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On Dirac's theory of the electron

II. Algebraic identities that contain differential quotients in Dirac's theory of the electron.

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Introduction and summary

The electromagnetic potentials are *real* in the **Dirac** equation for an electron in an external field. When one goes to complex conjugates, one will then obtain a second equation with the same potentials. When one, in turn, multiplies the Dirac equation with the 16 matrices of the total hypercomplex system of **Dirac** numbers – viz., I, α^1, α^2, α^3, α^4, $\alpha^{[23]}$, $\alpha^{[31]}$, $\alpha^{[12]}$, …, $\alpha^1\alpha^2\alpha^3\alpha^4$ – *one after the other*, and always takes the complex-conjugate equations, one will obtain 32 equations, from which one can eliminate the four potentials. The remaining 28 equations, which might go by the name of *reality relations* here, are differential relations that couple the quantities that arise in **Dirac**'s theory with each other. *They always exist, regardless of how the potentials might read in special cases.*

In this primitive form, the derivation will contain the 28 reality relations, but in addition to the interpretable quantities and their derivatives, there will also be *uninterpretable* quantities (Part III, § 1). The interpretable quantities might have the form, e.g., $(\psi^*, O\psi)$ or:

$$\frac{\partial}{\partial x_k}(\psi^*, O\psi) = \left(\frac{\partial \psi^*}{\partial x_k}, O\psi\right) + \left(\psi^*, O\frac{\partial \psi}{\partial x_k}\right),$$

while the uninterpretable ones might have the form, e.g.:

$$\left(\frac{\partial \psi^*}{\partial x_k}, O\psi\right) - \left(\psi^*, O\frac{\partial \psi}{\partial x_k}\right),$$

in which O means any operator from the **Dirac** matrix ring.

The objective of Part II is this: to produce the *mathematical tools* for the main problem of this paper – namely, the elimination of the uninterpretable quantities from the reality relations, in order to convert them from their original, uninterpretable form into an interpretable one (Part III, § 2).

Correspondingly, Part II will address the quantities $\left(\frac{\partial \psi^*}{\partial x_k}, O\psi\right)$ and $\left(\psi^*, O\frac{\partial \psi}{\partial x_k}\right)$. We refer to forms of those kinds as having been "backward" ("forward," resp.) differentiated. The uninterpretable quantities are combined in them; in order to convert them into interpretable ones, one must then consider these backward ands forward-differentiated quantities and express their *differences* in terms of *sums* of forward and backward-differentiated quantities. That problem will be solved in Part II.

Therefore, in § 1, 256 algebraic identities between the backward-differentiated quantities will be derived with the help of the first bilinear equation [I, eq. (4)], in which the two operators P and Q that enter into eq. (*) will run through the 16 matrices of the ring independently of each other. By partial application of the vector notation, they can be condensed into a collection of formulas (1)-(52). In § 2, the same process will be carried out with the forward-differentiated quantities, and a prescription will be given by which one can go from each formula for backward differentiation to a corresponding formula for forward differentiation. In § 3, it will be shown that the application of the basic equation in § 2 will imply nothing that was not obtained in the same way from the basic equation in § 1, as long as one only makes use of the symmetry properties of the matrix B. § 4 contains a different kind of application of the basic equation in § 1: The equations for the forward and backward-differentiated quantities that were presented in §§ 1 and 2 contain undifferentiated quantities as their coefficients; in § 4, by contrast, the basic equation in § 1 will be applied in such a way that only differentiated quantities will appear. Where the equations of §§ 1 and 2 are then linear in the backward (forward, resp.) differentiated quantities, those of § 4 will be bilinear (quadratic, resp.). In §§ 5-9, the identities that were presented in §§ 1-4 will be solved for the backward (forward, resp.) differentiated quantities. In § 5, one finds the solutions for ${}^k s_0$, ${}^k\mathfrak{s}$, ${}^k\hat{s}_0$, ${}^k\hat{\mathfrak{s}}$, while in § 6, one will find the solutions for s_0^k, $\mathfrak{s}^k$, $\hat{s}_0^k$, $\hat{\mathfrak{s}}^k$, and in § 8, one will the solutions for ${}^k\mathfrak{M}$, ${}^k\hat{\mathfrak{M}}$, $\mathfrak{M}^k$, $\hat{\mathfrak{M}}^k$. A very remarkable four-dimensional null vector will appear in these solutions – which one can also write as a three-dimensional complex unit vector $\mathfrak{q}$ ($\mathfrak{p}$, resp.) – and two initially-unknown parameters ξ and η. They will be explained in §§ 7 and 9. After those extensions, the solutions will be complete, and the reduction of the uninterpretable quantities to the interpretable ones will be carried out in § 10, to the extent that it is possible: viz., all uninterpretable quantities can be reduced to *a single* one [eqs. (87)-(93)].

By suitable combinations of differentiated and undifferentiated quantities under the reduction of the uninterpretable quantities, one will then arrive, much later, at nine identities, as in the case of the undifferentiated quantities. Seven of them will be independent then; e.g., Ω, $\mathfrak{s}$, $\hat{\mathfrak{s}}$. Here, there is only a single one; e.g., $i\,({}^k\Omega - \Omega^k)$.

At the conclusion, in § 11, one will find a collection of formulas for the conversion of different combinations of uninterpretable quantities into interpretable ones.

For the meaning of all the symbols that occur, cf., I, § 1.

§ 1. Application of the first bilinear equation to backward differentiation.

In order to give an application for the bilinear equation [I, (4)]:

$$\sum_{k=1}^{4} \gamma^k_{\rho\sigma}\gamma^k_{\bar\rho\bar\sigma} = \gamma^{[123]}_{\bar\rho\sigma}\gamma^{[123]}_{\rho\bar\sigma} - \gamma^{[123]}_{\rho\sigma}\gamma^{[123]}_{\bar\rho\bar\sigma} + \delta_{\bar\rho\sigma}\delta_{\rho\bar\sigma} - \gamma^5_{\bar\rho\sigma}\gamma^5_{\rho\bar\sigma} - \gamma^4_{\bar\rho\sigma}\gamma^4_{\rho\bar\sigma}$$

that was mentioned in I, § 2 to "backward" differentiation, (left) multiply it by $P_{\rho\bar\rho}Q_{\bar\rho\bar{\bar\rho}}$, in which P and Q might be two initially-arbitrary four-rowed matrices, and sum over $\bar\rho$ and $\bar{\bar\rho}$. One will then obtain:

$$\sum_{k=1}^{3} (P\gamma^k)_{\rho\sigma}(Q\gamma^k)_{\bar\rho\bar\sigma} + (P\gamma^{[123]})_{\rho\sigma}(Q\gamma^{[123]})_{\bar\rho\bar\sigma}$$
$$= P_{\rho\bar\sigma}Q_{\bar\rho\sigma} - (P\gamma^5)_{\rho\bar\sigma}(Q\gamma^5)_{\bar\rho\sigma} - (P\gamma^4)_{\rho\bar\sigma}(Q\gamma^4)_{\bar\rho\sigma} + (P\gamma^{[123]})_{\rho\bar\sigma}(Q\gamma^{[123]})_{\bar\rho\sigma}.$$

If φ_ρ, $\xi_{\bar\rho}$, χ_σ, $\psi_{\bar\sigma}$ mean a set of four functions then multiplying those functions and summing over the indices ρ, $\bar\rho$, σ, $\bar\sigma$ = 1, …, 4 will yield a bilinear equation with inner products:

$$\sum_{k=1}^{3} (\varphi, P\gamma^k\chi)(\xi, Q\gamma^k\psi) + (\varphi, P\gamma^{[123]}\chi)(\xi, Q\gamma^{[123]}\psi)$$
$$= (\varphi, P\psi)(\xi, Q\chi) - (\varphi, P\gamma^5\psi)(\xi, Q\,\gamma^5\chi) - (\varphi, P\,\gamma^4\psi)(\xi, Q\,\gamma^4\chi)$$
$$+ (\varphi, P\gamma^{[123]}\psi)(\xi, Q\,\gamma^{[123]}\chi).$$

If the four functions that are represented by χ are the same as the ones that are represented by ψ then the terms that contain $\gamma^{[123]}$ explicitly will drop away, and what will remain is an equation with a simple form:

$$\sum_{j=1}^{5} (\varphi, P\gamma^j\psi)(\xi, Q\gamma^j\psi) = (\varphi, P\psi)\,(\xi, Q\psi).$$

Ultimately, the substitutions $\varphi = \dfrac{\partial\psi^+}{\partial x_k}$ and $\xi = \psi^+$ will lead to the form:

$$(*) \qquad \boxed{\sum_{j=1}^{5}\left(\frac{\partial\psi^+}{\partial x_k}, P\gamma^j\psi\right)(\psi^+, Q\gamma^j\psi) = \left(\frac{\partial\psi^+}{\partial x_k}, P\psi\right)(\psi^+, Q\psi),}$$

which will be suitable for the applications to backward differentiation.

If one now lets P and Q run through the 16 matrices of the matrix ring of Dirac matrices then that will produce 256 (distinct) linear equations for the 16 inner products

${}^k\Omega$, ${}^k s_\mu$, ${}^k M_{lm}$, ${}^k M_{l0}$ that have been backward-differentiated with respect to x_k and that will have the undifferentiated inner products Ω, $\hat{\Omega}$, s_μ, , M_{lm}, M_{l0} (μ = 0, 1, 2, 3 and (l, m) = 1, 2, 3) for their coefficients. One will obtain these equations most simply from Tab. 1 on pp. 95 of the earlier publication (*), in which R is one of the operators P, Q, and the inner product (ψ^+, $R\ \gamma^j\ \psi$) is always found at the place whether a row and column intersects (**). In order to refer to the backward differentiation with respect to x_k, the first factor will always be provided with a left upper index k. The relations that one obtains from (*) in that way will be summarized in what follows, and the operators P and Q that lead to the equation will be given on the left. Vector notation will be used for one part of eq. (1)-(26), while for the other part (27)-(52), for which the vector notation is not suitable, one of the components will be written.

A. The first group of identities contains only the backward-differentiated quantities ${}^k s_0$, ${}^k\mathfrak{s}$, ${}^k\Omega$, ${}^k\hat{\Omega}$.

P	Q		
I	I	(1)	${}^k s_0\, s_0 - ({}^k\mathfrak{s}, \mathfrak{s}) = {}^k\Omega\Omega + {}^k\hat{\Omega}\hat{\Omega},$
I	γ^5	(2)	${}^k s_0\, \hat{s}_0 - ({}^k\mathfrak{s}, \hat{\mathfrak{s}}) = i({}^k\Omega\hat{\Omega} - {}^k\hat{\Omega}\Omega),$
I	$\gamma^{[14]}$	(3)	${}^k s_0\, \mathfrak{s} - {}^k\mathfrak{s}\, s_0 + i[{}^k\mathfrak{s}, \mathfrak{s}] = i\,({}^k\Omega\,\hat{\mathfrak{M}} - {}^k\hat{\Omega}\,\mathfrak{M}),$
I	$\gamma^{[23]}$	(4)	${}^k s_0\, \hat{\mathfrak{s}} - {}^k\mathfrak{s}\,\hat{s}_0 + i[{}^k\mathfrak{s}, \mathfrak{s}] = {}^k\Omega\,\mathfrak{M} + {}^k\hat{\Omega}\,\hat{\mathfrak{M}}),$
I	$\gamma^{[123]}$	(5)	$({}^k\mathfrak{s}, \mathfrak{M}) - {}^k\Omega\,\hat{s}_0 = i({}^k\hat{\Omega}\, s_0 - {}^k s_0\,\hat{\Omega}),$
I	$\gamma^{[234]}$	(6)	${}^k s_0\mathfrak{M} + [{}^k\mathfrak{s}, \hat{\mathfrak{M}}] - {}^k\Omega\,\hat{\mathfrak{s}} = i({}^k s_0\,\Omega - {}^k\Omega\, s_0),$
I	γ^4	(7)	$({}^k\mathfrak{s}_0, \hat{\mathfrak{M}}) - {}^k\Omega\,\hat{s}_0 = i({}^k s_0\,\Omega - {}^k\Omega\, s_0),$
I	γ^1	(8)	${}^k s_0\hat{\mathfrak{M}} - [{}^k\mathfrak{s}, \mathfrak{M}] - {}^k\hat{\Omega}\,\hat{\mathfrak{s}} = i({}^k\mathfrak{s}\,\Omega - {}^k\Omega\,\mathfrak{s}).$

B. The second group of identities contains only the backward-differentiated quantities ${}^k\hat{s}_0$, ${}^k\hat{\mathfrak{s}}$, ${}^k\Omega$, ${}^k\hat{\Omega}$:

(*) **W. Kofink**, Ann. Phys. (Leipzig) (5) **30** (1937), pp. 91.

(**) **Correction.** This table contains two printing errors. One must put $-\,i\,\hat{s}_0$ in place of $-\,\hat{s}_0$ at the point where the row γ^5 and column γ^4 intersect, and put $\hat{s}_0$ in place of $i\,\hat{s}_0$ at the point where the row E crosses the column $\gamma^{[123]}$.

P	Q		
γ^5	γ^5	(9)	${}^k\hat{s}_0\,\hat{s}_0 - ({}^k\hat{\mathfrak{s}},\hat{\mathfrak{s}}) = -({}^k\Omega\Omega + {}^k\hat{\Omega}\hat{\Omega}),$
γ^5	I	(2′)	${}^k\hat{s}_0\,\hat{s}_0 - ({}^k\hat{\mathfrak{s}},\mathfrak{s}) = -i\,({}^k\Omega\hat{\Omega} - {}^k\hat{\Omega}\Omega),$
γ^5	$\gamma^{[14]}$	(10)	${}^k\hat{s}_0\,\mathfrak{s} - {}^k\hat{\mathfrak{s}}\,s_0 + i[{}^k\hat{\mathfrak{s}},\hat{\mathfrak{s}}] = -({}^k\hat{\Omega}\,\hat{\mathfrak{M}} + {}^k\Omega\,\mathfrak{M}),$
γ^5	$\gamma^{[23]}$	(11)	${}^k\hat{s}_0\,\hat{\mathfrak{s}} - {}^k\hat{\mathfrak{s}}\,\hat{s}_0 + i[{}^k\hat{\mathfrak{s}},\mathfrak{s}] = i\,({}^k\hat{\Omega}\,\mathfrak{M} - {}^k\Omega\,\hat{\mathfrak{M}}),$
γ^5	$\gamma^{[123]}$	(12)	$({}^k\hat{\mathfrak{s}},\mathfrak{M}) - {}^k\Omega\,s_0 = i\,({}^k\hat{\Omega}\,\hat{s}_0 - {}^k\hat{s}_0\,\hat{\Omega}),$
γ^5	$\gamma^{[234]}$	(13)	${}^k\hat{s}_0\mathfrak{M} + [{}^k\hat{\mathfrak{s}},\hat{\mathfrak{M}}] - {}^k\Omega\,\mathfrak{s} = i({}^k\Omega\,\hat{\mathfrak{s}} - {}^k\hat{\mathfrak{s}}\,\hat{\Omega}),$
γ^5	γ^4	(14)	$({}^k\hat{\mathfrak{s}},\hat{\mathfrak{M}}) - {}^k\hat{\Omega}\,s_0 = i\,({}^k\hat{s}_0\,\Omega - {}^k\Omega\,\hat{s}_0),$
γ^5	γ^1	(15)	${}^k\hat{s}_0\hat{\mathfrak{M}} - [{}^k\hat{\mathfrak{s}},\mathfrak{M}] - {}^k\hat{\Omega}\,\mathfrak{s} = i\,({}^k\hat{\mathfrak{s}}\,\Omega - {}^k\Omega\,\hat{\mathfrak{s}}).$

When $P \neq Q$, another identity will arise from the ones that were specified by switching P and Q and shifting the symbol of backward differentiation k from the first factor to the second one. Eq. (2′) will go to eq. (2) in that way. We shall mention none of those identities besides that example. They are quite easy to arrive at from the ones that were written down.

C. The third group of identities contains only the backward-differentiated quantities ${}^k s_0$, ${}^k\hat{s}_0$, ${}^k\mathfrak{M}$, ${}^k\hat{\mathfrak{M}}$, ${}^k\Omega$, ${}^k\hat{\Omega}$:

P	Q		
$\gamma^{[123]}$	$\gamma^{[23]}$	(16)	${}^k\hat{\Omega}\,\hat{\mathfrak{s}} - {}^k\hat{s}_0\,\hat{\mathfrak{M}} - [{}^k\mathfrak{M},\mathfrak{s}] = i\,({}^k\mathfrak{M}\,\hat{s}_0 + {}^k\hat{s}_0\,\mathfrak{M}),$
$\gamma^{[123]}$	$\gamma^{[14]}$	(17)	${}^k\hat{\Omega}\,\mathfrak{s} - {}^k\hat{s}_0\,\hat{\mathfrak{M}} - [{}^k\mathfrak{M},\hat{\mathfrak{s}}] = i\,({}^k\mathfrak{M}\,s_0 - {}^k s_0\,\mathfrak{M}),$
γ^4	$\gamma^{[23]}$	(18)	${}^k\Omega\,\hat{\mathfrak{s}} - {}^k s_0\,\mathfrak{M} - [{}^k\hat{\mathfrak{M}},\mathfrak{s}] = -i\,({}^k\hat{\mathfrak{M}}\,\hat{s}_0 - {}^k\hat{s}_0\,\hat{\mathfrak{M}}),$
γ^4	$\gamma^{[14]}$	(19)	${}^k\Omega\,\mathfrak{s} - {}^k\hat{s}_0\,\mathfrak{M} - [{}^k\hat{\mathfrak{M}},\hat{\mathfrak{s}}] = -i\,({}^k\hat{\mathfrak{M}}\,s_0 - {}^k s_0\,\hat{\mathfrak{M}}),$
γ^4	γ^1	(20)	$[{}^k\hat{\mathfrak{M}},\mathfrak{M}] + {}^k s_0\mathfrak{s} - {}^k\hat{s}_0\hat{\mathfrak{s}} = i\,({}^k\Omega\,\hat{\mathfrak{M}} - {}^k\hat{\mathfrak{M}}\,\Omega),$
$\gamma^{[123]}$	$\gamma^{[234]}$	(21)	$-[{}^k\mathfrak{M},\hat{\mathfrak{M}}] + {}^k s_0\mathfrak{s} - {}^k\hat{s}_0\hat{\mathfrak{s}} = -i\,({}^k\hat{\Omega}\,\mathfrak{M} - {}^k\mathfrak{M}\,\hat{\Omega}),$
γ^4	$\gamma^{[234]}$	(22)	$-{}^k s_0\hat{\mathfrak{s}} + {}^k\hat{s}_0\mathfrak{s} + {}^k\hat{\mathfrak{M}}\,\hat{\Omega} + {}^k\Omega\,\mathfrak{M} = i[{}^k\hat{\mathfrak{M}},\hat{\mathfrak{M}}],$
$\gamma^{[123]}$	γ^1	(23)	$-{}^k s_0\hat{\mathfrak{s}} + {}^k\hat{s}_0\mathfrak{s} + {}^k\hat{\Omega}\,\hat{\mathfrak{M}} + {}^k\mathfrak{M}\,\Omega = i[{}^k\mathfrak{M},\mathfrak{M}],$
γ^4	$\gamma^{[123]}$	(24)	$({}^k\hat{\mathfrak{M}},\hat{\mathfrak{M}}) - {}^k\Omega\hat{\Omega} = i({}^k s_0\,\hat{s}_0 - {}^k s_0\,s_0),$
γ^4	γ^4	(25)	$({}^k\hat{\mathfrak{M}},\hat{\mathfrak{M}}) + {}^k\Omega\Omega - {}^k s_0\,s_0 + {}^k\hat{s}_0\,\hat{s}_0 = 0,$
$\gamma^{[123]}$	$\gamma^{[123]}$	(26)	$({}^k\mathfrak{M},\mathfrak{M}) + {}^k\hat{\Omega}\Omega - {}^k s_0\,s_0 + {}^k\hat{s}_0\,\hat{s}_0 = 0.$

The vector notation is not appropriate for the remaining identities. For them, we select one component from which the other two components will arise by cyclic permutation of the indices. We summarize them in groups of components with analogous structures.

D. Analogous to eq. [I, (25)-(29)]:

P	Q		
γ^1	$\gamma^{[23]}$	(27)	${}^k\Omega\,\hat{s}_0 - {}^k s_1 M_{23} - {}^k M_{31}\,s_2 - {}^k M_{12}\,s_3 = i\,({}^k\hat{s}_1 M_{10} - {}^k M_{10}\,\hat{s}_1),$
γ^1	$\gamma^{[14]}$	(28)	${}^k\Omega\,s_0 - {}^k\hat{s}_1 M_{23} - {}^k M_{31}\,\hat{s}_2 - {}^k M_{12}\,\hat{s}_3 = i\,({}^k s_1 M_{10} - {}^k M_{10}\,s_1),$
$\gamma^{[234]}$	$\gamma^{[23]}$	(29)	${}^k\hat{\Omega}\,\hat{s}_0 - {}^k s_1 M_{10} - {}^k M_{20}\,s_2 - {}^k M_{30}\,s_3 = -i\,({}^k\hat{s}_1 M_{23} - {}^k M_{23}\,\hat{s}_1),$
$\gamma^{[234]}$	$\gamma^{[14]}$	(30)	${}^k\hat{\Omega}\,s_0 - {}^k\hat{s}_1 M_{10} - {}^k M_{20}\,\hat{s}_2 - {}^k M_{30}\,s_3 = -i\,({}^k s_1 M_{23} - {}^k M_{23}\,s_1).$

E. Analogous to eqs. [I, (29)-(32)]:

$\gamma^{[124]}$	$\gamma^{[24]}$	(31)	${}^k\hat{\Omega}\,\hat{s}_1 - {}^k M_{10}\,s_0 + {}^k M_{12}\,s_2 - {}^k s_3 M_{31} = i\,({}^k\hat{s}_3 M_{20} - {}^k M_{20}\,\hat{s}_3),$
$\gamma^{[314]}$	$\gamma^{[34]}$	(32)	${}^k\hat{\Omega}\,s_1 - {}^k M_{10}\,s_0 - {}^k M_{31}\,\hat{s}_3 + {}^k s_2 M_{12} = -i\,({}^k\hat{s}_1 M_{30} - {}^k M_{30}\,\hat{s}_2),$
$\gamma^{[124]}$	$\gamma^{[31]}$	(33)	${}^k\hat{\Omega}\,\hat{s}_1 - {}^k M_{10}\,\hat{s}_0 + {}^k M_{12}\,\hat{s}_2 - {}^k\hat{s}_3 M_{30} = i\,({}^k s_3 M_{20} - {}^k M_{20}\,s_3),$
$\gamma^{[314]}$	$\gamma^{[12]}$	(34)	${}^k\hat{\Omega}\,s_1 - {}^k M_{10}\,\hat{s}_0 - {}^k M_{31}\,\hat{s}_3 + {}^k s_2 M_{12} = -i\,({}^k s_2 M_{30} - {}^k M_{30}\,s_2),$
γ^3	$\gamma^{[24]}$	(35)	${}^k\Omega\,\hat{s}_1 - {}^k M_{23}\,s_0 + {}^k M_{20}\,s_3 - {}^k s_2 M_{30} = i\,({}^k\hat{s}_3 M_{31} - {}^k M_{31}\,\hat{s}_3),$
γ^2	$\gamma^{[34]}$	(36)	${}^k\Omega\,\hat{s}_1 - {}^k M_{23}\,s_0 + {}^k M_{20}\,s_3 - {}^k s_2 M_{30} = -i\,({}^k\hat{s}_2 M_{12} - {}^k M_{12}\,\hat{s}_2),$
γ^3	$\gamma^{[31]}$	(37)	${}^k\Omega\,s_1 - {}^k M_{23}\,\hat{s}_0 - {}^k M_{30}\,\hat{s}_2 + {}^k\hat{s}_3 M_{20} = i\,({}^k s_3 M_{31} - {}^k M_{31}\,s_3),$
γ^2	$\gamma^{[12]}$	(38)	${}^k\Omega\,s_1 - {}^k M_{23}\,\hat{s}_0 + {}^k M_{20}\,\hat{s}_3 - {}^k\hat{s}_2 M_{30} = -i\,({}^k s_2 M_{12} - {}^k M_{12}\,s_2).$

F. Analogous to eqs. [I, (33), (34), and (40)]:

$\gamma^{[24]}$	$\gamma^{[12]}$	(39)	${}^k s_0\,s_1 - {}^k\hat{s}_1\,s_0 + {}^k M_{20}\,M_{12} - {}^k M_{31} M_{30} = -i\,({}^k s_2\,\hat{s}_3 - {}^k\hat{s}_3\,s_2),$
$\gamma^{[34]}$	$\gamma^{[31]}$	(40)	${}^k s_0\,s_1 - {}^k\hat{s}_0\,s_1 + {}^k M_{12}\,M_{20} + {}^k M_{30} M_{31} = -i\,({}^k\hat{s}_2 s_3 - {}^k s_3\,\hat{s}_2),$
γ^2	$\gamma^{[124]}$	(41)	${}^k s_2\,\hat{s}_3 - {}^k\hat{s}_2\,\hat{s}_3 + {}^k M_{23}\,\hat{\Omega} - {}^k\Omega\,M_{10} = i\,({}^k M_{20} M_{12} - {}^k M_{12}\,s_{20}),$
γ^3	$\gamma^{[314]}$	(42)	${}^k s_2\,\hat{s}_3 - {}^k\hat{s}_2\,\hat{s}_3 + {}^k\hat{\Omega}\,M_{23} - {}^k M_{10}\,\Omega = -i\,({}^k M_{30} M_{31} - {}^k M_{31}\,s_{30}),$
γ^2	γ^3	(43)	${}^k s_2\,s_3 - {}^k\hat{s}_2\,\hat{s}_3 + {}^k M_{12}\,M_{31} + {}^k M_{20} M_{30} = i\,({}^k M_{23}\,\Omega - {}^k\Omega\,M_{31}),$
$\gamma^{[314]}$	$\gamma^{[124]}$	(44)	${}^k s_2\,s_3 - {}^k\hat{s}_2\,\hat{s}_3 + {}^k M_{31}\,M_{12} + {}^k M_{30} M_{30} = i\,({}^k M_{10}\,\Omega - {}^k\hat{\Omega}\,M_{10}),$
$\gamma^{[12]}$	$\gamma^{[31]}$	(45)	${}^k s_2\,s_3 - {}^k\hat{s}_3\,\hat{s}_2 + {}^k M_{12}\,M_{31} + {}^k M_{30} M_{20} = -i\,({}^k s_1\hat{s}_0 - {}^k\hat{s}_0\,s_1),$
$\gamma^{[24]}$	$\gamma^{[34]}$	(46)	${}^k s_2\,s_3 - {}^k\hat{s}_3\,\hat{s}_2 + {}^k M_{31}\,M_{12} + {}^k M_{20} M_{30} = -i\,({}^k\hat{s}_1 s_0 - {}^k s_0\,\hat{s}_1).$

G. Analogous to eqs. [I, (22), (41), (17), and (42):

γ^1	$\gamma^{[234]}$	(47)	${}^k\Omega\hat{\Omega} - {}^k M_{10}\,M_{23} - {}^k M_{31}\,M_{20} - {}^k M_{12} M_{30} = i\,({}^k s_1\,\hat{s}_1 - {}^k\hat{s}_1\,s_1),$
γ^1	γ^1	(48)	${}^k\Omega\Omega + {}^k M_{10}\,M_{10} - {}^k M_{31}\,M_{31} - {}^k M_{12} M_{12} + {}^k s_1\,s_1 - {}^k\hat{s}_1\,\hat{s}_1 = 0,$
$\gamma^{[234]}$	$\gamma^{[234]}$	(49)	${}^k\hat{\Omega}\hat{\Omega} + {}^k M_{23}\,M_{23} - {}^k M_{20}\,M_{20} - {}^k M_{30} M_{30} + {}^k s_1\,s_1 - {}^k\hat{s}_1\,\hat{s}_1 = 0,$
$\gamma^{[23]}$	$\gamma^{[14]}$	(50)	$-{}^k\hat{s}_0\,s_0 + {}^k\hat{s}_1\,s_1 + {}^k\hat{s}_2\,s_2 + {}^k\hat{s}_3\,s_3 = i\,({}^k M_{23} M_{10} - {}^k M_{10}\,s_{23}),$
$\gamma^{[23]}$	$\gamma^{[23]}$	(51)	$-{}^k\hat{s}_0\,\hat{s}_0 + {}^k\hat{s}_1\,\hat{s}_1 + {}^k\hat{s}_2\,s_2 + {}^k\hat{s}_3\,s_3 - {}^k M_{23}\,M_{23} - {}^k M_{10} M_{10} = 0,$
$\gamma^{[14]}$	$\gamma^{[14]}$	(52)	${}^k s_0\,s_0 - {}^k s_1\,s_1 - {}^k\hat{s}_2\,\hat{s}_2 - {}^k\hat{s}_2\,\hat{s}_2 - {}^k M_{23}\,M_{23} - {}^k M_{10} M_{10} = 0.$

When reference is made later on to an identity that arises from the given one [eq. (x)] by switching P and Q, that will be denoted by affixing a prime (′) to the equation number [eq. (x′)]. Eq. (x′) emerges from eq. (x) by shifting the index k from the first to the second factor of our bilinear expression.

§ 2. Application of the first bilinear equation to forward differentiation.

Starting from the basic equation [I, (4)]:

$$\sum_{k=1}^{4} \gamma^k_{\bar\rho\bar\sigma}\gamma^k_{\bar\rho\bar{\bar\sigma}} + \gamma^{[123]}_{\bar\rho\bar\sigma}\gamma^{[123]}_{\bar\rho\bar{\bar\sigma}} = \delta_{\bar\rho\bar\sigma}\delta_{\rho\bar{\bar\sigma}} - \gamma^5_{\bar\rho\bar\sigma}\,\gamma^5_{\rho\bar{\bar\sigma}} - \gamma^4_{\bar\rho\bar\sigma}\,\gamma^4_{\rho\bar{\bar\sigma}} + \gamma^{[123]}_{\bar\rho\bar\sigma}\,\gamma^{[123]}_{\rho\bar{\bar\sigma}}\,,$$

in analogy to § 2, one will obtain four sets of four functions φ_ρ , χ_σ, $\xi_{\bar\rho}$, $\psi_{\bar\sigma}$ by left-multiplying it by the four-rowed matrices $P_{\bar\sigma\sigma}\,Q_{\bar{\bar\sigma}\bar\sigma}$, and summing over all indices σ, $\bar\sigma$, $\bar{\bar\sigma}$, $\bar{\bar{\bar\sigma}}$, ρ, $\bar\rho$ will give the bilinear equation in the inner products:

$$\sum_{k=1}^{3} (\varphi, \gamma^k P\chi)(\xi, \gamma^k Q\psi) + (\varphi,\, \gamma^{[123]}\, P\, \chi)(\xi,\, \gamma^{[123]}\, Q\, \psi)$$

$$= (\xi,\, P\chi)(\varphi,\, Q\psi) - (\xi,\, \gamma^4 P\chi)(\varphi,\, \gamma^4 Q\psi) + (\xi,\, \gamma^{[123]} P\chi)(\varphi,\, \gamma^{[123]} Q\psi).$$

In the special case $\varphi = \xi$, this equation will assume a simpler form, and for $\varphi = \xi = \psi^+$ and $\chi = \partial\psi / \partial x_k$, one will get:

$$(**) \qquad \boxed{\sum_{j=1}^{5} \left(\psi^+, \gamma^j P \frac{\partial\psi}{\partial x_k}\right)(\psi^+, \gamma^j Q\psi) = \left(\psi^+, P\frac{\partial\psi}{\partial x_k}\right)(\psi^+, Q\psi)}$$

from it. Comparing this with eq. (*) for the backward differentiation will show that the two equations coincide up to the sequence of factors γ^j and P (Q, resp.). Since P and Q run through the 16 matrices of the **Dirac** matrix ring, one will havc $\gamma^j\, P = \pm\, P\, \gamma^j$. Whether one chooses the upper or lower sign will be determined by whether P belongs to the subset I, γ^1, γ^2, γ^3, γ^4, γ^5 or the remaining subset of the matrix ring. When Q (or P) is one of the six matrices of the first set, the identity matrix must *certainly* occur among the $\gamma^j\, Q$; namely, for $Q = \gamma^j$. However, the associated inner product is Ω. When Q (or P) belongs to the ten matrices of the remaining set, the identity matrix will *certainly not* occur among the $\gamma^j\, Q$, so the associated inner product Ω will also be absent. The occurrence (non-occurrence, resp.) of Ω is a suitable criterion for the choice of sign for each factor of the bilinear expressions when one goes from eq. (*) to eq. (**): If Ω is present then the sign of Ω, $\hat\Omega$, s_0, s_1, s_2, s_3 will remain unchanged, while one must choose the opposite sign for $\hat s_0$, $\hat s_1$, $\hat s_2$, $\hat s_3$, M_{23}, M_{31}, M_{12}, M_{10}, M_{20}, M_{30} ; conversely, when Ω is absent, it will be the second set of quantities that keep their signs, while the first ones will

change. The same thing will also be true for the factors that contain a differential quotient, except that the backward differentiation will be switched simultaneously with the forward differentiation; e.g., ${}^k\hat{\Omega}$ will go to $\hat{\Omega}^k$ when ${}^k\Omega$ is present, and to $-\hat{\Omega}^k$ when ${}^k\Omega$ is absent. By applying this prescription to eqs. (1)-(52) of the last paragraph, one will see that all terms that contain the *imaginary unit explicitly* will invert their signs while the remaining signs will be preserved. This behavior is to be distinguished from a transition to complex conjugates when the Hermiticity of the Dirac matrices was not assumed, so one will have:

$$\left(\frac{\partial\psi^*}{\partial x_k}, A\psi\right)^* = \left(\psi^*, A^+\frac{\partial\psi}{\partial x_k}\right) \neq \left(\psi^*, A\frac{\partial\psi}{\partial x_k}\right),$$

in which $A^+ = \overline{A^*}$ is the adjoint of A, and A means a matrix from the complete **Dirac** matrix ring of α.

By contrast, if the matrices α are Hermitian then $A^+ = A$, and the transition from the backward to the forward differentiation will come about when one goes to conjugate complexes.

§ 3. Application of second equation.

From [I, (12)-(14)], the six matrices B, $B\gamma^\mu$ (μ = 1, 2, 3, 4, 5) are skew-symmetric, and the ten matrices $B\gamma^{[\mu\nu]}$, $B\gamma^{[\lambda\mu\nu]}$ are symmetric. That will imply that the inner products must be:

(53) $$(\psi, B\,\psi) = 0 \quad \text{and} \quad (\psi, B\gamma^\mu\,\psi) = 0,$$

while nothing simple can be said about $(\psi, B\,\gamma^{[\mu\nu]}\,\psi)$ and $(\psi, B\,\gamma^{[\lambda\mu\nu]}\psi)$. (53) makes it possible to partially apply the second fundamental equation when one makes the terms in eq. [I, (10)] that contain B vanish by taking suitable inner products. A discussion of the possible cases will yield the following:

a) As long as ψ_σ and $\psi_{\bar\sigma}$ are the same sets of functions, one can make the left-hand side, as well as the right-hand side, vanish by a direct application of eq. [I, (10)], so the exchange of ρ and $\bar\rho$ will then be meaningless.

b) By contrast, the multiplication of eq. [I, (10)] by two four-rowed matrices Γ and Γ' from the **Dirac** matrix ring and the forming of inner products with the help of four sets of four functions ψ_σ, $\chi_{\bar\sigma}$, $\varphi_{\bar\sigma}$, ξ_ρ will yield:

$$\begin{aligned}2\,(\psi, B\,\Gamma\chi)(\varphi, \Gamma\,B^{-1}\xi) = {} & (\xi, \psi)\,(\varphi, \Gamma'\,\Gamma\xi) - (\varphi, \Gamma'\psi)(\xi, \Gamma\chi)\\ & + (\xi, \gamma^5\psi)\,(\varphi, \Gamma'\,\gamma^5\,\Gamma\chi) - (\varphi, \Gamma'\gamma^5\psi)(\xi, \gamma^5\,\Gamma\chi)\\ & + (\xi, \gamma^4\psi)\,(\varphi, \Gamma'\,\gamma^4\,\Gamma\chi) - (\varphi, \Gamma'\gamma^4\psi)(\xi, \gamma^4\,\Gamma\chi)\\ & - (\xi, \gamma^{[123]}\psi)\,(\varphi, \Gamma'\,\gamma^{[123]}\,\Gamma\chi) + (\varphi, \Gamma'\gamma^{[123]}\psi)(\xi, \gamma^{[123]}\,\Gamma\chi).\end{aligned}$$

If *one* of the operators Γ or Γ' is chosen from the six matrices I, γ^1, γ^2, γ^3, γ^4, γ^5, while the other can remain unrestricted, and one then sets $\chi = \psi$ ($\varphi = \xi$, resp.), corresponding to that choice, then the left-hand side will vanish. When Γ means one of the matrices I, γ^1, γ^2, γ^3, γ^4, γ^5, the following equation will arise for $\chi = \psi$, $\varphi = \partial\psi^+ / \partial x_k$, $\xi = \psi^+$:

$$
\begin{aligned}
0 = {} & (\psi^+, \psi)\left(\frac{\partial\psi^+}{\partial x_k}, \Gamma'\Gamma\psi\right) - \left(\frac{\partial\psi^+}{\partial x_k}, \Gamma'\psi\right)(\psi^+, \Gamma\psi) \\
& + (\psi^+, \gamma^5\psi)\left(\frac{\partial\psi^+}{\partial x_k}, \Gamma'\gamma^5\Gamma\psi\right) - \left(\frac{\partial\psi^+}{\partial x_k}, \Gamma'\gamma^5\psi\right)(\psi^+, \gamma^5\Gamma\psi) \\
& + (\psi^+, \gamma^4\psi)\left(\frac{\partial\psi^+}{\partial x_k}, \Gamma'\gamma^4\Gamma\psi\right) - \left(\frac{\partial\psi^+}{\partial x_k}, \Gamma'\gamma^4\psi\right)(\psi^+, \gamma^4\Gamma\psi) \\
& - (\psi^+, \gamma^{[123]}\psi)\left(\frac{\partial\psi^+}{\partial x_k}, \Gamma'\gamma^{[123]}\Gamma\psi\right) + \left(\frac{\partial\psi^+}{\partial x_k}, \Gamma'\gamma^{[123]}\psi\right)(\psi^+, \gamma^{[123]}\Gamma\psi) .
\end{aligned}
$$

1. If Γ is set equal to I, γ^4, γ^5 inside of the allowed set then calculating the corresponding products will show that the right-hand side will also vanish in these three cases.

2. If Γ means one of the matrices γ^1, γ^2, γ^3 then one non-trivial equation will arise; e.g., for $\Gamma = \gamma^1$:

$$
(54) \quad \left\{
\begin{aligned}
0 = {} & (\psi^+, \psi)\left(\frac{\partial\psi^+}{\partial x_k}, \Gamma'\gamma^1\psi\right) - \left(\frac{\partial\psi^+}{\partial x_k}, \Gamma'\psi\right)(\psi^+, \gamma^1\psi) \\
& + i(\psi^+, \gamma^5\psi)\left(\frac{\partial\psi^+}{\partial x_k}, \Gamma'\gamma^{[234]}\psi\right) - i\left(\frac{\partial\psi^+}{\partial x_k}, \Gamma'\gamma^5\psi\right)(\psi^+, \gamma^{[234]}\psi) \\
& + i(\psi^+, \gamma^4\psi)\left(\frac{\partial\psi^+}{\partial x_k}, \Gamma'\gamma^{[14]}\psi\right) - i\left(\frac{\partial\psi^+}{\partial x_k}, \Gamma'\gamma^4\psi\right)(\psi^+, \gamma^{[14]}\psi) \\
& - (\psi^+, \gamma^{[123]}\psi)\left(\frac{\partial\psi^+}{\partial x_k}, \Gamma'\gamma^{[23]}\psi\right) + \left(\frac{\partial\psi^+}{\partial x_k}, \Gamma'\gamma^{[123]}\psi\right)(\psi^+, \gamma^{[23]}\psi).
\end{aligned}
\right.
$$

However, one can show that this equation also follows from the first fundamental equation {I, (4)] in the form [II, § 1, eq. (*)], when one chooses:

1. $Q = I$, $P = \Gamma'\gamma^1$, and
2. $Q = \gamma^{[23]}$, $P = \Gamma'\gamma^{[123]}$.

Subtracting the two equations that arise in that way will give (54). Since $\Gamma = \gamma^1$ is by no means distinguished from $\Gamma = \gamma^2$ or $\Gamma = \gamma^3$ as a choice, that will imply that the second bilinear equation [I, (10)] can lead to no other results than the first identities that are

implied by [I, (4)], as long as only the symmetry properties of the matrix B are used, but their explicit form is unknown.

§ 4. Further applications of the first bilinear equation.

Backward (forward, resp.) differentiation of both factors. Instead of setting $\xi = \psi^+$ in the foregoing eq. (*), one sets $\xi = \partial\psi^+ / \partial x_k$ and gets:

$$\sum_{j=1}^{5}\left(\frac{\partial\psi^+}{\partial x_k}, P\gamma^j\psi\right)\left(\frac{\partial\psi^+}{\partial x_k}, Q\gamma^j\psi\right) = \left(\frac{\partial\psi^+}{\partial x_k}, P\psi\right)\left(\frac{\partial\psi^+}{\partial x_k}, Q\psi\right). \tag{55}$$

If P and Q run through the matrix ring in this then the same algebraic identities will arise for the backward differentiated quantities that were quoted for the undifferentiated ones in I, § 3. One will then obtain them by affixing an upper index k to all of the identities in I, § 3. Two examples of this are:

$$({}^k s_0)^2 - ({}^k\mathfrak{s}, {}^k\mathfrak{s})^2 = ({}^k\Omega)^2 + ({}^k\hat{\Omega})^2 \quad \text{for} \quad P = Q = I, \tag{56}$$

$$({}^k\hat{s}_0)^2 - ({}^k\hat{\mathfrak{s}}, {}^k\hat{\mathfrak{s}}) = -({}^k\Omega)^2 - ({}^k\hat{\Omega})^2 \quad \text{for} \quad P = Q = \gamma^5. \tag{57}$$

Likewise, the replacement of ψ with $\partial\psi / \partial x_k$ in the second factor of (**) will lead to an analogous equation for the forward-differentiated quantities:

$$\sum_{j=1}^{5}\left(\psi^+, \gamma^j P\frac{\partial\psi}{\partial x_k}\right)\left(\psi^+, \gamma^j Q\frac{\partial\psi}{\partial x_k}\right) = \left(\psi^+, P\frac{\partial\psi}{\partial x_k}\right)\left(\psi^+, Q\frac{\partial\psi}{\partial x_k}\right). \tag{58}$$

Analogous to what was said about eq. (55), the same algebraic identities will arise from (58) for the forward-differentiated quantities as what arose from (55) for the backward-differentiated ones. One will get them when one affixes an upper index of k to all of the identities that were cited in I, § 3.

§ 5. Solution of the algebraic identities for backward differentiation.

A large number of algebraic identities that exist between the backward (forward, resp.) differentiated quantities and the undifferentiated ones were presented in the foregoing paragraphs. Our goal is now to reduce the backward (forward, resp.) differentiated quantities to the undifferentiated ones as much as is possible, in order to be able to eliminate them in later applications.

We know that the system of equations that was presented in § 4 is completely equal to the system of equations I, § 3 for the undifferentiated quantities. That will assure us that [analogous to I, (18) and (19)] the quantities ${}^k\mathfrak{M}$ and ${}^k\hat{\mathfrak{M}}$ can be expressed entirely in terms of the remaining ones ${}^k s_0$, ${}^k\hat{s}_0$, ${}^k\mathfrak{s}$, ${}^k\hat{\mathfrak{s}}_0$, ${}^k\Omega$, ${}^k\hat{\Omega}$. We shall then focus our attention

on the determination of these remaining quantities. We thus select the equations in §§ 1 and 4 that contain only those quantities, and indeed as few as possible. The system of equations (1)-(8), which contains only the backward-differentiated quantities ${}^k s_0$, ${}^k\mathfrak{s}$, ${}^k\Omega$, ${}^k\hat{\Omega}$, and the system of equations (9)-(15), which contains only the backward-differentiated quantities ${}^k\hat{s}_0$, ${}^k\hat{\mathfrak{s}}$, ${}^k\Omega$, ${}^k\hat{\Omega}$, will fulfill that demand.

Naturally, one cannot reduce all of the backward-differentiated quantities to undifferentiated ones. Two backward-differentiated quantities ${}^k\Omega$ and ${}^k\hat{\Omega}$ cannot be eliminated as long as one does not make explicit use of the fact that the index k means differentiation (§§ 5-8). If one employs that fact (§§ 9-11) then one can *still* eliminate one of those quantities – e.g., $i({}^k\hat{\Omega} - \hat{\Omega}^k)$ – by combined backward and forward differentiation. From the outset, we then regard the 14 backward-differentiated quantities ${}^k s_0$, ${}^k\mathfrak{s}$, ${}^k\hat{s}_0$, ${}^k\hat{\mathfrak{s}}$, ${}^k\mathfrak{M}$, ${}^k\hat{\mathfrak{M}}$, as *unknowns* in our equations, while the two backward-differentiated quantities ${}^k\Omega$, ${}^k\hat{\Omega}$, and regard the 16 undifferentiated quantities s_0, $\mathfrak{s}$, $\hat{s}_0$, $\mathfrak{M}$, $\hat{\mathfrak{M}}$, Ω, $\hat{\Omega}$ as *knowns.* In terms of components, we have 16 linear equations [eqs. (1)-(8)] in the four unknowns ${}^k s_0$, ${}^k\mathfrak{s}$ (${}^k s_1$, ${}^k s_2$, ${}^k s_3$) and one quadratic one [eq. (56)] that contained only ${}^k\Omega$ and ${}^k\hat{\Omega}$, in addition to undifferentiated quantities. The solution of that system of equations will show that the quadratic eq. (56) follows from the linear ones and that of the 16 linear eq. (1)-(8), only *three* of them are linearly-independent, such that despite the large number of equations, the *four* unknowns cannot be determined completely, since one unknown parameter ${}^k\xi$ will remain in the solutions. The 16 linear equations [eqs. (9), (2′), (10)-(15)], which were summarized in § 1 as the second group, as well as the quadratic eq. (57), behave analogously in regard to the four unknowns ${}^k\hat{s}_0$, ${}^k\hat{\mathfrak{s}}$ (${}^k\hat{s}_1$, ${}^k\hat{s}_2$, ${}^k\hat{s}_3$). A (second) unknown parameter ${}^k\eta$ will remain in their solutions. The parameters ${}^k\xi$, ${}^k\eta$ can be determined only after combining the backward and forward differentiated quantities and recalling the property of the index k that it means differentiation (§ 9).

A. *Solving the linear eqs.* (1)-(8) *and the quadratic one eq.* (56).

a) Proof that eq. (56) follows from eqs. (1), (3), (5), (7): If one scalar multiplies (3) by ${}^k\mathfrak{s}$ then that will imply:

$$ {}^k s_0\,({}^k\mathfrak{s}, \mathfrak{s}) - ({}^k\mathfrak{s}, {}^k\mathfrak{s})\, s_0 = i\,\{{}^k\Omega\,({}^k\mathfrak{s}, \hat{\mathfrak{M}}) - {}^k\hat{\Omega}\,({}^k\mathfrak{s}, \mathfrak{M})\}, $$

from which, eq. (56) will follow upon applying (1), (5), and (7).

b) Proof that eq. (5) follows from eqs. (1)-(4): If one scalar multiplies (3) by $s_0\,\mathfrak{M}$ and (4) by $-\,\hat{s}_0\,\mathfrak{M}$ and adds them then that will give:

$$ -(s_0^{\,2} - \hat{s}_0^{\,2})\,({}^k\mathfrak{s}, \mathfrak{M}) = i\, s_0\,\hat{\Omega}\,({}^k\mathfrak{s}, \mathfrak{s}) - i\,\hat{s}_0\,\hat{\Omega}\,({}^k\mathfrak{s}, \hat{\mathfrak{s}}) $$

$$- {}^k\Omega \; \{\hat{s}_0 (s_0^{\,2} - \hat{s}_0^{\,2} - \hat{\Omega}^2) - i\, s_0\, \Omega\, \hat{\Omega}\}$$
$$- {}^k\hat{\Omega} \{i\, s_0 (s_0^{\,2} - \hat{s}_0^{\,2} - \hat{\Omega}^2) + \hat{s}_0\, \Omega\, \hat{\Omega}\}\,.$$

(5) follows from this with an application of (1) and (2), since:

$$s_0^{\,2} - \hat{s}_0^{\,2} \neq 0,$$

in general.

c) Proof that eq. (7) follows from eqs. (1)-(4): Multiplying (3) by $-\, i\, s_0\, \mathfrak{M}$, with the use of (1), will yield:

$$\hat{s}_0\, ({}^k\mathfrak{s}\,, \hat{\mathfrak{M}}\,) = -\, i\, s_0\, ({}^k\mathfrak{s}, \mathfrak{M}) + {}^k s_0\, (i\;\hat{s}_0\, \Omega + s_0\;\hat{\Omega}\,) - {}^k\hat{\Omega}\, (s_0^{\,2} - \hat{s}_0^{\,2})\,,$$

which is an equation that couples (7) to (5). Substituting (5), which follows from eqs. (1)-(4), from *b*), also proves that (7) is a consequence of (1) to (4).

d) Proof that eq. (6) and (8) follow from eqs. (1)-(4): If one multiplies eq. (3) by $-\, i\, s_0$ and in addition vector-multiplies it by $-\hat{\mathfrak{s}}$, and then multiplies eq. (4) by $i\,\hat{s}_0$ and additionally vector-multiplies by $\mathfrak{s}$, and adds all four products then when one employs (1) and (2) and the algebraic identities [I, (15), (16), (29)-(32)], one will get:

(59)
$$\left\{ \begin{aligned} & {}^k s_0 [\mathfrak{s}, \hat{\mathfrak{s}}] - [\,{}^k\mathfrak{s}, s_0 \hat{\mathfrak{s}} - \hat{s}_0 \mathfrak{s}] = i\, {}^k\mathfrak{s}\, (\Omega^2 + \hat{\Omega}^2) \\ & \qquad - {}^k\Omega\, (i\mathfrak{s}\,\Omega + \hat{\mathfrak{s}}\,\hat{\Omega}) + {}^k\hat{\Omega}\, (-i\mathfrak{s}\,\hat{\Omega} + \hat{\mathfrak{s}}\,\Omega). \end{aligned} \right.$$

On the other hand, if one multiplies eq. (1) by $\hat{\mathfrak{s}}$, eq. (2) by $-\,\mathfrak{s}$, and adds them then that will give:

(60)
$${}^k s_0\, (s_0\, \hat{\mathfrak{s}} - \hat{s}_0\, \mathfrak{s}) + [{}^k\mathfrak{s}, [\mathfrak{s}, \hat{\mathfrak{s}}\,]] = {}^k\Omega\, (\,\hat{\mathfrak{s}}\,\Omega - i\, \mathfrak{s}\,\hat{\Omega}\,) + {}^k\hat{\Omega}\, (\hat{\mathfrak{s}}\,\hat{\Omega} + i\, \mathfrak{s}\Omega).$$

With the use of the algebraic identities [I, (18) and (19)], multiplying (59) by $-\hat{\Omega}$ and (60), by Ω and adding them will give eq. (6), while multiplication of (59) by Ω and (60) by $\hat{\Omega}$ and adding them will lead to eq. (8).

With that, we have shown that eqs. (56) and (5)-(8) follow from eqs. (10)-(4), such that the solution of (1)-(4) will now remain by itself.

e) Solution of eqs. (1)-(4). These equations will be satisfied with the Ansatz:

(61)
$$\boxed{\begin{aligned} {}^k s_0 &= {}^k A s_0 - i\, {}^k B\, \hat{s}_0 + {}^k\xi\, q_0, \\ {}^k\mathfrak{s} &= {}^k A \mathfrak{s} - i\, {}^k B\, \hat{\mathfrak{s}} + {}^k\xi\, \mathfrak{q}, \end{aligned}}$$

with

(62) $$ {}^kA = \frac{1}{\Omega^2+\hat{\Omega}^2}\,({}^k\Omega\,\Omega + {}^k\hat{\Omega}\,\hat{\Omega}), \qquad {}^kB = \frac{1}{\Omega^2+\hat{\Omega}^2}\,({}^k\Omega\,\hat{\Omega} - {}^k\hat{\Omega}\,\Omega), $$

and an arbitrary parameter ${}^k\xi$, in such a way that homogeneous, linear equations now remain for the new unknowns q_0 and $\mathfrak{q}$ (q_1, q_2, q_3). One can then set $q_0 = 1$ and obtain the equations for $\mathfrak{q}$:

(63) $$ \begin{cases} a) & (\mathfrak{s},\mathfrak{q}) = s_0, \\ b) & (\hat{\mathfrak{s}},\mathfrak{q}) = \hat{s}_0, \\ c) & s_0\mathfrak{q} - \mathfrak{s} = -i[\hat{\mathfrak{s}},\mathfrak{q}], \\ d) & \hat{s}_0\mathfrak{q} - \hat{\mathfrak{s}} = -i[\mathfrak{s},\mathfrak{q}]. \end{cases} $$

By vectorial multiplication of (63d) by $\mathfrak{s}$ and the employment of (63a, d) and the identity [I, (15)] or by vectorial multiplication of (63c) by $\hat{\mathfrak{s}}$ and the employment of (63b, c) and the identity [I, (16)], one will obtain in either case:

(64) $$ \mathfrak{q} = \frac{s_0\,\mathfrak{s} - \hat{s}_0\,\hat{\mathfrak{s}} + i[\mathfrak{s},\hat{\mathfrak{s}}]}{s_0^{\,2} - \hat{s}_0^{\,2} - (\Omega^2+\hat{\Omega}^2)}\,. $$

$\mathfrak{q}$ is a complex unit vector (in three dimensions) with $q_0 = 1$, together with a complex null vector (in four dimensions): One has $\sum_{i=1}^{3} q_i^2 = 1$, as one will recognize with the use of two formulas from [I, (36)] and the fact that $(s_0\,\mathfrak{s} - \hat{s}_0\,\hat{\mathfrak{s}}\,,\,[\mathfrak{s},\hat{\mathfrak{s}}\,]) = 0$.

B. One will find the solution for ${}^k\hat{s}_0$, ${}^k\hat{\mathfrak{s}}$ (${}^k\hat{s}_1$, ${}^k\hat{s}_2$, ${}^k\hat{s}_3$) from the second group of 16 linear equations § 1, eqs. (9), (2′), (10)-(15) in a completely analogous way:

(65) $$ \boxed{\begin{aligned} {}^k\hat{s}_0 &= {}^kA\,\hat{s}_0 - i\,{}^kB\,s_0 + {}^k\eta, \\ {}^k\hat{\mathfrak{s}} &= {}^kA\,\hat{\mathfrak{s}} - i\,{}^kB\,\mathfrak{s} + {}^k\eta\mathfrak{q}, \end{aligned}} $$

with the same meaning (62) for ${}^kA$ and ${}^kB$, the second arbitrary parameter ${}^k\eta$, and the same vector $\mathfrak{q}$ as in (64).

§ 6. Solution of the algebraic identities for forward differentiation.

Equations exist for the forward-differentiated quantities (§ 2) that arise from the backward-differentiated ones when all explicitly-appearing i are replaced with $-i$. One also obtains the solutions for forward differentiation from the solutions for backward differentiation by the same process. Let the unknown parameters that correspond to ${}^k\xi$ and ${}^k\eta$ under this transition be ξ^k and η^k, respectively, while the vector that is produced

by changing the sign of i in $\mathfrak{q}$ will be called $\mathfrak{p}$. One can then solve the corresponding equations for the forward differentiation by:

$$(66)\quad \begin{cases} s_0^k = A^k \cdot s_0 + i\,B^k \cdot \hat{s}_0 + \xi^k, \\ \mathfrak{s}^k = A^k \cdot \mathfrak{s} + i\,B^k \cdot \hat{\mathfrak{s}} + \xi^k \cdot \mathfrak{p}, \end{cases} \qquad (67)\quad \begin{cases} \hat{s}_0^k = A^k \cdot \hat{s}_0 + i\,B^k \cdot s_0 + \eta^k, \\ \hat{\mathfrak{s}}^k = A^k \cdot \hat{\mathfrak{s}} + i\,B^k \cdot \mathfrak{s} + \eta^k \cdot \mathfrak{p}, \end{cases}$$

with

$$(68)\qquad A^k = \frac{1}{\Omega^2 + \hat{\Omega}^2}(\Omega^k \cdot \Omega + \hat{\Omega}^k \cdot \hat{\Omega}), \qquad B^k = \frac{1}{\Omega^2 + \hat{\Omega}^2}(\Omega^k \cdot \hat{\Omega} + \hat{\Omega}^k \cdot \Omega),$$

and

$$(69)\qquad \mathfrak{p} = \frac{s_0\,\mathfrak{s} - \hat{s}_0\,\hat{\mathfrak{s}} - i[\mathfrak{s}, \hat{\mathfrak{s}}]}{s_0^{\,2} - \hat{s}_0^{\,2} - (\Omega^2 + \hat{\Omega}^2)}.$$

For *Hermitian* Dirac matrices, one will have $A^k = ({}^kA)^*$, $B^k = ({}^kB)^*$, $\mathfrak{p} = \mathfrak{q}^*$, $\xi^k = ({}^k\xi)^*$, $\eta^k = ({}^k\eta)^*$; this would not be true in the general case, since s_0, $\hat{s}_0$, $\mathfrak{s}$, $\hat{\mathfrak{s}}$, etc., would be real then. *In the general case,* the transition from backward to forward multiplication comes about by changing the sign in the explicitly-appearing i, replacing $\mathfrak{q}$ with $\mathfrak{p}$, and shifting the upper index from before to after.

The solutions for ${}^k\mathfrak{M}$, ${}^k\hat{\mathfrak{M}}$ ($\mathfrak{M}^k$, $\hat{\mathfrak{M}}^k$, resp.) that we obtained in § 8 are still lacking now.

§ 7. Some remarkable aspects of the complex vectors $\mathfrak{p}$ and $\mathfrak{q}$.

The complex unit vectors $\mathfrak{p}$ and $\mathfrak{q}$ admit the remarkable conversion in components:

$$\left.\begin{matrix} q_1 \\ p_1 \end{matrix}\right\} = \frac{s_0\,s_1 - \hat{s}_0\,\hat{s}_1 \pm i\,(s_2\,\hat{s}_3 - \hat{s}_2\,s_3)}{s_0^{\,2} - \hat{s}_0^{\,2} - \Omega^2 - \hat{\Omega}^2} = \frac{s_0\,\hat{s}_2 - \hat{s}_0\,s_2 \pm i\,(s_3\,s_1 - \hat{s}_3\,\hat{s}_1)}{s_1\,\hat{s}_2 - \hat{s}_1\,s_2 \pm i\,(s_0\,s_3 - \hat{s}_0\,\hat{s}_3)}.$$

In this, the upper sign always belongs to q_1, while the lower sign belongs to p_1. (One obtains q_2, q_3, p_2, p_3 by cyclic permutation of the indices 1, 2, 3.) In addition to (63), we shall mention the following formulas that will be applied, to some extent, in III and IV:

$$(70)\quad \begin{cases} a)\quad [\mathfrak{M}, \mathfrak{q}] = -\hat{\mathfrak{M}} + i\,\Omega\,\mathfrak{q}, \qquad e)\quad (\mathfrak{q}, \mathfrak{q}) = (\mathfrak{p}, \mathfrak{p}) = 1, \\ b)\quad [\hat{\mathfrak{M}}, \mathfrak{q}] = \mathfrak{M} + i\,\hat{\Omega}\,\mathfrak{q}, \qquad f)\quad (\mathfrak{p}, \mathfrak{q}) = \dfrac{s_0^{\,2} - \hat{s}_0^{\,2} + \Omega^2 + \hat{\Omega}^2}{s_0^{\,2} - \hat{s}_0^{\,2} - (\Omega^2 + \hat{\Omega}^2)}. \\ c)\quad (\mathfrak{M}, \mathfrak{q}) = -i\,\hat{\Omega}, \\ d)\quad (\hat{\mathfrak{M}}, \mathfrak{q}) = i\,\Omega, \end{cases}$$

The last one follows from:

(71) $$\hat{s}_k^{\,2} - s_k^{\,2} - \Omega^2 - \hat{\Omega}^2 = -\, p_k\, q_k\, (s_0^{\,2} - \hat{s}_0^{\,2} - \Omega^2 - \hat{\Omega}^2) \qquad k = 1, 2, 3.$$

Moreover, one has:

(72) $$s_0\, \mathfrak{s} - \hat{s}_0\, \mathfrak{s} = \frac{i}{2}(s_0^{\,2} - \hat{s}_0^{\,2} - \Omega^2 - \hat{\Omega}^2)\, [\mathfrak{p}, \mathfrak{q}],$$

(73) $$[\mathfrak{s}, \hat{\mathfrak{s}}\,] = \frac{i}{2}(s_0^{\,2} - \hat{s}_0^{\,2} - \Omega^2 - \hat{\Omega}^2)\, (\mathfrak{p} - \mathfrak{q}),$$

(74) $$s_0\, \mathfrak{s} - \hat{s}_0\, \hat{\mathfrak{s}} = \frac{i}{2}(s_0^{\,2} - \hat{s}_0^{\,2} - \Omega^2 - \hat{\Omega}^2)\, (\mathfrak{p} + \mathfrak{q}),$$

(74) $$s_1\, s_2 - \hat{s}_1\, \hat{s}_2 = \frac{i}{2}(s_0^{\,2} - \hat{s}_0^{\,2} - \Omega^2 - \hat{\Omega}^2)\, (q_1\, p_2 + p_1\, q_2).$$

§ 8. Agreement of solutions that are obtained in different ways.

Once the solutions for ${}^k s_0$, ${}^k\mathfrak{s}$, ${}^k\hat{s}_0$, ${}^k\hat{\mathfrak{s}}$ have been found (§ 5), the remaining quantities ${}^k\mathfrak{M}$ and ${}^k\hat{\mathfrak{M}}$, which we consider to be unknowns, can be calculated. As a result of the large number of equations in §§ 1 and 4, one can give these solutions different forms and verify their equivalence.

A. We first find ${}^k\mathfrak{M}$ and ${}^k\hat{\mathfrak{M}}$ in two forms, in which they are traced back to the quantities ${}^k s_0$, ${}^k\mathfrak{s}$, ${}^k\hat{s}_0$, ${}^k\hat{\mathfrak{s}}$. In order to do that, we eliminate ${}^k\hat{\mathfrak{M}}$ (${}^k\mathfrak{M}$, resp) from the equations:

(11′) $${}^k\mathfrak{M}\, \hat{\Omega} - {}^k\hat{\mathfrak{M}}\, \Omega = i\,({}^k\hat{s}_0\, \hat{\mathfrak{s}} - \hat{\mathfrak{s}}\, \hat{s}_0) - [\,{}^k\mathfrak{s}, \hat{\mathfrak{s}}]$$

and

(4′) $${}^k\mathfrak{M}\, \Omega + {}^k\hat{\mathfrak{M}}\, \hat{\Omega} = -(\,{}^k\hat{s}_0\, \mathfrak{s} - {}^k\hat{\mathfrak{s}}\, s_0) - i[\,{}^k\mathfrak{s}, \mathfrak{s}],$$

and obtain a first form:

(76) $$\begin{cases} {}^k\mathfrak{M}(\Omega^2 + \hat{\Omega}^2) = \Omega\{(s_0\, {}^k\hat{\mathfrak{s}} - {}^k\hat{s}_0\, \mathfrak{s}) + i[\mathfrak{s}, {}^k\mathfrak{s}]\} - \hat{\Omega}\{[\,{}^k\mathfrak{s}, \hat{\mathfrak{s}}] + i\,(\hat{s}_0\, {}^k\hat{\mathfrak{s}} - {}^k\hat{s}_0\, \hat{\mathfrak{s}})\}, \\ {}^k\hat{\mathfrak{M}}(\Omega^2 + \hat{\Omega}^2) = \hat{\Omega}\{(s_0\, {}^k\hat{\mathfrak{s}} - {}^k\hat{s}_0\, \mathfrak{s}) + i[\mathfrak{s}, {}^k\mathfrak{s}]\} - \Omega\{[\,{}^k\mathfrak{s}, \hat{\mathfrak{s}}] + i\,(\hat{s}_0\, {}^k\hat{\mathfrak{s}} - {}^k\hat{s}_0\, \hat{\mathfrak{s}})\}. \end{cases}$$

The second form contains terms with and without the imaginary unit in a mixture that is more suitable for a later application (§ 10). One gets it when one drops the quantities [${}^k\mathfrak{s}$, $\hat{\mathfrak{s}}$] and $s_0\, {}^k\hat{\mathfrak{s}} - {}^k\hat{s}_0\, \mathfrak{s}$ from (76) with the help of eqs. (3) and (10). It reads:

$$
(77)\quad \begin{cases}
{}^k\mathfrak{M}(\Omega^2+\hat{\Omega}^2) = ({}^k\Omega\,\Omega + {}^k\hat{\Omega}\,\hat{\Omega})\,\mathfrak{M} + ({}^k\Omega\,\Omega - {}^k\hat{\Omega}\,\hat{\Omega})\,\hat{\mathfrak{M}} \\
\qquad\qquad + i\,\Omega\{[\mathfrak{s}, {}^k\mathfrak{s}] - [\hat{\mathfrak{s}}, {}^k\hat{\mathfrak{s}}]\} \\
\qquad\qquad + i\,\hat{\Omega}(s_0\,{}^k\mathfrak{s} - {}^k s_0\,\mathfrak{s} + {}^k\hat{s}_0\,\hat{\mathfrak{s}} - \hat{s}_0\,{}^k\hat{\mathfrak{s}}), \\
{}^k\hat{\mathfrak{M}}(\Omega^2+\hat{\Omega}^2) = ({}^k\Omega\,\Omega + {}^k\hat{\Omega}\,\hat{\Omega})\,\hat{\mathfrak{M}} + ({}^k\hat{\Omega}\,\Omega - {}^k\Omega\,\hat{\Omega})\,\mathfrak{M} \\
\qquad\qquad + i\,\hat{\Omega}\{[\mathfrak{s}, {}^k\mathfrak{s}] - [\hat{\mathfrak{s}}, {}^k\hat{\mathfrak{s}}]\} \\
\qquad\qquad - i\,\Omega(s_0\,{}^k\mathfrak{s} - {}^k s_0\,\mathfrak{s} + {}^k\hat{s}_0\,\hat{\mathfrak{s}} - \hat{s}_0\,{}^k\hat{\mathfrak{s}}).
\end{cases}
$$

In this form, one sees that ${}^k\hat{\mathfrak{M}}$ will emerge from ${}^k\mathfrak{M}$ when one replaces Ω with $\hat{\Omega}$, $\hat{\Omega}$ with $-\Omega$, ${}^k\Omega$ with, and ${}^k\hat{\Omega}$ with $-{}^k\Omega$.

B. In (76) and (77), ${}^k\mathfrak{M}$ and ${}^k\hat{\mathfrak{M}}$ still contain ${}^k s_0$, ${}^k\mathfrak{s}$, ${}^k\hat{s}_0$, ${}^k\hat{\mathfrak{s}}$. If one introduces the Ansätze (61) and (65) for these quantities in (76) then one will get representations for ${}^k\mathfrak{M}$ and ${}^k\hat{\mathfrak{M}}$ in a form that is analogous to (61) and (65). Only the "known" quantities ${}^k\Omega$ and ${}^k\hat{\Omega}$ (according to the classification that was mentioned in § 5) and the undetermined parameters ${}^k\xi$ and ${}^k\eta$ will still remain then. If one recalls the identities [I, (34) and (35)] and the definition (62) for ${}^kA$ and ${}^kB$ then that will imply:

$$
(78)\quad \begin{cases}
{}^k\mathfrak{M} = {}^kA\,\mathfrak{M} + {}^kB\,\hat{\mathfrak{M}} + \dfrac{\Omega}{\Omega^2+\hat{\Omega}^2}\{({}^k\xi\,\hat{\mathfrak{s}} - {}^k\eta\,\mathfrak{s}) - \mathfrak{q}({}^k\xi\,\hat{s}_0 - {}^k\eta\,s_0)\} \\
\qquad\qquad + \dfrac{\hat{\Omega}}{\Omega^2+\hat{\Omega}^2}[\mathfrak{q}, ({}^k\xi\,\hat{\mathfrak{s}} - {}^k\eta\,\mathfrak{s})], \\
{}^k\hat{\mathfrak{M}} = {}^kA\,\hat{\mathfrak{M}} - {}^kB\,\hat{\mathfrak{M}} + \dfrac{\hat{\Omega}}{\Omega^2+\hat{\Omega}^2}\{({}^k\xi\,\hat{\mathfrak{s}} - {}^k\eta\,\mathfrak{s}) - \mathfrak{q}({}^k\xi\,\hat{s}_0 - {}^k\eta\,s_0)\} \\
\qquad\qquad + \dfrac{\Omega}{\Omega^2+\hat{\Omega}^2}[\mathfrak{q}, ({}^k\xi\,\hat{\mathfrak{s}} - {}^k\eta\,\mathfrak{s})].
\end{cases}
$$

Whereas only the undetermined parameter ${}^k\xi$ appears in the solutions for ${}^k s_0$, ${}^k\mathfrak{s}$, and only ${}^k\eta$ appears in the solutions for ${}^k\hat{s}_0$, ${}^k\hat{\mathfrak{s}}$, both of them occur together in ${}^k\mathfrak{M}$ and ${}^k\hat{\mathfrak{M}}$. This representation for ${}^k\mathfrak{M}$ and ${}^k\hat{\mathfrak{M}}$ is therefore less simple.

C. Finally, in § 4, one of the algebraic identities [I, (18)] {[I, (19)], resp.} gave an analogous representation for ${}^k\mathfrak{M}$ (${}^k\hat{\mathfrak{M}}$, resp.). For example, one then has:

$$
(79)\qquad {}^k\mathfrak{M}\,\{({}^k\Omega)^2 + ({}^k\hat{\Omega})^2\} = {}^k\Omega\,({}^k s_0\,{}^k\hat{\mathfrak{s}} - {}^k\hat{s}_0\,{}^k\mathfrak{s}) - {}^k\hat{\Omega}[{}^k\mathfrak{s}, {}^k\hat{\mathfrak{s}}]
$$

for ${}^k\mathfrak{M}$. One can establish the agreement of (79) with (78) by substituting the solutions (61) and (65) for ${}^k s_0$, ${}^k\mathfrak{s}$, ${}^k\hat{s}_0$, ${}^k\hat{\mathfrak{s}}$, in (79), dropping the explicitly-appearing i with the help of eqs. (63c and d), and observing that:

$$(80)\qquad \begin{cases} ({}^kA)^2 + ({}^kB)^2 = \dfrac{({}^k\Omega)^2 + ({}^k\hat{\Omega})^2}{{}^k\Omega^2 + {}^k\hat{\Omega}^2}, \quad {}^k\Omega\,{}^kA + {}^k\hat{\Omega}\,{}^kB = \Omega\dfrac{({}^k\Omega)^2 + ({}^k\hat{\Omega})^2}{{}^k\Omega^2 + {}^k\hat{\Omega}^2}, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad {}^k\Omega\,{}^kB + {}^k\hat{\Omega}\,{}^kA = \hat{\Omega}\dfrac{({}^k\Omega)^2 + ({}^k\hat{\Omega})^2}{{}^k\Omega^2 + {}^k\hat{\Omega}^2}. \end{cases}$$

Formulas that are analogous to (76)-(78) are valid for the forward-differentiated quantities $\mathfrak{M}^k$ and $\hat{\mathfrak{M}}^k$. In order to obtain them, one replaces all upper pre-indices with post-indices, changes the signs of all explicitly-appearing i's, and replaces $\mathfrak{q}$ with $\mathfrak{p}$.

§ 9. Determination of the parameters ξ and η and completion of the solutions.

Up to now, no use was made of the fact that the upper index k means differentiation with respect to x_k. Up to now, it was purely an index notation, and one could just as well replace ψ with an arbitrary function φ, instead of with $\partial\psi / \partial x_k$. Recalling its meaning will make it possible to determine the parameters ${}^k\xi$, ξ^k, ${}^k\eta$, η^k in the solutions (61), (65), (66), (67), and (78), which have been undetermined up to now.

A. When one combines eq. (3) for backward differentiation with the corresponding one for forward differentiation, one will get:

$$i\,({}^k\mathfrak{s} - \mathfrak{s}^k) = \frac{1}{s_0}\left\{ \mathfrak{s}\cdot i\,({}^k s_0 - s_0^{\,k}) + \left[\mathfrak{s}, \frac{\partial\mathfrak{s}}{\partial x_k}\right] - \mathfrak{M}\frac{\partial\hat{\Omega}}{\partial x_k} + \hat{\mathfrak{M}}\frac{\partial\Omega}{\partial x_k} \right\}.$$

If one introduces this expression into the likewise-extended eq. (1):

$$s_0 \cdot i\,({}^k s_0 - s_0^{\,k}) - (\mathfrak{s}, i\,({}^k\mathfrak{s} - \mathfrak{s}^k)) = \Omega \cdot i\,({}^k\Omega - \Omega^k) + \hat{\Omega}\cdot i\,({}^k\hat{\Omega} - \hat{\Omega}^k)$$

then if one recalls [I, (15), (25), (27)], one will get:

$$\begin{aligned}(\Omega^2 + \hat{\Omega}^2)\cdot i\,({}^k s_0 - s_0^{\,k}) &= s_0\,\{\,\Omega\cdot i\,({}^k\Omega - \Omega^k) + \hat{\Omega}\cdot i\,({}^k\hat{\Omega} - \hat{\Omega}^k)\,\} \\ &\quad + \hat{s}_0\left(\hat{\Omega}\frac{\partial\Omega}{\partial x_k} - \Omega\frac{\partial\hat{\Omega}}{\partial x_k}\right) + \left([\mathfrak{s}, \hat{\mathfrak{s}}], \frac{\partial\mathfrak{s}}{\partial x_k}\right).\end{aligned}$$

On the other hand, it follows from the solutions (61) and (66) that:

$$(\Omega^2 + \hat{\Omega}^2) \cdot i\,({}^k s_0 - s_0^{\,k}) = s_0\,\{\Omega \cdot i\,({}^k\Omega - \Omega^k) + \hat{\Omega} \cdot i\,({}^k\hat{\Omega} - \hat{\Omega}^k)\}$$
$$+\ \hat{s}_0 \left(\hat{\Omega}\frac{\partial \Omega}{\partial x_k} - \Omega \frac{\partial \hat{\Omega}}{\partial x_k} \right) + (\Omega^2 + \hat{\Omega}^2) \cdot i\,({}^k\xi - \xi^k)\,.$$

A comparison of the two expressions will then yield the imaginary part of ${}^k\xi$:

(81) $$\boxed{(\Omega^2 + \hat{\Omega}^2) \cdot i\,({}^k\xi - \xi^k) = \left([\mathfrak{s}, \hat{\mathfrak{s}}], \frac{\partial \mathfrak{s}}{\partial x_k} \right).}$$

B. The sum ${}^k s_0 + s_0^{\,k} = \partial s_0 / \partial x_k$ can likewise be constructed with the help of the solutions (61) and (66):

$$(\Omega^2 + \hat{\Omega}^2)\ \frac{\partial s_0}{\partial x_k} = \left(\Omega \frac{\partial \Omega}{\partial x_k} + \hat{\Omega} \frac{\partial \hat{\Omega}}{\partial x_k} \right) s_0$$
$$-\ \{\hat{\Omega} \cdot i\,({}^k\Omega - \Omega^k) - \Omega \cdot i\,({}^k\hat{\Omega} - \hat{\Omega}^k)\}\,\hat{s}_0 + (\Omega^2 + \hat{\Omega}^2)({}^k\xi + \xi^k).$$

On the other hand, one will get from eq. (2), when extended by forward differentiation:

$$\hat{\Omega} \cdot i\,({}^k\Omega - \Omega^k) - \Omega \cdot i\,({}^k\hat{\Omega} - \hat{\Omega}^k) = \hat{s}_0 \frac{\partial s_0}{\partial x_k} - \left(\hat{\mathfrak{s}}, \frac{\partial \mathfrak{s}}{\partial x_k} \right),$$

which one introduces above, that the real part of ${}^k\xi$ is:

$$(\Omega^2 + \hat{\Omega}^2)({}^k\xi + \xi^k) = (\Omega^2 + \hat{\Omega}^2 + \hat{s}_0^2)\ \frac{\partial s_0}{\partial x_k} - s_0 \left(\Omega \frac{\partial \Omega}{\partial x_k} + \hat{\Omega} \frac{\partial \hat{\Omega}}{\partial x_k} \right) - \hat{s}_0 \left(\hat{\mathfrak{s}}, \frac{\partial \mathfrak{s}}{\partial x_k} \right).$$

Use was made in this of the abbreviation:

(82) $$\left(\mathfrak{A}, \frac{\partial \mathfrak{B}}{\partial x_k} \right) = A_1 \frac{\partial B_1}{\partial x_k} + A_2 \frac{\partial B_2}{\partial x_k} + A_3 \frac{\partial B_3}{\partial x_k} = [\mathfrak{A}, \text{rot}\ \mathfrak{B}]_k + (\mathfrak{A}\ \text{grad})\ \mathfrak{B}_k\,.$$

The real part of ${}^k\xi$ can be written as follows, if one employs the identity [I, (15)] when differentiated with respect to x_k:

(83) $$\boxed{(\Omega^2 + \hat{\Omega}^2)({}^k\xi + \xi^k) = (\Omega^2 + \hat{\Omega}^2 + \hat{s}_0^2 - s_0^2) \frac{\partial s_0}{\partial x_k} + \left(s_0\ \mathfrak{s} - \hat{s}_0 \hat{\mathfrak{s}}, \frac{\partial \mathfrak{s}}{\partial x_k} \right).}$$

C. If one recalls the meanings (64) and (69) of the vectors $\mathfrak{q}$ and $\mathfrak{p}$ then one will get the parameters ${}^k\xi$ and ξ^k from (81) and (83):

$$(84)\qquad \begin{cases} {}^k\xi = \dfrac{s_0^{\,2} - \hat{s}_0^{\,2} - (\Omega^2 + \hat{\Omega}^2)}{2(\Omega^2 + \hat{\Omega}^2)} \left\{ -\dfrac{\partial s_0}{\partial x_k} + \left(\mathfrak{p}, \dfrac{\partial \mathfrak{s}}{\partial x_k} \right) \right\}, \\[2ex] \xi^k = \dfrac{s_0^{\,2} - \hat{s}_0^{\,2} - (\Omega^2 + \hat{\Omega}^2)}{2(\Omega^2 + \hat{\Omega}^2)} \left\{ -\dfrac{\partial s_0}{\partial x_k} + \left(\mathfrak{q}, \dfrac{\partial \mathfrak{s}}{\partial x_k} \right) \right\}. \end{cases}$$

One can also calculate ${}^k\eta$ and η^k in a completely analogous way from eqs. (11), (9), and (2′), when extended by forward differentiation and with an application of the identities [I, (16), (26), (28)]; one will get:

$$(85)\qquad \begin{cases} {}^k\eta = \dfrac{s_0^{\,2} - \hat{s}_0^{\,2} - (\Omega^2 + \hat{\Omega}^2)}{2(\Omega^2 + \hat{\Omega}^2)} \left\{ -\dfrac{\partial \hat{s}_0}{\partial x_k} + \left(\mathfrak{p}, \dfrac{\partial \hat{\mathfrak{s}}}{\partial x_k} \right) \right\}, \\[2ex] \eta^k = \dfrac{s_0^{\,2} - \hat{s}_0^{\,2} - (\Omega^2 + \hat{\Omega}^2)}{2(\Omega^2 + \hat{\Omega}^2)} \left\{ -\dfrac{\partial \hat{s}_0}{\partial x_k} + \left(\mathfrak{q}, \dfrac{\partial \hat{\mathfrak{s}}}{\partial x_k} \right) \right\}. \end{cases}$$

Substituting ${}^k\xi$, ξ^k, ${}^k\eta$, η^k in formulas (61), (65), (66), (67), and (78) will complete the solutions ${}^ks_0$, ${}^k\mathfrak{s}$, ${}^k\hat{s}_0$, ${}^k\mathfrak{M}$, ${}^k\hat{\mathfrak{M}}$. Only two uninterpretable quantities ${}^k\Omega$ and ${}^k\hat{\Omega}$ will appear in the completed solutions, in addition to the interpretable ones.

The solutions that were given satisfy the entire list of eqs. (1)-(52). They contain no algebraic identity that does not trace back to the ones that were mentioned before.

§ 10. Tracing the uninterpretable quantities back to interpretable ones.

The results of the foregoing paragraphs make it possible to trace all of the uninterpretable quantities – viz., $i\,({}^ks_0 - s_0^{\,k})$, $i\,({}^k\mathfrak{s} - \mathfrak{s}^k)$, $i\,({}^k\hat{s}_0 - \hat{s}_0^{\,k})$, $i\,({}^k\hat{\mathfrak{s}} - \hat{\mathfrak{s}}^k)$, $i\,({}^k\mathfrak{M} - \mathfrak{M}^k)$, $i\,({}^k\hat{\mathfrak{M}} - \hat{\mathfrak{M}}^k)$, $i\,({}^k\Omega - \Omega^k)$, $-i\,({}^k\hat{\Omega} - \hat{\Omega}^k)$ to a single one; e.g., $i\,({}^k\Omega - \Omega^k)$. With the notation:

$$(86)\qquad U_k = \Omega \cdot i\,({}^k\Omega - \Omega^k) + \hat{\Omega} \cdot i\,({}^k\hat{\Omega} - \hat{\Omega}^k),$$

one can give them the following representation, if one preserves two uninterpretable quantities:

$$i({}^k s_0 - s_0^{\,k})(\Omega^2 + \hat{\Omega}^2) = s_0 U_k + \hat{s}_0 \left(\hat{\Omega} \frac{\partial \Omega}{\partial x_k} - \Omega \frac{\partial \hat{\Omega}}{\partial x_k} \right) + \left([\mathfrak{s}, \hat{\mathfrak{s}}], \frac{\partial \mathfrak{s}}{\partial x_k} \right), \tag{87}$$

$$i({}^k \mathfrak{s} - \mathfrak{s}^k)(\Omega^2 + \hat{\Omega}^2) = \mathfrak{s} U_k + \hat{\mathfrak{s}} \left(\hat{\Omega} \frac{\partial \Omega}{\partial x_k} - \Omega \frac{\partial \hat{\Omega}}{\partial x_k} \right) + [\mathfrak{s}, \hat{\mathfrak{s}}] \frac{\partial s_0}{\partial x_k} + \left[s_0 \hat{\mathfrak{s}} - \hat{s}_0 \mathfrak{s}, \frac{\partial \mathfrak{s}}{\partial x_k} \right], \tag{88}$$

$$i({}^k \hat{s}_0 - \hat{s}_0^{\,k})(\Omega^2 + \hat{\Omega}^2) = \hat{s}_0 U_k + s_0 \left(\hat{\Omega} \frac{\partial \Omega}{\partial x_k} - \Omega \frac{\partial \hat{\Omega}}{\partial x_k} \right) + \left([\mathfrak{s}, \hat{\mathfrak{s}}], \frac{\partial \hat{\mathfrak{s}}}{\partial x_k} \right), \tag{89}$$

$$i({}^k \hat{\mathfrak{s}} - \hat{\mathfrak{s}}^k)(\Omega^2 + \hat{\Omega}^2) = \hat{\mathfrak{s}} U_k + \mathfrak{s} \left(\hat{\Omega} \frac{\partial \Omega}{\partial x_k} - \Omega \frac{\partial \hat{\Omega}}{\partial x_k} \right) + [\mathfrak{s}, \hat{\mathfrak{s}}] \frac{\partial s_0}{\partial x_k} + \left[s_0 \hat{\mathfrak{s}} - \hat{s}_0 \mathfrak{s}, \frac{\partial \hat{\mathfrak{s}}}{\partial x_k} \right], \tag{90}$$

$$\begin{aligned} i({}^k \mathfrak{M} - \mathfrak{M}^k)(\Omega^2 + \hat{\Omega}^2) &= \mathfrak{M} U_k + \hat{\mathfrak{M}} \left\{ s_0 \frac{\partial \hat{s}_0}{\partial x_k} - \left(\mathfrak{s}, \frac{\partial \hat{\mathfrak{s}}}{\partial x_k} \right) \right\} - \Omega \left(\left[\mathfrak{s}, \frac{\partial \mathfrak{s}}{\partial x_k} \right] - \left[\hat{\mathfrak{s}}, \frac{\partial \hat{\mathfrak{s}}}{\partial x_k} \right] \right) \\ &\quad - \hat{\Omega} \left(s_0 \frac{\partial \mathfrak{s}}{\partial x_k} - \hat{s}_0 \frac{\partial \hat{\mathfrak{s}}}{\partial x_k} - \mathfrak{s} \frac{\partial s_0}{\partial x_k} + \hat{\mathfrak{s}} \frac{\partial \hat{s}_0}{\partial x_k} \right), \end{aligned} \tag{91}$$

$$\begin{aligned} i({}^k \hat{\mathfrak{M}} - \hat{\mathfrak{M}}^k)(\Omega^2 + \hat{\Omega}^2) &= \hat{\mathfrak{M}} U_k + \mathfrak{M} \left\{ \hat{s}_0 \frac{\partial s_0}{\partial x_k} - \left(\hat{\mathfrak{s}}, \frac{\partial \mathfrak{s}}{\partial x_k} \right) \right\} - \hat{\Omega} \left(\left[\mathfrak{s}, \frac{\partial \mathfrak{s}}{\partial x_k} \right] - \left[\hat{\mathfrak{s}}, \frac{\partial \hat{\mathfrak{s}}}{\partial x_k} \right] \right) \\ &\quad + \Omega \left(s_0 \frac{\partial \mathfrak{s}}{\partial x_k} - \hat{s}_0 \frac{\partial \hat{\mathfrak{s}}}{\partial x_k} - \mathfrak{s} \frac{\partial s_0}{\partial x_k} + \hat{\mathfrak{s}} \frac{\partial \hat{s}_0}{\partial x_k} \right). \end{aligned} \tag{92}$$

This notation has the advantage that is it symmetric in comparison to the one that one obtains when one ultimately makes use of the remaining relationship:

$$\left\{ \begin{aligned} &\hat{\Omega} \cdot i({}^k \Omega - \Omega^k) - \Omega \cdot i({}^k \hat{\Omega} - \hat{\Omega}^k) \\ &\qquad = \hat{s}_0 \frac{\partial s_0}{\partial x_k} - \left(\hat{\mathfrak{s}}, \frac{\partial \mathfrak{s}}{\partial x_k} \right) = -s_0 \frac{\partial \hat{s}_0}{\partial x_k} + \left(\mathfrak{s}, \frac{\partial \hat{\mathfrak{s}}}{\partial x_k} \right) \end{aligned} \right. \tag{93}$$

in order to eliminate – e.g., $i({}^k \hat{\Omega} - \hat{\Omega}^k)$ – from (87)-(92). However, one sees from this that it is possible to reduce all of the uninterpretable quantities to a single one.

Derivation of eqs. (87)-(93). One will get eqs. (87)-(90) from the solutions (61) and (66), (65) and (67) by combining the corresponding backward and forward-differentiated quantities. In this, one must consider that one has:

$$
(94)\qquad \begin{cases}
i\,({}^{k}A - A^{k})(\Omega^2 + \hat{\Omega}^2) = U_k, \quad ({}^{k}B + B^{k})(\Omega^2 + \hat{\Omega}^2) = \hat{\Omega}\dfrac{\partial \Omega}{\partial x_k} - \Omega\dfrac{\partial \hat{\Omega}}{\partial x_k}, \\[2ex]
i\,({}^{k}\xi - \xi^{k})(\Omega^2 + \hat{\Omega}^2) = \left([\mathfrak{s}, \hat{\mathfrak{s}}], \dfrac{\partial \mathfrak{s}}{\partial x_k}\right), \\[2ex]
i\,({}^{k}\xi\mathfrak{q} - \mathfrak{p}\xi^{k})(\Omega^2 + \hat{\Omega}^2) = [\mathfrak{s}, \hat{\mathfrak{s}}]\dfrac{\partial s_0}{\partial x_k} + \left[s_0\,\hat{\mathfrak{s}} - \hat{s}_0\,\mathfrak{s}, \dfrac{\partial \mathfrak{s}}{\partial x_k}\right], \\[2ex]
i\,({}^{k}\eta - \eta^{k})(\Omega^2 + \hat{\Omega}^2) = \left([\mathfrak{s}, \hat{\mathfrak{s}}], \dfrac{\partial \hat{\mathfrak{s}}}{\partial x_k}\right), \\[2ex]
i\,({}^{k}\xi\mathfrak{q} - \mathfrak{p}\xi^{k})(\Omega^2 + \hat{\Omega}^2) = [\mathfrak{s}, \hat{\mathfrak{s}}]\dfrac{\partial \hat{s}_0}{\partial x_k} + \left[s_0\,\hat{\mathfrak{s}} - \hat{s}_0\,\mathfrak{s}, \dfrac{\partial \hat{\mathfrak{s}}}{\partial x_k}\right].
\end{cases}
$$

These expressions agree with ones that one can obtain by the combinations of uninterpretable quantities in the next paragraph. For example, one gets (87) by multiplying [§ 11, eq. (7)] by Ω, [§ 11, eq. (14)] by $\hat{\Omega}$ and adding them, (88), by multiplying [§ 11, eq. (8) by Ω, [§ 11, eq. (6)] by $\hat{\Omega}$ and adding them, (89), by multiplying [§ 11, eq. (14)] by Ω, [§ 11, eq. (12)] by $\hat{\Omega}$ and adding them, and (90), by multiplying [§ 11, eq. (15)] by Ω, [§ 11, eq. (13)] by $\hat{\Omega}$ and adding them. Eqs. (91) and (92) arise by applying (93) to the formulas (77) for ${}^{k}\mathfrak{M}$ and ${}^{k}\hat{\mathfrak{M}}$ while combining them with a corresponding expression in the forward-differentiated quantities. Eq. (93) itself emerges from a combination of eq. (2) or (2′) and the corresponding ones for the forward-differentiated quantities.

§ 11. Combinations of uninterpretable quantities.

As we mentioned before, the total system of formulas § 1, eqs. (1)-(52) for backward differentiation will go to a system of equations that takes combinations of uninterpretable quantities into interpretable ones by a suitable coupling with a corresponding formulas for forward differentiation. In the following summary, the most important combinations of uninterpretable quantities will be written down, namely, the ones that will be applied in III, and with the same numbers as their original equations in § 1.

$$
(2)\qquad \begin{cases}
i\,({}^{k}\Omega - \Omega^{k})\,\hat{\Omega} - i\,({}^{k}\hat{\Omega} - \hat{\Omega}^{k})\,\Omega \\[1ex]
= \hat{s}_0\dfrac{\partial s_0}{\partial x_k} - \left(\mathfrak{s}, \dfrac{\partial \mathfrak{s}}{\partial x_k}\right) = -s_0\dfrac{\partial \hat{s}_0}{\partial x_k} + \left(\mathfrak{s}, \dfrac{\partial \hat{\mathfrak{s}}}{\partial x_k}\right),
\end{cases}
$$

$$(5)\quad \begin{cases} i\,({}^k s_0 - s_0^{\,k})\,\hat{\Omega} - i\,({}^k\hat{\Omega} - \hat{\Omega}^k)\,\Omega \\ = \hat{s}_0 \dfrac{\partial s_0}{\partial x_k} - \left(\hat{\mathfrak{s}}, \dfrac{\partial \mathfrak{s}}{\partial x_k}\right) = -s_0 \dfrac{\partial \hat{s}_0}{\partial x_k} + \left(\mathfrak{s}, \dfrac{\partial \hat{\mathfrak{s}}}{\partial x_k}\right), \end{cases}$$

$$(6)\quad \begin{cases} i\,({}^k \mathfrak{s} - \mathfrak{s}^k)\,\hat{\Omega} - i\,({}^k\hat{\Omega} - \hat{\Omega}^k)\,\mathfrak{s} \\ = \left[\mathfrak{s}, \dfrac{\partial \hat{\mathfrak{M}}}{\partial x_k}\right] + s_0 \dfrac{\partial \mathfrak{M}}{\partial x_k} - \Omega \dfrac{\partial \hat{\mathfrak{s}}}{\partial x_k} = \left[\hat{\mathfrak{M}}, \dfrac{\partial \mathfrak{s}}{\partial x_k}\right] - \mathfrak{M} \dfrac{\partial s}{\partial x_k} + \hat{\mathfrak{s}} \dfrac{\partial \Omega}{\partial x_k}, \end{cases}$$

$$(7)\quad \begin{cases} i\,({}^k s_0 - s_0^{\,k})\,\Omega - i\,({}^k\Omega - \Omega^k)\,s_0 \\ = \left(\hat{\mathfrak{M}}, \dfrac{\partial \mathfrak{s}}{\partial x_k}\right) - \hat{s}_0 \dfrac{\partial \hat{\Omega}}{\partial x_k} = -\left(\mathfrak{s}, \dfrac{\partial \hat{\mathfrak{M}}}{\partial x_k}\right) + \Omega \dfrac{\partial \hat{s}_0}{\partial x_k}, \end{cases}$$

$$(8)\quad \begin{cases} i\,({}^k \mathfrak{s} - \mathfrak{s}^k)\,\Omega - i\,({}^k\Omega - \Omega^k)\,\mathfrak{s} \\ = \left[\mathfrak{s}, \dfrac{\partial \mathfrak{M}}{\partial x_k}\right] - s_0 \dfrac{\partial \hat{\mathfrak{M}}}{\partial x_k} + \hat{\Omega} \dfrac{\partial \hat{\mathfrak{s}}}{\partial x_k} = \left[\mathfrak{M}, \dfrac{\partial \mathfrak{s}}{\partial x_k}\right] + \hat{\mathfrak{M}} \dfrac{\partial s_0}{\partial x_k} - \hat{\mathfrak{s}} \dfrac{\partial \hat{\Omega}}{\partial x_k}, \end{cases}$$

$$(12)\quad \begin{cases} i\,({}^k \hat{s}_0 - \hat{s}_0^{\,k})\,\hat{\Omega} - i\,({}^k\hat{\Omega} - \hat{\Omega}^k)\,\hat{s}_0 \\ = -\left(\mathfrak{M}, \dfrac{\partial \hat{\mathfrak{s}}}{\partial x_k}\right) + s_0 \dfrac{\partial \Omega}{\partial x_k} = \left(\hat{\mathfrak{s}}, \dfrac{\partial \mathfrak{M}}{\partial x_k}\right) - \Omega \dfrac{\partial s_0}{\partial x_k}, \end{cases}$$

$$(13)\quad \begin{cases} i\,({}^k \hat{\mathfrak{s}} - \hat{\mathfrak{s}}^k)\,\hat{\Omega} - i\,({}^k\hat{\Omega} - \hat{\Omega}^k)\,\hat{\mathfrak{s}} \\ = -\Omega \dfrac{\partial \mathfrak{s}}{\partial x_k} + \left[\hat{\mathfrak{s}}, \dfrac{\partial \hat{\mathfrak{M}}}{\partial x_k}\right] + \hat{s}_0 \dfrac{\partial \mathfrak{M}}{\partial x_k} = \mathfrak{s} \dfrac{\partial \Omega}{\partial x_k} + \left[\hat{\mathfrak{M}}, \dfrac{\partial \hat{\mathfrak{s}}}{\partial x_k}\right] - \mathfrak{M} \dfrac{\partial \hat{s}_0}{\partial x_k}, \end{cases}$$

$$(14)\quad \begin{cases} i\,({}^k \hat{s}_0 - \hat{s}_0^{\,k})\,\Omega - i\,({}^k\Omega - \Omega^k)\,\hat{s}_0 \\ = \left(\hat{\mathfrak{M}}, \dfrac{\partial \hat{\mathfrak{s}}}{\partial x_k}\right) - s_0 \dfrac{\partial \hat{\Omega}}{\partial x_k} = -\left(\hat{\mathfrak{s}}, \dfrac{\partial \hat{\mathfrak{M}}}{\partial x_k}\right) + \hat{\Omega} \dfrac{\partial s_0}{\partial x_k}, \end{cases}$$

$$(15)\quad \begin{cases} i\,({}^k \hat{\mathfrak{s}} - \hat{\mathfrak{s}}^k)\,\Omega - i\,({}^k\Omega - \Omega^k)\,\hat{\mathfrak{s}} \\ = \hat{\Omega} \dfrac{\partial \mathfrak{s}}{\partial x_k} + \left[\hat{\mathfrak{s}}, \dfrac{\partial \mathfrak{M}}{\partial x_k}\right] - \hat{s}_0 \dfrac{\partial \hat{\mathfrak{M}}}{\partial x_k} = -\mathfrak{s} \dfrac{\partial \hat{\Omega}}{\partial x_k} + \left[\mathfrak{M}, \dfrac{\partial \hat{\mathfrak{s}}}{\partial x_k}\right] + \hat{\mathfrak{M}} \dfrac{\partial \hat{s}_0}{\partial x_k}, \end{cases}$$

$$(20)\quad \begin{cases} i\,({}^k\hat{\mathfrak{M}} - \hat{\mathfrak{M}}^k)\,\Omega - i\,({}^k\Omega - \Omega^k)\,\hat{\mathfrak{M}} \\ = \left[\hat{\mathfrak{M}}, \dfrac{\partial \mathfrak{M}}{\partial x_k}\right] + s_0 \dfrac{\partial \mathfrak{s}}{\partial x_k} - \hat{s}_0 \dfrac{\partial \hat{\mathfrak{s}}}{\partial x_k} = \left[\mathfrak{M}, \dfrac{\partial \hat{\mathfrak{M}}}{\partial x_k}\right] - \mathfrak{s} \dfrac{\partial s_0}{\partial x_k} + \hat{\mathfrak{s}} \dfrac{\partial \hat{s}_0}{\partial x_k}, \end{cases}$$

$$(21)\quad \begin{cases} i\,({}^k\mathfrak{M} - \mathfrak{M}^k)\,\hat{\Omega} - i\,({}^k\hat{\Omega} - \hat{\Omega}^k)\,\mathfrak{M} \\ = \left[\mathfrak{M}, \dfrac{\partial \hat{\mathfrak{M}}}{\partial x_k}\right] - s_0 \dfrac{\partial \mathfrak{s}}{\partial x_k} + \hat{s}_0 \dfrac{\partial \hat{\mathfrak{s}}}{\partial x_k} = \left[\hat{\mathfrak{M}}, \dfrac{\partial \mathfrak{M}}{\partial x_k}\right] + \mathfrak{s} \dfrac{\partial s_0}{\partial x_k} - \hat{\mathfrak{s}} \dfrac{\partial \hat{s}_0}{\partial x_k}, \end{cases}$$

$$(43)\quad \begin{cases} i\,({}^k\mathfrak{M} - \mathfrak{M}^k)\,\Omega - i\,({}^k\Omega + \Omega^k)\,\mathfrak{M} \\ = \dfrac{1}{2}\left\{\left[\mathfrak{M}, \dfrac{\partial \mathfrak{M}}{\partial x_k}\right] - \left[\hat{\mathfrak{M}}, \dfrac{\partial \hat{\mathfrak{M}}}{\partial x_k}\right] - \left[\mathfrak{s}, \dfrac{\partial \mathfrak{s}}{\partial x_k}\right] + \left[\hat{\mathfrak{s}}, \dfrac{\partial \hat{\mathfrak{s}}}{\partial x_k}\right]\right\}, \end{cases}$$

$$(44)\quad \begin{cases} i\,({}^k\hat{\mathfrak{M}} - \hat{\mathfrak{M}}^k)\,\hat{\Omega} - i\,({}^k\hat{\Omega} - \hat{\Omega}^k)\,\hat{\mathfrak{M}} \\ = \dfrac{1}{2}\left\{-\left[\mathfrak{M}, \dfrac{\partial \mathfrak{M}}{\partial x_k}\right] + \left[\hat{\mathfrak{M}}, \dfrac{\partial \hat{\mathfrak{M}}}{\partial x_k}\right] - \left[\mathfrak{s}, \dfrac{\partial \mathfrak{s}}{\partial x_k}\right] + \left[\hat{\mathfrak{s}}, \dfrac{\partial \hat{\mathfrak{s}}}{\partial x_k}\right]\right\}. \end{cases}$$

In these formulas, use was made of the abbreviated vector notations $\left(\mathfrak{A}, \dfrac{\partial \mathfrak{B}}{\partial x_k}\right)$ [definition eq. (82)] and $\left[\mathfrak{A}, \dfrac{\partial \mathfrak{B}}{\partial x_k}\right]$, whose first component means, e.g., $\left[\mathfrak{A}, \dfrac{\partial \mathfrak{B}}{\partial x_k}\right]_1 = A_2 \dfrac{\partial B_3}{\partial x_k} - A_3 \dfrac{\partial B_2}{\partial x_k}$.

The equations arise from only the algebra of the **Dirac** matrices, regardless of whether they contain differential quotients. They require either a special representation or the Hermiticity of those matrices.

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