"Ueber die Auflösung der Gleichungen, auf welche man bei der Untersuchung der linearen Vertheilung galvanische Ströme gefuhrt wird," Ann. Phys. Chem. **72** (1847), 497-508.

On the solution of the equations to which one will be led in the study of linear distributions of galvanic currents

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If a system of n wires 1, 2, 3, … is given that are connected to each other in some way, and if an arbitrary electromotive force exists in each of them then if one wishes to determine the intensities of the currents that flow through the wires I_1, I_2, I_3, …, one will find the necessary number of equations by employing the following two theorems ([1]):

I. – If the wires k_1, k_2, … define a closed figure and w_k denotes the resistance of the wire k, while E_k denotes the electromotive force that exists in it (positively-directed in the same sense as I_k), then in the event that I_{k1}, I_{k2}, … are all regarded as positive in *one direction*, one will have:

$$w_{k1}\, I_{k1} + w_{k2}\, I_{k2} + \ldots = E_{k1} + E_{k2} + \ldots$$

II. – If the wires λ_1, λ_2, … come together at a point and $I_{\lambda 1}$, $I_{\lambda 2}$, … are all considered to be positive at that point then:

$$I_{\lambda 1} + I_{\lambda 2} + \ldots = 0\ .$$

I would now like to show that the solutions of the equations that one obtains by applying those theorems for I_1, I_2, I_3, …, while assuming that the given system of wires cannot be separated into several other complete ones, can be given in general in the following way:

Let m be the number of crossing points that are present (i.e., the points at which two or more wires come together), and let $\mu = n - m + 1$.

The common denominator of all quantities I will then be the sum of those combinations of w_1, w_2, …, w_n into μ elements w_{k1}, w_{k2}, …, $w_{k\mu}$ that have the property that *no* closed figure will remain after one removes the wires k_1, k_2, …, k_μ and

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([1]) Bd. 64, pp. 513, these Annalen.

The numerator of I_λ is the sum of those combinations of $w_1, w_2, \ldots, w_n$ into $\mu - 1$ elements $w_{k1}, w_{k2}, \ldots, w_{k\mu}$ that have the property that *one* closed figure will remain after removing $k_1, k_2, \ldots, k_\mu$, and λ will appear in it. Each combination will be multiplied by the sum of the electromotive forces that are found in the associated closed figure. The electromotive forces are then regarded as positive when they are in the same direction as the one that makes I_λ positive.

For the sake of ease of understanding, I would like to devote an individual section to the proof that I give to that theorem.

1.

Let μ be the *smallest* number that gives how many wires one must remove in order to destroy all closed figures. μ will also be the number of mutually-independent equations that one can derive by applying Theorem I then.

One can, in fact, exhibit μ equations that are mutually-independent, and each one of them that follows from Theorem I can be derived in the following way:

Let $1, 2, \ldots, \mu - 1, \mu$ be μ wires such that no closed figure remains after they are removed. *One* closed figure will then remain after removing $\mu - 1$ of them. One then applies Theorem I to the closed figures that remain when one removes:

$$\begin{array}{llllll} & 2, & 3, & \ldots, & & \mu \\ 1, & & 3, & \ldots, & & \mu \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 1, & 2, & 3, & \ldots, & & \mu - 1, \end{array}$$

in sequence.

One of the μ equations that are constructed in that way can be a consequence of the other ones, since each of them contains one unknown that is not generally present in all of the others; the first one includes I_1, the second one, I_2, etc. However, one can also construct another equation from those equations that can be derived with the help of Theorem I, because it must be possible to construct an equation that follows from a closed figure that is composed of several others from the equations that follow from the latter (by addition or subtraction), and as we would like to show, each closed figure can be composed of those μ figures. The totality of closed figures (namely, the given system, which we would like to denote by S) can be decomposed into ones that include the wire μ and ones that are included in the system S', which will arise from S when the wire μ is removed. If we assume that all figures that belong to the second class can be composed of the first $\mu - 1$ of those μ figures then we will see that each figure in the system S can be composed of those μ figures, because an arbitrary figure that includes μ can be composed of one particular one that includes μ and ones that do not include μ . However, the assumption that was made about the system S' can be once more reduced to a similar one in regard to S'', if S'' is the system that arises from S when one removes μ and $\mu - 1$, namely, to the assumption that all of the closed figures that are included in S'' are composed of the first $\mu - 2$ of those μ figures. Upon continuing that

argument, we will finally come to the system $S^{(\mu-1)}$. Since it includes only one closed figure, the validity of the assumption that we must make in regard to it in order to recognize the truth of our assertion is obvious.

2.

Since Theorems I and II must produce the number of equations that are necessary for one to determine $I_1, I_2, \ldots, I_n$, from what we have proved, they must be the following ones:

$$\alpha_1^1 w_1 I_1 + \alpha_2^1 w_2 I_2 + \cdots + \alpha_n^1 w_n I_n = \alpha_1^1 E_1 + \alpha_2^1 E_2 + \cdots + \alpha_2^1 E_2 ,$$

$$\alpha_1^2 w_1 I_1 + \alpha_2^2 w_2 I_2 + \cdots + \alpha_n^2 w_n I_n = \alpha_1^2 E_1 + \alpha_2^2 E_2 + \cdots + \alpha_2^2 E_2 ,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\alpha_1^\mu w_1 I_1 + \alpha_2^\mu w_2 I_2 + \cdots + \alpha_n^\mu w_n I_n = \alpha_1^\mu E_1 + \alpha_2^\mu E_2 + \cdots + \alpha_2^\mu E_2 ,$$

$$\alpha_1^{\mu+1} I_1 + \alpha_2^{\mu+1} I_2 + \cdots + \alpha_n^{\mu+1} I_n = 0 ,$$

$$\alpha_1^{\mu+2} I_1 + \alpha_2^{\mu+2} I_2 + \cdots + \alpha_n^{\mu+2} I_n = 0 ,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\alpha_1^n I_1 + \alpha_2^n I_2 + \cdots + \alpha_n^n I_n = 0 ,$$

in which some of the quantities α are + 1, some of them are – 1, some of them are 0, and μ has the same meaning as before.

It emerges from this that the common denominator of the quantities I (i.e., the determinant of those equations) is a homogeneous function of degree μ in $w_1, w_2, \ldots, w_n$, which include each individual w only linearly and includes only numbers besides the w's. We can also express that result in the following way: The common denominator of the I's is the sum of the combinations of the $w_1, w_2, \ldots, w_n$ into those μ elements with each combination multiplied by a numerical coefficient. One likewise sees that the numerator of I is the sum of the combinations of $w_1, w_2, \ldots, w_n$ into those $\mu - 1$ elements with each combination multiplied by a linear homogeneous function of the quantities $E_1, E_2, \ldots, E_n$ whose coefficients are numbers.

3.

In order to determine the numerical coefficients of the denominator and numerator of the quantities I, we remark that it makes no difference whether we make the resistance $w_\kappa = \infty$ or cut through or remove the wire κ. Thus, under the substitution $w_\kappa = \infty$, the expressions for the I's must go to the solutions of those equations that we will obtain by applying Theorems I and II to the

system of wires that arise from the given one when we remove the wire κ. I_κ itself must vanish for $w_\kappa = \infty$.

We would like to divide the numerator and the denominator of the I's by $w_1 \cdot w_2 \cdots w_{\mu-1}$ and then set $w_1 = \infty$, $w_2 = \infty$, …, $w_{\mu-1} = \infty$; I_λ will go to (I_λ) in that way. If we then denote the function of the E's that is multiplied by

$$w_{\kappa 1} \cdot w_{\kappa 2} \cdots w_{\kappa\mu-1}$$

in the numerator of I_λ by $A^{\lambda}_{\kappa_1,\kappa_2,\ldots,\kappa_{\mu-1}}$ and denote the coefficients of $w_{\kappa 1} \cdot w_{\kappa 2} \cdots w_{\kappa\mu-1}$ in the denominator by $a_{\kappa 1}$, $a_{\kappa 2}$, …, $a_{\kappa\mu-1}$ then we will have:

$$(I_\lambda) = \frac{A^{\lambda}_{\kappa_1,\kappa_2,\ldots,\kappa_{\mu-1}}}{a_{1,2,\ldots,\mu-1,\mu} \cdot w_\mu + a_{1,2,\ldots,\mu-1,\mu+1} \cdot w_{\mu+1} + \cdots + a_{1,2,\ldots,\mu-1,n} \cdot w_n} \ .$$

As a result of the remark that was made:

$$(I_\lambda) = 0$$

when λ is included in 1, 2, …, $\mu - 1$, and:

$$(I_\lambda) = I'_\lambda$$

when λ is not included in 1, 2, …, $\mu - 1$, where I'_λ denotes the intensity of the current that will flow through the wire λ when the wires 1, 2, …, $\mu - 1$ are removed.

We imagine that we have exhibited the equations for the determination of I'_μ, $I'_{\mu+1}$, …, I'_n that are produced by applying Theorems I and II to the remaining system of wires. Theorem I will then imply μ' mutually-independent equations. The common denominator of the quantities I' will then be a function of degree μ' in the w_μ, $w_{\mu+1}$, …, w_n , and their numerator will be a function of degree $\mu' - 1$ relative to the same arguments. Due to the definition of μ, μ' will either be equal to one or greater than one. If $\mu' > 1$ then in order for the equation $(I_\lambda) = I'_\lambda$ to be true, either the numerator and denominator of I'_λ would need to have a common factor of degree $\mu' - 1$ relative to w_μ, $w_{\mu+1}$, …, w_n , or one would need to have $(I_\lambda) = 0$ and $I'_\lambda = 0$, or finally (I_λ) would need to assume the form 0 / 0 . If one of the quantities (I) is represented in the form 0 / 0 then all of them must have that form, since they have a common denominator, and none of them can be ∞. Should that case not occur, then the denominator and numerator of each I' would need to have have a common factor of degree $\mu' - 1$, and indeed those factors must be the same for all of the quantities I'. However, that is impossible, as one can show in the following way:

We assume that there is a factor of the indicated kind that includes the quantity w_κ . κ must then be a wire that lies in a closed figure, since otherwise w_κ could not enter into the equations for I_μ, $I_{\mu n}$, … at all. Since the numerator and the denominator of the quantities I' are linear with respect to each w, we will obtain expressions for them by taking away those factors that are free of w_κ. If

we substitute them in one of the equations that includes $w_\kappa I'_\kappa$ then it will become an identity. By partially differentiating with respect to w_κ, we will get:

$$I'_\kappa = 0 .$$

However, is might always be possible for that equation to not be true. Should that be the case then it would also need to remain correct when one sets arbitrarily many of the quantities w to ∞, i.e., when one removes arbitrarily many of the wires. However, if one removes enough wires that only *one* closed figure remains that includes κ then it would be impossible for I'_κ to vanish for arbitrary values of the quantities E.

We then see that when $\mu' > 1$, (I_μ), $(I_{\mu+1})$, …, (I_n) must be represented in the form 0 / 0, or since we have found that $(I_1) = 0$, $(I_2) = 0$, …, $(I_{\mu-1}) = 0$, when more than one closed figure remains after removing the wires 1, 2, …, $\mu - 1$, the product:

$$w_1 \cdot w_2 \cdots w_{\kappa-1}$$

can enter into either a numerator or a denominator of the quantities $I_1, I_2, \ldots, I_n$.

4.

We would now like to seek to determine the factors with which the product $w_1 \cdot w_2 \cdots w_{\kappa-1}$ in the numerator and in the denominator of the I's will be multiplied if the condition that only one closed figure must remain after removing 1, 2, …, $\mu - 1$ is fulfilled.

The remaining figure includes the wires $\lambda_1, \lambda_2, \ldots, \lambda_\nu$. If λ does not occur among them then:

$$I'_\lambda = 0 ,$$

and if λ does occur among them:

$$I'_\lambda = \frac{E_{\lambda 1} + E_{\lambda 2} + \cdots + E_{\lambda \nu}}{w_{\lambda 1} + w_{\lambda 2} + \cdots + w_{\lambda \nu}} ,$$

in which $E_{\lambda 1}, E_{\lambda 2}, \ldots$ are considered to be positive in the direction that makes I_λ positive.

The denominator of that value can differ from the denominator of the quantity (I_λ), i.e., the expression:

$$a_{1,2,\ldots,\mu-1,\mu}\, w_\mu + a_{1,2,\ldots,\mu-1,\mu+1}\, w_{\mu+1} + \ldots + a_{1,2,\ldots,\mu-1,n}\, w_n ,$$

only by a numerical factor. Therefore, all of the quantities $a_{1,2,\ldots,\mu-1,\mu}$, $a_{1,2,\ldots,\mu-1,\mu+1}$, … must vanish, except for:

$$a_{1,2,\ldots,\mu-1,\lambda 1} , a_{1,2,\ldots,\mu-1,\lambda 2} , \ldots, a_{1,2,\ldots,\mu-1,\lambda \nu} ,$$

and they must be equal to each other. We conclude from this that the coefficient of the combination $w_{\kappa 1} \cdot w_{\kappa 2} \cdots w_{\kappa\mu}$ in the denominator of the quantities I can be non-zero only when all closed figures will be destroyed by removing the lines $\kappa_1, \kappa_2, \ldots, \kappa_\mu$, and that all combinations that fulfill that condition and which include $\mu - 1$ common factors w must have the same coefficient.

With the help of that, one can prove that any two combinations:

$$w_{\kappa 1} \cdot w_{\kappa 2} \cdots w_{\kappa\mu} \qquad \text{and} \qquad w_{\kappa' 1} \cdot w_{\kappa' 2} \cdots w_{\kappa'\mu}$$

in the denominator of the I's must have the same coefficients when all closed figures are destroyed by removing the wire $\kappa_1, \kappa_2, \ldots, \kappa_\mu$, along with the wire $\kappa'_1, \kappa'_2, \ldots, \kappa'_\mu$.

In order to be able to follow through on that proof, we preface it with the following remarks:

All closed figures might be destroyed by removing the wire $\kappa_1, \kappa_2, \ldots, \kappa_\mu$. Each of those wires must occur in at least one closed figure then.

However, at least one of those wires must occur in each closed figure: We then know of the wire κ' that it lies in a closed figure, so it must lie in the same closed figure as at least one of the wires $\kappa_1, \kappa_2, \ldots, \kappa_\mu$.

Furthermore, each of the wires $\kappa_1, \kappa_2, \ldots, \kappa_\mu$ must occur in a closed figure in which the other $\mu - 1$ wires do not occur; e.g., κ_μ will lie in the ones that remain after removing $\kappa_1, \kappa_2, \ldots, \kappa_{\mu-1}$, and which we would like to denote by $f_{\kappa\mu}$. If the wire κ'_μ also lies in $f_{\kappa\mu}$ then all closed figures will also be destroyed by removing $\kappa_1, \kappa_2, \ldots, \kappa_{\mu-1}, \kappa'_\mu$. One easily sees with the help of those remarks that when we select any closed figure f, $\mu - 1$ wires can always be found such that f remains as the only closed figure after removing those f. Namely, if, say, $\kappa_1, \kappa_2, \kappa_3$ are present in f from among the wires $\kappa_1, \kappa_2, \ldots, \kappa_\mu$, and if κ'_2 is a wire that occurs in $f_{\kappa 2}$, but not in f, and κ'_3 is a wire that occurs in $f_{\kappa 3}$, but not in f then $\kappa'_2, \kappa'_3, \kappa_4, \ldots, \kappa_\mu$ will be wires of the desired kind.

We would now like to carry out that proof in such a way that we assume that the coefficients of two combinations of the desired kind are equal to each other when they have ν common factors w in common and then prove that the coefficients of two combinations that have only $\nu - 1$ common factors must also be equal to each other. If we have succeeded in doing that then we will have exhibited the truth of our assertion.

The form of the proof will remain the same no matter which value of ν we choose in it. We would therefore like to follow it through for only one value of ν, namely, $\nu = 3$. We would then like to prove that the two combinations:

$$w_{\kappa 1} \cdot w_{\kappa 2} \cdot w_{\kappa 3} \cdots w_{\kappa\mu} \qquad \text{and} \qquad w_{\kappa' 1} \cdot w_{\kappa' 2} \cdot w_{\kappa' 3} \cdots w_{\kappa'\mu}$$

must have the same coefficients.

In the system of wires that arises from the given one when one removes κ_1 and κ_2, none of the closed figures can be destroyed by removing less than $\mu - 2$ wires. It will be destroyed by removing $\kappa_3, \kappa_4, \ldots, \kappa_\mu$, and it will follow from the removal of $\kappa'_3, \kappa'_4, \ldots, \kappa'_\mu$ that κ'_3 will lie in the same closed figure with at least one of the wires; we shall assume that it is κ_3. It will remain as the only

one when one removes $\kappa''_4, \kappa''_5, \ldots, \kappa''_\mu$. It will then remain as the only one from the original system when one removes $\kappa_1, \kappa_2, \kappa''_4, \kappa''_5, \ldots, \kappa''_\mu$. It follows from this that the two combinations:

$$w_{\kappa 1} \cdot w_{\kappa 2} \cdot w_{\kappa 3} \cdot w_{\kappa''4} \cdot w_{\kappa''5} \cdots w_{\kappa''\mu}$$

and

$$w_{\kappa 1} \cdot w_{\kappa 2} \cdot w_{\kappa'3} \cdot w_{\kappa''4} \cdot w_{\kappa''5} \cdots w_{\kappa''\mu}$$

which have $\mu - 1$ common factors w, must have the same coefficients. However, from our assumption, the combinations:

$$w_{\kappa 1} \cdot w_{\kappa 2} \cdot w_{\kappa 3} \cdot w_{\kappa 4} \cdots w_{\kappa\mu} \quad \text{and} \quad w_{\kappa 1} \cdot w_{\kappa 2} \cdot w_{\kappa 3} \cdot w_{\kappa''4} \cdots w_{\kappa''\mu}$$
$$w_{\kappa 1} \cdot w_{\kappa 2} \cdot w_{\kappa'3} \cdot w_{\kappa'4} \cdots w_{\kappa'\mu} \quad \text{and} \quad w_{\kappa 1} \cdot w_{\kappa 2} \cdot w_{\kappa'3} \cdot w_{\kappa''4} \cdots w_{\kappa''\mu}$$

will also have the same coefficients pair-wise. Therefore, the coefficients of:

$$w_{\kappa 1} \cdot w_{\kappa 2} \cdot w_{\kappa 3} \cdot w_{\kappa 4} \cdots w_{\kappa\mu} \quad \text{and} \quad w_{\kappa 1} \cdot w_{\kappa 2} \cdot w_{\kappa'3} \cdots w_{\kappa'\mu}$$

will also be equal to each other.

In that way, we have proved that the common denominator of the I's is the sum of those combinations of $w_1, w_2, \ldots, w_n$ into μ elements $w_{\kappa 1} \cdot w_{\kappa 2} \cdots w_{\kappa\mu}$ that have the property that no closed figure will remain after removing the wires $\kappa_1, \kappa_2, \ldots, \kappa_\mu$. That sum is multiplied by a numerical coefficient. We can then set the numerical coefficient equal to unity and then determine the numerator of the I's afterwards.

Now, that numerator is very easy to find. Namely, it will follow from the equations:

$$(I_\lambda) = 0 \qquad \text{and} \qquad (I_\lambda) = I'_\lambda,$$

the first of which is true when $\lambda \geq \mu - 1$ and the second, when $\lambda > \mu - 1$, that:

$$A^\lambda_{1,2,\ldots,\mu-1} = E_{\lambda 1} + E_{\lambda 2} + \ldots + E_{\lambda\nu}$$

for the case in which λ occurs among $\lambda_1, \lambda_2, \ldots, \lambda_\nu$ and:

$$A^\lambda_{1,2,\ldots,\mu-1} = 0$$

for the opposite case.

The coefficient of the term $w_1 \cdot w_2, \ldots w_{\mu-1}$, which we have shown previously can be non-zero only when a single closed figure remains after removing 1, 2, …, $\mu - 1$, is then zero when λ does not occur in that figure. If λ does occur in it then it will be equal to the sum of the electromotive forces that are found in it. It is considered to be positive in the direction that makes I_λ positive.

5.

In order to have proved our theorem as we have expressed it, we must now show that $\mu = n - m + 1$. That assertion is true only when the given system of wires does not decompose into several mutually-independent ones, while the considerations that were made up to now do not require such an assumption.

As we have seen, μ is the number of mutually-independent equations that can be derived with the help of Theorem I. The number of mutually-independent equations that Theorem II implies must then be $n - \mu$. However, it can be shown that under that assumption, that number will be $m - 1$, from which it would then follow that $\mu = n - m + 1$.

We cannot derive more than $m - 1$ mutually-independent equations with the help of Theorem II, because if we apply it to all m crossing points then each I will occur twice in the equations that arise in that way, once with the coefficient + 1 and once with the coefficient – 1. The sum of all equations will then give the identity 0 = 0. However, the equations that one obtains by applying that theorem to $m - 1$ arbitrary crossing points are mutually-independent, because they have the property that when we select arbitrarily many of them arbitrarily one or more of the unknowns will occur in them just once. Namely, if we let (κ, λ) denote a wire that connects two of the crossing points 1, 2, …, m to each other – say, κ and λ – then the unknown $I_{(\kappa 1, \lambda)}$ will occur only once in the equations that are derived by considering the points $\kappa_1, \kappa_2, \ldots, \kappa_\nu$ when one of them – say, κ_1 – is connected to another other λ, along with the points that occur among $\kappa_2, \ldots, \kappa_\nu$. However, one of the points $\kappa_2, \ldots, \kappa_\nu$ must be connected with a point λ, along with other ones, when the wires that connect the points $\kappa_1, \kappa_2, \ldots, \kappa_\nu$ to each other do not define a closed system.

Allow me to make a few remarks in regard to the theorem that was just proved.

If one arranges the terms in the numerator of I_λ according to the quantities $E_1, E_2, \ldots, E_n$ then the coefficient of E_κ will be the sum of the combinations (some of which are positive and some of which are negative) of the $w_1, w_2, \ldots, w_n$ into $\mu - 1$ of them that will occur in the denominator of the I's multiplied by w_λ, as well as w_κ. They are, in fact, precisely those combinations $w_{\kappa 1} \cdot w_{\kappa 2} \cdots w_{\kappa \mu - 1}$ that have the properties that only one closed figure will remain after removing the wires $\kappa_1, \kappa_2, \ldots, \kappa_{\mu-1}$ and that λ will occur in it, as well as κ. $w_{\kappa 1} \cdot w_{\kappa 2} \cdots w_{\kappa \mu - 1}$ is taken to be positive when the positive direction of I_λ coincides with the direction of E_κ in the remaining figure and negative in the opposite case.

It emerges from this that, among other things, when we select two wires from an arbitrary system, the current intensity that that is produced in one of them by an electromotive force in the second one will be precisely the same as the intensity of the current that is produced in the second one by an electromotive force in the first one that is just as large.

As one easily sees, the condition that we have found for the occurrence of a combination in the denominators of the I's can also be expressed in the following way: The combinations $w_{\kappa 1} \cdot w_{\kappa 2} \cdots w_{\kappa \mu}$ occur when the equations that produced Theorem I are independent of $I_{\kappa 1}, I_{\kappa 2}, \ldots, I_{\kappa \mu}$. It can be shown that this condition agrees with the one that no equation exists between $I_{\kappa 1}, I_{\kappa 2}, \ldots,$

$I_{\kappa\mu}$ or some of those quantities that can be derived from the equations that arise by applying Theorem II. That remark frequently makes it easier to exhibit the combinations that are missing from the denominator of the I's. For example, if the wires 1, 2, 3 intersect at one point, 3, 4, 5, at a second, and 5, 6, 7, at a third [as in Fig. 4, Tab. V (†)], then all of the combinations that include:

$$w_1 \cdot w_2 \cdot w_3 \quad , \quad w_3 \cdot w_4 \cdot w_5 \quad , \quad w_5 \cdot w_6 \cdot w_7 \, ,$$
$$w_1 \cdot w_2 \cdot w_4 \cdot w_5 \quad , \quad w_3 \cdot w_4 \cdot w_5 \cdot w_7 \, ,$$
$$w_1 \cdot w_2 \cdot w_4 \cdot w_6 \cdot w_7$$

will be missing.

The denominator of the I's for the combination of wires that is represented in Figure 5, Table V is therefore the sum of all combinations of $w_1, w_2, \ldots, w_6$ into three elements, with the exception of the following ones:

$$w_1 \cdot w_2 \cdot w_4 \, , \quad w_1 \cdot w_3 \cdot w_5 \, , \quad w_2 \cdot w_3 \cdot w_6 \, , \quad w_4 \cdot w_5 \cdot w_6 \, .$$

(†) Translator: The figures were not available to me at the time of translation.