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On the characteristic of a system of functions

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As a result of investigations that I mentioned in the lecture that I presented eight days ago, I have discovered a new fundamental property of the characteristic of systems of functions of several variables that I introduced in my communication on 4 March 1869 and expressed by a multiple integral there. The new property that I shall present here will be achieved by varying the system of functions and can be suitably employed for the numerical determination of the characteristic.

As in my treatises in the Monatsberichten on March and August 1869, let $z_1, z_2, \ldots, z_n$ denote real variables, and let $F_{00}, F_{10}, \ldots, F_{n0}$ denote single-valued real functions of them that also satisfy the conditions that were given there, as well. Furthermore, F_{gh} shall mean the derivative of F_{g0} with respect to z_h , and $[a]$ means positive or negative unity or zero according to whether the real value a is positive, negative, or zero, respectively. Hence, the progression principle that was developed in *loc. cit.* will imply the equation:

$$\text{(I)} \qquad \sum [|\, F_{ik} \,|] = 0 \qquad (i, k = 1, 2, \ldots, n),$$

in which the summation refers to all values of the variables z for which all n functions $F_{10}, F_{20}, \ldots, F_{n0}$ vanish. If one replaces F_{m0} with the product $F_{00} \cdot F_{m0}$ then it will follow from equation (I) that:

$$\text{(II)} \qquad -\tfrac{1}{2}\sum [|\, F_{gh} \,|] \qquad (g, h = 0, 1, 2, \ldots, n)$$

will have one and the same value regardless of which of the $n + 1$ systems of conditions:

$$\text{(III)} \qquad F_{g0} = 0 \qquad (g = 1, 2, 3, \ldots, n, \text{ except for } g = m)$$

that correspond to the values of $m = 0, 1, \ldots, n$ is chosen for the summation. The value in (II) is a positive or negative whole number, since the number of terms in the sum in the case of $m = 0$ coincides with the obviously *even* number of terms in the sum (I), and in the aforementioned treatise, I have referred to that number as *the characteristic of the system of* $n + 1$ *functions* $F_{00}, F_{10}, \ldots, F_{n0}$. Therefore, it is given also by the sum:

$$\sum [|\, F_{ik} \,|] = 0 \qquad (i, k = 1, 2, \ldots, n)$$

when the conditions:

$$F_{00} < 0,\ F_{10} = 0,\ F_{20} = 0,\ \ldots,\ F_{n0} = 0$$

are established for the summation. That expression for the characteristic shows that any number of systems of values will be determined by it that are defined simultaneously by equations and inequalities, and therein lies the fact that the integral for the characteristic that was presented in the Monatsbericht in March 1869, which first appeared there as a winding number, appeared in many other questions, such as in the total curvature of surfaces, the total density of ray systems, the mutual linking of curves, and their knotting. That is because in all of those questions, one deals with only the problem of ascertaining a number of systems of values for variables that are determined by equations and in that way subjected to certain inequality conditions.

If one imagines varying the $n + 1$ functions F in some way, but such that their behavior at infinity remains unchanged, then one will see from the integral expression:

$$-\frac{1}{\varpi}\int |\, F_{gh} \,| \cdot \frac{dw}{S^n\,\mathfrak{S}} \qquad (g, h = 0, 1, \ldots, n)$$

that I give for the characteristic in the Monatsbericht on March 1869, as well as from the expression that I denoted by (II) above, that *the characteristic can experience a change only when the variation makes one pass through a system of functions that all vanish for one and the same system of values* $(z_1, z_2, \ldots, z_n)$. In order to explain that in more detail, I remark first of all that for every system of values (z) that satisfy the conditions (III), the determinant $|\, F_{gh} \,|$ will be equal to the product of F_{mn} and the functional determinant of the remaining n functions F_{g0} . The vanishing of the determinant $|\, F_{gh} \,|$ for one of the systems of values (z) is then required for the continuity of such a thing to break under that variation of the functions F, and as a result, it will be required for a change in the number of terms in the expression (II), as well as for the changes in the sign of one of those terms, so for any change in the characteristic at all. However, if one replaces:

$$F_{i0} \qquad \text{with} \qquad F_{i0} - v_i\, F_{00}$$

for $i = 1, 2, 3, \ldots, n$, as one can do without damaging the value of the characteristic, and in which $v_1, v_2, \ldots, v_n$ are understood to mean variable quantities, and if one then takes $m = 0$ in the conditions (III), those conditions will be converted into the following ones:

$$\text{(III}'\text{)} \qquad F_{i0} = v_i\, F_{00} \qquad (i = 1, 2, \ldots, n),$$

by means of which, one will have:

$$\text{(IV)} \qquad F_{00} \cdot |\, F_{ik} - v_i\, F_{0k} \,| = |\, F_{gh} \,| \qquad \begin{pmatrix} g = 0, i_1, i_2, \ldots, i_\mu; & i = i_1, i_2, \ldots, i_\mu \\ h = 0, k_1, k_2, \ldots, k_\mu; & k = k_1, k_2, \ldots, k_\mu \end{pmatrix}$$

when $i_1, i_2, \ldots, i_n$ and $k_1, k_2, \ldots, k_n$ mean any of the numbers $1, 2, \ldots, n$. Now, since differentiating (III′) will yield the n equations:

$$\sum_{k=1}^{n} (F_{ik} - v_i\, F_{0k})\, dz_k = F_{00}\, dv_i \qquad (i = 1, 2, \ldots, n),$$

and the variables v are all independent of each other, it will follow from any m of those equations for the case in which all determinants (IV) vanish for $\mu = m$ that either $F_{00} = 0$ or all determinants (IV) will vanish for $\mu = m - 1$, and in that way, one will arrive at the condition $F_{00} = 0$ from the condition above that is required for the characteristic to change, namely, that $| F_{gh} | = 0$ for $\mu = n$; i.e., when one considers the simultaneous existence of equations (III′), one will come to the $n + 1$ conditions:

$$\text{(V)} \qquad F_{g0} = 0 \qquad (g = 0, 1, \ldots, n).$$

If the variation of the functions F_{g0} makes one pass a system for which those conditions can be fulfilled then naturally the determinant:

$$| F_{gh} | \qquad (g, h = 0, 1, \ldots, n)$$

will also be equal to zero for the relevant system of values $(z_1, z_2, \ldots, z_n)$, and *according to whether it goes from positive to negative or conversely, the characteristic will increase or decrease, respectively,* when, say, the location that is passed in not singular. For those values of the variable z for which $F_{10} = 0$, $F_{20} = 0$, …, $F_{n0} = 0$, the determinant $| F_{gh} |$ will reduce to:

$$F_{00} \cdot | F_{ik} | \qquad (i, k = 1, 2, \ldots, n),$$

and it will therefore be that product whose change when one goes through zero is definitive of the change in the value of the characteristic.

If one considers the functions $F(z_1, z_2, \ldots, z_n)$ to depend upon ν real parameters $x_1, x_2, \ldots, x_\nu$ then each point of the ν-fold manifold (x) will correspond to a certain system of functions $(F_{00}, F_{10}, \ldots, F_{n0})$, and therefore to a certain characteristic for it, as well. Now, if the points (x) that correspond to those special systems for which the conditions (V) can be satisfied fill up a $(\nu - 1)$-fold manifold:

$$R(x_1, x_2, \ldots, x_\nu) = 0$$

then the domains in the ν-fold manifold (x) in which the characteristic has different values will be separate from each other in that way, and from the deduction above, *when one goes through* $R(x_1, x_2, \ldots, x_\nu) = 0$, *one will add:*

$$-[| F_{ik} | \, \delta F_{00}] \qquad (i, k = 1, 2, \ldots, n)$$

to the characteristic when one denotes the change in the point $(x_1, x_2, \ldots, x_\nu)$ *by* δ *and substitutes the system of values* $(z_1, z_2, \ldots, z_n)$ *that satisfy the* $n + 1$ *equations* (V) *in* F_{00}, F_{ik} . Thus, the characteristic can change when one goes from one system of functions to another, and therefore when the characteristic of just a single system of functions is known, one can determine the characteristics of all the systems of functions that correspond to the different points. Therefore, when one goes over to a single dependent variable x, that determination is found to reduce to ascertaining the characteristic of a system of two functions of one variable, which can result from **Sturm**'s process in the case of algebraic functions.

If the assumption that was made above that the points of:

$$R(x_1, x_2, \ldots, x_\nu) = 0$$

define an $(\nu - 1)$-fold manifold does not hold true, and those points all lie on a manifold that is at most $(\nu - 2)$-fold extended then the characteristic will have one and the same value for all systems of functions, and can therefore be found by examining just one of the systems.

If the functions F are rational functions of the variables z then $R = 0$ will be resultant of the $n + 1$ equations $F = 0$, and the sign of:

$$- F_{00} \cdot | F_{ik} | \qquad (i, k = 1, 2, \ldots, n)$$

in the neighborhood of the transition through $R = 0$ will be equal to that of the expression:

$$\sum \frac{-1}{F_{00} \, | F_{ik} |} \qquad (i, k = 1, 2, \ldots, n)$$

when the summation is extended over all real and complex systems of values (z) that satisfy the equations:

$$F_{k0} = 0 \qquad (k = 1, 2, \ldots, n).$$

Now, if G_0 means an entire function of the variable z that coincides with $- R / F_{00}$ for all of those systems of values, and if one sets:

$$R_1 = \sum \frac{G_0}{| F_{ik} |} \qquad (i, k = 1, 2, \ldots, n),$$

in which the summation again refers to all of those systems of values, then G_0 will belong to those multipliers for which (*):

$$\sum_{h=0}^{n} (-1)^{h+1} G_h \, F_{h0} = R,$$

and an increase or decrease in the characteristic will result at those places where $R = 0$, just like the product $R\,R_1$. One then obtains the expression for the total change in the characteristic along a path from one system of functions F to another in the signed sum:

$$\sum \, [R_1 \, \delta R]$$

when one refers the summation to all of the places (x) where $R = 0$. To ascertain the value of that signed sum, one can appeal to **Sturm**'s process, in which one then starts from the two expressions R and R_1 . Above all, it is the system of those two functions as the coefficients of $F_{00}, F_{10}, \ldots, F_{n0}$ that clarifies the essence of the characteristic of such systems of algebraic functions, and it is also therefore gives an almost immediate extension of **Sturm**'s theorem in regard to the number of equations, as well as in regard to the numbers of variables in the coefficients. I shall reserve a more detailed explanation of that until later, as well as carrying out the general developments above in

(*) Cf., my treatise that was published in the Monatsbericht of Dec. 1865.

particular. Here however, it might ultimately be recalled that for a special choice of the function F_{00}, the characteristic means essentially the number of real systems of values (z) that satisfy the n equations $F_{10} = 0$, $F_{20} = 0$, …, $F_{n0} = 0$, and that the foregoing considerations will lead to the establishment of that number accordingly.

In order to establish that in one of the simplest cases, as is Section IX of my oft-cited treatise on 4 March 1869, let the number n be *even*, and indeed let it be equal to $2m$. Furthermore, let f_1, f_2, …, f_m be entire rational functions of the m complex variables $y_1, y_2, \ldots, y_m$, and let $f'_1, f'_2, \ldots, f'_m$ be conjugate to $f_1, f_2, \ldots, f_m$, resp. Finally, for $k = 1, 2, \ldots, m$, let:

$$y_k = z_k + i\, z_{m+k}\,, \qquad 2\, F_{k0} = f_k + f'_k\,, \qquad 2i\, F_{m+k,\,0} = f_k - f'_k\,.$$

Let the functions f be arranged so that the m aggregates of terms of highest dimension do not vanish simultaneously for non-zero values of the variables y, and the variation of the functions f might result in only such a way that the terms of highest order remain unchanged by it, but the absolute values of the coefficients of the remaining terms will always remain below a limit γ. Having assumed that, a function $F_{00}(z_1, z_2, \ldots, z_n)$ can always be determined, e.g., in the form:

$$F_{00} = \sum_k z_k^2 - r^2 \qquad (k = 1, 2, \ldots, n),$$

such that in the (external) region $F_{00} > 0$, neither the m functions f of the system from which one starts nor the m functions of any of the varied systems will be zero simultaneously. Namely, if one lets λ denote the dimension of one of the functions f and lets r_k denote the absolute value of y_k then the absolute value of any term of dimension less than λ will be less than $\gamma\, r_1^{\lambda-1}$ when the remaining quantities $r_2, r_3, \ldots, r_m$ are less than (or at least not greater than) r_1 . Each of those terms will then have the form:

$$\rho\, \gamma\, r_1^{\lambda-1}\, e^{\nu i} \qquad (0 < \rho < 1),$$

and when the number of them is denoted by μ, the function f_k itself can be represented by the expression:

$$(\varphi_k + \psi_k\, i)\, y_1^\lambda + \frac{\gamma}{r_1} u_k (\rho_k + \sigma_k\, i)\, y_1^\lambda \qquad \begin{pmatrix} 0 < \rho_k < 1 \\ 0 < \sigma_k < 1 \end{pmatrix},$$

in which the first part subsumes all of the terms of highest dimension. For sufficiently large values of r_1 (i.e., since one has:

$$r_1^2 = z_1^2 + z_{m+2}^2 \geq z_k^2 + z_{m+k}^2 \qquad (k = 1, 2, \ldots, m)$$

in the region $F_{00} > 0$ when the value of r in:

$$F_{00} = \sum_k z_k^2 - r^2 \qquad (k = 1, 2, \ldots, m)$$

is assumed to be sufficiently large), the $2m$ equations:

$$\varphi_k + \frac{\gamma}{r_1}\mu_k\,\rho_k = 0, \quad \psi_k + \frac{\gamma}{r_1}\mu_k\,\sigma_k = 0 \qquad (k = 1, 2, \ldots, m)$$

cannot all be fulfilled then. That is because, by assumption, the m functions $\varphi_k + \psi_k\, i$ cannot vanish simultaneously, so the sum:

$$\sum(\varphi_k^2 + \psi_k^2) \, ,$$

which extends over $k = 1, 2, \ldots, m$ must always be greater than a certain quantity, and thus, for a sufficiently large value of r_1, one will also have:

$$\sum(\varphi_k^2 + \psi_k^2) > 2\frac{\gamma^2}{r_1^2}\sum\mu_k^2 > \frac{\gamma^2}{r_1^2}\sum\mu_k^2(\rho_k^2 + \sigma_k^2) \, .$$

Since the functional determinant:

$$| F_{ik} | \qquad (i, k = 1, 2, \ldots, n)$$

is always positive in the present case, the characteristic of the system of functions (F_{00}, F_{10}, …, F_{n0}), with the determination of F_{00} above, will be equal to the total number of systems of values (z_1, z_2, …, z_n) that satisfy the n equations $F_{10} = 0$, $F_{20} = 0$, …, $F_{n0} = 0$ or the m equations $f_1 = 0$, $f_2 = 0$, …, $f_m = 0$. From the foregoing discussion, that number will then remain unchanged when one restricts $m - 1$ of the functions f to merely their terms of highest dimension, but assumes that in the remaining m^{th} function f there is a term that is free of all the variables y, in addition. The fact that for one such system of equations:

$$f_1 = 0, \;\; f_2 = 0, \;\ldots, \qquad f_m = 0 \, ,$$

the number of systems of values that satisfy it is equal to the product of the dimensions of the m functions f follows quite directly when one assumes the relevant property for the case of only m – 1 complex variables y. However, in the case $m = 1$, the foregoing development will lead directly to the "fundamental theorem of the theory of algebraic equations," which defines the actual essence of the derivation that **Gauss** gave of it in his 1849 treatise when he showed that for two systems of curves that are represented by any algebraic equation $f(x + y\, i) = 0$, the configuration of their intersection points inside of a circle that is chosen to be sufficiently large is no different from what it would be for those systems of curves that emerge from a "pure" equation of the same degree. Moreover, one can recognize from **Gauss**'s argument that, it is actually only the highest power of $x + y\, i$ that one considers in that way, along with the fact that of the coefficients of the remaining terms in the equations, only the property that their absolute values lies below a certain limit is considered, such that the argument will not affect a change in the coefficients that is admissible in that way. However, such a change was also employed directly by **Weierstrass** in order to prove the fundamental theorem of algebra, which he announced in July 1868, but which has not been published up to now.

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