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Natural geometry of Minding bendings of ray surfaces

By

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In my article "Zur Differentialgeometrie der Strahlflächen und Raumkurven," [Sitzber. Öst. Akad. Wiss. Wien, math.-nat. Kl., Abt. IIa, **157** (1949)], I developed a natural geometry of ray surfaces (= ruled surfaces) that was distinguished by the fact that it showed how the theory of space curves was a special case of the theory of ray surfaces in an especially simple way. The way of constructing the connection between ray surfaces and space curves that it implied, some of which has still not been noticed up to now, was the subject of that paper and two further ones "Strahlflächen als Verallgemeinerungen der Cesàro-Kurven," Monatsh. f. Math. **52** (1948) and "Das Analogon zu einem Satz von Cesàro über Bertrand-Kurven im Bereich der Strahlflächen," *ibidem* **54** (1950).

It will now be shown that the theory of ray surfaces that was developed in the first-mentioned paper ([1]) is also especially suited to a treatment of **Minding** bendings of ray surfaces. **X. Antonari** (Thèse, Paris, 1894) has already suggested a natural geometry of ray surfaces and their **Minding** bendings, but the choice of fundamental invariants of motion and invariant parameters that he made was not ideal.

The right-handed system $\mathfrak{e}$, $\mathfrak{n}$, $\mathfrak{z}$ of unit vectors that consists of the generator, the central normal, and central tangent, is the sliding dreibein of the general e (u) that we shall represent as depending upon the arc-length u along the line of striction. If $\mathfrak{e}$ and $\mathfrak{z}$ are interpreted as points on the unit sphere, and u_1 and u_3 are the arc-lengths on the spherical images of $\mathfrak{e}$ (u) and $\mathfrak{z}$ (u) then one can, as in the theory of curves, refer to $\kappa = du_1 : du$ as the *curvature* and $\kappa_1 = du_3 : du$ as *torsion* of the ray surface along e (u). Moreover, if $\mathfrak{t}$ (u) is the tangent vector to the line of striction then $\angle(\mathfrak{e}, \mathfrak{t}) = \sigma(u)$ will be the *striction* of e.

κ, κ_1, σ, as functions of the arc-length u of the line of striction, can be serve as the basis for a natural geometry of ray surfaces in which all concepts and lemmas are expressed in a coordinate-

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([1]) It will be cited as *loc. cit.* in what follows.

free way in terms of κ, κ_1, σ, and their derivatives. If $\mathfrak{s}$ (u) is the position vector for the points on the line of striction then the ray surface Φ can be represented by:

$$\mathfrak{x} = \mathfrak{s}\,(u) + v\,\mathfrak{e}\,(u)\,. \tag{1}$$

For the tangent vector $\mathfrak{t}$ that was explained above, one has $\mathfrak{t} = \mathfrak{e}\cos\sigma + \mathfrak{z}\sin\sigma$, such one can write:

$$\mathfrak{x} = \int (\mathfrak{e}\cos\sigma + \mathfrak{z}\sin\sigma)\,du + v\,\mathfrak{e}\,(u)\,, \tag{2}$$

instead of (1). The differential equations for $\mathfrak{e}$, $\mathfrak{n}$, $\mathfrak{z}$:

$$\dot{\mathfrak{e}} = \kappa\,\mathfrak{n}\,, \qquad \dot{\mathfrak{n}} = -\kappa\,\mathfrak{n} + \kappa_1\,\mathfrak{z}\,, \qquad \dot{\mathfrak{z}} = -\kappa_1\,\mathfrak{n} \tag{3}$$

coincide formally with the **Frenet** formulas in the theory of curves and will be identical to them when $\sigma = 0$, i.e., when the ray surface is the tangent surface of a space curve.

If one orients a generator e and its central normal n then the sliding dreibein $\mathfrak{e}$, $\mathfrak{n}$, $\mathfrak{z}$ will be determined for all generators in a neighborhood of e. Φ has obtained an *orientation* by that. κ is either positive or negative corresponding to the orientation that is chosen. By contrast, κ_1 is independent of the orientation. It is convenient to restrict the striction to the interval $-\pi/2 < \sigma \leq \pi/2$. One can say the following about the inversion of the signs of κ, κ_1, σ : The ray surfaces (κ, κ_1, σ) and ($-\kappa$, κ_1, $-\sigma$) are not different, except for their orientations. The ray surface (κ, $-\kappa_1$, $-\sigma$) [i.e., ($-\kappa$, $-\kappa_1$, σ), as well] is congruent to the ray surface (κ, κ_1, σ), but in the opposite sense. By contrast, the surfaces Φ_1 ($+\kappa$, κ_1, $\mp\sigma$) and Φ_2 ($\pm\kappa$, $-\kappa_1$, $\pm\sigma$) are essentially different from Φ (κ, κ_1, σ), in which the upper or the lower sign is chosen according to the orientation.

One speaks of a *Minding bending* ([1]) of a ray surface Φ when it can be mapped isometrically onto another one Φ^* in such a way that the generators of Φ go to the generators of Φ^*. Φ^* is then a *Minding bending surface of* Φ. We will ignore the trivial case of an isometry by a congruent transformation in the same or opposite sense ([2]).

From (2) and (3), one finds that the square of the arc-length differential for a curve on Φ is:

$$ds^2 = (1 + v^2\,\kappa^2)\,du^2 + 2\cos\sigma\,du\,dv + dv^2\,. \tag{4}$$

From (4), ds^2 *is independent of* κ_1 *and insensitive to the changes of sign of* κ *and* σ. It follows from this that every replacement of κ_1 with another function κ_1^*, as well as the inversions of the signs of κ or σ is a **Minding** bending of Φ (κ, κ_1, σ) into $\Phi_{1,2}^*(\kappa, \kappa_1^*, \pm\sigma)$.

([1]) **F. Minding**, J. f. Math. **18** (1836), pp. 297, 365.
([2]) A survey presentation of the theory of **Minding** bendings in a coordinate-geometry treatment is in **G. Darboux**, t. III (1894), pp. 293-316.

The most-general equations for the mapping of one ray surface Φ to another one Φ^* that takes the generators of Φ to generators of Φ^* congruently read $u = \varphi(u^*)$, $v = v^* + \psi(u^*)$. If one imposes the condition on it that (4) should remain invariant under it then that will imply that $u = u^*$, $v = v^*$. It will then follow that:

One will get all Minding bending surfaces from $\Phi(\kappa, \kappa_1, \sigma)$ from the following two mutually-independent operations:

1. *Replacing κ_1 with any other function κ_1^*.*
2. *Inverting the sign of κ or σ.*

Therefore, $|\kappa|$ and $|\sigma|$ are invariants of **Minding** bendings. Since it also follows from $v = 0$ that $v^* = 0$, the lines of striction will always be associated curves under all **Minding** bendings.

The problems of Minding bending can be treated in an entirely unified way within the natural geometry of ray surfaces that is based upon κ, κ_1, σ. They can then be divided into two classes:

A. Problems whose solution is based upon only the invariant of $|\kappa|$ and $|\sigma|$.
B. Problems that also require the calculation of the torsion κ_1^* of the bending surface Φ^*.

Example of A: *Under a Minding bending, the magnitudes of:*

a) *The twist p,*
b) *The polar radius of curvature r,*
c) *The geodetic curvature K_g of the line of striction*

will remain invariant.

That follows immediately from the formulas $p\,\kappa = \sin\sigma$ [*loc. cit.*, **§ 1** (11)], $r\,\kappa = \cos\sigma$ (*loc. cit.*, pp. 151), $K_g = \dot{\sigma}$ [*loc. cit.*, (5)].

The following theorem goes back to **E. Laguerre** ([1]):

If a ray surface can be taken to a hyperboloid of revolution by a Minding bending then its line of striction is a Bertrand curve.

Proof: Since κ and σ are constant for a hyperboloid of rotation, that will also be true for the **Minding** bending surfaces, and the theorem will therefore be an immediate consequence of the theorem (*loc. cit.*, **§ 2**, Theorem 7): The line of striction of a ray surface with κ = const. and σ = const. is a **Bertrand** curve.

([1]) Bull. Soc. math. **11** (1871), pp. 279.

Example of B: *A skew ray surface Φ is subjected to a Minding bending into a ray surface* Φ^* *in such a way that a curve c, v = v (u) that is given on Φ goes to an osculating tangent curve on* Φ^* ([1]).

As was shown in *loc. cit.*, § 10 (5), the differential equation of the osculating tangent curve $v = v(u)$ of the bending surface $\Phi^*(\kappa, \kappa_1^*, \sigma)$ reads:

$$2\kappa\dot{v}\sin\sigma = \kappa^2\kappa_1^* v^2 + (\kappa\dot{\sigma}\cos\sigma - \dot{\kappa}\sin\sigma)v + (\kappa_1^*\sin\sigma - \kappa\cos\sigma)\sin\sigma . \tag{5}$$

If one replaces the v in (5) with the given function $v(u)$ then (5) will imply the torsion κ_1^* of the desired bending surface $\Phi_{1,2}^*(\kappa, \kappa_1^*, \pm\sigma)$ of $\Phi(\kappa, \kappa_1, \sigma)$.

A skew ray surface Φ is bent by a Minding bending in such a way that a given geodetic line of Φ goes to a line ([2]).

A curve $v = v(u)$ that lies on a ray surface and has the vector representation:

$$\mathfrak{x} = \mathfrak{s} + v(u)\,\mathfrak{e} \tag{6}$$

is a line if and only if $\dot{\mathfrak{x}}$ and $\ddot{\mathfrak{x}}$ are linearly dependent, so when the matrix:

$$\left\| \begin{matrix} \cos\sigma + \dot{v} & v\kappa & \sin\sigma \\ \ddot{v} - v\kappa^2 - \dot{\sigma}\sin\sigma & 2\dot{v}\kappa + v\dot{\kappa} + \kappa\cos\sigma - \kappa_1\sin\sigma & v\kappa\kappa_1 + \dot{\sigma}\cos\sigma \end{matrix} \right\| \tag{7}$$

has rank 1. That condition can be expressed in terms of two independent equations in infinitely-many ways. If (i.) for i = 1, 2, 3 are the three columns in (7) then we can choose those two equations to be:

$$|\,(1.), (3.)\,| = 0 \qquad \text{and} \qquad |\,(1.), v\,\kappa\,(2.) + \sin\sigma\,(3.)\,| = 0 .$$

The first of them reads:

$$\ddot{v}\sin\sigma - \dot{v}(v\kappa\kappa_1 + \dot{\sigma}\cos\sigma) - v\kappa(\kappa\sin\sigma + \kappa_1\cos\sigma) - \dot{\sigma} = 0 \tag{8}$$

when written out. The second one is the *differential equation for the geodetic lines* on a ray surface [*loc. cit.*, § 11 (3)]:

$$2\kappa^2 v v^2 + \dot{v}\{\kappa\dot{\kappa}v^2 + \cos\sigma(3\kappa^2 v + \dot{\sigma}\sin\sigma) + (1+\kappa^2 v^2)(\kappa^2 v + \dot{\sigma}\sin\sigma) + \kappa\dot{\kappa}\cos\sigma = 0 . \tag{9}$$

([1]) **E. Beltrami**, Ann. di mat. (1) **7** (1866), pp. 112.
([2]) **E. Beltrami**, *loc. cit.*, pp. 109.

Since the given curve $v = v(u)$ is a geodetic line, (9) is fulfilled. Therefore, if one replaces v in (9) with the given function $v(u)$ and writes κ_1^* instead of κ_1 then one will have the equations from which κ_1^*, and therefore the desired bending surface $(\kappa, \kappa_1^*, \pm\sigma)$, can be obtained.

A skew ray surface Φ *shall be bent by a Minding bending in such a way that a given space curve* $v = v(u)$ *on it will become planar* ([1]).

A curve (6) is planar if and only if $(\dot{\mathfrak{x}}\,\ddot{\mathfrak{x}}\,\dddot{\mathfrak{x}}) = 0$. When we write out that condition for the bending surface $\Phi^*(\kappa, \kappa_1^*, \sigma)$, it will lead to an equation of the form:

$$G\,\dot{\kappa}_1^* + F_1\,\kappa_1^{*3} + F_2\,\kappa_1^{*2} + F_3\,\kappa_1^* + F_4 = 0\,, \tag{10}$$

in which G, $F_{1,\,2,\,3,\,4}$ are independent of κ_1^*. Any solution of (10) for κ_1^* will produce the bending surface $(\kappa, \kappa_1^*, \pm\sigma)$.

(10) simplifies in the following special cases: The function G that appears in G proves to be the left-hand side of the differential equation (9) $G = 0$ for the geodetic lines on Φ. Thus, one has:

The problem of bending a skew ray surface by a Minding bending that geodetic line on it will be planar leads to the cubic equation:

$$F_1\,\kappa_1^{*3} + F_2\,\kappa_1^{*2} + F_3\,\kappa_1^* + F_4 = 0\,. \tag{11}$$

The function F_1 in (10) is:

$$F_1 = (v^2\,\kappa^2 + \sin^2\sigma)(\dot{v} + \cos\sigma)\,. \tag{12}$$

$\dot{v} + \cos\sigma = 0$ [*loc. cit.*, § 8 (2)] is the condition for the curve $v = v(u)$ to cut the generators at right angles. Since $F_1 = 0$, from (13), it will then follow from (10) that:

The problem of bending a skew ray surface by a Minding bending such that given curves on it that are orthogonal to the generators will become planar leads to the Riccati differential equation:

$$G\,\dot{\kappa}_1^* + F_2\,\kappa_1^{*2} + F_3\,\kappa_1^* + F_4 = 0\,. \tag{13}$$

A problem that is similar to the aforementioned one, but still seems to have not been treated, is the following one:

([1]) **E. Beltrami**, *loc. cit.*, pp. 119.

Take a skew ray surface Φ *to a ray surface* Φ^* *under a Minding bending such that a given curve c*, $v = v(u)$ *on* Φ *goes to a curve on* Φ^* *along which the mean curvature of* Φ^* *vanishes.*

In order to solve that problem, one must set the expression [*loc. cit.*, § 10 (7)] for the mean curvature of a ray surface equal to zero, once one has replaced the v in it with the given function $v(u)$ and replaced κ_1 with κ_1^*. Thus, one will have the following equation for the κ_1^* of the bending surface $\Phi_{1,2}^*(\kappa, \kappa_1^*, \pm\sigma)$:

$$\sin\sigma(\kappa\cos\sigma + \kappa_1^*\sin\sigma) - v(\dot{\sigma}\sin\sigma - \kappa\dot{\sigma}\cos\sigma) + v^2\kappa^2\kappa_1^* = 0 . \tag{14}$$
