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# On systems of functions of several variables 

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Kronecker read a treatise "Über Systeme von Functionen mehrer Variabeln." Of the results that were contained in it, only a brief excerpt of the most important ones shall be communicated here, as well as only some suggestions as to the methods that were employed.

## I.

Let $F_{0}, F_{1}, F_{2}, \ldots, F_{n}$ be single-valued real functions of the $n$ real quantities $z_{1}, z_{2}, \ldots, z_{n}$, whose variability is unrestricted, and indeed in such a way that an $n$-fold infinitude of them will assume positive or negative values. Moreover, the functions $F$ will generally be assumed to be continuous and differentiable with respect to the individual variables, and finally, it shall be assumed that none of the $n+1$ functional determinants vanish simultaneously with the functions in question for infinitely-many systems of values $z$. In order to also fix the signs of those $n+1$ functional determinants, of the $n$ columns that are defined by the $n$ partial derivatives, I shall select one, whose $n+1$ elements might by be denoted by $F_{00}, F_{10}, \ldots, F_{n 0}$, resp., and define the $(n+1)^{2}$ elements of the determinant that results in that way when it is ordered by the sequence of indices. Thus, the partial differential quotient of that determinant with respect to $F_{k 0}$ will also be, up to sign, the well-defined functional determinant of the $n$ functions $F_{0}, F_{1}, \ldots, F_{k-1}, F_{k+1}, \ldots, F_{n}$, which shall be denoted by $\Delta$.

When one sets any $n-1$ of the functions $F$ equal to zero, the variability of the variables $z$ will be restricted to a simple infinitude or manifold. The system of values of the $n$ quantities $z$ that belongs to it will define a continuous sequence, and one can then refer to that system of values as a "point" and their continuous sequence as a "line." In order to fix the sense of advance along the line:

$$
F_{i}=0, \quad \text { for } i \text { set equal to all indices, except for two of them }(h \text { and } k)
$$

at an arbitrary point of it, I shall replace either $F_{h}$ or $F_{k}$ with any single-valued function and determine the advance in such a way that at the location considered, $d \Phi$ will keep the same sign that the functional determinant of the $n-1$ functions $F_{i}$, plus $\Phi$, has at the point considered. With that convention, the sense of advance will differ according to whether $\Phi$ enters in place of $F_{h}$ or $F_{k}$, and the line itself shall be denoted by [ $h k$ ] or [ $k h$ ], accordingly. However, the sense of advance is fixed precisely and independent of the choice of function $\Phi$, moreover, at all points of the line
that are not double points. Since the sense of advance is established up to an arbitrary neighborhood of the double point, the applicability of that determination will not exclude the occurrence of double points. The "principle of advance" that is put forth here defines the actual foundation of my investigations into systems of functions of several variables.

## II.

It shall now be assumed that all of the $\frac{1}{2} n(n+1)$ lines that arise from the system of functions $F$ in the given way are closed lines, and that the number of points that are determined by every set of $n$ equations $F=0$ is finite. If one now considers the line [ $h k$ ] with the sense of advance that is fixed by that symbol at those locations where it intersects the ( $n-1$ )-fold manifold $F_{h}=0$ then (when one does not likewise have $F_{k}=0$ ) it will go from a region where $F_{h} \cdot F_{k}$ is negative to one in which $F_{h} \cdot F_{k}$ is positive, or conversely. Insofar as one can refer to the former region as an internal one and the latter as an external one, a point of intersection of $[h k]$ with $F_{h}=0$ can be regarded as an exit or an entrance of the line [ $h \mathrm{k}$ ], respectively, and that conception of things, as based upon the principle of advance, has an essential significance in the natural interpretation of the analytical relations, as will be shown. The total number of entrances and exits is an even number. Hence, when one subtracts the exits from the entrances, one-half of the difference that is defined will be a whole number that can be positive, negative, or zero. One has the fundamental theorem that this number is constant, regardless of how one might choose the indices $h$ and $k$. The number is then characteristic of the entire system of functions and shall, in turn, be called its characteristic.

The entrance and exit points of the $\frac{1}{2} n(n+1)$ different lines of the system of functions can also be regarded as the common points of each set of $n$ functions $F$. Each of those points will then belong to a system of only $n$ functions that remain after omitting any one of the $F_{k}$. If one adds another function $\mathfrak{F}_{k}$ to those $n$ functions in place of $F_{k}$ that has the property that it is negative for only one of the points that are determined by:

$$
F_{0}=F_{1}=\ldots=F_{k-1}=F_{k+1}=\ldots=F_{n}=0,
$$

but positive for all of the other ones, then the characteristic of that new system of $n+1$ functions will have an absolute value of unity, with the same sign as the functional determinant $\Delta_{k}$ at each point, assuming that $\Delta \neq 0$. The characteristic shall be called the character of the point and will be denoted by $\chi\left({ }^{1}\right)$. For every point $\left(\zeta_{1}^{(k)}, \zeta_{2}^{(k)}, \ldots, \zeta_{n}^{(k)}\right)$ that is simply defined by the equations:

[^0]$$
F\left(\zeta_{1}^{(k)}, \zeta_{2}^{(k)}, \ldots, \zeta_{n}^{(k)}\right)=0, \quad(i=0,1, \ldots, k-1, k+1, \ldots, n)
$$
one will then have:
$$
\chi\left(\zeta_{1}^{(k)}, \zeta_{2}^{(k)}, \ldots, \zeta_{n}^{(k)}\right) \cdot \Delta_{k}\left(\zeta_{1}^{(k)}, \zeta_{2}^{(k)}, \ldots, \zeta_{n}^{(k)}\right)>0
$$
and therefore, one will have:
$$
\chi\left(\zeta^{(k)}\right) \cdot d F_{h}>0
$$
at every point $\zeta^{(k)}$, by means of the principle of advance along the line $[h k]$.
It follows immediately from this that the algebraic sum of the characters of all points $\zeta^{(k)}$ will be equal to zero, and furthermore, that the algebraic sum of the characters of all of those points $\zeta^{(k)}$ for which one has:
$$
F\left(\zeta_{1}^{(k)}, \zeta_{2}^{(k)}, \ldots, \zeta_{n}^{(k)}\right)<0
$$
will coincide with one-half the difference between the number of entrance points and exit points of the line $[h k]$; i.e., with the characteristic of the system of functions.

That simple relationship between the characters of the points $\zeta$ and the characteristic of the system $F$ shows that the latter will remain unchanged for all lines $[h k$ ] with a constant index $k$. However, the same relationship will also lead to the proof of the invariance of the characteristic under a permutation of the indices $h$ and $k$, and thus to a complete proof of the theorem on the constancy of the characteristic. Namely, if one considers all of the points $\zeta^{(h)}$ and $\zeta^{(k)}$ to be defined by the $n$ equations:

$$
F_{0}=\ldots=F_{h-1}=F_{h} \cdot F_{h}=F_{k-1}=0, \quad F_{k+1}=F_{k+2}=\ldots=F_{n}=0,
$$

whose functional determinant is:

$$
F_{k} \cdot \Delta_{k}-F_{h} \cdot \Delta_{h},
$$

then the definition of the character of a point $\zeta^{(k)}$ relative to this system of equations will be:

$$
\varepsilon \cdot \chi\left(\zeta^{(k)}\right), \quad \text { when } \quad \varepsilon= \pm 1 \quad \text { and } \quad \varepsilon \cdot F_{k}\left(\zeta^{(k)}\right)>0
$$

and that of the character of a point $\zeta^{(h)}$ will be:

$$
-\varepsilon \cdot \chi\left(\zeta^{(h)}\right), \text { when } \quad \varepsilon= \pm 1 \quad \text { and } \quad \varepsilon \cdot F_{h}\left(\zeta^{(h)}\right)>0
$$

and for that reason:

$$
\sum \varepsilon \cdot \chi\left(\zeta^{(k)}\right)-\sum \varepsilon \cdot \chi\left(\zeta^{(h)}\right)=0 .
$$

Moreover, one has:

$$
\sum \chi\left(\zeta^{(k)}\right)=0, \quad \sum \chi\left(\zeta^{(h)}\right)=0
$$

one hand, the clarity of the representation will improve quite immensely in that way, and on the other hand, because I am free to revert to the original terminology whenever the expression should introduce possible complications.
in which the summation extends over all points $\zeta^{(k)}$ and $\zeta^{(h)}$, resp., and it will ultimately follow from this that:

$$
\sum \chi\left(\zeta^{(k)}\right)=\sum \chi\left(\zeta^{(h)}\right)
$$

in which the former sum extends only over all of the points $\zeta^{(k)}$ for which $F_{k}<0$ and the latter extends only over the points $\zeta^{(h)}$ for which $F_{h}<0$.

The foregoing definitions (with certain modifications) will also retain their validity for some special cases that can be regarded as limiting cases of the general ones. For example, when the functional determinant of the $n$ equations $F=0$ likewise vanishes at the points that are defined by them, which will make the characters of the relevant points different from unity. However, the cited theorems can also be justified, in essence, for the case in which the variables $z$ are restricted to an $n$-fold infinitude of discrete points. Nonetheless, rather lengthy discussions would be necessary for us to go into that case, and only some remarks that are entirely essential for the following applications of the stated theorems shall be added in order to free the functions $F$ from certain restrictions. Namely, it is by no means necessary that the functions $F$ should always be coupled by one and the same analytical law, and each of the functions $F$ can be replaced with any other one, moreover, as long as the latter is chosen such that it always have the same sign as that function $F$ along the lines considered. As a result of that, e.g., systems of functions that do not yield closed lines can be replaced with ones that do.

## III.

The characteristic of the system $F$ itself will take on a special significance when one considers the functional determinant of $n$ functions $F$. Namely, if the functional determinant of $F_{1}, F_{2}, \ldots$, $F_{n}$ is denoted by $\Delta_{0}$, as above, then that characteristic will be equal to the difference that one obtains when one subtracts from the number of points for which:

$$
\Delta_{0} \cdot F_{0}<0, \quad F_{1}=F_{2}=\ldots=F_{n}=0,
$$

the number of points for which:

$$
\Delta_{0} \cdot F_{0}>0, \quad F_{1}=F_{2}=\ldots=F_{n}=0 .
$$

If one then lets:

$$
\left(\zeta_{1}, \zeta_{2}, \ldots, \zeta_{n}\right), \quad\left(\zeta^{\prime} 1, \zeta^{\prime} 2, \ldots, \zeta_{n}^{\prime}\right), \quad \ldots
$$

denote all of the common systems of values for the equations:

$$
F_{1}=0, F_{2}=0, \ldots, F_{n}=0
$$

then the characteristic of the system:

$$
\left(\Delta_{0} \cdot F_{0}^{\prime}, F_{1}, F_{2}, \ldots, F_{n}\right)
$$

will give the excess of the number of points ( $\zeta$ ) that lie inside of $F_{0}^{\prime}=0$ over the number of points that lie outside of $F_{0}^{\prime}=0$. Finally, when the region $F_{0}^{\prime}<0$ includes another region $F^{\prime \prime}<0$ completely, the number of points $(\zeta)$ that lie between the two limits $F^{\prime}{ }_{0}=0$ and $F^{\prime \prime}{ }_{0}=0$ will be expressed by one-half the difference of the two characteristics:

$$
\left(\Delta_{0} \cdot F_{0}^{\prime}, F_{1}, F_{2}, \ldots, F_{n}\right) \quad \text { and } \quad\left(\Delta_{0} \cdot F^{\prime \prime}, F_{1}, F_{2}, \ldots, F_{n}\right) .
$$

In addition to that connection between the number of systems $(\zeta)$ and the characteristic, some special relationships will exist for the case in which (when $n=2 m$ ) the $n$ functions $F_{1}, F_{2}, \ldots, F_{n}$ are the $n$ divisors of $m$ functions of just as many complex variables. Namely, if $F_{0}$ is then chosen to be a function that vanishes only for finite values of the variables $z$ and is arranged, moreover, such that one likewise has $F_{0}<0$ for all points ( $\zeta$ ) then there will only entrances and no exits at all in that region $\left(F_{0}<0\right)$. The same thing is also true for the boundary $F_{0}=0$, and indeed there will be twice as many intersection points of every line that is defined, with the exception of one of the remaining functions $F$, as there are inside of it. If $F_{0}^{\prime}<0$ is a region that is included in $F_{0}<0$ then the characteristic of the system:

$$
\left(F^{\prime}, F_{1}, F_{2}, \ldots, F_{n}\right)
$$

will ultimately give virtually the number of points ( $\zeta$ ) that lie in the region $F_{0}^{\prime}<0$. However, that assumes that the concept of characteristic that corresponds to the lines in the region $F_{0}>0$ that branch off to infinity can be replaced with closed lines in the case considered.

## IV.

When $F_{0}, F_{1}, \ldots, F_{n}$ are entire rational functions of the $n$ variables $z$, one can apply a process to them that is analogous to a continued fraction development with the help of an interpolation formula that I presented in the Monatsbericht in December 1865. The series of functions that it yields defines a generalization of the Sturm sequence and can serve as a way of ascertaining the characteristic of the system $F$. In order to show that in the simplest case, let $z_{1}=x, z_{2}=y$, and first of all let:

$$
F_{0}=y, \quad F_{1}=f(x)-y, \quad F_{2}=f_{1}(x)-y,
$$

where $f$ and $f_{1}$ mean entire functions of degrees $2 v$ and $2 v-1$, resp., in which the coefficients of the highest powers of $x$ are positive. If one now constructs a Sturm sequence:

$$
f(x), \quad f_{1}(x), \quad f_{2}(x), \ldots
$$

in the known way by a continued fraction development of the quotient of the two functions $f$ and $f_{1}$ then the reduction in sign reversals that this sequence suffers upon going from $x=-\infty$ to $x=+$ $\infty$ can be interpreted in a simpler and more intuitive way on the basis of the discussions above. Namely, that reduction agrees precisely (also in sign) with the characteristic of the system ( $F_{0}, F_{1}$,
$F_{2}$ ). Furthermore, when $f$ and $f_{1}$ have the same degree, but $f-f_{1}$ has lower degree, and one constructs the Sturm sequence:

$$
f(x), \quad f(x)-f_{1}(x), \quad f_{2}(x), \ldots,
$$

and first sets $x$ equal to any value $a$ in it and then to a larger value $b$, the difference between the number of sign changes (assuming that all sign changes can be counted twice, with the exception of the one between the first two terms) will be equal to the difference between the characteristics of the two systems:

$$
\left(y \cdot(x-a), f(x)-y, f_{1}(x)-y\right), \quad\left(y \cdot(x-b), f(x)-y, f_{1}(x)-y\right) .
$$

When one then moves from $b$ to $a$ along the abscissa axis and refers to the places that $F_{1}$ passes through as entrances and exits according to whether one enters or exits a region that is enclosed by $F_{1}$ and $F_{2}$ there, respectively, then the difference between the number of those entrances and exits will be determined by the reduction in sign changes that the Sturm sequence suffers between $x=a$ and $x=b$. The simplest case that was mentioned here was, to my knowledge, first treated by Sylvester and interpreted in a similar, if also less intuitive, way. Sylvester had added some remarks (Philosophical Transactions, Part III, 1853, pp. 495) from which it emerged that he had suggested that might be an extension of some of his theorems to functions of several variables. It seems to me that this suggestion corresponds to the theorem on the constancy of the characteristic that was given above in number II. The difficulties in the generalization that Sylvester mentioned probably lie mostly in the restriction to algebraic structures that he had established. As soon as I gained the insight that all relevant considerations belong exclusively to that general realm that is referred to as the "geometry of position" for the case where $n=2$ or 3 , that gave me the simplest means for overcoming the opposing difficulties.

## V.

If a manifold that is defined by the $n$ real variables $x_{1}, x_{2}, \ldots, x_{n}$ is related to the variables $z$ by the equations:

$$
x_{1}=F_{1}\left(z_{1}, z_{2}, \ldots, z_{n}\right), x_{2}=F_{2}\left(z_{1}, z_{2}, \ldots, z_{n}\right), \ldots, x_{n}=F_{n}\left(z_{1}, z_{2}, \ldots, z_{n}\right)
$$

then every point $z$ will correspond to a point $x$, but in general, such that one also has that one and the same point $x$ can belong to different points $z$, so the $(n-1)$-fold manifold:

$$
F_{0}\left(z_{1}, z_{2}, \ldots, z_{n}\right)=0
$$

will then correspond to an $(n-1)$-fold manifold:

$$
\Phi\left(x_{1}, x_{2}, \ldots, x_{n}\right)=0
$$

It will always be assumed of the function $F_{0}$ that it is negative for only finite values of the variables $z$. Now if one sets all of the variables $x$ (except for only $x_{h}$ and $x_{k}$ ) equal to zero then one will get a simply-infinite sequence of systems of values of $x$ that is contained in the $(n-1)$-fold manifold $\Phi$ $=0$; i.e., a line. That line will wind about the origin exactly as many times as the characteristic of the system of functions $F$ would yield, and in that way, the sign of the characteristic will also indicate the sense of the winding. That is because every entrance of the line $[h k]$ through $F_{h}=0$ corresponds to a crossing of $x_{h}=0$ in one sense of rotation, while every exit corresponds to a crossing of $x_{h}=0$ in the opposite sense, and from one entrance to the following one, one will complete one-half of a winding. Hence, when one regards "moving forward" in the former sense of rotation, one will often likewise move forward when there is an entrance and likewise move backwards when an exit is present, such that actual forward motion through one-half a winding will be given by twice the number that was defined to be the "characteristic." Since the indices $k$ and $h$ are chosen arbitrarily and furthermore, from the remark that was made at the conclusion of number II, the functions $F$ can be replaced with certain other ones, e.g.:

$$
F_{1}-\lambda_{1} \sqrt{F_{k}^{2}+F_{h}^{2}} \text { for } F_{1}, \quad F_{2}-\lambda_{2} \sqrt{F_{k}^{2}+F_{h}^{2}} \text { for } F_{2}, \ldots,
$$

the meaning of the characteristic as a winding number will not remain restricted to certain lines that belong to $\Phi=0$, but it will also apply to the entire manifold that is represented by $\Phi=0$. Similarly, that manifold $(\Phi=0)$ will have a well-defined winding number relative to any arbitrary point $\left(\xi_{1}, \xi_{2}, \ldots, \xi_{n}\right)$, just as it does relative to the origin, and the winding number in question will be equal to the characteristic of the system:

$$
\left(F_{0}, F_{1}-\xi_{1}, F_{2}-\xi_{2}, \ldots, F_{n}-\xi_{n}\right) .
$$

Now, the manifold $x$ can be divided into different regions by the values of that characteristic such that one and the same region will be defined by all of the points $\xi$ for which that characteristic has the same value. The transition from one region to another will then result in the manifold $\Phi=0$. However, I would like to pass over the deeper details of that in order to move on to the most important application of the foregoing applications, namely, expressing the characteristic in terms of an ( $n-1$ )-fold integral to which one will be led directly by it.

Namely, let $d w$ be the positively-taken element of the ( $n-1$ )-fold manifold that is represented by any equation $F\left(z_{1}, z_{2}, \ldots, z_{n}\right)=0$ at all, and as is known, one might abbreviate the value of its integral by:

$$
\int d w, \quad \text { which extends over } \quad z_{1}^{2}+z_{2}^{2}+\cdots+z_{n}^{2}=1
$$

which is expressible in powers of $\pi$, by $\varpi$, and furthermore, the positive values of:

$$
\sqrt{F_{1}^{2}+F_{2}^{2}+\cdots+F_{n}^{2}} \text { and } \sqrt{F_{01}^{2}+F_{02}^{2}+\cdots+F_{0 n}^{2}}
$$

will be denoted by $S$ and $\mathfrak{S}$, resp., and finally one sets:

$$
R=\frac{1}{\mathfrak{S}} \cdot\left|\begin{array}{ccccc}
0 & F_{01} & F_{02} & \cdots & F_{0 n} \\
F_{1} & F_{11} & F_{12} & \cdots & F_{1 n} \\
F_{2} & F_{21} & F_{22} & \cdots & F_{2 n} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
F_{n} & F_{n 1} & F_{n 2} & \cdots & F_{n n}
\end{array}\right| .
$$

The characteristic of the system $F$ is then expressed by:

$$
-\frac{1}{\varpi} \int \frac{R}{S^{n}} d w,
$$

in which the integration extends over the $(n-1)$-fold manifold $F_{0}=0$.
From discussion that is contained in no. III, the boundary function $F_{0}$ can be chosen such that number of points $\zeta$ that are contained in a certain region can be represented by the integral expression that is given here, so in the case $n=2$, for two curves $F_{1}=0, F_{2}=0$, it will be the number of real intersection points that lie inside of a given region.

I remark that one has the following equation for the element $d w$ :

$$
d w= \pm \frac{\mathfrak{S}}{F_{0 k}} d z_{1} \ldots d z_{k-1} \cdot d z_{k+1} \ldots d z_{n}
$$

Namely, if one constructs the well-known volume determinant for $n$ points that belong to the manifold $F_{0}=0$ and are infinitely close to the point $\left(z_{1}^{0}, z_{2}^{0}, \ldots, z_{n}^{0}\right)$ and an $(n+1)^{\text {th }}$ point $\left(z_{1}, z_{2}\right.$, $\ldots, z_{n}$ ) that lies outside of it and then divides by:

$$
\sqrt{\left(z_{1}^{0}-z_{1}\right)^{2}+\left(z_{2}^{0}-z_{2}\right)^{2}+\cdots+\left(z_{n}^{0}-z_{n}\right)^{2}}
$$

then that quotient will approach the expression that is given for the element $d w$ when the point $\left(z_{1}\right.$, $\left.z_{2}, \ldots, z_{n}\right)$ goes out to infinity.

When one denotes all of the points $z$ that belong to the manifold $F_{0}=0$ by $z^{0}$, such that $z_{1}^{0}, z_{2}^{0}$, $\ldots, z_{n}^{0}$ are coupled to each other by the equation:

$$
F_{0}\left(z_{1}^{0}, z_{2}^{0}, \ldots, z_{n}^{0}\right)=0,
$$

and when one then relates, so to speak, any point $z$ to the manifold $F_{0}=0$ by the equations:

$$
z_{k}=z_{k}^{0}+p \cdot \frac{F_{0 k}}{\mathfrak{S}}
$$

(cf., Gauss's "Allgemeine Lehrsätze, etc.," art. 23), the variable $p$ will represent a quantity that corresponds to the distance from the point $z^{0}$ to the point $z$ in the normal direction for $n=3$ and whose sign will agree with that of $F_{0}$. The quotient $F_{0 k} / \mathfrak{S}$ will then take on the meaning that it is equal to the partial differential quotient $\partial z_{k} / \partial p$ for $z_{k}=z_{k}^{0}$, and corresponding to the interpretation, one can set:

$$
\frac{F_{0 k}}{\mathfrak{S}}=z_{k, p}^{0}
$$

The determinant $R$ will then take the form:

$$
R=\left|\begin{array}{ccccc}
0 & z_{1 p}^{0} & z_{2 p}^{0} & \cdots & z_{n p}^{0} \\
F_{1} & F_{11} & F_{12} & \cdots & F_{1 n} \\
F_{2} & F_{21} & F_{22} & \cdots & F_{2 n} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
F_{n} & F_{n 1} & F_{n 2} & \cdots & F_{n n}
\end{array}\right|,
$$

and $\mathfrak{S}$ will coincide with the partial differential quotient $\partial F_{0} / \partial p$.
When one transforms the integral that expresses the characteristic from the manifold $z$ to the manifold $x$, it will take on an intuitive form, because the element itself will then generalize the spatial angle between the element at the origin and an element of the surface $\Phi=0$ for $n=3$. The integral of the characteristic will then represent a generalization of the integral that Gauss gave in his Theoria attractionis corporum sphaeroidicormum ellipticorum, art. 6, in two respects, because it is the restriction to three integration variables, and even for $n=3$, the restriction to the boundaries of simple bodies is lifted, so the case of a mutual interpenetration of bodies in not excluded.

## VI.

The aforementioned Gaussian integral is a special case of the one that is employed in the derivation of the potential equation. That general integral goes to the latter when the density is constant. That fact led to a suspicion that is confirmed by results, namely, that the reference point in potential theory can serve as a means for achieving a general representation of arbitrary functions of a point $\zeta$ that is defined by a system of equation $F=0$, and therefore also arrive at a generalization of the so-called Cauchy integral.

If $\mathfrak{F}$ means a single-valued function of the variables $z$, and one sets the element of an $n$-fold manifold equal to $d v$, to abbreviate, i.e.:

$$
d z_{1} \cdot d z_{2} \ldots d z_{n}
$$

and further lets $S(\xi)$ denote the positive value of the square root:

$$
\left(F_{1}-\xi_{1}\right)^{2}+\left(F_{2}-\xi_{2}\right)^{2}+\ldots+\left(F_{n}-\xi_{n}\right)^{2},
$$

then the integral:

$$
\int \frac{\mathfrak{F} \cdot \Delta_{0}}{(n-2) \cdot S(\xi)^{n-2}} d v,
$$

which is extended over the region $F_{0}<0$, will be a function of the $n$ variables $\xi$ that analogous to a potential, and it shall be denoted by $\Pi(\xi)$. However, for the case of $n=2$, one must take $-\log$. $S(\xi)$ instead of the divisor under the integral sign. The integral $\Pi$, which I will briefly call a potential, will take the usual form when one transforms it from the manifold $z$ to the manifold $x$, since $\Delta_{0}$ is the functional determinant of the $n$ functions $F$. However, in the manifold $x, \mathfrak{F}$ is no longer single-valued function of the point $x$, in general, but a multivalued function, such that the potential $\Pi$ will also represent a potential of a multivalued density or multiple covering of space for the case of $n=3$.

We will establish the previous convention for the functions $F$ that they are (at least inside of the region considered $F_{0} \leq 0$ ) generally continuous, differentiable, and finite, although that convention can be modified. The region $F_{0}<0$ shall contain only finite values of the variables $z$. The function $\mathfrak{F}$ shall be arranged such that $\mathfrak{F} \cdot \Delta_{0}$ remains finite and continuous inside of the integration region, and that $\mathfrak{F}$ itself will possess that property in the immediate neighborhood of the points $\zeta$, whose number will be assumed to be finite, as before. Furthermore, the function $\mathfrak{F}$ shall be differentiable in all variables. If one now lets $\Delta \Pi(\xi)$ denote the sum of the second derivatives of $\Pi$ with respect to the variables $\xi$, in the usual way, then one can, with no loss of generality, restrict to the case where all $\xi$ can be set equal to zero after differentiation, and one will then obtain the fundamental equation:

$$
\begin{equation*}
\Delta \Pi(0)=-\varpi \sum \chi(\zeta) \cdot \mathfrak{F}(\zeta) \tag{A}
\end{equation*}
$$

in which the summation refers to all of the points $\zeta$ that are defined by the conditions:

$$
F_{0}<0, \quad F_{1}=0, F_{2}=0, \ldots, F_{n}=0 .
$$

The special meaning of the potential $\Pi$, as well as the importance of the differing characters of the points, emerges most clearly in that remarkable equation. One can also explain that by saying that [when one simple points $\zeta$ are present, and when one sets $\mathfrak{F} \cdot \Delta_{0}=\Psi(z)$ and denotes the absolute - i.e., positive - value of $\Delta_{0}(\zeta)$ by $\Delta(\zeta)$ ] the equation $(\mathfrak{A})$ will go to:

$$
\Delta \Pi(0)=-\varpi \sum \frac{\Psi(\zeta)}{\Delta(\zeta)}
$$

However, the relationship to the results for the case of $n=2$ is first shown when one converts $\Delta \Pi$ into the difference of two integrals, by means of which I have arrived at the fundamental equation.

## VII.

When one introduces the variables $x$ into potential $\Pi$, one will obtain an approximation to the integral:

$$
\frac{1}{n-2} \int \frac{K(x)}{\left(\sum\left(x_{k}-\xi_{k}\right)^{2}\right)^{(n-2) / 2}} d v
$$

which is extended over the region $\Phi<0$ (cf., no. V), namely:

$$
\mathfrak{F}(\zeta) \cdot \int \frac{R}{S^{n}} \cdot d w
$$

i.e., when one considers what was said in no. V, the limit:

$$
-\varpi \cdot \chi(\zeta) \cdot \mathfrak{F}(\zeta)
$$

It will then follow from this that one will have:

$$
\begin{equation*}
-V+W=-\varpi \cdot \sum \chi(\zeta) \cdot \mathfrak{F}(\zeta), \tag{A}
\end{equation*}
$$

when the integrations in $V$ and $W$ are extended over regions that are determined by arbitrary functions, and then the summation on the right refers to all points $\zeta$ for which $F_{0}(\zeta)<0$, except that one assumes that the function $F_{0}$ itself does not vanish for any point $\zeta$.

For the case of $n=3$ and $F_{1}=z_{1}, F_{2}=z_{2}, F_{3}=z_{3}$, the two integrals $V$ and $W$, which are both taken to be negative, will go to the ones that Gauss denoted by $M$ and $N$ in his "Allgemeine Lehrsätze, etc.," pp. 14 (Gauss Werke, Bd. V, pp. 209). For an arbitrary $n$ and linear functions $F$, the integral $W$ will be converted into the integral that Jacobi treated (Journal für Mathematik, Bd. XIV, pp. 51, et seq.). However, I shall confine myself to merely that citation here, without going into more details about its content and the meaning of the aforementioned Jacobi treatise.

When $\mathfrak{F}=1, V$ will vanish, and the relation (A) will yield:

$$
-\frac{1}{\varpi} W=\sum \chi(\zeta) ;
$$

i.e., the sum of the point characters, which is equivalent to the characteristic of the system ( $F_{0}, F_{1}$, $\ldots, F_{n}$ ), expressed by the integral $W$, which coincides with the content of section V. However, for the general case, equation (A), in analogy to the so-called Cauchy theorem, is an integral expression for the algebraic sum of all values that a given function $\mathfrak{F}\left(z_{1}, z_{2}, \ldots, z_{n}\right)$ will assume when the system of values of $z$ that is defined by the conditions:

$$
F_{0}<0, \quad F_{1}=0, F_{2}=0, \ldots, F_{n}=0
$$

is substituted in it, but assuming that one provides each of those function values with the same sign in the summation that the functional determinant of $F_{0}, F_{1}, \ldots, F_{n}$ has for the relevant system of values. Hence, when the functional determinant keeps one and the same sign for all those systems of values $z$, equation (A) will represent the sum of the function values $\mathfrak{F}$ by an integral expression, precisely as in Cauchy's theorem.

## IX. (sic)

When the functions $F_{1}, F_{2}, \ldots, F_{n}$ are the $n$ divisors of $m$ functions of the complex variables:

$$
y_{1}=z_{1}+i z_{m+1}, \quad y_{2}=z_{2}+i z_{m+2}, \quad \ldots, \quad y_{m}=z_{m}+i z_{2 m}
$$

for an even number $n=2 m$, the functional determinant will always have a single sign. However, in that case, the analogy between equation (A) and Cauchy's theorem will not be represented completely, since the aggregate of the two integrals $V$ and $W$ can be converted into a single ( $n-$ 1)-fold or boundary integral then. In order to specify that remarkable conversion more precisely, let $f_{1}, f_{2}, \ldots, f_{n}$ be functions of $y_{1}, y_{2}, \ldots, y_{m}$, and for each index $k$, let $f_{k}^{\prime}$ be conjugate to $f_{k}$; similarly, let $\mathfrak{f}$ be a function of the complex variables $y$, and let $\mathfrak{f}^{\prime}$ be conjugate to $\mathfrak{f}$. Furthermore, let $f_{k h}$ be the derivative of $f_{k}$ with respect to $y_{h}$, and let $f_{k h}^{\prime}$ be conjugate to $f_{k h}$. Finally, let:

$$
2 F_{k}=f_{k}+f_{k}^{\prime}, \quad 2 i F_{m+k}=f_{k}-f_{k}^{\prime}
$$

for $k=1,2, \ldots, m$, and:

$$
\mathfrak{F}=\varepsilon \mathfrak{f}+\varepsilon^{\prime} \mathfrak{f}^{\prime},
$$

in which one sets either $\varepsilon=\varepsilon^{\prime}=+1$ or $\varepsilon=-\varepsilon^{\prime}=i$. I shall now introduce the following notations, to abbreviate:

$$
F_{0 k}-i F_{0, m+k}=2 f_{0 k}, \quad F_{0 k}+i F_{0, m+k}=2 f_{0 k}^{\prime},
$$

for $k=1,2, \ldots, m$, and:

$$
\begin{aligned}
& \mathfrak{R}_{1}=\left|\begin{array}{ccccc}
0 & \mathfrak{f}_{1} & \mathfrak{f}_{2} & \cdots & \mathfrak{f}_{m} \\
f_{1} & f_{11} & f_{12} & \cdots & f_{1 m} \\
f_{2} & f_{21} & f_{22} & \cdots & f_{2 m} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
f_{m} & f_{m 1} & f_{m 2} & \cdots & f_{m m}
\end{array}\right| \text { and } \mathfrak{R}_{2} \text { is conjugate to } \mathfrak{R}_{1}, \\
& R_{1}=\frac{1}{\mathfrak{S}} \cdot\left|\begin{array}{ccccc}
0 & f_{01} & f_{02} & \cdots & f_{0 m} \\
f_{1} & f_{11} & f_{12} & \cdots & f_{1 m} \\
f_{2} & f_{21} & f_{22} & \cdots & f_{2 m} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
f_{m} & f_{m 1} & f_{m 2} & \cdots & f_{m m}
\end{array}\right| \text { and } R_{2} \text { is conjugate to } R_{1},
\end{aligned}
$$

$$
D_{1}=\left|\begin{array}{cccc}
f_{11} & f_{12} & \cdots & f_{1 m} \\
f_{21} & f_{22} & \cdots & f_{2 m} \\
\vdots & \vdots & \vdots & \vdots \\
f_{m 1} & f_{m 2} & \cdots & f_{m m}
\end{array}\right| \text { and } D_{2} \text { is conjugate to } D_{1} \text {. }
$$

One will then have:

$$
\begin{aligned}
& \Re=\varepsilon \Re_{1} D_{2}+\varepsilon^{\prime} \mathfrak{R}_{2} D_{1}, \\
& R=R_{1} D_{2}+R_{2} D_{1} ;
\end{aligned}
$$

i.e., the determinants $\mathfrak{R}$ and $R$ can both be represented as second-order determinants whose elements are determinants of order $m+1$ and $m$, respectively. Moreover, the $n$-fold integral $V$ can be transformed into an $(n-1)$-fold integral for regions that contain no points $\zeta$, and in fact, one will have:

$$
V=\int\left(\varepsilon \mathfrak{f} R_{1} D_{2}+\varepsilon^{\prime} \mathfrak{f}^{\prime} R_{2} D_{1}\right) \cdot \frac{d w}{S^{n}},
$$

while one has:

$$
W=\int\left(\varepsilon \mathfrak{f}+\varepsilon^{\prime} \mathfrak{f}^{\prime}\right)\left(R_{1} D_{2}+R_{2} D_{1}\right) \cdot \frac{d w}{S^{n}},
$$

in which both integrations extend over the entire boundary. It will then follow from this that since $-V+W$ is equal to zero, one will have:

$$
\varepsilon \int \mathfrak{f} \cdot \frac{R_{2} D_{1}}{S^{n}} d w+\varepsilon^{\prime} \int \mathfrak{f}^{\prime} \cdot \frac{R_{1} D_{2}}{S^{n}} d w=0 .
$$

Now, since one sets $\varepsilon=\varepsilon^{\prime}=+1$, on the one hand, and $\varepsilon=-\varepsilon^{\prime}=i$, on the other, it will ultimately follow that each of the two mutually-conjugate integrals must vanish individually. Now, the integral that appears here:

$$
\int \mathfrak{f} \cdot \frac{R_{2} D_{1}}{S^{n}} d w
$$

is the one that can be considered to be a generalization of Cauchy's integral, because whereas it will vanish whenever the boundary $F_{0}=$ encloses no point $\zeta$, when $F_{0}<0$, it will reduce to:

$$
\mathfrak{f}(\zeta) \cdot \int \frac{R_{2} D_{1}}{S^{n}} d w
$$

in the immediate neighborhood of an isolated point $\zeta$. However, it can be shown that:

$$
\int\left(R_{2} D_{1}-R_{1} D_{2}\right) \cdot \frac{d w}{S^{n}}=0
$$

for such a boundary, so one will have:

$$
\int \frac{R_{2} D_{1}}{S^{n}} d w=\frac{1}{2} \int\left(R_{1} D_{2}+R_{2} D_{1}\right) \cdot \frac{d w}{S^{n}} .
$$

Now since from what was said in no. V, the right-hand side of this equation represents the character of an enclosed point, multiplied by $-\frac{1}{2} \varpi$, one will get:

$$
\int \mathfrak{f} \cdot \frac{R_{2} D_{1}}{S^{n}} d w=-\frac{1}{2} \varpi \cdot \chi(\zeta) \cdot \mathfrak{f}(\zeta)
$$

for the chosen boundary, and therefore one will finally have:

$$
\begin{equation*}
\int \mathfrak{f} \cdot \frac{R_{2} D_{1}}{S^{n}} d w=-\frac{1}{2} \varpi \sum \chi(\zeta) \cdot \mathfrak{f}(\zeta) \tag{B}
\end{equation*}
$$

for an arbitrary boundary $F_{0}$, where the integration extends over the $(n-1)$-fold manifold $F_{0}=0$, and the summation extends over all points $\zeta$ for which $F_{0}$ assumes a negative value.

## X.

The characters of all points $\zeta$ that satisfy the $n$ equations $F_{0}=0$ are positive and equal to unity for simple points. When one sets $v_{k}=\zeta_{k}+\mathrm{i} \zeta_{m+k}$, the right-hand side of equation (B) will then become:

$$
-\frac{1}{2} \varpi \sum \mathfrak{f}(\eta)
$$

assuming that one takes each value $\mathfrak{f}(\eta)$ in the summation just as many times as the character of the point in question would imply. Moreover, one has:

$$
S^{n}=\left(\sum f_{k} \cdot f_{k}^{\prime}\right)^{m}
$$

and (cf., no. V):

$$
2 f_{0 k}^{\prime}=F_{0 k}+i F_{0, m+k}=\left(z_{k, p}^{0}+i z_{m+k, p}^{0}\right) \cdot \mathfrak{S} .
$$

Hence, when one sets:

$$
y_{k}^{0}=z_{k}^{0}+i z_{m+k}^{0}
$$

and

$$
y_{k p}^{0}=z_{k p}^{0}+i z_{m+k, p}^{0}
$$

accordingly, where $y_{k p}^{0}$ means (so to speak) the partial differential quotient of $y_{k}$ with respect to the normal at the point $y_{0}$ that belongs to the manifold $F_{0}=0$, one will have:

$$
\frac{f_{0 k}^{\prime}}{\mathfrak{S}}=\frac{1}{2} y_{k p}^{0} .
$$

Finally, let:

$$
\theta=\left|\begin{array}{ccccc}
0 & y_{1 p}^{0} & y_{2 p}^{0} & \cdots & y_{m p}^{0} \\
f_{1}^{\prime} & f_{11}^{\prime} & f_{12}^{\prime} & \cdots & f_{1 m}^{\prime} \\
f_{2}^{\prime} & f_{21}^{\prime} & f_{22}^{\prime} & \cdots & f_{2 m}^{\prime} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
f_{m}^{\prime} & f_{m 1}^{\prime} & f_{m 2}^{\prime} & \cdots & f_{m m}^{\prime}
\end{array}\right|
$$

and $\mathfrak{f} \cdot D_{1}=\phi$, where $D_{1}$ means the functional determinant of the functions $f$ that we have used up to now. After introducing those notations, equation (B) will go to:

$$
\begin{equation*}
\int \frac{\phi \theta}{\left(\sum f f^{\prime}\right)^{m}} d w=-\varpi \cdot \sum \frac{\phi(\eta)}{D_{1}(\eta)}, \tag{C}
\end{equation*}
$$

in which the integration extends over the manifold $F_{0}=0$, and indeed always in the sense that $\int d w$ remains positive, while the summation on the right refers to all points that satisfy the conditions:

$$
F_{0}<0, \quad f_{1}=0, f_{2}=0, \ldots, \quad f_{m}=0
$$

but every point is counted as many times as its character would imply.
For $m=1$, equation (C) goes to Cauchy's formula, because one will then have:

$$
\varpi=2 \pi, \quad \sum f f^{\prime}=f_{1} f_{1}^{\prime}, \quad \theta=-y_{1 p}^{0} \cdot f_{1}^{\prime}
$$

and when one introduces the notations $x_{0}, y_{0}$ in place of $z_{1}^{0}, z_{2}^{0}$ for the coefficients of the point on the curve $F_{0}=0$ :

$$
y_{1 p}^{0}=\frac{\partial\left(x_{0}+y_{0} i\right)}{\partial p} .
$$

If one now takes $d w$ (i.e., the element of the increasing arc-length along the curve $F_{0}=0$ ) to be such that when one advances in that sense, the negative values of $p$ (and thus the negative values of $F_{0}$, as well) keep to the left then one will have:

$$
\frac{\partial x_{0}}{\partial p}=\frac{\partial y_{0}}{\partial w}, \quad \frac{\partial y_{0}}{\partial p}=-\frac{\partial x_{0}}{\partial w}
$$

so:

$$
y_{1 p}^{0}=-i \cdot \frac{\partial\left(x_{0}+y_{0} i\right)}{\partial w}
$$

and equation (C) will then take the known form:

$$
\int \frac{\phi\left(x_{0}+y_{0} i\right)}{f\left(x_{0}+y_{0} i\right)} d\left(x_{0}+y_{0} i\right)=2 \pi i \sum \frac{\phi(\xi+\eta i)}{f^{\prime}(\xi+\eta i)}
$$

(when one sets $f_{1}, f_{11}$ equal to $f, f^{\prime}$, resp., and sets $\eta$ equal to $\xi+\eta i$ ), which then, in fact, represents a special form of equation (C).

## XI.

As was already pointed out above, those systems of values or points ( $\left.\zeta_{1}, \zeta_{2}, \ldots, \zeta_{n}\right)$ for which the functional determinant vanishes, along with the functions $F_{1}, F_{2}, \ldots, F_{n}$, have characters whose absolute values can be different from unity and even equal to zero. The characters of such points can then be regarded as being defined by the integral of the characteristic (cf., no. V), so by:

$$
-\frac{1}{\varpi} \int \frac{R}{S^{n}} d w
$$

in which the integration extends over a region $F_{0}=0$ that is arbitrary, but it can enclose no other points beside the point $\zeta$ considered. In the same way, for the special case that was treated in sections IX and X of functions of complex variables, the expression:

$$
\begin{equation*}
-\frac{1}{\varpi} \int \frac{\theta D_{1}}{\left(\sum f f^{\prime}\right)^{m}} d w \tag{D}
\end{equation*}
$$

can serve as the definition of the character of a point, i.e., the integral (D) will give how many times a system of values $\left(\eta_{1}, \eta_{2}, \ldots, \eta_{m}\right)$ that satisfies the equations:

$$
f_{1}\left(\eta_{1}, \eta_{2}, \ldots, \eta_{m}\right)=0, \quad f_{2}\left(\eta_{1}, \eta_{2}, \ldots, \eta_{m}\right)=0, \quad \ldots, \quad f_{m}\left(\eta_{1}, \eta_{2}, \ldots, \eta_{m}\right)=0
$$

is counted, as long as the integration extends over an $(n-1)$-fold manifold $F_{0}=0$ for which the region $F_{0}<0$ contains only that single point ( $\eta$ ).

The formula (C) will remain applicable when the equations $f=0$ are fulfilled for discrete points $\eta$, as well as for a simple or multiple manifold of points, except that the boundary $F_{0}$ must then be chosen in such a way that point sequences will remain excluded from the region $F_{0}<0$. Now, if $F_{0}$ is likewise determined in that case in such a way that the region $F_{0}<0$ includes all discrete points $\eta$ then the number of them will be expressed by the integral (D). For that reason, one might expect that a more detailed discussion of that integral will give some information about the aforementioned remarkable cases, as well as the ones in which a multiplicity of points appears, which are cases that have rarely been discussed up to now, even for systems of algebraic equations. However, one should not underestimate the fact that ascertaining the value of the integral (D) for the indicated cases will raise difficulties that are inextricably linked with their general validity. That is already explained by the fact that all of the integrals considered here will assume an entirely different form when one replaces the system of equations $F_{k}=0$ or $f_{k}=0$ with other, but equivalent
ones, i.e., ones that determine exactly the same points $\zeta$ or $\eta$ (along with the characters), while naturally the value of the integral must thereby remain unchanged. On the other hand, simplifications should emerge that might allow one to ascertain the value of the integral directly. Namely, one can replace the given boundary functions $F_{0}$ with another analytical representation of an available boundary that is as convenient and suitable as possible, since it only matters that it should agree in relation to the enclosed points ( $\eta$ ).

I would now like to make some explanatory remarks in connection with the determination of the value of the integral:

$$
\int \psi\left(z_{1}^{0}, z_{2}^{0}, \ldots, z_{n}^{0}\right) d w
$$

that was touched upon here. The introduction of such boundary integrals was entirely imperative on the occasion of my investigations, and also brought with it the same advantages of greater simplicity and intuitiveness as in the cases where $n$ had the value 2 or 3 , i.e., as in the cases of the plane and space. However, such boundary integrals are not immediately suited to calculation, but to that end, one must first transform them in some appropriate way. The domain of integration for the aforementioned integral is $F_{0}=0$-i.e., an $(n-1)$-fold manifold that is taken from the $n$-fold manifold $(z)$ - and the element on it $d w$ is found (cf., no. V) to be expressed by:

$$
\pm \frac{\mathfrak{S}}{F_{0 n}} d z_{1} \cdot d z_{2} \ldots d z_{n-1}
$$

while the element of an $(n-1)$-fold manifold $\left(z_{1}, z_{2}, \ldots, z_{n-1}\right)$ that is considered by itself will be given by the product:

$$
d z_{1} \cdot d z_{2} \ldots d z_{n-1}
$$

The difference in the nature of $v$-fold manifolds that appears here according to whether one considers them by themselves or as something that is removed from a higher-order manifold cannot be emphasized enough. Such distinctions are also quite common in the cases that are less accessible to geometric interpretations. A $v$-fold manifold that is considered by itself or flat (according to Riemann) has an element that is the product of the elements of the $v$ simple manifolds from which the $v$-fold one is derived. One can now briefly characterize the transformation that is required for the evaluation of the boundary integral by the fact that it puts the $(n-1)$-fold manifold $F_{0}=0$ that is removed into a one-to-one correspondence with an intrinsically considered or flat ( $n-1$ )-fold manifold, and it must be transformed into the latter. The possibility of such a transformation emerges from the following considerations, among others:

When one replaces the variables $z$ with new variables $r, u_{1}, u_{2}, \ldots, u_{n}$ that are defined by the equations:

$$
\begin{aligned}
& r \cdot u_{1}=z_{1}-Z_{1}, \quad r \cdot u_{2}=z_{2}-Z_{2}, \quad \cdots, \quad r \cdot u_{n}=z_{n}-Z_{n}, \\
& u_{1}^{2}+u_{2}^{2}+\cdots+u_{n}^{2}=1
\end{aligned}
$$

and correspond to polar coordinates, in which $r$ is taken to be positive and $\left(Z_{1}, Z_{2}, \ldots, Z_{n}\right)$ is understood to mean any fixed point, the equation $F_{0}=0$ can determine the radius vector $r$ as a single-valued function of the variables $u$ by means of that transformation. In that case, the region $F_{0}<0$ shall be called a "sector" (Bezirk) of the point Z (or relative to the point Z). Since the $n$ variables $u$ are uniquely expressible in terms of $n-1$ variables, moreover, every simple sequence of values of a variable $x$ between arbitrary limits:

$$
\frac{a}{a^{\prime}}<x<\frac{b}{b^{\prime}}
$$

will also be transformed uniquely into the total infinite sequence of values of the variable $y$ by a substitution, e.g., when one sets:

$$
x=\frac{a e^{-y}+b e^{+y}}{a^{\prime} e^{-y}+b^{\prime} e^{+y}},
$$

that will explain the fact that as long as only $F_{0}<0$ defines a "sector" for any point $Z$, every boundary integral that extends over the ( $n-1$ )-fold manifold $F_{0}=0$ can be converted into one for which the integration extends over a totally planar $(n-1)$-fold manifold. However, when $F_{0}<0$ does not define a sector, one can subdivide it into sectors, assuming that the radius vector $r$, which is multivalued in the present case, nonetheless has a number of values everywhere that does not exceed a certain number. Namely, if one starts from any point $Z$ of the region $F_{0}<0$ then the lines $r$ will fill up a certain subset of the region $F_{0}<0$ that will then define a sector of the point $Z$ when one continues those lines only as far as the smallest value of $r_{0}$ that belongs to the boundary $F_{0}<$ 0 , but with a modification that is easy to rectify. That modification will become intuitive in the case of the plane $(n=2)$ when one exhibits a boundary that contains a line segment and in that way assumes that the starting point of the radii vectores is such that one of them coincides with that straight line. One can then treat the resulting distinction between a sector in the region $F_{0}<0$ and the remaining subsets in the same way until the entire region $F_{0}=0$ is exhausted. The subdivision of a region into sectors offers certain advantages for the present purposes. However, under the assumptions that were made, one can also arrive at a unique determination of the different values of $r_{0}$ without it when one distinguishes them as the first, second, third, etc. of those quantities. For the transformation of multiple integrals, I shall ultimately refer to discussion of them that Lipschitz gave in Borchardt's Journal, Bd. 66, pp. 281, et seq.

## XII.

In the investigations whose course of development I have outlined here, I started from Sturm's theorem. An extension of it to systems of equations was already given long ago by Hermite, but it seemed to me, moreover, that the continued fraction development that was the basis for Sturm's developments itself could be generalized, and once that had been done, the more-general results that were obtained in that way could be interpreted naturally. In that way, I was led to those considerations that define the content of the first four sections of the present notice, and which I
communicated to my friend Weierstrass in a conversation at the time. From it, I was compelled to pursue my investigations further from the new viewpoint that had been achieved, and in fact in the direction in which not merely an extension of Sturm's theorem would be obtained, but also that of Cauchy. The work that I then undertook and the results that I thereby achieved are found to be set down in sections $V$ to X , and indeed in essence in the same sequence as the one that they presented to me in my investigation. I have chosen that genetic representation in the present extracted communication, as well as in the thorough treatise that is reserved for a memorandum, because in that way, one's understanding of the connection between the various results will be lightened. However, I cannot leave it unmentioned that one can arrive at another verification of some of the results that were found along a simpler path that is more concise. The methods that are applied in that way are mostly based upon the partial integration of multiple integrals and their variation at the limits. For that reason, I would like to present the relevant formulas here in conclusion, and especially the methods that must be employed, which I have suggested in present notice.

When $P, Q, Q^{(1)}, Q^{(2)}, \ldots, Q^{(n)}$ mean real, single-valued, and generally continuous functions of $n$ variables $z$, and their derivatives with respect to the individual variables are denoted by the addition of corresponding lower indices, one will have the following formula for partial integration:

$$
\begin{equation*}
\int \sum P_{k} Q^{(k)} \cdot d v+\int P \cdot \sum Q_{k}^{(k)} d v=\int P \cdot \sum Q^{(k)} \cdot z_{k p}^{0} d w \tag{1}
\end{equation*}
$$

The summations are extended from $k=1$ to $k=n$ in all cases. For the two $n$-fold integrals on the left-hand side, the common integration domain is considered to be given. The integration on the right-hand side is then extended over the total $(n-1)$-fold manifold $F_{0}=0$ that defines the natural boundary of that domain. Namely, the expression "natural boundary" shall be understood to mean the total $(n-1)$-fold manifold that includes or completes the discontinuities of the functions to be integrated (i.e., isolated points, as well as simply-extended or multiply-extended point sequences) infinitely closely. The general sense of the expression "natural boundary" also includes the given boundary of the domain of integration of the $n$-fold integral, as long as one assumes that the values of the functions being integrated go to zero suddenly at the given boundary when one extends that domain. Moreover, it should be pointed out that (1) can vanish for certain subsets of the natural boundary and that one is then justified in omitting those subsets.

As in section $V$, the quantity $z_{k p}^{0}$ on the right in formula (1) is determined from the equations:

$$
\sqrt{F_{01}^{2}+F_{02}^{2}+\cdots+F_{0 n}^{2}}=\mathfrak{S}, \quad z_{k p}^{0}=\frac{F_{0 k}}{\mathfrak{S}},
$$

where $z_{1}, z_{2}, \ldots$ are replaced with their values $z_{1}^{0}, z_{2}^{0}, \ldots$ on the boundary in $F_{01}, F_{02}, \ldots$
If $J$ denotes the $n$-fold integral:

$$
\int P d v \quad\left[\text { which is extended over } F_{0}\left(z_{1}, z_{2}, \ldots, z_{n}, t\right)<0\right]
$$

then one will have the following formula for the differentiation of it with respect to the parameter $t$ that is contained in the boundary function:

$$
\begin{equation*}
\frac{\partial J}{\partial t}=-\int \frac{P}{\mathfrak{S}} \cdot \frac{\partial F_{0}}{\partial t} \cdot d w \tag{2}
\end{equation*}
$$

where the integral on the right extended over the boundary $F_{0}=0$.
If one sets $\mathfrak{S} \cdot Q^{(k)}=F_{0 k}$ in equation (1), corresponding to a formula of Borchardt (cf., Liouville's Journal, t. XIX, pp. 383) then the right-hand side will go to:

$$
\int P d w
$$

and the equation can then be employed to express the boundary integral in terms of an $n$-fold integral.

If one further sets $Q^{(k)}=Q_{k}$ in equation (1) then one will have (cf., no. V):

$$
\sum Q^{(k)} \cdot z_{k p}^{0}=\sum Q_{k} \frac{\partial z_{k}^{0}}{\partial p}
$$

where the right-hand side is nothing but the partial differential quotient of $Q$ with respect to $p$ at a point $\left(z^{0}\right)$. With the assumption that was made, equation (1) will then go to the following one:

$$
\begin{equation*}
\int \sum P_{k} Q_{k} d v+\int \sum P \cdot \sum Q_{k k} d v=\int P \cdot \frac{\partial Q}{\partial p} \cdot d w \tag{3}
\end{equation*}
$$

Finally, the following two formulas for functions of complex variables will result from equation (1), when one preserves the notations that were introduced in section IX:

$$
y_{h}=z_{h}+i z_{m+h}, \quad \mathfrak{f}\left(z_{1}, z_{2}, \ldots, z_{n}\right)=\mathfrak{f}, \quad \frac{\partial \mathfrak{f}}{\partial y_{h}}=\mathfrak{f}_{h}
$$

and introduces $m$ functions $g^{(h)}$, moreover, which can depend upon the variables $y$, as well as their conjugates $y^{\prime}$ :

$$
\begin{align*}
\int \sum\left(\mathfrak{f}_{h} \cdot g^{(h)}+\mathfrak{f} \cdot g_{h}^{(h)}\right) d v & =\frac{1}{2} \int \mathfrak{f} \cdot \sum\left(F_{0 h}-i F_{0, m+h}\right) \cdot g^{(h)} \cdot \frac{d w}{\mathfrak{S}},  \tag{4}\\
\int \mathfrak{f}^{\prime} \cdot \sum g_{h}^{(h)} d v & =\frac{1}{2} \int \mathfrak{f}^{\prime} \cdot \sum\left(F_{0 h}-i F_{0, m+h}\right) \cdot g^{(h)} \cdot \frac{d w}{\mathfrak{S}} . \tag{5}
\end{align*}
$$

The summations in this extend from $h=1$ to $h=m$, one understands $\mathfrak{f}^{\prime}$ to mean the conjugate function to $\mathfrak{f}$, and finally $g_{h}$ means the partial derivative of $g$ with respect to $y_{h}$, while the same determinations are true for the domain of integration as in formula (1).

The potential theorems that go back to Gauss and Green can be adapted to the potential $\Pi$ with the help of the formulas that were given, but we shall not go further into that here, but only make one application of the formulas above that is closer to the topic that is being treated here.

If one replaces the $n$ integration variables $z$ with $n$ variables $\xi$ in formula (3), and to distinguish $d v^{\prime}$ from $d v$, one has:

$$
d v^{\prime}=d \xi_{1} d \xi_{2} \ldots d \xi_{n}
$$

and one further sets:

$$
P=1, Q=\Pi(\xi)
$$

then one will get:

$$
\begin{equation*}
\int \Delta \Pi(\xi) d v^{\prime}=\int \frac{\partial \Pi}{\partial p^{\prime}} \cdot d w^{\prime} \tag{6}
\end{equation*}
$$

where the integration on the left extends over a region:

$$
U_{0}\left(\xi_{1}, \xi_{2} \ldots \xi_{n}\right)<0
$$

and the one on the right extends over its boundary $U_{0}=0$. Now, since when one sets:

$$
\mathfrak{S}^{\prime}=\sqrt{U_{01}^{2}+U_{02}^{2}+\cdots+U_{0 n}^{2}},
$$

in analogy to the previous notation, one will have:

$$
\frac{\partial \Pi}{\partial p^{\prime}}=\sum \frac{\partial \Pi}{\partial \xi_{k}} \cdot \frac{U_{0 k}}{\mathfrak{S}^{\prime}},
$$

when one recalls the expression for $\Pi(\xi)$ that was given in section VI, the integral on the righthand side of equation (6) will be converted into:

$$
\int \mathfrak{F} \Delta_{0} d v=\int \sum U_{0 k}\left(F_{k}-\xi_{k}\right) \cdot \frac{d w^{\prime}}{S^{n} \cdot \mathfrak{S}^{\prime}}
$$

From section V, the innermost of those two integrals represents the characteristic of the system of functions of the variables $\xi$ :

$$
\left(U_{0}, \xi_{1}-F_{1}, \xi_{2}-F_{2}, \ldots, \xi_{n}-F_{n}\right),
$$

multiplied by $-\varpi$, and that characteristic is obviously one or zero according to whether:

$$
U_{0}\left(F_{1}, F_{2}, \ldots, F_{n}\right)
$$

has a negative or positive value, respectively. Under further integration over $d v$, the integral with the element $d w^{\prime}$ that we speak of will then take on the role of a discontinuous factor and will exclude all of the systems of values for which $U_{0}(F)>0$, such that the remarkable formula:

$$
\begin{equation*}
\int \Delta \Pi(\xi) d v^{\prime}=-\varpi \int \sum \mathfrak{F} \cdot \Delta_{0} \cdot d v \tag{7}
\end{equation*}
$$

will finally result from equation (6). The integrations in this extend over the region $U_{0}\left(\xi_{1}, \xi_{2}, \ldots\right.$, $\left.\xi_{n}\right)<0$ on the left and over all of the points $z$ that correspond to those points $\xi$ on the right. while the relationship between the manifolds of $x$ and $z$ is defined by the equations:

$$
\xi_{k}=F_{k}\left(z_{1}, z_{2}, \ldots, z_{k}\right) \quad(k=1,2, \ldots, n)
$$

The fundamental equation ( $\mathfrak{A}$ ) that was presented in section VI can be considered to be a limiting case of formula (7), namely, when the region that is defined by $U_{0}<0$ is restricted to the immediate vicinity of the point $\left(\xi_{k}=0\right)$. Since formula (7) is not just arrived at in a simpler way, but also without the assumption that derivatives of the function $\mathfrak{F}$ exist, it would seem quite preferable to employ formula (7) as the basis for equation ( $\mathfrak{A}$ ). However, it is necessary in that to either suppose the existence of second differential quotients of $\Pi(\xi)$ from the outset [as we did in the derivation of formula (7) above] or to verify that the sequence of the two boundary operations can be changed, one of which consists of the transition from a finite region $U<0$ to a single point that is contained in it, while the other consists of the transition from a differential quotient:

$$
\frac{1}{\delta_{k}}\left\{\Pi_{k}\left(\ldots, \xi_{k}+\delta_{k}, \ldots\right)-\Pi_{k}\left(\ldots, \xi_{k}, \ldots\right)\right\}
$$

that I would like to denote by $D_{k}$, to the corresponding differential quotient $\Pi_{k k}$. Without the assumption that $\Pi_{k k}$ exists, the method above will imply only that:

$$
\lim \cdot \int \sum D_{k} \cdot d v^{\prime}=-\varpi \int \sum \mathfrak{F} \cdot \Delta_{0} \cdot d v
$$

when the quantities $\delta_{k}$ that are contained in $D_{k}$ all approach zero, and it is, after all, remarkable that such a relation exists that can be interpreted intuitively for the case of ordinary mass potentials and corresponds entirely to the partial differential equation for the potential, in whose derivation, no assumption about the density function is required except that the potential function itself must have a well-defined meaning. Should it be possible to replace the potential equation with the relation in question in applications to physics, then one would freed from the necessity of making an assumption that is not available in the case of nature, namely, the assumption that density function is differentiable. Of course, even in the derivation of the potential equation, one does not necessarily need that assumption, as it might seem on first glance in Gauss's presentation of it. Rather, it can be modified in an essentially restrictive way, and in Clausius's derivation, one makes use of only the differential quotients of the mean density of a radius vector that issues from an attracting point. However, one must now defer any further investigations into resolving the questions of whether one can base the partial differential equation of the potential without making any assumption on the density function and whether that differential equation will retain its validity at all in such a context, like the relationship that corresponds to it above.


[^0]:    $\left({ }^{1}\right)$ I have not in any way tried to hide the fact that there is something awkward about going from the term "characteristic of a system of functions" to the expression "character of a point," because whereas it is entirely harmless to make the characteristic depend upon the sequence and signs of the function $F$, as a property of the system $\left(F_{0}, F_{1}, \ldots, F_{n}\right)$, it is nonetheless conceivable that such a property might also be something that refers to the system of values that is determined by the equations $F_{k}=0$. However, the odd or unsavory character of such a term is not shared by the concept of the character of the point, since the latter is indeed completely identical to that of a characteristic, but only in its expression, and after a thorough analysis, I have decided upon the use of that expression because on the

