Electrodynamics and wave mechanics from the standpoint of the correspondence principle

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Abstract: After a brief overview of the basic concepts of the wave mechanics of the one-electron problem in § **1**, expressions will be presented in § **2** that can serve as the relativistic generalization of the wave-mechanical expressions that **Schrödinger** gave for the electric density and current vector. Starting from that, the evaluation of the **Maxwell-Lorentz** theory in quantum theory on the basis of the corresponding principle will be discussed in § **3** and explained in terms of simple examples in § **4**. The perturbation of an atom by external forces and the **Compton** effect will be discussed in § **5** as further examples of that way of regarding the theory. Finally, some remarks on five-dimensional wave mechanics will be imparted in § **7**.

Introduction. – Under the influence of quantum theory, the well-known difficulties that obstruct the application of classical theories to the description of atomic processes have led to a revision of our mechanical conception of things that drew upon the known analogy between point mechanics and wave theory that is at the basis of **Hamilton**'s theory. We can thank **de Broglie** for taking the first step in that direction when he compared the motion of a particle with the propagation of waves in a dispersive medium, and thus arrived at a geometric interpretation of the quantum conditions for periodic systems. In that way, **Schrödinger** then succeeded in developing a general wave mechanics. The many significant results of that theory aroused the hope that with its help, one could avoid the discontinuities that were formulated within the postulates of **Bohr**'s theory of atomic structure and were characteristic of the quantum theory, and in that way create a true continuum theory in space and time. However, such a way of looking at things will encounter unsolved difficulties whose roots lie deep, and in the present state of science, an adequate description of phenomena might only be achieved by using the correspondence principle that **Bohr** established. The basis for such an evaluation of wave theory would also be constructed from the connection between non-relativistic wave mechanics and **Heisenberg**'s quantum mechanics that **Schrödinger** discovered. As is known, upon referring to a matrix representation of the mechanical quantities using **Heisenberg**'s procedure, a rational corresponding evaluation of point mechanics in the sense of **Bohr**'s basic postulates was already achieved before the creation of **Schrödinger**'s theory. The possibility of realizing an even more direct relationship between wave mechanics and the postulates of quantum theory was

emphasized by **Born** especially, in conjunction with his treatment of the collision phenomena that are so important to atomic theory.

The following treatment of radiation processes starts from the field equations of the **Maxwell-Lorentz** theory and seeks simply to evaluate wave mechanics from the standpoint of **Bohr**'s correspondence principle.   In that way, one will arrive spontaneously at a description that satisfies the requirements of special relativity.   The representation is naturally connected with the relativistic generalization of the expressions for the electric density and current vector that **Schrödinger** presented.   In so doing, we restrict ourselves to the one-electron problem (simplified by ignoring the proper rotation of the electron), in which it was possible up to now to create only a theory that satisfied the principle of relativity.   Even in that problem, the demand that the matrix theory should be relativistic raised some peculiar problems that seemed to be based in the nature of things.   However, here one must recall the interesting treatment of the **Compton** effect that **Dirac** gave with the help of his symbolic representation of matrix mechanics.

As the author hopes to show soon, the theory can be extended in the sense of the general theory of relativity.   In that way, one will get a representation of the quantum-mechanical equations of motion using the correspondence principle that is an immediate expression for the conservation of energy and impulse, which defines just the necessary condition for the coupling of wave mechanics with **Einstein**'s field equations.   In connection with that, in the last paragraphs of this article, some remarks will be made that will go more deeply into five-dimensional wave mechanics in connection with a representation of general relativity that **Kaluza** had previously attempted.   That form of wave theory starts from the aspiration to arrive at a way of describing things that, despite the unfamiliar path of introducing a new dimension, would correspond to the classical theory more closely than the current representation using the correspondence principle that seems to be inevitable in a space-time description of phenomena.

The following paper can thank Prof. **N. Bohr** for its existence almost entirely and the friendly and animated interest that he has shown in the author's work for some years now.   Not only has he provided me with the invaluable advantage of belonging to his circle of colleagues, but he has also actively contributed to this work in with his advice and criticism, which made the relationship between wave mechanics and the postulates of quantum theory much clearer to me, in particular.   Professor **Bohr** hoped to return to that question soon the context of a general discussion of the questions of quantum theory.   At this point, I would also like to acknowledge some detailed discussions about general and special problems in wave mechanics with Prof. **H. A. Lorentz**, Prof. **P. Ehrenfest**, and other Dutch physicists that were made possible by a kind invitation for me to go to Leiden by the **H. A. Lorentz** foundation.

§ 1. Foundations of wave mechanics. – We consider the motion of an electron in an electromagnetic field according to the mechanics of the special theory of relativity.   Let the charge of the electron be $-\varepsilon$, let its rest mass be μ, and let its position be defined with respect rectangular coordinates (x, y, z), and let the time (t) be measured by a clock at rest in that coordinate system.   We describe the electromagnetic field by the four-potential $\mathfrak{A}$ and the scalar potential V, and we impose the usual condition upon it:

$$\mathrm{div}\,\mathfrak{A} + \frac{1}{c}\frac{\partial V}{\partial t} = 0, \tag{1}$$

in which c denotes the speed of light.

The following **Hamilton-Jacobi** differential equation for the action function S can be regarded as the expression for the motion of the electron:

$$\frac{1}{2\mu}\left\{\left(\mathrm{grad}\,S + \frac{\varepsilon}{c}\mathfrak{A}\right)^2 - \frac{1}{c^2}\left(\frac{\partial S}{\partial t} - \varepsilon V\right)^2\right\} + \frac{1}{2}\mu c^2 = 0. \tag{2}$$

Certain "ray equations" belong to this equation, which corresponds to the differential equation for the wave surface in optics according to **Hamilton**'s theory, and those ray equations represent precisely the relativistic equations of motion of the electron. Those equations can be written in canonical form as follows:

$$\left.\begin{array}{llll}
\dfrac{dx}{d\tau} = \dfrac{\partial H}{\partial p_x}, & \dfrac{dy}{d\tau} = \dfrac{\partial H}{\partial p_y}, & \dfrac{dz}{d\tau} = \dfrac{\partial H}{\partial p_z}, & \dfrac{dt}{d\tau} = \dfrac{\partial H}{\partial p_t}, \\[3mm]
\dfrac{dp_x}{d\tau} = -\dfrac{\partial H}{\partial x}, & \dfrac{dp_y}{d\tau} = -\dfrac{\partial H}{\partial y}, & \dfrac{dp_z}{d\tau} = -\dfrac{\partial H}{\partial z}, & \dfrac{dp_t}{d\tau} = -\dfrac{\partial H}{\partial t},
\end{array}\right\} \tag{3}$$

with

$$H = \frac{1}{2\mu}\left\{\left(p_x + \frac{\varepsilon}{c}\mathfrak{A}_x\right)^2 + \left(p_y + \frac{\varepsilon}{c}\mathfrak{A}_y\right)^2 + \left(p_z + \frac{\varepsilon}{c}\mathfrak{A}_z\right)^2 - \frac{1}{c^2}\left(p_t - \varepsilon V\right)^2\right\} + \frac{1}{2}\mu c^2. \tag{4}$$

If follows from (4) and (3) that:

$$p_x = \mu\frac{dx}{d\tau} - \frac{\varepsilon}{c}\mathfrak{A}_x, \quad p_y = \mu\frac{dy}{d\tau} - \frac{\varepsilon}{c}\mathfrak{A}_y, \quad p_z = \mu\frac{dz}{d\tau} - \frac{\varepsilon}{c}\mathfrak{A}_z, \quad p_t = \mu c^2\frac{dt}{d\tau} + \varepsilon V. \tag{5}$$

As a result, we can write H as:

$$H = \frac{\mu}{2}\left\{\left(\frac{dx}{d\tau}\right)^2 + \left(\frac{dy}{d\tau}\right)^2 + \left(\frac{dz}{d\tau}\right)^2 - \frac{1}{c^2}\left(\frac{dt}{d\tau}\right)^2\right\} + \frac{1}{2}\mu c^2,$$

such that relation that follows from (2):
$$H = 0$$

will be fulfilled when we set $d\tau$ equal to the proper time that is associated with the electron thus:

$$d\tau = \sqrt{dt^2 - \frac{dx^2 + dy^2 + dz^2}{c^2}}. \tag{6}$$

The quantities p_x, p_y, p_z are then precisely the momenta that enter into the phase integral:

$$\int (p_x\, dx + p_y\, dy + p_z\, dz)\,,$$

which is important in quantum theory, while $- p_t$ is a measure of the energy of the electron (viz., its rest energy $\mu\, c^2$).

We will now get the usual quantum theory of periodic systems when look for the stationary states of those solutions of equation (2) for which $e^{\frac{2\pi i}{h} S}$, in which h denotes **Planck**'s constant, is a single-valued function of the position in space in the case of a static force field. Following **L. de Broglie** ([1]), we perceive an interference relation in the last condition, which is equivalent to the usual quantum conditions, of the kind that arises in the determination of eigen-oscillations and by which the quantum numbers will take on the meaning of node numbers and the quantum conditions will be organically coupled with the laws of motion. Furthermore, we connect the known problems in the usual quantum theory of periodic systems that refer to deviations from ordinary mechanics, according to **Schrödinger** ([2]), with the fact that the aforementioned method for calculation the eigen-oscillations will only lead to results that are approximately correct in optics, as well, when the curvature of the light ray is close to one wavelength (viz., high quantum numbers), and in the general case it must be replaced with the consideration of the second-order linear equation. Correspondingly, we replace the second-degree, first-order equation (2) with the following second-order linear equation:

$$-\frac{h^2}{4\pi^2}\Box\varphi + 2\frac{h}{2\pi i}\frac{\varepsilon}{c}\left[(\mathfrak{A}\ \text{grad}\ \varphi) + \frac{V}{c}\frac{\partial\varphi}{\partial t}\right] + \left[\mu^2 c^2 + \frac{\varepsilon^2}{c^2}(\mathfrak{A}^2 - V^2)\right]\varphi = 0, \qquad (7)$$

where φ means a function of time and position that corresponds to $e^{\frac{2\pi i}{h} S}$, and where:

$$\Box = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\,,$$

means the **d'Alembert** wave operator ([3]).

With the Ansatz:

$$\varphi = e^{\frac{2\pi i}{h} S}\,, \qquad (8)$$

we will get from (7) that:

([1]) **L. de Broglie**, Ann. de physique (10) **3** (1925), 22 (1924 thesis).

([2]) **E. Schrödinger**, Ann. Phys. (Leipzig) **79** (1926), pp. 361, 489, 734; *ibid.* **80** (1926), 437; *ibid.* **81** (1926), 109.

([3]) Like the **Hamilton-Jacobi** equation (2), that equation will give a class of solutions for which the energy proves to be negative, and which have no direction relationship to the motion of the electron. Naturally, they will be excluded from consideration.

$$\left(\mathrm{grad}\, S + \frac{\varepsilon}{c}\mathfrak{A} \right)^2 - \frac{1}{c^2}\left(\frac{\partial S}{\partial t} - \varepsilon V \right)^2 + \mu^2 c^2 + \frac{h}{2\pi i}\square\, S = 0. \tag{9}$$

In fact, for $h = 0$, the **Hamilton-Jacobi** equation (2) will yield the corresponding transition from wave optics to geometric optics.

Equation (7), which was presented from various angles, represents the direct relativistic generalization of **Schrödinger**'s wave equation for the one-electron problem. For the comparison with **Schrödinger**'s results, we shall refer to the fact that his non-relativistic equation can be obtained from (7) by the Ansatz:

$$\varphi = \xi\, e^{-\frac{2\pi i}{h}\mu c^2 t} \tag{10}$$

when we assume that $\dfrac{h}{2\pi i}\dfrac{\partial \xi}{\partial t}$, $\varepsilon V \xi$, $\varepsilon\,|\,\mathfrak{A}\,|\,\xi$ can be treated as if they were infinitely small of first order compared to $\mu c^2\, \xi$. It will then follow that:

$$\Delta\xi + \frac{8\pi^2 \mu}{h^2}\left\{ \varepsilon V - \frac{h}{2\pi i}\left[\frac{\varepsilon}{\mu c}(\mathfrak{A}\,\mathrm{grad}) + \frac{\partial}{\partial t} \right] \right\}\xi = 0, \tag{11}$$

which agrees with the equation that **Schrödinger** gave.

In order to explain equation (7), we now consider the case of a force-field that is static in a well-defined coordinate system. We can then set φ equal to:

$$\varphi = \Phi\, e^{2\pi i T t}, \tag{12}$$

where Φ no longer depends upon time, and T refers to a constant. That will then imply that:

$$-\frac{h^2}{4\pi^2}\Delta\Phi + 2\frac{h}{2\pi i}\frac{\varepsilon}{c}(\mathfrak{A}\,\mathrm{grad}\,\Phi) + \left[\mu^2 c^2 + \frac{\varepsilon^2}{c^2}\mathfrak{A}^2 - \frac{1}{c^2}(hT - \varepsilon V)^2 \right]\Phi = 0. \tag{13}$$

From the foregoing, we see that we will achieve a natural connection to the usual quantum conditions for periodic systems when we consider the eigen-oscillations that belong to that equation to be representatives of the stationary state of the atom. The replacement of the quantum conditions with the eigenvalue problem that belongs to equation (11) will now bring with it the immediate advantage that the problem in question generally has well-defined discrete solutions, such that fundamental complication arises that originated in the usual theory of stationary states by the exceptional role that is played by periodic systems in ordinary mechanics.

In the limit of ordinary relativistic mechanics, it would follow from (8) that hT will go to the quantity p_t , which measures the energy, when taken with the negative sign. From **Bohr**'s frequency condition, we would expect that the associated eigenvalue T would represent the spectral term. However, the frequency condition itself is likewise foreign to

the representation of ordinary quantum theory that is based in the phase integral. Here, we stand at precisely the starting point for **Bohr**'s correspondence principle, and as we will see in the next paragraph, it is, in fact, possible to arrive at a logical association of the quantum-theoretic postulate with the demands of classical electrodynamics in connection with the wave-mechanical interpretation of the frequency condition that **Schrödinger** discovered, and that would correspond to the spirit of the correspondence principle.

When the force fields that enter into (7) vary in time, that equation can still be employed for the solution of the quantum problem by means of the correspondence principle, in contrast to the **Hamilton-Jacobi** equation (2). That fact is closely linked with the linearity of that equation, which will imply properties of its solutions that will correspond to the transitions between stationary states of the associated "virtual" oscillators. As in **Heisenberg**'s theory, that will define a bridge between the theory of periodic systems and dispersion theory in terms of the correspondence principle, as it was developed by **Ladenburg** and **Kramers**.

§ 2. Wave-mechanical expressions for the electric density and current vector. – In **Lorentz**'s theory of electrons, the electromagnetic field is known to be determined by the electric density ρ and the (electrostatically-measured) current vector $\mathfrak{J}$ in the following way:

$$\left.\begin{array}{c} \operatorname{div} \mathfrak{E} = 4\pi \rho, \\[2mm] \operatorname{rot} \mathfrak{H} - \dfrac{1}{c}\dfrac{\partial \mathfrak{E}}{\partial t} = \dfrac{4\pi}{c}\mathfrak{J}. \end{array}\right\} \tag{14}$$

In this, $\mathfrak{E}$ denotes the electric field vector and $\mathfrak{H}$ denotes the magnetic one. The following relations exist between the quantities $\mathfrak{E}$ and $\mathfrak{H}$, on the one hand, and V and $\mathfrak{A}$, on the other, which correspond to the second pair of **Maxwell** equations:

$$\mathfrak{E} = -\left(\operatorname{grad} V + \dfrac{1}{c}\dfrac{\partial \mathfrak{A}}{\partial t}\right), \qquad\qquad \mathfrak{H} = \operatorname{rot} \mathfrak{A}. \tag{15}$$

The law of conservation of electricity, namely:

$$\operatorname{div} \mathfrak{J} + \dfrac{\partial \rho}{\partial t} = 0, \tag{16}$$

follows from (14) in the well-known way.

If we would now like to evaluate equations of the form (14) with the help of wave mechanics using the correspondence principle, it would be, above all, necessary to form expressions from the solutions of the wave equation that fulfill the relationship (16).

To that end, along the lines of some arguments that **Schrödinger** recently communicated, we consider equation (7). Since $i = \sqrt{-1}$ enters into that equation

explicitly, the following equation will exist that is equivalent to the latter equation and in which i is switched with $-i$:

$$-\frac{h^2}{4\pi^2}\Box\psi - 2\frac{h}{2\pi i}\frac{\varepsilon}{c}\left[(\mathfrak{A}\,\mathrm{grad}\,\psi) + \frac{V}{c}\frac{\partial\psi}{\partial t}\right] + \left[\mu^2 c^2 + \frac{\varepsilon^2}{c^2}(\mathfrak{A}^2 - V^2)\right]\psi = 0, \qquad (7a)$$

in which ψ denotes a function of position and time that be the complex conjugate of the function φ in (7), in particular. Just as the **Schrödinger** equation (11) follows from (7), using the Ansatz:

$$\psi = \eta\, e^{\frac{2\pi i}{h}\mu c^2 t}, \qquad (10a)$$

we will get the following equation from (7a):

$$\Delta\eta + \frac{8\pi^2\mu}{h^2}\left\{\varepsilon V + \frac{h}{2\pi i}\left[\frac{\varepsilon}{\mu c}(\mathfrak{A}\,\mathrm{grad}) + \frac{\partial}{\partial t}\right]\right\}\eta = 0. \qquad (11a)$$

Likewise, when we set $h = 0$, the Ansatz:

$$\psi = e^{-\frac{2\pi i}{h}S} \qquad (8a)$$

will yield the **Hamilton-Jacobi** equation (2) for the function S. In analogy with (12), in the case of a static force field, one can ultimately set:

$$\psi = \Psi\, e^{-2\pi i T t}, \qquad (12a)$$

which will imply the equation Ψ:

$$-\frac{h^2}{4\pi^2}\Delta\psi - 2\frac{h}{2\pi i}\frac{\varepsilon}{c}(\mathfrak{A}\,\mathrm{grad}\,\psi) + \left[\mu^2 c^2 + \frac{\varepsilon^2}{c^2}\mathfrak{A}^2 - \frac{1}{c^2}(hT - \varepsilon V)^2\right]\psi = 0, \qquad (13a)$$

which differs from equation (13) only by the sign i.

We now multiply equation (7) by ψ and equation (7a) by φ and subtract. After a simple calculation, in which use is made of the condition (1), we will get:

$$\mathrm{div}\left\{\frac{h}{2\pi i}(\psi\,\mathrm{grad}\,\varphi - \varphi\,\mathrm{grad}\,\psi) + 2\frac{\varepsilon}{c}\mathfrak{A}\,\varphi\psi\right\}$$

$$+\frac{\partial}{\partial t}\left\{-\frac{h}{2\pi i}\frac{1}{c^2}\left(\psi\frac{\partial\varphi}{\partial t} - \varphi\frac{\partial\psi}{\partial t}\right) + \frac{2\varepsilon}{c^2}V\varphi\psi\right\} = 0. \qquad (17)$$

In fact, we have an equation with the form of the continuity equation (16) before in (17). When we multiply the expressions in brackets by $-\varepsilon / 2\mu$, on grounds that will become clear later, we would like to set:

$$\rho = -\frac{\varepsilon}{2\mu c^2}\left\{-\frac{h}{2\pi i}\left(\psi\frac{\partial\varphi}{\partial t}-\varphi\frac{\partial\psi}{\partial t}\right)+2\varepsilon V\varphi\psi\right\},$$

$$\mathfrak{J} = -\frac{\varepsilon}{2\mu}\left\{\frac{h}{2\pi i}\left(\psi\,\mathrm{grad}\,\varphi-\varphi\,\mathrm{grad}\,\psi\right)+2\frac{\varepsilon}{c}\mathfrak{A}\,\varphi\psi\right\}.$$

(18)

When we neglect relativity in these expressions with the help of (10) and (10a), and in addition, consider the magnetic field to be so weak that the term in $\mathfrak{J}$ that is proportional to $\mathfrak{A}$ can be dropped, we will get precisely the expressions for the electric density and current vector that **Schrödinger** gave, namely:

$$\rho = -\varepsilon\,\xi\,\eta,$$

$$\mathfrak{J} = -\frac{\varepsilon}{2\mu}\left\{\frac{h}{2\pi i}\left(\psi\,\mathrm{grad}\,\varphi-\varphi\,\mathrm{grad}\,\psi\right)\right\}.$$

(19)

In order to make the expressions (18) more intuitive, we would like to go to the limit $h = 0$. In order to do that, we substitute the expressions (8) and (8a) for φ and ψ in (18), which will yield:

$$\rho = \frac{\varepsilon}{\mu c^2}\left(\frac{\partial S}{\partial t}-\varepsilon V\right),$$

$$\mathfrak{J} = -\frac{\varepsilon}{\mu}\left(\mathrm{grad}\,S+\frac{\varepsilon}{h}\mathfrak{A}\right).$$

(20)

With the help of the "ray equations" (3), we would further like to express the differential quotients of S in (20) in terms of the components of the dx / dt, dy / dt, dz / dt of the "ray velocity" $\mathfrak{v}$, which will yield:

$$\rho = -\frac{\varepsilon}{\sqrt{1-\mathfrak{v}^2/c^2}},$$

$$\mathfrak{J} = -\frac{\varepsilon\,\mathfrak{v}}{\sqrt{1-\mathfrak{v}^2/c^2}}.$$

(21)

The superficial similarity between these expressions and the corresponding formulas of the classical theory of electrons might help illuminate the fact that on first glance, the potential that enters into (18) seems rather foreign to those equations. However, it must be emphasized that the passage to the limit that is considered here must be regarded as purely formal and does not answer the deeper question of how one can arrive at the results of the classical theory of electrons continuously from the properties of the wave-mechanical electron model. In that regard, **Schrödinger** attempted to find a connection by comparing a particle with a "wave packet." However, it is known that it is not

possible to achieve cohesion in the electron in that way. It seems as if the coupling will come about here by precisely the correspondence principle, which will seem all the more natural when one seeks to consider the existence of the particle to be a quantum problem ([1]).

Before we go on to a discussion of the applications of the relations (18), we would like to derive the following auxiliary equation:

$$\frac{\partial}{\partial t}\int \rho \, dv = 0 \tag{22}$$

by multiplying equation (16) with a volume element dv and integrating over the entire region in which ρ and $\mathfrak{J}$ exist, under the assumption that the electric current vanishes on the boundary surface. Moreover, as is known, (16) implies that the integral $\int \rho \, dv$ is also invariant under **Lorentz** transformation.

§ 3. Evaluation of wave mechanics in the case of a static force field using the correspondence principle.

– We now turn to the evaluation of the expressions (18) for quantum theory, and we would first like to consider the case of a static force field and in so doing, assume that we are dealing with a nondegenerate system, which will imply no essential restriction of generality. In that case, the general solutions to equations (7) and (7a) will be linear combinations of the solutions that belong to the individual eigen-oscillations, such that from (12) and (12a), we can set:

$$\varphi = \sum_n \Phi_n \, e^{2\pi i \, T_n t}, \qquad \psi = \sum_n \Psi_n \, e^{-2\pi i \, T_n t}, \tag{23}$$

in which Φ_n and Ψ_n denotes a pair of eigenfunctions of the equations (13) and (13a), and in which all T_n are different. With the help of those expressions, the quantities ρ and $\mathfrak{J}$ will assume the following form:

$$\rho = \sum_n \sum_m \rho_{nm},$$

$$\mathfrak{J} = \sum_n \sum_m \mathfrak{J}_{nm}, \tag{24}$$

where

$$\rho_{mn} = -\frac{\varepsilon}{\mu c^2}\left[-\frac{h(T_n + T_m)}{2} + \varepsilon V\right]\Phi_n \Psi_m e^{2\pi i(T_n - T_m)t}$$

$$\mathfrak{J}_{mn} = -\frac{\varepsilon}{2\mu}\left[\frac{h}{2\pi i}(\Psi_m \,\mathrm{grad}\,\Phi_n - \Phi_n \,\mathrm{grad}\,\Psi_m) + 2\frac{\varepsilon}{c}\Phi_n \Psi_m\right]e^{2\pi i(T_n - T_m)t}. \tag{25}$$

([1]) Cf., **O. Klein**, Nature **118** (1926), 516.

If we consider the electromagnetic field that belongs to the expressions (24), according to the field equations (14), then that will point to a striking similarity with the external behavior of an atom in a way that corresponds to the postulates of **Bohr**'s theory, and that will endow the representation of stationary states by eigen-oscillations with an essential meaning. As we see, part of the field in question consists of static terms that belong to the individual eigen-oscillation and part of it consists of purely harmonic oscillations that belong to the pairs of two different eigen-oscillations. The frequency that belongs to the pair of numbers *n*, *m* is then given by:

$$\nu = \mid T_n - T_m \mid \,, \tag{26}$$

such that the entire spectrum will satisfy the **Rydberg-Ritz** combination principle. At the same time, **Bohr**'s frequency condition is also fulfilled, so as we have seen, the wave-mechanical picture will lead to the following relation between the energy *E* of the atom and the quantity *T* :

$$E = - h \, T. \tag{27}$$

It was exactly that significant agreement between the results of wave mechanics and the demands of quantum theory that **Schrödinger** considered to be the basis for his program of a purely wave-mechanical theory of atomic processes, in which the postulates of quantum theory would no longer appear explicitly, although they would already be required in order to explain the combination principle in the theory of periodic systems that is based in classical mechanics. In the meantime, wave mechanics has not helped us get over the fundamental problems that can come about in the generation of spectra, *inter alia*, and which find their expressions in the postulates. By contrast, the wave-mechanical expressions for ρ and $\mathfrak{J}$ permit a quantitative formulation of the correspondence between the demands of electrodynamics and the description of atomic processes that is based upon **Bohr**'s postulates, which is a correspondence that might be better understood with the help of the classical theory of electrons, which only seems to allow one to give an asymptotically-quantitative expression, however ([1]).

Pursuing the correspondence, we associate the terms in the wave-mechanical expressions (24) for the density and current vector in the following way:

$$\left.\begin{aligned}
\rho = \sum_n \rho_n, \quad \rho_n = \rho_{nn} + \sum_m^{E_m < E_n} (\rho_{mn} + \rho_{nm}), \\
\mathfrak{J} = \sum_n \mathfrak{J}_n, \quad \mathfrak{J}_n = \mathfrak{J}_{nn} + \sum_m^{E_m < E_n} (\mathfrak{J}_{mn} + \mathfrak{J}_{nm}),
\end{aligned}\right\} \tag{28}$$

in which the summations in the expressions for ρ_n and $\mathfrak{J}_n$ is extended over all of those eigen-oscillations for which $E_m < E_n$. We first see that the quantities ρ and $\mathfrak{J}$ will be real only when each Ψ_n is assumed to be the complex conjugate of the corresponding Φ_n (up to a factor that is common for all *n*, and which we naturally set equal to one on the

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([1]) See **N. Bohr**, "Über die Quantentheorie der Linienspektren," Braunschweig (1923).

grounds of symmetry). With that assumption, the quantities $(\rho_{nm} + \rho_{mn})$ and $(\mathfrak{J}_{nm} + \mathfrak{J}_{mn})$ will become individually real. If we now consider the quantities ρ_n and $\mathfrak{J}_n$ then we will see that precisely all of the radiation frequencies are represented in them that belong to spontaneous transitions from that stationary state whose energy is equal to E_n from the postulates of quantum theory.

We will then attempt to describe the electromagnetic effects that originate in an atom in a stationary state with the help of the quantities ρ_n and $\mathfrak{J}_n$. To that end, we must first demand that the total charge that belongs to the density ρ_n is equal to the charge $-\varepsilon$ of the electron. The fact that it is possible to make such a demand is based in the fact that the total charge is independent of time, from the continuity equation (16), which is fulfilled for each pair of quantities ρ_{nm} and $\mathfrak{J}_{nm}$. When n and m are different, it follows simply from (22) that:

$$\int \rho_{nm}\, dv = 0,$$

which is a relation that goes to the well-known orthogonality condition for eigenfunctions when one neglects relativity. In fact, it follows from (10) and (10a) that:

$$\int \xi_n \eta_m\, dv = 0. \tag{29}$$

From (28), we then have:

$$\int \rho_n\, dv = \int \rho_{nn}\, dv,$$

such that fixing the total charge by (25) will lead to the following relation:

$$-\frac{1}{\mu c^2}\int (hT - \varepsilon V)\Phi_n \Psi_n\, dv = 1, \tag{30}$$

which will normalize each eigenfunction in a well-defined way. If we neglect relativity then that will imply that:

$$\int \xi_n \eta_n\, dv = 1, \tag{31}$$

which agrees with the normalization condition that **Schrödinger** imposed upon the eigenfunctions, and is especially significant for the derivation of the connection between wave mechanics and matrix mechanics. In fact, if we neglect relativity and normalize all eigenfunctions in the aforementioned way then the quantities ρ_{nm} will be the elements of a matrix from which the representation of the mechanical quantities of the motion of electrons will follow in a manner that corresponds to **Heisenberg**'s theory.

Since we seek to evaluate wave mechanics in direct analogy to the electromagnetic field equations, we will assume that the quantities ρ_n and $\mathfrak{J}_n$ correspond to electromagnetic phenomena, in the sense of **Bohr**'s correspondence principle, that give a quantitative expression for the presence of observed effects that are coupled with an atom in the stationary state in question. In that, **Einstein**'s probability coefficients for spontaneous transitions are to be ascertained in the usual way under the assumption of

conservation of energy. We see that our assumption about that leads to an association of ρ and $\mathfrak{J}$ with the totality of electromagnetic phenomena that belong to an imagined simultaneous presence of electrons that are bound with no interaction in the same static field in such a way that each possible stationary will be represented by an electron. It was by precisely that association that we obtained a connection between the wave-mechanical description and the quantum-theoretical picture of the way that individual atoms behave. To the extent that those considerations deal with sums of harmonic oscillator terms, they are linked with the asymptotic representation of the correspondence principle that is based upon the classical theory of electrons. However, it should be remarked that it is just in the nature of wave theory that the coupling in the limit where the relative difference between stationary states vanishes does not take an especially simple form, as was mentioned before. Moreover, that is precisely the quantum-theoretic aspect of the correspondence that **Bohr** emphasized and which defines the nucleus of our analysis here.

The evaluation of wave mechanics using the correspondence principle that was described allows not only the demands of relativity to be satisfied, which are, as mentioned, the source of the difficulties in matrix theory, but the direct connection with the field equations also makes it possible to simplify the treatment of the radiation problem. That problem was taken up in the context of matrix mechanics by **Dirac** ([1]), in particular, and he succeeded in deriving an expression for the probability coefficients of the transitions that are induced by external radiation that coincided with the calculation of the probability coefficients for spontaneous transition that was described above when one appeals to general relation that **Einstein** gave. In regard to that, it should be pointed out that in **Dirac**'s calculation, as in **Born**'s collision theory, the wave equation was employed in a way that is essentially different from the way that it was used here. Whereas in our presentation, it is in the nature of things that the properties of an electron are always coupled with normalized eigenfunctions, the aforementioned theory dealt with arbitrary amplitudes whose changes were considered to be a measure of the probability of the transition processes that were stimulated by external agencies.

We shall now go into somewhat deeper detail and consider the external behavior of an atom that should be expected from the argument above. For that, we first turn to the static field that belongs to a *single stationary state*. The quantities ρ and $\mathfrak{J}$ that come under consideration in it will then be independent of time, such that we can solve the field equation (14) by the usual method with the following expressions:

$$\left.\begin{aligned} V &= \int \frac{\rho \, d\upsilon}{r_{PQ}}, \\ \mathfrak{A} &= \frac{1}{c} \int \frac{\mathfrak{J} \, d\upsilon}{r_{PQ}}, \end{aligned}\right\} \tag{32}$$

in which r_{PQ} denotes the distance from the source point Q to the reference point P. We would now like to assume that the force field in which the electron moves is essentially centrally-symmetric, which would correspond to an actual atom. Furthermore,

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([1]) **P. A. M. Dirac**, Proc. Roy. Soc. (A) **112** (1926), 661.

corresponding to the usual experimental conditions, we would like to place the reference point so far from the atom that the expressions (32) for the potentials will take on appreciable magnitudes only within a distance from the center point of the atom that is very small in comparison with the distance r to the reference point.

Now let $\mathfrak{n}$ be a unit vector that gives the direction from the center of the atom to the reference point, and let $\mathfrak{r}$ be the radius vector from the center to the source point. Consistent with what was just said, we would like to replace the quantity $1 / r_{PQ}$ with the approximate value $\dfrac{1}{r} + \dfrac{(\mathfrak{n}\,\mathfrak{r})}{r^2}$, which would then yield:

$$
\left.
\begin{aligned}
V &= \frac{1}{r}\int \rho\,dv + \frac{1}{r^2}\int (\mathfrak{n}\mathfrak{r})\rho\,dv, \\[2mm]
\mathfrak{A} &= \frac{1}{cr}\int \rho\,dv + \frac{1}{cr^2}\int (\mathfrak{n}\mathfrak{r})\,\mathfrak{J}\,dv.
\end{aligned}
\right\}
\tag{33}
$$

The meaning of the first of those formulas is brought to light immediately. The first term gives the potential of the total charge of the electron, which was just set equal to $-\varepsilon$, and as we saw, that corresponds to the normalization of the eigenfunctions. The second term gives the potential of the electric moment of the charge distribution. The vector of the electric moment $\mathfrak{D}$ in this is given by the following expression:

$$
\mathfrak{D} = \int \mathfrak{r}\,\rho\,dv .
\tag{34}
$$

We can give the expression for $\mathfrak{A}$ the following form by a simple calculation:

$$
\mathfrak{A} = \left[\frac{\mathfrak{n}}{2c\,r^3}\int [\mathfrak{J}\,\mathfrak{r}]\,dv \right].
\tag{35}
$$

That expression represents the vector potential of a magnet whose magnetic moment is given by the vector:

$$
\mathfrak{B} = \left[\frac{1}{2c}\int [\mathfrak{J}\,\mathfrak{r}]\,dv \right].
\tag{36}
$$

In fact, the expression for the field vector that following from (35):

$$
\mathfrak{H} = \operatorname{rot}\mathfrak{A}
$$

can be transformed into the expression below by a known rule:

$$
\mathfrak{H} = -\operatorname{grad}\frac{(\mathfrak{n}\cdot\mathfrak{B})}{r^2} .
$$

We shall now go on to the consideration of the radiation field that is associated with a *transition process* by way of (25). Here, we can set:

$$\left.\begin{array}{l} \rho = \rho_0\, e^{2\pi i v t}, \\[2mm] \mathfrak{J} = \mathfrak{J}_0\, e^{2\pi i v t}, \end{array}\right\} \tag{37}$$

in which ρ_0 and $\mathfrak{J}_0$ are independent of time, and v denotes the frequency that belongs to the transition process. As in the classical theory of radiation, we solve the field equations (14) by retarded potentials, in which consider only the part of the field that is proportional to $1/r$. That will then imply that:

$$V = \frac{1}{r} e^{2\pi i v (t - r/c)} \int \rho_0\, e^{\frac{2\pi i v}{c}(\mathfrak{n}\,\mathfrak{r})}\, dv,$$

$$\tag{38}$$

$$\mathfrak{A} = \frac{1}{cr} e^{2\pi i v (t - r/c)} \int \mathfrak{J}_0\, e^{\frac{2\pi i v}{c}(\mathfrak{n}\,\mathfrak{r})}\, dv.$$

Since we would like to consider only dipole radiation, we can replace $e^{\frac{2\pi i v}{c}(\mathfrak{n}\,\mathfrak{r})}$ with one in $\mathfrak{A}$ and with $1 + \dfrac{2\pi i v}{c}\,(\mathfrak{n}\,\mathfrak{r})$ in V. From the continuity equation, which now reads:

$$\mathrm{div}\ \mathfrak{J}_0 + 2\pi i\, v\, \rho_0 = 0,$$

we see that:

$$\int \rho_0\, dv = 0$$

and

$$\int \mathfrak{J}_0\, dv = 2\pi i v \int \mathfrak{r}\, \rho_0\, dv,$$

which then implies that:

$$\left.\begin{array}{l} V = \dfrac{2\pi i v}{c\,r}\, e^{2\pi i v (t - r/c)} \displaystyle\int (\mathfrak{n}\,\mathfrak{r})\rho_0\, dv, \\[4mm] \mathfrak{A} = \dfrac{2\pi i v}{c\,r}\, e^{2\pi i v (t - r/c)} \displaystyle\int \mathfrak{r}\rho_0\, dv. \end{array}\right\} \tag{39}$$

If we now calculate the field vectors $\mathfrak{E}$ and $\mathfrak{H}$ that describe the radiation field directly then (38) will simply yield that:

$$\left.\begin{array}{l} \mathfrak{E} = -\dfrac{2\pi i v}{c\,r}(\mathfrak{A} - \mathfrak{n}V), \\[4mm] \mathfrak{H} = -\dfrac{2\pi i v}{c\,r}[\mathfrak{n}\,\mathfrak{r}] \end{array}\right\} \tag{40}$$

when we consider only the terms that are proportional to $1 / r$. If we then set:

$$\mathfrak{D}_0 = \int \mathfrak{r}\, \rho_0 \, dv$$

then (39) will imply that:

$$
\left.
\begin{aligned}
\mathfrak{E} &= \frac{4\pi^2 v^2}{c^2 r} e^{2\pi i v(t - r/c)} [\mathfrak{D}_0 - (\mathfrak{n}\,\mathfrak{D}_0)\,\mathfrak{n}], \\[2mm]
\mathfrak{H} &= \frac{4\pi^2 v^2}{c^2 r} e^{2\pi i v(t - r/c)} [\mathfrak{n}\,\mathfrak{D}_0].
\end{aligned}
\right\}
\tag{41}
$$

A spherical electromagnetic wave is described by formulas (41) in a well-known way. One sees immediately in them that the electric force, in magnitude as well as direction, is determined by the components of the electric moment in the plane that is perpendicular to $\mathfrak{n}$, and that the magnetic field is perpendicular to $\mathfrak{n}$, just like $\mathfrak{E}$, and has the same magnitude as $\mathfrak{E}$.

In that, we should recall that the radiation field that corresponds to the transition process $n \to m$, where $E_n > E_m$, will be described by the real quantities $\mathfrak{E} + \overline{\mathfrak{E}}$, $\mathfrak{H} + \overline{\mathfrak{H}}$, in which $\overline{\mathfrak{E}}$ is complex conjugate to $\mathfrak{E}$ and $\overline{\mathfrak{H}}$ is complex conjugate to $\mathfrak{H}$, when the quantity $(E_n - E_m) / h$ is substituted for v and the amplitude of:

$$\mathfrak{D}_{nm} = \int r\, \rho_{nm} \, dv \tag{42}$$

is substituted for $\mathfrak{D}_0$. Just as the quantity $\mathfrak{D}_{nm}$ means the static electric moment of the atom in the state n, from (34), we will refer to $\mathfrak{D}_{nm} + \mathfrak{D}_{mn}$ as the electric moment that belongs to the transition.

§ 4. Illustrative examples from the theory of atomic structure. – In this paragraph, we would like to clarify the arguments that were made in the previous paragraphs with some simple examples. We first turn to the simple and most important case in which the electron moves in a pure centrally-symmetric field. Since the absolute direction in space plays no role in that system, we are dealing with a degenerate case here in which only two quantum numbers n and k are necessary to characterize the stationary state. We introduce a polar coordinate system r, ϑ, α, in which r is the length of the radius vector $\mathfrak{r}$, ϑ means the angle that $\mathfrak{r}$ makes with a fixed axis, and α is the angle between the projection of $\mathfrak{r}$ onto the plane that is perpendicular to that axis and a fixed line that lies in that plane. As is known, we can then set an eigenfunction Φ of the equation (13) equal to:

$$\Phi = X(r)\, Y(\vartheta, \alpha), \tag{43}$$

in which $Y(\vartheta, \alpha)$ denotes a spherical function – i.e., an eigenfunction of the following equation:

$$\Delta^* Y + \lambda\, Y = 0, \tag{44}$$

where Δ^* represents the two-dimensional Laplace operator that refers to the outer surface of the unit ball. The eigenvalues of that equation are known to the be the quantities $k\,(k+1)$, where k means a whole number, and the associated solutions are linear combinations of expressions like:

$$Y_{k,m} = e^{im\alpha}\, P_{k,m}(\cos\vartheta), \tag{45}$$

in which m is a positive or negative whole number whose absolute value is equal to at most k, and $P_{k,m}$ means the following polynomial:

$$P_{k,m}(s) = \sqrt{1-s^2}\, \frac{d^m}{ds^m} P_k(s), \tag{46}$$

where

$$P_k(s) = \frac{1}{2!\,k!} \frac{d^k}{ds^k} (s^2 - 1)^k \tag{47}$$

means a **Legendre** polynomial.

   In order to investigate the light radiation that belongs to that system, we form the component of the amplitude $\mathfrak{D}_0$ in the direction of the polar axis, which reads:

$$\int r \cos\vartheta\, \rho_0\, dv.$$

From (43), that integral will split into two factors, one of which represents an integral over just r and the other of which defines an integral that is extended over exclusively the angles ϑ and α. The first integral is non-zero, in general, and changes from one central system to another. The second integral, which is common to all central systems, is non-zero only under special conditions, and expresses the well-known selection rule for the changes in the number k, which largely implies the characteristic picture of ordinary series spectra.

   In fact, from (25) and (43), we can write ρ_0 as:

$$\rho_0 = F(r)\, Y'(\vartheta, \alpha)\, Y''(\vartheta, \alpha),$$

in which $F(r)$ means a function of r that is of no interest here, while Y' and Y'' should be two spherical functions that belong to different k-values. From (45), the integral that depends upon the angles, which belongs to a well-defined transition that is characterized by the numbers k' and k'', can always be a linear combination of a number of integrals of the type:

$$\int_0^{2\pi} e^{i(m'-m'')\alpha}\, d\alpha \int_{-1}^{+1} s\, P_{k'.m'}(s)\, P_{k''.m''}(s)\, ds,$$

in which we have introduced the variable:

$$s = \cos \vartheta,$$

in place of ϑ. In order for that integral to be non-vanishing, as one sees, one must first have m' equal to m'', such that we only need to examine the following integral:

$$\int_{-1}^{+1} s\, P_{k'.m}(s)\, P_{k''.m}(s)\, ds .$$

Now, the polynomial $P_{k,m}$ fulfills the following relation:

$$s\, P_{k,m} = \frac{k-m+1}{2k+1} P_{k+1,m} + \frac{k+m}{2k+1} P_{k-1,m}. \tag{48}$$

We then get:

$$\int_{-1}^{+1} s\, P_{k'.m}\, P_{k''.m}\, ds = \frac{k'-m+1}{2k'+1} \int_{-1}^{+1} P_{k''.m}\, P_{k'+1.m}\, ds + \frac{k'+m}{2k'+1} \int_{-1}^{+1} P_{k''.m}\, P_{k'-1.m}\, ds .$$

Since the quantities $P_{k,m}$ satisfy the orthogonality condition:

$$\int_{-1}^{+1} P_{k'.m}\, P_{k''.m}\, ds = 0 \qquad \text{for} \qquad k' \neq k'', \tag{49}$$

we see that the component of $\mathfrak{D}_0$ that we consider will be non-zero only when:

$$k' - k'' = \pm 1. \tag{50}$$

Since the axis refers to a direction that is completely arbitrary with respect to the central field, that result must be true for each component of $\mathfrak{D}_0$. From (41), we will then get a finite intensity of the dipole radiation only when the relation (50) is fulfilled, which coincides with exactly the selection rule for the auxiliary quantum number k that **Bohr** derived from the correspondence principle.

Let us take another simple example of the case of an axially-symmetric system. Here, we would first like to calculate the magnetic moment of the system in a stationary state when we neglect relativity as an example of an application of formula (35). We introduce cylindrical coordinates z, a, α, in which z denotes the projection of $\mathfrak{r}$ onto the axis, a is its projection onto the plane that is perpendicular to the axis, and α is the angle between that projection and a fixed line in that plane. We can set the eigenfunctions of equation (13) to be:

$$\Phi = e^{im\alpha}\, \theta(z, a) \tag{51}$$

in this case, in which m means a positive or negative whole number, and θ is a function of z and a. When we ignore the terms that depend upon $\mathfrak{A}$, we will get:

$$\mathfrak{J} = - \frac{\varepsilon}{2\mu} \frac{h}{2\pi i} (\Psi \ \text{grad} \ \Phi - \Phi \ \text{grad} \ \Psi) \tag{52}$$

for the current vector, in which Ψ is complex conjugate to Φ, and in which, from (26), one will have:

$$\int \Phi \Psi \, dv = 1 \tag{53}$$

when one ignores relativity.

On symmetry grounds, it is clear that the magnetic moment will point in the z-direction, such that we need to calculate only the z-component of the vector $[\mathfrak{r} \ \mathfrak{J}]$, which is obviously equal to $a \, \mathfrak{J}_\alpha$, where $\mathfrak{J}_\alpha$ is the component of $\mathfrak{J}$ in the direction that points to an increase in α. We now get:

$$a \, \mathfrak{J}_\alpha = - \frac{\varepsilon}{2\mu} \frac{h}{2\pi i} \left(\Psi \frac{\partial \Phi}{\partial \alpha} - \Phi \frac{\partial \Psi}{\partial \alpha} \right)$$

from (52), and then:

$$a \, \mathfrak{J}_\alpha = - \frac{\varepsilon h}{2\pi\mu} m \Phi \Psi$$

from (51), and finally:

$$\mathfrak{B}_z = - m \frac{\varepsilon h}{4\pi\mu c} \tag{54}$$

from (35) and (53). According to whether m is positive or negative in (51) [which would correspond to the positive or negative direction of rotation around the z-axis, respectively, according to (8)], we will then get a magnetic moment that points in the direction of the negative or positive z-axis, resp., which would correspond to the negative charge of the electron, and its magnitude would be an m-fold multiple of the **Bohr** magneton. We would like to return to that question soon in connection with the action of a weak perturbing magnetic field on the atom.

If we now consider the radiation properties of that atom then we will see from (51) that the z-component of the electric moment that belongs to a transition (m', m'') includes a factor of the form:

$$\int_0^{2\pi} e^{i\,(m'-m'')\alpha} d\alpha$$

that is non-zero only when $m' = m''$. Hence, it is only in that case that the light wave (42) will contain a component whose direction of polarization coincides with the direction of the z-axis. Likewise, the components of $\mathfrak{D}_0$ that are perpendicular to the z-axis will contain a factor of the form:

$$\int_0^{2\pi} e^{i\,(m'-m''\pm1)\alpha} d\alpha \,,$$

such that the corresponding part of the radiation will appear only when:

$$m' - m'' = \pm 1.$$

We then come to the well-known selection or polarization rules for the quantum number.

As a special axially-symmetric system, we consider an electron that moves in an axially-symmetric electrostatic field, over which a weak, homogeneous magnetic field that is directed along the axis of the system is superimposed. We can then set:

$$\mathfrak{A} = \tfrac{1}{2} [\mathfrak{H} \; \mathfrak{r}].$$

In the aforementioned cylindrical coordinate system, the operator $(\mathfrak{A} \text{ grad})$ is simply equal to $\tfrac{1}{2} | \mathfrak{H} | \dfrac{\partial}{\partial \alpha}$ then. Since the electrostatic potential V is independent of t, as well as α, by introducing a new variable:

$$\alpha' = \alpha - \frac{\varepsilon | \mathfrak{H} |}{2 \mu c} t \tag{55}$$

and neglecting quantities of second order in $| \mathfrak{H} |$, we can reduce the relativistic wave equation (11) to the corresponding equation with no magnetic field. In fact, that transformation, which corresponds precisely to **Larmor**'s theorem, will replace the operator $\dfrac{\varepsilon | \mathfrak{H} |}{2 \mu c} \dfrac{\partial}{\partial \alpha} + \dfrac{\partial}{\partial t}$ with the operator $\dfrac{\partial}{\partial t}$, while Δ goes to the operator Δ' that arises from Δ by replacing α with α'. Equation (11) then assumes the form:

$$\Delta' \xi + \frac{8 \pi^2 \mu}{h^2} \left(\varepsilon V - \frac{h}{2 \pi i} \frac{\partial}{\partial t} \right) \xi = 0,$$

in which the magnetic field vanishes. Now, from (10), (12), and (51), the solutions to those equations can be written in the form:

$$\xi = \theta(z, a) e^{-\frac{2 \pi i}{h} E t + i m \alpha'},$$

in which E represents the energy (rest energy of the electron = 0) of the atom with no magnetic field. From (55), one will then have:

$$\xi = \theta(z, a) e^{-\frac{2 \pi i}{h} \left(E + \frac{\varepsilon | \mathfrak{H} |}{4 \pi \mu c} \right) t + i m \, \alpha'}. \tag{56}$$

Therefore, the energy will change by $\dfrac{\varepsilon | \mathfrak{H} |}{4 \pi \mu c}$ as a result of the magnetic field, which will lead to a normal **Zeeman** effect in conjunction with the selection rule for m that was derived above.

In the special axially-symmetric system that arises from the perturbation of a centrally-symmetric system by a weak constant magnetic field, the eigenfunctions will be given by (43) and (45) in the first approximation, where the axis lies in the direction of the magnetic field. In that case, the maximum value of *m* will then be given by *k*. We can therefore say (if we ignore relativity) that the number *k* determined the number of magnetons of the magnetic moment of a central system whose direction is arbitrary due to degeneracy.

§ 5. Perturbation of an atom by external forces. – The action of a weak perturbing force field on an atom in a stationary state will serve as a further case of the application of the evaluation of wave mechanics by the correspondence principle. For the sake of simplicity, we would like to ignore the influence of relativity in it and assume that the force field is purely electrostatic in the unperturbed state. If we denote the potential that belongs to that field by V_0 then, from (11), we will have the following equation for the determination of the stationary states of the unperturbed atom:

$$\Delta \xi + \frac{8\pi^2 \mu}{h^2}\left(-\frac{h}{2\pi i}\frac{\partial}{\partial t} + \varepsilon V_0\right)\xi = 0, \tag{57}$$

with the eigen-solutions:

$$\xi_n = u_n\, e^{-\frac{2\pi i}{h}E_n t}, \tag{58}$$

in which u_n is independent of time, and in which we have introduced the energy value of the stationary state in place of the term that is defined in the usual way (viz., rest energy = 0). In this, we would like to assume that the unperturbed system is nondegenerate; i.e., each eigenvalue belongs to *just one* eigenfunction (up to a constant factor of modulus one that defines the phase).

We would like to denote the perturbing force field by σV and $\sigma \mathfrak{A}$, in which σ is a constant parameter that can be treated as a first-order infinitesimal. We now seek a solution to equation (11) with the form $\xi_n + \sigma f_n$, which would give us the following perturbation equation for the function f_n :

$$\Delta f_n + \frac{8\pi^2 \mu}{h^2}\left(-\frac{h}{2\pi i}\frac{\partial}{\partial t} + \varepsilon V_0\right)f_n = -\frac{8\pi^2 \mu}{h^2}\left(\varepsilon V - \frac{h}{2\pi i}\frac{\varepsilon}{\mu c}(\mathfrak{A}\ \text{grad})\right)\xi_n. \tag{59}$$

In order to solve that equation, following **Schrödinger**, we develop the function f_n , as well as the quantities on the right-hand side of the equation, in terms of functions ξ_n of the unperturbed atoms. We then set:

$$f_n = \sum_s f_{ns}\,\xi_s, \qquad V\xi_n = \sum_s V_{ns}\,\xi_s, \qquad -\frac{h}{2\pi i}(\mathfrak{A}\ \text{grad}\ \xi_n) = \sum_s A_{ns}\,\xi_s, \tag{60}$$

with

$$V_{ns} = \int V \, \xi_n \, \eta_s \, dv \qquad \text{and} \qquad A_{ns} = - \frac{h}{2\pi i} \int (\mathfrak{A} \, \text{grad} \, \xi_n) \eta_s \, dv, \tag{61}$$

in which the quantities f_{ns}, V_{ns}, and A_{ns} are functions of time, but not the position coordinates, and η_s is the complex conjugate of ξ_s. If we introduce those expressions into (59) then when one starts with (57) and identifies the coefficients of ξ_s on both sides, that will yield the equation:

$$\frac{h}{2\pi i} \frac{df_{ns}}{dt} = \varepsilon V_{ns} + \frac{\varepsilon}{\mu c} A_{ns}. \tag{62}$$

We would now like to apply these general perturbation equations to the case in which *the perturbing field is static*, and therefore in which all of the considerations of § **3** prove to be true. Since the quantities V_{ns} and A_{ns} include time only in the factor $e^{-\frac{2\pi i}{h}(E_n - E_s) t}$, we can solve equations (62) by the following expressions:

$$f_{ns} = - \frac{\varepsilon V_{ns} + \dfrac{\varepsilon}{\mu c} A_{ns}}{E_n - E_s} \quad (s \neq n), \qquad f_{nn} = \frac{2\pi i}{h} \left(\varepsilon V_{nn} + \frac{\varepsilon}{\mu c} A_{nn} \right) t. \tag{63}$$

We then get:

$$\xi_n + \sigma f_n = \left[1 + \sigma \frac{2\pi i}{h} \left(\varepsilon V_{nn} + \frac{\varepsilon}{\mu c} A_{nn} \right) t \right] \xi_n - \sigma \sum_{s \neq n} \frac{\varepsilon V_{nn} + \dfrac{\varepsilon}{\mu c} A_{nn}}{E_n - E_s} \xi_s,$$

and since the expression is correct only to the first order of magnitude, we can also write this as:

$$\xi_n + \sigma f_n = e^{\frac{2\pi i}{h} \sigma \left(\varepsilon V_{nn} + \frac{\varepsilon}{\mu c} A_{nn} \right) t} \left[\xi_n - \sigma \sum_{s \neq n} \frac{\varepsilon V_{nn} + \dfrac{\varepsilon}{\mu c} A_{nn}}{E_n - E_s} \right]. \tag{64}$$

Since, from (58) and (61), the expression in brackets includes time in the common factor $e^{\frac{2\pi i}{h} E_n t}$, that solution can be considered to be an eigenfunction of the perturbed atom in which only a single frequency appears, in a sense. Moreover, one easily shows that $\xi_n + \sigma f_n$ also fulfills the orthogonality condition.

If we now consider the expression (64) somewhat more closely then we will see that the energy of the state that is denoted by n has increased by:

$$\gamma_n = - \sigma \left(\varepsilon V_{nn} + \frac{\varepsilon}{\mu c} A_{nn} \right). \tag{65}$$

From (61), we can write this as:

$$\gamma_n = -\,\sigma\,\varepsilon \int V\,\xi_n\,\eta_n\,d\upsilon + \sigma\,\frac{\varepsilon}{\mu c}\,\frac{h}{2\pi i}\int (\mathfrak{A}\ \mathrm{grad}\ \xi_n)\eta_n\,d\upsilon,$$

or, from (19) and (66):

$$\gamma_n = \sigma \int V\,\rho_{nn}\,d\upsilon - \frac{\sigma}{c}\int (\mathfrak{A}\ \mathfrak{J}_{nn})\,d\upsilon. \tag{66}$$

This formula for the change in energy of the stationary state has a simple meaning that corresponds completely to **Bohr**'s theory of the perturbation of a periodic system ([1]). As one can see, the first term means the potential energy of the density distribution that is symbolized by ρ_{nn} relative to the perturbing electrostatic field, while the second term is equal to the interaction energy of the magnetic field of the atom with the perturbing magnetic field, taken with the negative sign. In order to make the last point emerge more clearly, we convert the second term of (66) in such a way that we replace the quantities $\mathfrak{J}_{nn}$ with $\dfrac{c}{4\pi}$ rot $\mathfrak{H}_n$, on the grounds of (14), in which $\mathfrak{H}_n$ denotes the magnetic field vector that belongs to $\mathfrak{J}_{nn}$. We then get:

$$-\,\frac{\sigma}{4\pi}\int (\mathfrak{A}\ \mathrm{rot}\ \mathfrak{H}_n)\,d\upsilon,$$

which is equal to:

$$-\,\frac{\sigma}{4\pi}\int (\mathrm{rot}\ \mathfrak{A}\ \mathfrak{H}_n)\,d\upsilon,$$

or equal to:

$$-\,\frac{1}{4\pi}\int (\mathfrak{H}\ \mathfrak{H}_n)\,d\upsilon,$$

in which $\mathfrak{H}$ denotes the field vector that belongs to $\sigma\,\mathfrak{A}$. However, the magnetic energy of our system is:

$$\frac{1}{8\pi}\int (\mathfrak{H} + \mathfrak{H}_n)^2\,d\upsilon = \frac{1}{8\pi}\int \mathfrak{H}^2\,d\upsilon + \frac{1}{4\pi}\int (\mathfrak{H}\ \mathfrak{H}_n)\,d\upsilon + \frac{1}{8\pi}\int \mathfrak{H}_n^2\,d\upsilon,$$

in which the middle term represents precisely the interaction energy, so the assertion is then proved.

In the special case where the perturbing field is homogeneous, we can bring (66) into an especially simple form. If we choose the zero-point of the potential to be the center of atom then we can then set the electrostatic potential σV equal to:

$$\sigma V = -\,(\mathfrak{E}\ \mathfrak{r}),$$

([1]) **N. Bohr**, *loc. cit.*, pp. 123.

in which $\mathfrak{E}$ denotes the electric field vector of the perturbing field, while the vector potential will assume the following form by way of the corresponding convention:

$$\sigma \mathfrak{A} = - \tfrac{1}{2} [\mathfrak{H} \ \mathfrak{r}].$$

We then get:

$$\gamma_n = - \mathfrak{E} \cdot \int \mathfrak{r} \, \rho_{nn} \, dv - \frac{1}{2c} \int [\mathfrak{A} \ \mathfrak{r}] \mathfrak{J}_{nn} \, dv \, ,$$

or, from (34) and (36):

$$\gamma_n = - (\mathfrak{E} \ \mathfrak{D}_n) - (\mathfrak{H} \ \mathfrak{B}_n), \tag{67}$$

in which $\mathfrak{D}_n$ and $\mathfrak{B}_n$ refer to the electric and magnetic moment vectors that belong to the state n . That expression shows that the atom in a stationary state also behaves like a dipole of moment $\mathfrak{D}_n$ (a magnet of moment $\mathfrak{B}_n$, resp.) in terms of energy. For the case considered above of an axially-symmetric atom, when the magnetic field is assumed to be parallel to the axis, formula (67) will lead to the usual expression for the energy under the normal **Zeeman** effect.

In order to now find the influence of the perturbing force field on the radiation that the atom emits, we form the electric moment that is associated with a pair of stationary states:

$$\mathfrak{D}_{n'n''} + \sigma \mathfrak{d}_{n'n''} = - \varepsilon \int \mathfrak{r} \, (\xi_{n'} + \sigma f_{n'})(\eta_{n''} + \sigma g_{n''}) dv \, ,$$

which is definitive of the radiation, from (41). It then follows from (64) that the moment that is provoked by the perturbation is:

$$\sigma \mathfrak{d}_{n'n''} = - \sigma \sum_s \left[\frac{\left(\varepsilon V_{n's} + \dfrac{\varepsilon}{\mu c} A_{n's} \right) \mathfrak{D}_{sn''}}{E_{n'} - E_s} + \frac{\left(\varepsilon \overline{V}_{n''s} + \dfrac{\varepsilon}{\mu c} \overline{A}_{n''s} \right) \mathfrak{D}_{n's}}{E_{n''} - E_s} \right]. \tag{68}$$

In order to explain the latter formula, we consider the case in which the perturbing field is homogeneous and purely electrostatic. From (61), one then has:

$$\sigma \varepsilon V_{ns} = (\mathfrak{E} \ \mathfrak{D}_{ns}), \qquad A_{ns} = 0,$$

such that we will get:

$$\sigma \mathfrak{d}_{n'n''} = - \sum_s \left[\frac{(\mathfrak{E} \ \mathfrak{D}_{n's}) \mathfrak{D}_{sn''}}{E_{n'} - E_s} + \frac{(\mathfrak{E} \ \mathfrak{D}_{sn''}) \mathfrak{D}_{n's}}{E_{n''} - E_s} \right]. \tag{69}$$

That equation gives immediate information of the change in intensity of the radiation that belongs to a well-defined transition process. For example, it gives an expression for the appearance of new combination lines in the series spectra under the influence of an electric field that **Bohr** inferred from the asymptotic form of the correspondence principle. As one sees immediately, the frequencies that appear in (69) will then be sums

and differences of the frequencies of the spectral lines that belong to the unperturbed atom. Finally, that formula will yield the static electric moment that is induced by the electric force for n' equal to n''.

We now turn to the question of how to evaluate wave mechanics on the basis of the correspondence principle in the case of a *time-varying perturbing field*. As an example, we will consider the important case of the scattering of light by an atom more closely here. For a monochromatic light wave, we can set the scalar potential equal to zero, such that the electric vector of the light wave will be given by:

$$\mathfrak{E} = - \frac{\sigma}{c} \frac{\partial \mathfrak{A}}{\partial t}, \tag{70}$$

in which $\mathfrak{A}$ must be chosen in such a way that its temporal mean vanishes at each point. The expression for $\mathfrak{E}$ obviously has order of magnitude $\sigma |\mathfrak{A}| / \lambda$ then, where λ means the wavelength of light. Now, that might be very large in comparison to atomic dimensions, as in ordinary experiments. Since the electric field strength is equal to the magnetic field strength, $\sigma |\mathfrak{A}|$ will have order of magnitude of $\lambda |\mathfrak{H}|$, and as a result of the aforementioned assumption, we can neglect the variation of $\mathfrak{A}$ with position in space and simply introduce its value at the center of the atom into equation (62). If we then set:

$$\mathfrak{A} = \mathfrak{C}\, e^{2\pi i v t} + \overline{\mathfrak{C}}\, e^{-2\pi i v t},$$

in which $\mathfrak{C}$ and $\overline{\mathfrak{C}}$ denote two complex conjugate vectors, then we will have:

$$\mathfrak{E} = - \sigma \frac{2\pi i v}{c} (\mathfrak{C}\, e^{2\pi i v t} - \overline{\mathfrak{C}}\, e^{-2\pi i v t}). \tag{72}$$

After a simple calculation, we will get:

$$\frac{\varepsilon}{\mu c} A_{ns} = \frac{1}{c} \int (\mathfrak{A}\, \mathfrak{J}_{ns})\, dv = \frac{1}{c} \int (\mathfrak{C}\, e^{2\pi i v t} + \overline{\mathfrak{C}}\, e^{-2\pi i v t})\, \overset{o}{\mathfrak{J}}_{ns}\, e^{-\frac{2\pi i}{h}(E_n - E_s)t}\, dv,$$

in which $\overset{o}{\mathfrak{J}}_{ns}$ denotes the amplitude of $\mathfrak{J}_{ns}$. With some conversions, this will imply that:

$$\frac{\varepsilon}{\mu c} A_{ns} = - 2\pi i \frac{E_n - E_s}{c h} \int \mathfrak{r}\, (\mathfrak{C}\, e^{2\pi i v t} + \overline{\mathfrak{C}}\, e^{-2\pi i v t})\, \overset{o}{\mathfrak{J}}_{ns}\, e^{-\frac{2\pi i}{h}(E_n - E_s)t}\, dv$$

or

$$\frac{\varepsilon}{\mu c} A_{ns} = - 2\pi i \frac{v_{ns}}{c} [(\mathfrak{C}\, \overset{o}{\mathfrak{D}}_{ns})\, e^{-2\pi i (v_{ns} - v)t} + (\overline{\mathfrak{C}}\, \overset{o}{\mathfrak{D}}_{ns})\, e^{-2\pi i (v_{ns} + v)t}], \tag{73}$$

in which we have denoted the quantities $(E_n - E_s) / h$ by ν_{ns} . From (62), our perturbation equations will now read:

$$\frac{h}{2\pi i}\frac{df_{ns}}{dt} = \frac{\varepsilon}{\mu c}A_{ns}$$
(74)

If we exclude the case of resonance and assume that ν is different from all of the frequencies ν_{ns} then we can set:

$$f_{ns} = \frac{2\pi i}{hc}\nu_{ns}\left(\frac{\mathfrak{C}\, e^{-2\pi i\,(\nu_{ns}-\nu)t}}{\nu_{ns}-\nu} + \frac{\overline{\mathfrak{C}}\, e^{-2\pi i\,(\nu_{ns}+\nu)t}}{\nu_{ns}+\nu}\right)\overset{\mathrm{o}}{\mathfrak{D}}_{ns} ,$$
(75)

and as a result:

$$\left.\begin{aligned}
f_n &= \frac{2\pi i}{hc}\sum_s\nu_{ns}\left(\frac{\mathfrak{C}\, e^{-2\pi i\,(\nu_{ns}-\nu)t}}{\nu_{ns}-\nu} + \frac{\overline{\mathfrak{C}}\, e^{-2\pi i\,(\nu_{ns}+\nu)t}}{\nu_{ns}+\nu}\right)\overset{\mathrm{o}}{\mathfrak{D}}_{ns}\,\xi_s ,\\[2ex]
g_n &= -\frac{2\pi i}{hc}\sum_s\nu_{ns}\left(\frac{\overline{\mathfrak{C}}\, e^{-2\pi i\,(\nu_{ns}-\nu)t}}{\nu_{ns}-\nu} + \frac{\mathfrak{C}\, e^{-2\pi i\,(\nu_{ns}+\nu)t}}{\nu_{ns}+\nu}\right)\overset{\mathrm{o}}{\mathfrak{D}}_{ns}\,\eta_s .
\end{aligned}\right\}$$
(76)

In order to evaluate the result of the perturbation calculation so as to ascertain the external effects of the atom, similarly to what we did in § **3**, we would now like to exhibit the electric densities that are associated with a well-defined stationary state of the atom. If we denote that density for the state n by $\rho_n + \sigma\, P_n$, in which ρ_n means the corresponding density in the absence of the perturbing radiation, then we will come to the following expression for the quantity P_n by an argument that is similar to the one in § **3** :

$$P_n = P_{nn} + \sum_m^{E_m < E_n \pm h\nu} (P_{nm} + P_{mn}) .$$
(77)

In this:

$$P_{nm} = \xi_n\, g_m + \eta_m\, f_n ,$$
(78)

and in the summation, one includes only those terms in the quantities $P_{nm} + P_{mn}$ for which the condition $E_m < E_n + h\nu$ ($E_m < E_n - h\nu$, resp.) is fulfilled. The expression for P_n , which satisfies the requirement that the total charge should be equal to $-\varepsilon$ (when ξ_n and η_n are normalized eigenfunction of the unperturbed system), represents the analogue of part of the Fourier development of the motion of the electron that is perturbed by the light wave. In fact, we see from (78) that the term P_{nn} represents a harmonic oscillation whose frequency coincides with the frequency of the incident light and thus corresponds to a coherent scattered ray of the kind that one assumes in order to explain ordinary dispersion phenomena. As one sees, the remaining terms in P_n represent harmonic oscillation terms whose frequencies are sums or differences of spectral frequencies with the frequency of the incident light. Those terms correspond to the incoherent scattered radiation that was proposed by **Smekal** on the grounds of **Einstein**'s light quantum hypothesis and by **Kramers** and **Heisenberg** on the basis of **Bohr**'s correspondence principle. From quantum theory, radiation with frequencies $\nu_{nm} + \nu$ and $\nu_{nm} - \nu$ that appear P_{nm} can only be coupled with transitions from one of the two stationary states that are denoted by the

symbols n and m to the other one that is induced by the light radiation. In that way, the question of whether the transition starts from the state n or the state m will be dictated by the sign of the quantity $v_{nm} + v$ [$= (E_n - E_m + h\nu) / h$] or $v_{nm} - v$ [$= (E_n - E_m - h\nu) / h$], resp., and indeed the state that is denoted by n will be the initial state when the quantity in question is positive and conversely. In fact, as we have reasoned in the presentation of the expression for P_n, that arrangement will correspond to the argument that **Smekal** ([1]) developed on the basis of the light quantum hypothesis.

If we would like to compare the total expression for the density with the imagined system in which each stationary state is represented by an electron then we would collide with a certain complication that originates in the fact that the different quantities P_{nn} cancel each other in the summation over all n. That complication, which, on first glance, obstructs a single-valued definition of the quantities P_n, since quantities enter into each P_{nn} that refer to all possible states, is physically connected with the fact we should not expect any actual dispersion from a system of the type that we have in mind. The total absorption will then vanish, since, according to **Einstein**, transitions between a given pair of stationary states that is induced by the radiation will happen at the same rate. Here, we are obviously dealing with a case that is analogous to the degenerate one. In fact, we can regard the equality of the frequencies of ξ_n and η_n as a kind of degeneracy, and in that way the problem can be surmounted that we do not set the combination frequency v_{nn} of ξ_n with η_n equal to zero from the outset, which would make the various terms P_{nn} separate from each other.

From the foregoing, we will now get information about the presence of atoms in the state n by the radiation that is stimulated by incident light when we consider the quantity:

$$\sigma \mathfrak{d}_n = \sigma \int \mathfrak{r}\, P_n \, dv, \tag{79}$$

which amounts to the change in the electric moment of the state in question that is due to the radiation. From (77), we can write the quantity $\mathfrak{d}_n$ as:

$$\mathfrak{d}_n = \mathfrak{d}_{nm} + \sum_{m}^{E_m < E_m \pm h\nu} (\mathfrak{d}_{nm} + \mathfrak{d}_{mn}), \tag{80}$$

in which:

$$\mathfrak{d}_{nm} = \int \mathfrak{r}\, P_{nm}\, dv = \sum_{s} (f_{ns}\, \mathfrak{D}_{sm} + g_{ms}\, \mathfrak{D}_{ns}), \tag{81}$$

or, from (77):

$$\mathfrak{d}_{nm} = \frac{2\pi i}{hc} \sum_{s} \left\{ \left[v_{ns}\, \frac{(\mathfrak{C}\,\overset{o}{\mathfrak{D}}_{ns})\,\overset{o}{\mathfrak{D}}_{sm}}{v_{ns} - v} - v_{ms}\, \frac{(\mathfrak{C}\,\overset{o}{\mathfrak{D}}_{sm})\,\overset{o}{\mathfrak{D}}_{ns}}{v_{ms} + v} \right] e^{-2\pi i\,(v_{nm} - v)\,t} \right.$$

$$\left. + \left[v_{ns}\, \frac{(\overline{\mathfrak{C}}\,\overset{o}{\mathfrak{D}}_{ns})\,\overset{o}{\mathfrak{D}}_{sm}}{v_{ns} + v} - v_{ms}\, \frac{(\overline{\mathfrak{C}}\,\overset{o}{\mathfrak{D}}_{sm})\,\overset{o}{\mathfrak{D}}_{ns}}{v_{ms} - v} \right] e^{-2\pi i\,(v_{nm} + v)\,t} \right\}. \tag{82}$$

([1]) **A. Smekal**, Nature **11** (1923), 873.

As one easily shows, the term $\mathfrak{d}_{nm}$, which is responsible for the dispersion, is identical to the formula for the corresponding part of the scattered electric moment of a radiating atom that **Kramers** ([1]) derived on the basis of an ingenious application of the correspondence principle to the result of a classical-mechanical perturbation calculation, which was a formula that also remained true in matrix mechanics. The derivation above is formally close to the wave-mechanical derivation that **Schrödinger** ([2]) gave. Meanwhile, the deep-rooted difference between the more-classical picture that **Schrödinger** presented and the viewpoint that is assumed here on the basis of the correspondence principle will become clearer when we consider those terms in (82) that belong to incoherent scattered radiation. In fact, the expression for $\sigma \, \mathfrak{d}_n$ coincides with the complete formula that **Kramers** and **Heisenberg** ([3]) gave for the scattered electric moment of an atom, such that the objections that **Schrödinger** raised against the reality of the incoherent scattered radiation will drop out from our presentation.

Along those lines, let it be remarked that just as formula (82) yields the induced static electric moment of the atom in the limit $\nu = 0$ when $n = m$, it will also go over to the expression (72) for the new combination of lines that is excited by an external electric field in that limit when $n \neq m$, which is a fact that **Pauli** ([4]) has already employed in order to calculate the intensity of such combination lines before the construction of a rational quantum mechanics.

§ 6. Interaction of radiation with free electrons. – The examples that were considered in the foregoing paragraphs are characterized by the fact that the force field in which the electron moves has a significant influence. In the language of wave mechanics, that says that the wave function can have noticeably values only at a distance from a certain spatial point (e.g., the atomic nucleus) that is small compared to the light wavelengths that come under consideration. In contrast to such a "bound" electron, we shall now consider an example in the form of the **Compton** effect, in which one is dealing with a "free" electron. Here, we will get a picture that corresponds to the experimental conditions when we assume that the electron is available in a force-free region whose dimensions are large in comparison to the wave length of light, and in which the influence of the magnitude and form of the region on the light that the electron emits is therefore vanishingly small. The wave equation (7) assumes the simple form here:

$$-\frac{h^2}{4\pi^2}\Box\varphi + \mu^2 c^2 \, \varphi = 0. \tag{83}$$

When the volume of the region is equal to υ, we can solve this and the corresponding equation for ψ that follows from (7a) by the following pair of wave functions:

([1]) **H. A. Kramers**, Nature **113** (1924), 673; **114** (1924), 310.
([2]) **E. Schrödinger**, Ann. Phys. (Leipzig) **81** (1926), 109.
([3]) **H. A. Kramers** and **W. Heisenberg**, Zeit. Phys. **31** (1925), 681.
([4]) **W. Pauli**, Det. Kgl. Danske Videnskabernes Selskab. Math.fys. Med. **7** (1925), 3.

$$\varphi = \frac{1}{\sqrt{\upsilon}}\,e^{\frac{2\pi i}{h}[-Et+(\mathfrak{M}\mathfrak{r})]}, \qquad \psi = \frac{1}{\sqrt{\upsilon}}\,e^{-\frac{2\pi i}{h}[-Et+(\mathfrak{M}\mathfrak{r})]}, \tag{84}$$

in which $\mathfrak{r}$ means the radius vector from a fixed point that lies in the region and the point considered. Those expressions, in which E represents the value of the energy, and $\mathfrak{M}$ represents the impulse vector in a well-defined state of the electron, correspond to the **de Broglie** waves for a free electron. On the grounds of (83), the relationship:

$$\mathfrak{M}^2 - E^2/c^2 + \mu^2 c^2 = 0 \tag{85}$$

exists between E and $\mathfrak{M}$, which coincides with the relationship between energy and impulse of a free electron in ordinary relativistic mechanics.

A plane, monochromatic light wave now falls on the electron, which we would like to describe by the following Ansatz for the potentials:

$$\sigma\mathfrak{A} = \sigma\left[\mathfrak{C}e^{-2\pi i\nu\left(t-\frac{(\mathfrak{n}\mathfrak{r})}{c}\right)} + \overline{\mathfrak{C}}e^{-2\pi i\nu\left(t-\frac{(\mathfrak{n}\mathfrak{r})}{c}\right)}\right], \qquad \sigma V = 0, \tag{86}$$

in which σ again denotes a small constant parameter, while $\mathfrak{n}$ means the unit vector that defines the direction of radiation.

In order to consider the effect of the light wave on the electron, we would like to content ourselves with the first approximation in σ. If we denote the solution to equation (7) that belongs to a well-defined state by $\varphi + \sigma f$ then we will get the following perturbation equation for f:

$$-\frac{h^2}{4\pi^2}\square\varphi + \mu^2 c^2\,\varphi = -\,2\frac{h}{2\pi i}\frac{\varepsilon}{c}\left[\mathfrak{C}e^{-2\pi i\nu\left(t-\frac{(\mathfrak{n}\mathfrak{r})}{c}\right)} + \overline{\mathfrak{C}}e^{-2\pi i\nu\left(t-\frac{(\mathfrak{n}\mathfrak{r})}{c}\right)}\right]\text{grad }\varphi. \tag{87}$$

From (84), we can solve that equation with the following expression:

$$f = \frac{\varepsilon\,\mathfrak{M}}{\sqrt{\upsilon}\,h\nu\,[E/c-(\mathfrak{M}\mathfrak{n})]}\left[\mathfrak{C}e^{-\frac{2\pi i}{h}\left\{-(E-h\nu)t+\left(\mathfrak{M}-\mathfrak{n}\frac{h\nu}{c}\right)\mathfrak{r}\right\}} + \overline{\mathfrak{C}}e^{\frac{2\pi i}{h}\left\{-(E+h\nu)t+\left(\mathfrak{M}+\mathfrak{n}\frac{h\nu}{c}\right)\mathfrak{r}\right\}}\right]. \tag{88}$$

If we denote the corresponding solution of (7a) by $\psi + \sigma g$ then we can write g as:

$$g = \frac{1}{\sqrt{\upsilon}\,h\nu}\frac{\varepsilon\,\mathfrak{M}}{[E/c-(\mathfrak{M}\mathfrak{n})]}\left[\overline{\mathfrak{C}}e^{\frac{2\pi i}{h}\left\{-(E-h\nu)t+\left(\mathfrak{M}-\mathfrak{n}\frac{h\nu}{c}\right)\mathfrak{r}\right\}} + \mathfrak{C}e^{-\frac{2\pi i}{h}\left\{-(E+h\nu)t+\left(\mathfrak{M}+\mathfrak{n}\frac{h\nu}{c}\right)\mathfrak{r}\right\}}\right]. \tag{88a}$$

In order to find the scattered radiation, we can now proceed in a manner that is similar to what one does for light scattered by an atom and exhibit a general expression for the density that belongs to a well-defined initial state of the system. However, we would not like to go into the quantitative side of the intensity question here ([1]), but only examine the dependency of the frequency on the direction of observation, which corresponds formally to the selection rule for the appearance of spectral lines. We can then be content to consider an expression for density that (corresponding to one of the quantities ρ_{nm}) emerges from the general expression when we introduce a well-defined solution $\varphi + \sigma f$ of the wave equation (7) that is given by (84) and (88) in place of φ and a solution $\psi + \sigma g$ that belong to another state and is given by (84) and (88a) in place of ψ. We thus consider the following expression:

$$\rho = -\frac{\varepsilon}{2\mu c^2}\left\{-\frac{h}{2\pi i}\left[\psi'\frac{\partial\varphi}{\partial t} - \varphi\frac{\partial\psi'}{\partial t} + \sigma\left(\psi'\frac{\partial f}{\partial t} - f\frac{\partial\psi'}{\partial t} + g'\frac{\partial\varphi}{\partial t} - \varphi\frac{\partial g'}{\partial t}\right)\right]\right\}. \qquad (89)$$

It is good to point out that this expression represents two transition probabilities, each of which starts from the state in question. In fact, the two frequencies $E - E' + h\nu$ and $E' - E + h\nu$ enter into them, and from quantum theory, the former corresponds to a transition from the state with energy E, while the latter corresponds to a transition from the state with energy E'. When choose the first state to be the initial state, we would only like to bring terms with the frequency $E - E' + h\nu$ into consideration.

As one sees, the expressions (89) are composed of terms of the form:

$$a\, e^{\,2\pi i\,[\omega t + (\mathfrak{s}\,\mathfrak{r})]},$$

in which a and ω are constants, and $\mathfrak{s}$ denotes a constant vector. When we form the corresponding expressions for the potentials, we would like to assume that the dimensions of the region in which the electron exists are small in comparison to the distance r between the fixed point and the reference point. From (38), the potentials have the following form:

$$\frac{a}{r}e^{2\pi i\omega(t-r/c)}\int e^{2\pi i\left(\mathfrak{s}+\frac{\omega\mathfrak{n}'}{c}\right)\mathfrak{r}}\,dv,$$

in which the unit vector $\mathfrak{n}'$ points in the direction of observation. Obviously, for given $\mathfrak{s}$ and ω, that integral will yield appreciable contributions to the field only for those directions for which the exponent that contains $\mathfrak{r}$ lies very close to zero, so:

$$\mathfrak{n}' = -\frac{c}{\omega}\,\mathfrak{s}. \qquad (90)$$

([1]) See **P. A. M. Dirac**, Proc. Roy. Soc. (A) **3** (1926), 405, in which a thorough treatment of that question is given that is based in matrix theory. Cf., also **Breit**, Phys. Rev. **27** (1926), 362, in which the intensity problem is treated on the grounds of the correspondence principle in conjunction with the classical theory of electrons.

Since $\mathfrak{n}'$ means a unit vector, that condition can be fulfilled only when:

$$c^2 \, \mathfrak{s}^2 = \omega^2 . \tag{91}$$

From (89), we will first address those terms that belong to the unperturbed electron, and in which:

$$\omega = \frac{E - E'}{h} , \qquad \mathfrak{s} = -\frac{\mathfrak{M} - \mathfrak{M}'}{h} .$$

From (91), one then have:

$$c^2 \, (\mathfrak{M} - \mathfrak{M}')^2 = (E - E')^2,$$

here, and on the grounds (85), that condition will be fulfilled only when:

$$\mathfrak{M} = \mathfrak{M}', \qquad E = E',$$

which then express precisely the idea that a free electron cannot radiate. For the radiation terms, we get:

$$\omega = \frac{E - E'}{h} + \nu, \qquad \mathfrak{s} = -\frac{1}{h}\left(\mathfrak{M} - \mathfrak{M}' + \mathfrak{n}\frac{h\nu}{c} \right),$$

and when we denote the frequency ω of the scattered light by ν', it will then follow that:

$$\mathfrak{M} + \mathfrak{n}\frac{h\nu}{c} = \mathfrak{M}' + \mathfrak{n}'\frac{h\nu'}{c}, \qquad E + \nu = E' + h\nu' \tag{92}$$

are exactly the well-known conditions that **Compton** and **Debye** gave for the relationship between the frequencies and directions of the primary and secondary light under the **Compton** effect.

As one sees, the presentation of the **Compton** effect that was sketched out in the foregoing pages has a great formal similarity to the theory of lattice reflection, in which the combination of two **de Broglie** waves will lead to a charge distribution on the lattice that will selectively reflect incident light. With that presentation, we arrived at an interpretation in terms of the correspondence principle of the aforementioned peculiar coupling of the directions of the incident and scattered light and the electron that is liberated by the photoelectric effect by means of **Einstein**'s light quantum hypothesis, which was verified experimentally by **Geiger** and **Bothe** and **Compton**, and which is based upon assumptions that are similar to the description of the ordinary spectrum that is emitted by an atom that uses the correspondence principle.

§ **7. Five-dimensional wave mechanics.** – In two articles that appeared recently, the author [1] attempted to connect the formalism of quantum theory with the five-

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[1] **O. Klein**, Zeit. Phys. **37** (1926), 895; Nature **118** (1926), 516.

dimensional generalization of **Einstein**'s theory of relativity that **Kaluza** had proposed, and recently **Fock** ([1]) has also attempted to express similar endeavors. Some remarks about that five-dimensional wave mechanics shall follow here in connection with the questions that were touched upon in the present treatise, and in that way, we will show how it is possible to shed some light upon the appearance of two different wave functions φ and ψ in the treatment of wave mechanics from that standpoint using the correspondence principle.

   In order to define the basis for five-dimensional wave mechanics, we shall start from the fact that the **Hamilton-Jacobi** differential equation (2) for the relativistic motion of electrons has the form of the characteristic equation for a five-dimensional wave equation ([2]). In fact, it will emerge from the following homogeneous quadratic equation:

$$\left(\frac{\partial \Omega}{\partial x} - \mathfrak{A}_x \frac{\partial \Omega}{\partial x_0}\right)^2 + \left(\frac{\partial \Omega}{\partial y} - \mathfrak{A}_y \frac{\partial \Omega}{\partial x_0}\right)^2 + \left(\frac{\partial \Omega}{\partial z} - \mathfrak{A}_x \frac{\partial \Omega}{\partial x_0}\right)^2 - \frac{1}{c^2}\left(\frac{\partial \Omega}{\partial t} + c\,V \frac{\partial \Omega}{\partial x_0}\right)^2 + \frac{\mu^2 c^4}{\varepsilon^2}\left(\frac{\partial \Omega}{\partial x_0}\right)^2$$

$$= 0, \tag{93}$$

in which x_0 means the coordinate of the fifth dimension, by the Ansatz:

$$\Omega = -\frac{\varepsilon}{c} x_0 + S\,(x,\,y,\,z,\,t)\;. \tag{94}$$

The simplest wave equation that belongs to the characteristic equation (93) reads:

$$\square U - 2\left(\mathfrak{A}\,\frac{\partial\,\mathrm{grad}\,U}{\partial x_0}\right) - 2\frac{V}{c}\frac{\partial^2 U}{\partial t\,\partial x_0} + \frac{c^2}{\varepsilon^2}\left[\mu^2 c^2 + \frac{\varepsilon^2}{c^2}(\mathfrak{A}^2 - V^2)\right]\frac{\partial^2 U}{\partial x_0^2} = 0. \tag{95}$$

From the known properties of characteristics, we know from the outset that the wave equation (95) will be replaced by precisely the **Hamilton-Jacobi** equation (93) in the limit of geometrical optics.

   The coefficients of the wave equation (95) are independent of the quantity x_0 . For that reason, we can summarize the general solution to that equation as particular solutions with the form:

$$\varphi\,e^{-i\omega x_0} + \psi\,e^{i\omega x_0}\,,$$

in which φ and ψ do not include x_0 , and ω denotes an arbitrary constant. If we now try to evaluate equation (95) for quantum theory then from the arguments in § **1** and (94), we would expect that the solutions that come under consideration in the quantum problem could be represented approximately by:

([1])   **V. Fock**, Zeit. Phys. **39** (1926), 226.
([2])   For the meaning of a characteristic equation, see, e.g., **J. Hadamard**, *Leçons sur la propagation des ondes et les équations de l'hydrodynamique*, Paris, 1903.

$$U = e^{\pm \frac{2\pi i}{h} \Omega} \qquad (96)$$

in the vicinity of the "geometrical optics" limit, in which Ω is a solution of (93). From (94), the solutions in question are harmonic in x_0 with periods $h\,c\,/\,\varepsilon$ in this case. Since that property, in this form, has nothing to do with the limit, we will be led to generally assume that:

$$\omega = \frac{2\pi \varepsilon}{hc}, \qquad (97)$$

and thus to set:

$$U = \varphi\, e^{-\frac{2\pi i\,\varepsilon}{h\,c} x_0} + \psi\, e^{\frac{2\pi i\,\varepsilon}{h\,c} x_0} \qquad (98)$$

in order to get the most general solution of (95) that comes under consideration. If we introduce that expression into (95) then that equation will split into two equations, one for φ that agrees with (7), and one for ψ that agrees with (7a). The fact that only those solutions of these equations should be considered in this that correspond to a positive energy means that in the five-dimensional representation of the motion of electrons, only those waves should come under consideration that have a well-defined direction of propagation relative to the fifth dimension.

The periodicity in x_0 that is expressed in (98), by which **Planck**'s constant is introduced into the wave equation (95), admits a simple geometric interpretation by way of the assumption that the five-dimensional space is closed in the direction of x_0 , which makes the solution (98) correspond to the ground state oscillation in x_0 . That is closely related to the problem of relating the non-appearance of a fifth coordinate in our ordinary physical equations to this picture, and of considering those equations to be mean values over the fifth coordinate of more general equations that include the fifth coordinate. Correspondingly, we must first take the mean value over the fifth dimension in the formation of expressions that have degree two in U (such as the electric density) that we have to put in the right-hand side of the ordinary field equations in the corresponding representation of behavior of an electron. As we would like to show in the simplest example of the function $U^{\,2}$, in that way, we will come to exactly those expressions in which the two functions φ and ψ enter bilinearly, as they do in (18). Namely, from (98), one has:

$$U^{\,2} = \varphi^2\, e^{-\frac{4\pi i\,\varepsilon}{h\,c} x_0} + 2\varphi\psi + \psi^2 e^{\frac{4\pi i\,\varepsilon}{h\,c} x_0} ,$$

such that taking the mean over x_0 will, in fact, yield:

$$\overline{U^{\,2}} = 2\varphi\,\psi . \qquad (99)$$

We can see how the "observable" physical quantities that enter into the usual physical description contain exactly the products of the conjugate wave functions, and the characteristic duality of quantum phenomena then finds an expression in that fact. The assumption of such a closedness in the fifth dimension will then not only imply the possibility of introducing **Planck**'s constant into the theory in a way that is naturally

connected with the world-view of the theory of relativity, but it will also lead us immediately to the four-dimensional corresponding representation of the one-electron problem on the basis of wave mechanics.

Copenhagen, Univers. Institut for teoretisk Fysik, 4 Dec. 1926.

Added in correction.

After the present article was submitted, **Gordon**'s thorough treatment of the **Compton** effect based upon **Schrödinger**'s theory [Zeit. Phys. **40** (1926), 117] was brought to my attention, and in it, he also arrived at the relativistic expressions for the electric density and current vector that were developed in § **2**. In conjunction with it, **Schrödinger** then gave a simple geometric interpretation of the wave-mechanical theory of the **Compton** effect in a treatise that just appeared [Ann. Phys. (Leipzig) **82** (1927), 257], and which was very close to the arguments that were present here in § **6**, without relating it to the general questions of quantum theory, however. The latter is also true for **Schrödinger**'s simultaneously-appearing treatise [Ann. Phys. (Leipzig) **82** (1927), 265] on the wave-mechanical energy-impulse principle, in which questions were addressed that were similar to the work of the author that was announced in the introduction.

I would also like to take this opportunity to point out the fact that **Epstein** [Proc. Nat. Acad. **12** (1926), 634] has treated the normal **Zeeman** effect in a way that is similar to what was done above in § **4**, and that **Fermi** [Nature, 18 Dec. 1926] has published a calculation of the magnetic moment of a centrally-symmetric atom that is found in a magnetic field that is close to the calculations in § **4**.

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