the summation symbol τ in the second sum and the σ in the third sum with ρ then I will get the following equation:

$$\sum_\rho \left(\xi_\rho \frac{\partial(A_{\kappa\lambda} / A)}{\partial x_\rho} - \frac{A_{\kappa\rho}}{A} \frac{\partial \xi_\lambda}{\partial x_\rho} - \frac{A_{\lambda\rho}}{A} \frac{\partial \xi_\kappa}{\partial x_\rho} \right) = 0 , \tag{10}$$

which might be worthy of attention in its own right, due to its similarity with (1).

Here, one might replace the symbol ρ with σ in the last two sums and then introduce new summation symbols in place of κ and λ and multiply by:

$$\begin{bmatrix} \iota\,\mu \\ \sigma \end{bmatrix} \begin{bmatrix} \kappa\,\lambda \\ \tau \end{bmatrix} - \begin{bmatrix} \iota\,\kappa \\ \sigma \end{bmatrix} \begin{bmatrix} \lambda\,\mu \\ \tau \end{bmatrix}.$$

In that way, the sum over ρ, σ, τ in (9) will take the form:

$$\sum_{\rho,\sigma,\tau} \xi_\rho \frac{\partial(A_{\sigma\tau} / A)}{\partial x_\rho} \left(\begin{bmatrix} \iota\,\mu \\ \sigma \end{bmatrix} \begin{bmatrix} \kappa\,\lambda \\ \tau \end{bmatrix} - \begin{bmatrix} \iota\,\kappa \\ \sigma \end{bmatrix} \begin{bmatrix} \lambda\,\mu \\ \tau \end{bmatrix} \right),$$

and equation (9) will assume the following simple form:

$$\sum_\rho \left[\xi_\rho \frac{\partial(\iota\lambda\kappa\mu)}{\partial x_\rho} + \frac{\partial \xi_\rho}{\partial x_\iota} (\rho\lambda\kappa\mu) + \frac{\partial \xi_\rho}{\partial x_\lambda} (\iota\rho\kappa\mu) + \frac{\partial \xi_\rho}{\partial x_\kappa} (\iota\lambda\rho\mu) + \frac{\partial \xi_\rho}{\partial x_\mu} (\iota\lambda\kappa\rho) \right] = 0 . \tag{11}$$

That equation, which is obtained from (1) by developing the integrability conditions, has a great outward similarity with an equation that we obtain from several equations (1) by merely adding and and multiplying. Namely, if we multiply equation (1) by $a_{\lambda\mu}$, then form equation (1) for the indices λ and μ and multiply by $a_{\iota\kappa}$, and then add those equations and subtract ones that are similarly formed then we will arrive at an equation whose similarity with (11) will then emerge quite clearly when we introduce the abbreviation:

$$[\iota\lambda\kappa\mu] = a_{\iota\kappa}\, a_{\lambda\mu} - a_{\iota\mu}\, a_{\lambda\kappa} . \tag{12}$$

The new equation will then become:

$$\sum_\rho \left[\xi_\rho \frac{\partial[\iota\lambda\kappa\mu]}{\partial x_\rho} + \frac{\partial \xi_\rho}{\partial x_\iota} [\rho\lambda\kappa\mu] + \frac{\partial \xi_\rho}{\partial x_\lambda} [\iota\rho\kappa\mu] + \frac{\partial \xi_\rho}{\partial x_\kappa} [\iota\lambda\rho\mu] + \frac{\partial \xi_\rho}{\partial x_\mu} [\iota\lambda\kappa\rho] \right] = 0 . \tag{13}$$

A comparison of equations (11) and (13) strongly suggests that one has the following equation for any four indices:

$$(\iota\lambda\kappa\mu) = M \cdot [\iota\lambda\kappa\mu] , \tag{14}$$

in which M denotes a constant. Before we go on to attempt to justify that suspicion, we recall the relations that **Christoffel** and **Lipschitz** exhibited for the expressions $(\iota\lambda\kappa\mu)$. They were given by the four equations:

$$(15)\qquad \begin{cases} (\iota\lambda\kappa\mu) = -(\lambda\iota\kappa\mu), \\ (\iota\lambda\kappa\mu) = -(\iota\lambda\mu\kappa), \\ (\iota\lambda\kappa\mu) = \quad (\kappa\mu\iota\lambda), \\ (\iota\lambda\kappa\mu) + (\iota\kappa\mu\lambda) + (\iota\mu\lambda\kappa) = 0. \end{cases}$$

The same relations exist between the quantities $[\iota\lambda\kappa\mu]$ that were defined in (12).

We shall next prove the validity of equation (14) by forming the expressions $(\iota\lambda\kappa\mu)$ and $[\iota\lambda\kappa\mu]$ for the same fixed point in general position and showing that M will not change when we replace the indices ι, λ, κ, μ with any other four indices. To that end, we return to equation (11) and consider the fact that in the previous section we learned about the initial terms in the individual components of $n(n-1)/2$ mutually-independent infinitesimal transformations. We leave the point x_μ fixed and denote the values that the $a_{\iota\kappa}$ assume at that point by $a_{\iota\kappa}^0$, and then develop the components $\xi_1, \ldots, \xi_n$ in powers of $x_1 - x_1^0, \ldots, x_n - x_n^0$. A transformation $X_{\alpha\beta}f$ then has components that include ξ_α and ξ_β to only degree one, and only one of degree two and higher in all the other ones. We get:

$$\xi_\alpha = \sum_\alpha a_{\beta\sigma}^0 (x_\sigma - x_\sigma^0) + \cdots, \quad \xi_\beta = \sum_\sigma a_{\alpha\sigma}^0 (x_\sigma - x_\sigma^0) + \cdots,$$

while every ξ_γ begins with terms of order two for any γ that is different from α and β. We substitute those values of $\xi_1, \ldots, \xi_n$ in equation (11) and then let $x_\sigma = x_\sigma^0$. In that way, every ξ_ρ will vanish, $\partial\xi_\alpha / \partial x_\iota$ will become $a_{\beta\iota}^0$, etc. At the same time, we must also substitute the value $x_\sigma^0 = x_\sigma$ in the expressions for $(\rho\lambda\mu\nu)$, etc. However, the value that is obtained will be true for every x^0 then, and we can thus drop the upper index. In that way, we will arrive at the equation:

$$(16)\qquad \begin{cases} \quad a_{\beta\iota}(\alpha\lambda\kappa\mu) - a_{\alpha\iota}(\beta\lambda\kappa\mu) + a_{\beta\lambda}(\iota\alpha\kappa\mu) - a_{\alpha\lambda}(\iota\beta\kappa\mu) \\ + a_{\beta\kappa}(\iota\lambda\alpha\mu) - a_{\alpha\kappa}(\iota\lambda\alpha\mu) + a_{\beta\mu}(\iota\lambda\kappa\alpha) - a_{\alpha\mu}(\iota\lambda\kappa\beta) = 0. \end{cases}$$

The foregoing equation will give no relation between the $(\iota\lambda\kappa\mu)$ and $a_{\iota\kappa}, \ldots$ when $\alpha = \beta$ or when $\iota = \lambda$ or when $\kappa = \mu$. For $\iota = \kappa$, $\lambda = \mu$, we will get from (6) that:

$$(16.a)\qquad a_{\beta\iota}(\alpha\lambda\iota\lambda) - a_{\alpha\iota}(\beta\lambda\iota\lambda) + a_{\beta\lambda}(\iota\alpha\lambda\iota) - a_{\alpha\lambda}(\iota\lambda\iota\beta) = 0,$$

and for $\iota = \kappa$:

$$(16.b)\qquad \begin{cases} \quad a_{\beta\iota}[(\alpha\lambda\iota\mu) + (\iota\lambda\alpha\mu)] - a_{\alpha\iota}[(\beta\lambda\iota\mu) + (\iota\lambda\beta\mu)] \\ + a_{\beta\lambda}(\iota\alpha\iota\mu) - a_{\alpha\lambda}(\iota\beta\iota\mu) + a_{\beta\mu}(\iota\lambda\iota\alpha) - a_{\alpha\mu}(\iota\lambda\iota\beta) = 0. \end{cases}$$

In equation (16), I shall initially restrict myself to three different indices, which I choose to be 1, 2, 3. I set $\alpha = 2$, $\beta = 3$, $\iota = 2$, $\lambda = 1$ in (16.a) and obtain:

$$(\alpha) \qquad a_{32}\,(1212) - a_{22}\,(1213) - a_{12}\,(2321) = 0\,.$$

When I multiply that equation by a_{33} and then multiply the one that is obtain the from it by switching 2 and 3 by a_{22}, I will get:

$$(\beta) \qquad a_{23}\,[a_{32}\,(1212) - a_{22}\,(1313)] - a_{12}\,a_{33}\,(2321) - a_{13}\,a_{22}\,(2331) = 0\,.$$

I set $\iota = 1$, $\lambda = 2$, $\mu = 3$, $\alpha = 2$, $\beta = 3$ in (16.b) and then obtain an equation from which, in conjunction with the one that is obtained by raising the indices, it will follow immediately that:

$$a_{11}\,(2323) + a_{23}\,(1213) = a_{22}\,(3131) + a_{31}\,(2321) = \ldots$$

I can then remove the square bracket from (β) and obtain:

$$(2321) = M\,[2321]\,, \quad (3132) = M\,[3132]\,,$$

and therefore (1213) = M [1213], as well. The corresponding formulas for (1212), (2323), (3131) will follow when that value is substituted in (α).

However, that only proves that M does not change when one preserves the three indices 1, 2, 3. Nonetheless, when one applies the equations above to the indices 1, 2, 4, it will follow that:

$$(1212) = M'[1212]\,, \quad (1414) = M'[1414]\,,$$

among other things, so $M = M'$ when (1212) and [1212] do not both vanish. However, one will get the same result when one sets the indices α, β, ι, λ in (16.a) equal to 1, 2, 3, 4 in an arbitrary sequence. Nonetheless, if one were to set ι, λ, κ, μ = 1, 2, 3, 4 in (16) then one would not arrive at any new results. However, if one chooses $\alpha = 1$, $\beta = 2$, $\iota = 3$, $\lambda = 2$ in (16.b) then it would follow that:

$$(1234) + (1432) = M\,\{[1234] + [1432]\}\,.$$

Thus:

$$(\gamma) \qquad (1234) = M\,[1234] + Q\,, \quad (1432) = M\,[1432] - Q\,.$$

Now, from (15), one has:

$$(1234) - (1432) = (1234) + (1423) = -\,(1342)\,.$$

so

$$(1342) = M\,[1342] - 2Q\,,$$

from which it follows that:

$$(1243) = M\,[1243] + 2Q\,,$$

corresponding to equations (γ), while the first equation in (γ) implies that:

$$(1243) = M\,[1243] - Q\,,$$

such that one must have $Q = 0$.

We still have to show that M will not change by adding further indices either. One recognizes that proof very easily and gets:

$$(\iota\lambda\kappa\mu) = M\,(x)\cdot[\iota\lambda\kappa\mu]\,,$$

in full generality. If one multiplies equation (13) by M (x) and subtracts it from (11) then it will follow that:

$$\sum \xi_\rho \frac{\partial M(x)}{\partial x_\rho} = 0\,.$$

Here, one might set $\xi_1, \ldots, \xi_n$ equal to the components of any infinitely-small motion of the space form that is possible. Thus, if the foregoing equation is true for a variable M (x) then it will remain invariant under all transformations of the group, so the group would be intransitive. Since that is impossible, M must only be a constant.

Since the quantities $a_{\alpha\beta}\,a_{\gamma\delta} - a_{\alpha\delta}\,a_{\gamma\beta}$, which we have denoted by $[\alpha\gamma\beta\delta]$, for brevity, cannot all vanish, the constant M must be finite. We must distinguish three cases, namely, that M is equal to zero, positive, and negative, respectively. In the case $M = 0$, all of the expressions $(\iota\lambda\kappa\mu)$ will vanish identically. **Lipschitz** had then proved that the form $\sum a_{\iota\kappa}\,dx_\iota\,dx_\kappa$ can be converted into one with constant coefficients. We can then choose the coordinates such that form will be $\sum dx_\iota^2$. Should that form remain unchanged under the transformation $\delta x_\kappa = \delta t\cdot F_\kappa(x)$, then we would need to have:

$$\sum dx_\iota \cdot dF_\kappa = \sum_{\iota,\kappa} dx_\iota \cdot \frac{\partial F_\iota}{\partial x_\kappa}\,dx_\kappa = 0\,,$$

so

$$\frac{\partial F_\iota}{\partial x_\iota} = 0, \quad \frac{\partial F_\iota}{\partial x_\kappa} + \frac{\partial F_\kappa}{\partial x_\iota} = 0\,,$$

from which, it would follow that:

$$\frac{\partial^2 F_\lambda}{\partial x_\kappa^2} = 0, \quad \frac{\partial^2 F_\kappa}{\partial x_\iota\,\partial x_\lambda} + \frac{\partial^2 F_\lambda}{\partial x_\iota\,\partial x_\kappa} = 0\,.$$

One can form new equations from the last one by permuting the indices ι, κ, λ, and it will then follow that $\frac{\partial^2 F_\lambda}{\partial x_\iota\,\partial x_\kappa} = 0$, such that F_λ will be a linear function $\sum_\rho m_{\lambda\rho} x_\rho + q_\lambda$ with the condition that $m_{\lambda\rho} + m_{\rho\lambda} = 0$.

That leads to the n (n + 1) / 2 infinitesimal transformations:

$$p_\iota\,, \qquad x_\iota\, p_\kappa - x_\kappa\, p_\iota \qquad \text{for} \qquad \iota, \kappa = 1, \ldots, n\,.$$

It is known that the Euclidian space form is determined in that way.

When M is positive and equal to $1 / k^2$, from **Lipschitz**'s analysis, one can bring the quadratic expression into the form:

$$\frac{\sum dx_\iota^2}{1 + \frac{1}{4k^2}\sum x_\iota^2}\,.$$

That form already implies some very simple equations of motion. However, one will arrive at even simpler ones when one sets:

$$x_\kappa = \frac{2u_\kappa}{1+u_0}\,, \quad k^2\,u_0^2 + u_1^2 + \cdots + u_n^2 = k^2\,,$$

or inversely:

$$u_0 = \frac{4k^2 - \sum x_\kappa^2}{4k^2 + \sum x_\kappa^2}\,, \quad u_\alpha = \frac{4k^2 x_\alpha}{4k^2 + \sum x_\kappa^2}\,.$$

The differential expression will go to $k^2u_0^2 + u_1^2 + \cdots + u_n^2$ with that. Hence, when one sets $\partial f / \partial u_\alpha = q_\alpha$, the infinitely-small motions can be represented symbolically by $u_\iota\, q_\kappa - u_\kappa\, q_\iota$ and $k^2\, u_0\, q_\kappa - u_\kappa\, q_0$. Those are the space forms of constant positive curvature, which I have distinguished between (this journal, Bd. 86, pp. 72, *et seq.*) as the **Riemannian** space form and its polar form.

If M is negative then one can let k^2 be negative. The formulas will then remain unchanged, and one will arrive at the **Lobachevskian** space form.

§ 12. – Direct determination of the associated group.

We would now like to see if we can arrive at the results that were obtained in the previous section by means of the integrability conditions in a different way. In order to at least draw attention to a method of proof that can be applied here, we express the main result of § **10** in words as:

Any infinitely-small region of any proper space form can be considered to be Euclidian.

Starting from that result, one can derive the planar and spherical structures in a purely-geometrical way. For a three-dimensional space, one will then arrive at all of the laws that **Euclid** assumed implicitly or explicitly, with the exception of the so-called eleventh postulate, and the results will be entirely similar for a multi-dimensional space. One will come to the different types of space upon starting from that.

However, corresponding to the path that was followed in § **10**, we would prefer to apply the theory of transformation groups even further. Above all, we shall consider the result that the group of those motions that are possible for a fixed point will be determined by $n\,(n-1)/2$ infinitesimal transformations $X_{\iota\kappa} f$, and that the following relations exist between them:

(1) $$(X_{\iota\kappa}\, X_{\kappa\lambda}) = X_{\iota\lambda}\, f\,, \qquad (X_{\iota\kappa}\, X_{\lambda\mu}) = 0$$

for unequal values of $\iota, \kappa, \lambda, \mu$. That group is simple for $n = 3, 5, 6, \ldots$; it is only for $n = 4$ that it can be defined by a mere composition of two simple groups. One now asks how one must add the n further transformations $X_1 f, \ldots, X_n f$ that are necessary to determine the group. In order to be able to answer that question, we consider the fact that no structure must remain fixed for a foxed point, so the group belongs to all of the ones that **Lie** referred to as *asystatic*. He proved the following theorem about them (*Transformationsgruppen I*, pp. 520):

A transitive group is asystatic if and only if the subgroup that is associated with a point in general position is not invariant under any larger subgroup (much less under the group itself).

In our case, the group of $X_{\iota\kappa}$ can be represented as an invariant subgroup of either the group of all motions in space or in a subgroup with more than $n\,(n-1)\,/\,2$ parameters. As long as the $X_\alpha f$ are only independent of each other and the $X_{\iota\kappa}$, it might not be possible to also choose the $X_\alpha f$ in such a way that the composition of all $X_{\iota\kappa}$ with some of the X_α can be expressed in terms of the $X_{\iota\kappa}$ alone.

With that, we pose the following theorem:

If a group is not its own principal subgroup and possesses a simple or semi-simple subgroup G_e then it will also always contain a subgroup whose transformations commute with all of the ones in G_e .

However, when the principal subgroup contains less than $n\,(n+1)\,/\,2$ parameters in our case, there will be at least one transformation in the group that commutes with all transformations of the group $X_{\iota\kappa}$. If one adds it to the $X_{\iota\kappa}$ then one will get a subgroup under which the $X_{\iota\kappa}$ are invariant, which was excluded by the statements above. That implies that the group must be its own principal subgroup. Since there can be no subgroup of it that commutes with all of the transformations $X_{\iota\kappa}$, from the investigations into the composition of groups, only three cases will remain: Either the group is itself simple (semi-simple for $n = 3$, resp.), or it is composed from the group of $X_{\iota\kappa}$ and an invariant subgroup of rank zero, or the simple subgroup that it contains has more than $n\,(n-1)\,/\,2$ parameters. Of those three cases, the last one is excluded. Namely, should the subgroup that is formed according to the rule (1) occur in a simple group then the number of mutually-independent transformations that enter into it would have to amount to at least n. [However, if one would like for a semi-simple group to suffice, but demand that the group (1) must not be an invariant subgroup, which is necessary here, then that number must be required.] In the first two cases, the infinitesimal transformations $X_\iota f$ must be chosen such that $(X_\iota\, X_{\iota\kappa}) = X_\kappa$, and for the simple (semi-simple for $n = 3$, resp.) groups, we will then get the following relations:

(2) $$(X_\iota\, X_{\iota\kappa}) = X_\kappa f\,, \qquad (X_\iota\, X_{\kappa\lambda}) = 0\,, \quad (X_\iota\, X_\kappa) = -\frac{1}{k^2}\, X_{\iota\kappa} f\,,$$

in which k^2 is a positive or negative constant. That group can obviously be represented in an n-dimensional space in the following way: One chooses $n + 1$ variables $x_0, x_1, \ldots, x_n$ and establishes the following relation between them:

$$k^2 x_0^2 + x_1^2 + \cdots + x_n^2 = k^2, \tag{3}$$

in which one might assume that $x_1, \ldots, x_n$ might take on all systems of values from (0, 0, …, 0) up to a certain limit, and x_0 might vary continually in the manner that was prescribed by equation (3). At the same time, one sets:

$$X_{\iota\kappa} = x_\iota\, p_\kappa - x_\kappa\, p_\iota, \quad X_\iota = x_0\, p_\iota - \frac{x_\iota\, p_0}{k^2}. \tag{4}$$

Obviously, the point ($x_0 = 1$, $x_1 = x_2 = \ldots = x_n = 0$) is a point in general position. The associated subgroup has the given composition. Therefore, any group that satisfies the given conditions will be similar to the foregoing one. We will then obtain only two essentially-different groups according to whether k^2 is positive or negative, and once more arrive at the space forms of constant positive or negative curvature.

If we likewise apply the general theorems on the form of groups to the ones that are composed of the group (1) and an invariant subgroup of rank zero then we will find that the transformations of the invariant subgroup all commute with each other. If one chooses any n of them in order to exhibit a coordinate system then that will yield the infinitesimal transformations in the form:

$$X_1 f = p_1, \qquad X_2 f = p_2, \qquad \ldots, \qquad X_n f = p_n.$$

However, since they determine an invariant subgroup:

$$(X_\alpha\, X_{\iota\kappa}) = \frac{\partial X_{\iota\kappa}}{\partial x_\alpha}$$

must be obtained from $p_1, \ldots, p_\kappa$ by multiplying by constant quantities and adding. Thus, the $X_{\iota\kappa}$ occur in the $x_1, \ldots, x_n$ only linearly. Now, should the linear form $K(dx_1, \ldots, dx_n)$ contain nothing but squares then the infinitesimal transformations would be:

$$p_\iota, \qquad x_\iota\, p_\kappa - x_\kappa\, p_\iota \qquad \text{for} \qquad \iota, \kappa = 1, \ldots, n. \tag{5}$$

All properties of Euclidian geometry follow immediately from that; We then get the theorem:

There are only three distinct groups that belong to the proper space forms. They must all be their own principal subgroup. In order to determine them, one employs $n(n + 1) / 2$ *infinitesimal transformations and chooses them in such a way that one always has:*

$$(X_{\iota\kappa}\ X_{\kappa\lambda}) = X_{\iota\lambda}, \qquad (X_{\iota\kappa}\ X_{\lambda\mu}) = 0, \qquad (X_\iota\ X_{\iota\kappa}) = X_\kappa f.$$

Along with that, one has either:

$$(X_\iota\, X_\kappa) = -\frac{1}{k^2} X_{\iota\kappa} f,$$

in which k^2 is a positive or negative constant, or:

$$(X_\iota\ \ X_\kappa) = 0 .$$

One can support the theorem that the invariant subgroup of rank zero that can exist in group that belongs to a proper space form contains nothing but commuting transformations with a theorem that one can express as follows, when one applies a terminology that I introduced:

If all of the auxiliary roots are implied by a single one then the transformations of the invariant subgroup will commute with each other.

Namely, it follows from this theorem that:

If the $Y_\iota f$ determine a simple subgroup and the $Z_\alpha f$ determine the invariant subgroup of rank zero, and if one arrives at all $Z_\alpha f$ each time upon starting from one $Z_\alpha f$ by composing it with all $Y_\iota f$ (at least step-by-step) then all transformations of the invariant subgroup will commute with each other.

In that case, one can choose the $X_{\iota\kappa} f$ in such a way that the zero point remains fixed. When we suitably choose n directions from them, we can assume that the $X_{\iota\kappa} f$ take the form:

$$X_{\iota\kappa} f = x_\iota\, p_\kappa - x_\kappa\, p_\iota + \ldots,$$

in which the $p_1, \ldots, p_n$ will be multiplied by functions of $x_1, \ldots, x_n$ of degree two and higher in the further terms. We can likewise assume that p_ι occurs in each $X_\iota f$ as the single term without $x_1, \ldots, x_n$. It follows from this that:

$$(X_\iota\ \ X_{\iota\kappa}) = X_\kappa f + [X_{\rho\sigma} f] ,$$

in which the bracketed expression shall denote a linear homogeneous function of the $X_{\rho\sigma}$. It must vanish when the $X_\iota f, \ldots, X_\kappa f$ are assumed to belong to the invariant subgroup. One then, in fact, arrives at all $X_\kappa f$ upon starting from any arbitrary $X_\iota f$ by the operation $(X_\iota\ \ X_{\iota\kappa})$.

Although the foregoing proof is quite simple, it still has the disadvantage that it assumes many theorems that can only be first obtained by a deeper study of the theory of transformations groups. It would then seem appropriate to add a proof that assumes only the simplest principles of this theory, namely, formulas (1)-(4) in § **9**.

To that end, we once more start with the representation of the $n\,(n + 1) / 2$ infinitesimal transformations that we said were possible above; it has the form:

(6) $$\begin{cases} X_\iota f = p_\iota + \sum_{\rho,\sigma} h^\iota_{\rho\sigma} x_\rho p_\sigma + \cdots, \\ X_{\iota\kappa} f = x_\iota p_\kappa - x_\kappa p_\iota + \sum_{\rho,\sigma} h^\iota_{\rho\sigma,\tau} x_\rho x_\sigma p_\tau + \cdots \end{cases}$$

The implies the composition formulas:

(7) $$\begin{cases} (X_\iota \, X_{\iota\kappa}) = X_\kappa + \sum_{(\rho\sigma)} c^\iota_{\iota\kappa,\rho\sigma} \, X_{\rho\sigma} + \cdots, \\ (X_\iota \, X_{\kappa\lambda}) = \sum_{(\rho\sigma)} c^\iota_{\kappa\lambda,\rho\sigma} \, X_{\rho\sigma} + \cdots, \end{cases}$$

along with the one that was derived above $(X_{\iota\kappa} \, X_{\kappa\lambda}) = X_{\iota\lambda}$. In (7), as in all of what follows, the summation over ρ, σ extends over only the various combinations of two unequal indices in the series 1, …, n.

We now add a linear expression in the $X_{\rho\sigma}$ to each $X_\iota f$, namely:

$$\sum_{(\rho\sigma)} m^\iota_{\rho\sigma} \, X_{\rho\sigma} \, ,$$

and we would like to show that the $\binom{n}{1}\binom{n}{2}$ coefficients $m^\iota_{\rho\varrho}$ can be chosen such that the following equations will be true:

(8) $$\begin{cases} (X_\iota + \sum_{(\rho\sigma)} m^\iota_{\rho\sigma} \, X_{\rho\sigma}, X_{\iota\kappa}) = X_\kappa + \sum_{(\rho\sigma)} m^\iota_{\rho\sigma} \, X_{\rho\sigma}, \\ (X_\iota + \sum_{(\rho\sigma)} m^\iota_{\rho\sigma} \, X_{\rho\sigma}, X_{\kappa\lambda}) = 0 \, . \end{cases}$$

Now, one has:

$$(X_\iota + \sum_{\rho\sigma} m^\iota_{\rho\sigma} \, X_{\rho\sigma}, X_{\iota\kappa}) = X_\kappa + \sum_{(\rho\sigma)} c^\iota_{\iota\kappa,\rho\sigma} \, X_{\rho\sigma} + \sum_\tau (m^\iota_{\iota\tau} X_{\kappa\tau} - m^\iota_{\kappa\tau} X_{\iota\tau}) \, ,$$

in which the summation over t is performed over just the indices ι and κ that are distinct. One likewise has:

$$(X_\iota + \sum_{\rho\sigma} m^\iota_{\rho\sigma} \, X_{\rho\sigma}, X_{\kappa\lambda}) = \sum_{(\rho\sigma)} c^\iota_{\kappa\lambda,\rho\sigma} \, X_{\rho\sigma} + \sum_\tau (m^\iota_{\kappa\tau} X_{\lambda\tau} - m^\iota_{\lambda\tau} X_{\kappa\tau}) \, ,$$

in which τ must now be different from κ and λ .

That yields the following conditions for the c :

(9) $$c^\iota_{\kappa\lambda,\kappa\lambda} = 0, \quad c^\iota_{\kappa\lambda,\mu\nu} = 0, \quad c^\iota_{\kappa\lambda,\iota\mu} = 0,$$

and the following equations for determining the m :

$$m^\iota_{\iota\kappa} = c^\iota_{\kappa\lambda,\lambda\iota} \, , \quad m^\iota_{\kappa\mu} = -c^\iota_{\kappa\lambda,\lambda\mu}, \quad c^\kappa_{\kappa\iota} = c^\iota_{\iota\kappa,\iota\kappa}, \quad m^\kappa_{\lambda\mu} = c^\iota_{\iota\kappa,\lambda\mu},$$
$$m^\iota_{\kappa\lambda} + m^\kappa_{\iota\lambda} = c^\iota_{\iota\kappa,\iota\lambda} \, , \quad m^\iota_{\iota\lambda} - m^\kappa_{\kappa\lambda} = -c^\iota_{\iota\kappa,\kappa\lambda} .$$

When we next compare the expressions for the $m^\iota_{\iota\kappa}$ with each other, that will imply the further conditions for the c :

(10) $$c^\iota_{\kappa\lambda,\lambda\iota} = c^\iota_{\kappa\mu,\kappa\mu} = c^\kappa_{\iota\kappa,\lambda\kappa}, \quad c^\iota_{\iota\kappa,\kappa\lambda} - c^\lambda_{\kappa\lambda,\kappa\lambda} + c^\lambda_{\iota\lambda,\iota\lambda} = 0 .$$

As a result, one should add the following relations for $m^\iota_{\lambda\kappa}$:

(11) $$\begin{cases} c^\iota_{\iota\kappa,\iota\lambda} = c^\kappa_{\kappa\iota,\kappa\lambda}, \quad c^\iota_{\iota\kappa,\iota\lambda} + c^\kappa_{\kappa\lambda,\kappa\iota} + c^\lambda_{\lambda\iota,\lambda\kappa} = 0, \quad c^\iota_{\kappa\mu,\mu\lambda} + c^\kappa_{\iota\mu,\mu\lambda} + c^\iota_{\iota\kappa,\iota\lambda} = 0, \\ c^\iota_{\kappa\mu,\lambda\mu} = c^\iota_{\kappa\nu,\lambda\nu} = c^\mu_{\mu\iota,\kappa\lambda} = c^\nu_{\nu\iota,\kappa\lambda}. \end{cases}$$

One can infer the fact that these relations are all true very easily from equation (4) of § **9** (viz., the **Jacobi** identity). We next form them for X_ι , $X_{\iota\kappa}$, $X_{\kappa\lambda}$, which will give us the equation:

$$((X_\iota\ X_{\iota\kappa})\ X_{\kappa\lambda}) - (X_\iota\ X_{\iota\lambda}) - ((X_\iota\ X_{\kappa\lambda})\ X_{\iota\kappa}) = 0 .$$

When we substitute the values (7) in that, we will get:

$$(X_\iota\ X_{\kappa\lambda}) - (X_\iota\ X_{\iota\lambda}) =$$

$$c^\iota_{\iota\kappa,\iota\lambda}\ X_{\iota\kappa} - c^\iota_{\iota\kappa,\iota\kappa}\ X_{\iota\lambda} - c^\iota_{\kappa\lambda,\kappa\lambda}\ X_{\iota\kappa} + c^\iota_{\iota\kappa,\iota\lambda}\ X_{\kappa\lambda} + \sum_\tau (c^\iota_{\kappa\lambda,\iota\tau}\ X_{\kappa\tau} + c^\iota_{\iota\kappa,\lambda\tau}\ X_{\kappa\tau} - c^\iota_{\kappa\lambda,\kappa\tau}\ X_{\iota\tau} - c^\iota_{\iota\kappa,\kappa\tau}\ X_{\lambda\tau}),$$

in which the summation over τ extends over only the distinct indices ι, κ, λ . The implies the following equations:

(12) $$\begin{cases} c^\iota_{\iota\kappa,\iota\lambda} + c^\iota_{\iota\lambda,\iota\kappa} - c^\kappa_{\kappa\lambda,\iota\kappa} = 0, \qquad c^\iota_{\iota\lambda,\kappa\lambda} + c^\iota_{\kappa\lambda,\iota\lambda} + c^\kappa_{\kappa\lambda,\kappa\lambda} = 0, \\ c^\iota_{\iota\kappa,\iota\kappa} - c^\iota_{\iota\lambda,\iota\lambda} + c^\kappa_{\kappa\lambda,\iota\lambda} + c^\iota_{\kappa\lambda,\kappa\lambda} = 0, \qquad c^\iota_{\iota\lambda,\mu\nu} = c^\kappa_{\kappa\lambda,\mu\nu}, \\ c^\iota_{\iota\lambda,\iota\mu} - c^\kappa_{\kappa\lambda,\lambda\mu} - c^\iota_{\kappa\lambda,\kappa\mu} = 0, \quad c^\iota_{\iota\kappa,\lambda\mu} + c^\iota_{\kappa\lambda,\iota\mu} + c^\iota_{\iota\lambda,\kappa\mu} - c^\kappa_{\kappa\lambda,\kappa\mu} = 0. \end{cases}$$

One likewise for equation (4) in **9** for X_ι , $X_{\iota\kappa}$, $X_{\lambda\mu}$ and gets:

$$c^\kappa_{\lambda\mu,\iota\kappa} = 0, \quad c^\kappa_{\iota\mu,\lambda\mu} = 0, \quad c^\kappa_{\lambda\mu,\lambda\mu} = c^\iota_{\iota\kappa,\mu\nu}, \quad c^\iota_{\iota\kappa,\lambda\nu} + c^\kappa_{\lambda\mu,\mu\nu} = 0.$$

If one would not like to derive the relations:

$$c^\iota_{\kappa\mu,\nu\rho} = 0, \quad c^\iota_{\kappa\mu,\iota\kappa} = c^\iota_{\lambda\mu,\iota\lambda}$$

from the relations that were formed then one can write them down immediately from the identity that was defined for X_ι , $X_{\iota\kappa}$, $X_{\lambda\mu}$.

With that, most of the relations that are implied by equations (9), (10), (11) have been proved directly. However, a few of them require a more concise derivation. If one compares the first equation in (12) with the one that emerges from it by switching the indices ι and κ then it will follow that $c^{\iota}_{\iota\kappa,\iota\lambda} = c^{\kappa}_{\kappa\iota,\kappa\lambda}$, and if one replaces the second term in that equation with $c^{\lambda}_{\lambda\iota,\lambda\kappa}$ then one will get the second equation in (11). The last equation (10) will be obtained by merely altering the indices in the third equation in (12). (Which shall suggest only small alterations that the derivation experiences for $n = 3$.) With that, it has been proved that the $m^{\iota}_{\rho\varrho}$ can always be determined in the prescribed way (for $n = 3$, but in such a way that some quantity remains undetermined). We can then assume that the following relations are valid:

$$(X_\iota\, X_{\iota\kappa}) = X_\kappa f\,, \qquad (X_\iota\, X_{\kappa\lambda}) = 0\,.$$

If one now forms the oft-used equation (4) in § **9** for X_ι, X_κ, $X_{\iota\kappa}$ then it will follow that $(X_\iota\, X_\kappa)$ commutes with $X_{\iota\kappa}$, so in addition to $X_{\iota\kappa}$, it can include only those X_λ, $X_{\lambda\mu}$ for which λ, μ are different from ι and κ. Likewise, $(X_\iota\, X_\kappa\, X_{\lambda\mu})$ implies the theorem that $(X_\iota\, X_\kappa)$ also commutes with $X_{\lambda\mu}$. For $n > 4$, it will then follow that $(X_\iota\, X_\kappa) = e_{\iota\kappa}\, X_{\iota\kappa}$. For $n = 3$, that yields:

$$(X_\iota\, X_\kappa) = e_{\iota\kappa}\, X_{\iota\kappa} + g_\lambda\, X_\lambda\,,$$

but then the remaining indeterminacy that likewise results when one applies the relation employed to X_1, X_2, X_3 will serve to make the g_λ vanish. For $n = 4$, it then follows that:

$$(X_\iota\, X_\kappa) = e_{\iota\kappa}\, X_{\iota\kappa} + g_{\iota\kappa}\, X_{\lambda\mu}\,,$$

but the following investigation makes $g_{\iota\kappa}$ vanish. Finally, the **Jacobi** identity for $(X_\iota\, X_\kappa\, X_{\kappa\lambda})$ or the equation:

$$((X_\iota\, X_\kappa)\, X_{\kappa\lambda}) = (X_\iota\, X_\lambda)$$

says that the coefficient c in $(X_\iota\, X_\kappa) = c\, X_{\iota\kappa}$ is independent of the indices ι, κ. We can then once more set c equal to either zero or $-1 / k^2$ and can, in turn, find that in regard to the composition, only the given cases are possible. However, the fact that we obtain only one space form for $c = 0$ has been shown already very easily above. Therefore, we still have to prove that for a non-vanishing $c = -1 / k^2$ and a suitable choice of coordinate system, the composition will correspond to a single representation that emerges from the form (6) for infinitely-small values of $x_1, \ldots, x_n$. However, since the representation (6) assumes only that the infinitely-small values of $x_1, \ldots, x_n$ are well-defined, that will come down to determining the continuation that arises from the simplest-possible formulas. The path was taken in § **7** will be less suitable for that purpose. A small alteration of it would also lead us to that goal, in general. Meanwhile, I would like to point out a different path that is also quite useful with corresponding changes to it.

The representation (6) shows that for infinitely-small values of the variables, the structure $x_\iota = 0$ will be preserved for those subgroups that are obtained when one restricts the indices α, β in $X_\alpha f$, $X_\beta f$ to the $n - 1$ indices $1, 2, \ldots, i - 1, i + 1, \ldots, n$, with the exception of ι. Therefore, a structure will also generally move with itself under that subgroup, and we generally set $x_\iota = 0$ for its points. However, along with that structure, a family of such things will also move within itself, and it would seem apt that each of them should keep the value of x_ι constant. The $n - 1$ structures $x_1 = 0$,

$x_2 = 0, \ldots, x_{n-1} = 0$ that are determined in that way, and therefore their intersections, as well, will be displaced into themselves by the one-parameter subgroup $X_n f$. One employs that motion to associate each point of that line with a number y_n according to the conventions that were encountered in § **7**. One then sets $x_n = k \sin (y_n / k)$, and since the same value x_n will be valid for all points of the aforementioned structure, one can establish n coordinates for all points in a finite neighborhood of the zero point. Now, the representation (6) is valid for infinitely-small values. However, one also sees very easily that, in general, one has the representation:

$$X_\iota f = p_\iota \sqrt{\frac{k^2 - x_1^2 - x_2^2 - \cdots - x_n^2}{k^2}}, \quad X_{\iota\kappa} f = x_\iota p_\kappa - x_\kappa p_\iota ,$$

so it would not be necessary to go any further into the details of the proof.

§ 13. – The space forms in two dimensions.

Some time ago, **Lie** gave all of the transformation groups for two variables (Math. Ann., Bd. XVI). For them, one only had to appeal to reality in order to be able to easily decide which groups satisfied our last assumption. Later on, and independently of that (Programm, 1884), I exhibited all three-parameter groups, and since our assumption obviously yields three degrees of mobility for two dimensions, it would also suffice to select the suitable systems from the various ones that are found in that way. Nonetheless, I prefer a direct development.

The condition that a moving point will remain on a line while a point is fixed is by no means sufficient for one to also obtain only one narrowly-restricted class of space forms. That is due to the fact that one has assumed an invariant between two points. However, when only one variable in it is essential, that will require yet another invariant, and it can be assumed to exist between arbitrarily-many points. It is only when one assumes that the line is closed that one will arrive at very few space forms, as **von Helmholtz** already proved in 1868. However, since we have embarked upon a different path for a large number of dimensions, we must also pursue it here, as well.

If we keep one point fixed then the motions that are possible will be determined by a one-parameter group. The same thing will also be true for the infinitely-close points then, and when we use the notation that was applied in § **10**, we can assume that the infinitesimal transformation for the change is $\sum m_{\iota\kappa} z_\iota r_\kappa$ for $\iota, \kappa = 1, 2$. In that way, two (real or imaginary) lines, or just one, will continue to cover the starting point. Since the latter case cannot happen, the invariant will be:

$$(M_1\, dx_1 + M_2\, dx_2)^\alpha \cdot (N_1\, dx_1 + N_2\, dx_2)^\beta,$$

in which the two linear expressions must be complex conjugates. Here, M_1, M_2, N_1, N_2 are functions of x_1 and x_2 . Corresponding to the investigations in § **11**, we can then ask when the present form does not change under transformations. That question will be resolved in an especially-simple way when the imaginary parts of α and β do not vanish, but we shall not go further into that here. By contrast, § **11** does not change for $\alpha = \beta$, and we will get $a_{11} dx_1^2 + 2a_{12} dx_1 dx_2 + a_{22} dx_2^2$. Eq. (11) in § **11** is not altered then, and since the equation $(1212) = M(x) \cdot [1212]$ is self-explanatory, the constancy of $M(x)$ will follow immediately.

If we now apply the method that was developed in § **12** to our present case then we can assume that:

$$(1) \qquad X_1 f = p_1 + \ldots, \qquad X_2 f = p_2 + \ldots, \qquad X_3 f = (a\, x_1 + x_2)\, p_1 + (-x_1 + \alpha\, x_2)\, p_2 + \ldots,$$

from which it follows that their compositions are:

$$(X_1 X_3) = \alpha X_1 - X_2 + \beta X_3 , \quad (X_2 X_3) = X_1 + \alpha X_2 + \gamma X_3 , \quad (X_1 X_2) = c_1 X_1 + c_2 X_2 + c_3 X_3 .$$

It now follows from equation (4) in § **9** that:

$$(2) \qquad (c_1 - \gamma)\,(X_1 X_3) + (c_2 + \beta)\,(X_2 X_3) + 2\alpha\,(X_1 X_2) = 0 .$$

However, since:

$$(X_1 + \kappa X_3 , X_2 + \lambda X_3) = (X_1 X_2) + \lambda\,(X_1 X_3) - \kappa\,(X_2 X_3) ,$$

one can make that expression equal to zero for non-vanishing α when one sets $\kappa = (c_2 + \beta) / 2\alpha$, $\lambda = -(c_1 - \gamma) / 2\alpha$. However, one can also make the coefficients of X_1 and X_2 in the last expression vanish for $\alpha = 0$. If one then replaces $X_1 + \kappa X_3$ with X_1 and $X_2 + \lambda X_3$ with X_2 then β and γ must also vanish, due to the independence of $(X_1 X_3)$ and $(X_2 X_3)$. We then obtain the following possibilities:

$$(a) \qquad (X_1 X_3) = -X_2 , \qquad (X_2 X_3) = X_1 , \qquad (X_1 X_2) = c\, X_3 ,$$

in which c will be positive or negative or equal to zero, or:

$$(b) \qquad (X_1 X_3) = \alpha X_1 - X_2 , \qquad (X_2 X_3) = \alpha X_1 + X_2 , \qquad (X_1 X_2) = 0 ,$$

in which α shall be non-zero.

In order to represent the group (a), one again starts from the form (1) and arrives at those space forms that correspond entirely to the ones that were found in the previous section. By contrast, the space form (b) exhibits essentially-different properties. We can represent the infinitely-small transformations by:

$$X_1 = p_1 , \qquad X_2 = p_2 , \qquad X_3 = (\alpha\, x_1 + x_2)\, p_1 + (-x_1 + \alpha\, x_2)\, p_2 .$$

The most general motion is obtained from the equations:

$$x' - a = e^{at}\,[(x - a) \cos t - (y - b) \sin t)] ,$$

$$y' - b = e^{at}\,[(x - a)\sin t + (y - b)\cos t)]\ ,$$

with the use of three arbitrary constants a, b, t .

Under a rotation about a point, any two points will move a certain spiral. In that way, any direction will return to the initial position, but the individual points will not assume their original positions again. If we let the interior of a simply-bounded region of this space rotate about a point, and indeed perform a complete rotation ($t = 2\pi$), then the region of space that it will then cover either contains the previous one as a subset or it will be contained within it. Under an unlimited continuation of that motion, an arbitrarily-chosen body (where the word is being used in the sense of our postulates) can be made to contain each arbitrarily-chosen (finite) region of space once, and for other directions of rotation, it can be made to arrive completely inside an arbitrarily-small region of space that is chosen about the fixed point. One can then easily state postulate VIII in a way that excludes that space form completely. In order to represent it, one can choose all of the lines that are parallel to a fixed direction in a three-fold-extended **Lobachevski** space form, as long as one assumes that every rotation around a line of the family is combined with a displacement along that line that is characterized by α .