“Die Existenzbedingungen des verallgemeinerten kinetischen Potentials,” Math. Ann. **62** (1906), 148-155.

The existence conditions for the generalized kinetic potential

By

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Let $F_1, F_2, \ldots, F_n$ be given functions that include n unknown functions $y_1, y_2, \ldots, y_n$, in addition to the m independent functions $x_1, x_2, \ldots, x_m$, along with a finite number of derivatives:

$$y_1^{(1)} = \frac{dy_1}{dx_1}, \quad \ldots, \quad y_h^{(i)} = \frac{dy_h}{dx_i}, \quad \ldots, \quad y_h^{(i_1 i_2 \cdots i_\nu)} = \frac{d^\nu y_h}{dx_{i_1}\, dx_{i_2} \cdots dx_{i_\nu}}, \quad \ldots$$

If there is a function:

$$f\,(x_1, \ldots, x_m; y_1, \ldots, y_n; y_1^{(1)}, \ldots, y_h^{(i)}, \ldots, y_h^{(i_1 i_2 \cdots i_\nu)}, \ldots)$$

such that every F_h can be represented in the form:

$$\begin{aligned} F_h = V_h\,(f) = \frac{\partial f}{\partial y_h} &- \frac{d}{dx_1}\frac{\partial f}{\partial y_h^{(1)}} - \frac{d}{dx_2}\frac{\partial f}{\partial y_h^{(2)}} - \cdots \\ &+ \frac{d^2}{dx_1^2}\frac{\partial f}{\partial y_h^{(11)}} + \frac{d^2}{dx_1\, dx_2}\frac{\partial f}{\partial y_h^{(22)}} + \cdots \\ &- \frac{d^3}{dx_1^3}\frac{\partial f}{\partial y_h^{(111)}} - \frac{d^3}{dx_1^2\, dx_2}\frac{\partial f}{\partial y_h^{(112)}} - \cdots \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

then we, with **Leo Koenigsberger**, will say that the system $F_1, F_2, \ldots, F_n$ possesses the *generalized kinetic potential* f.

Our problem shall be to show the general validity of the elegant existence conditions for such a potential that **Arthur Hirsch** presented (*) but proved only for $m = 1$ in some very special cases. I shall follow precisely the same path that **Hirsch** pursued in deriving the necessity of those

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(*) **A. Hirsch**, “Über eine characteristische Eigenschaft der Differentialgleichungen der Variationsrechnung,” Math. Ann. **49** (1897), 49-72.

“Die Existenzbedingen des verallgemeinerten kinetischen Potential,” Math. Ann. **50** (1898), 429-441.

conditions. By contrast, I had to look for a new way of proving the sufficiency of the conditions that are known to be necessary.

1. – Before we turn to the topic itself, we shall first present some considerations about systems of linear differential expressions that are adjoint to each other.

If we denote n undetermined functions of $x_1, x_2, \ldots, x_m$ by:

$$u_1, u_2, \ldots, u_m,$$

and let:

$$P_{hk}(u) = \alpha_{hk}(x_1, \ldots, x_m)\,u + \beta_{hk}(x_1, \ldots, x_m)\frac{du}{dx_1} + \cdots + \mu_{hk}(x_1, \ldots, x_m)\frac{d^\nu u}{dx_{i_1}\,dx_{i_2}\cdots dx_{i_\nu}} + \cdots$$

$$(h, k = 1, 2, \ldots, n)$$

be n^2 linear homogeneous differential expressions, from which we compose the n sums:

$$\sum_{k=1}^{n} P_{1k}(u_k), \qquad \sum_{k=1}^{n} P_{2k}(u_k), \qquad \ldots, \qquad \sum_{k=1}^{n} P_{nk}(u_k), \tag{1}$$

then an obvious question to ask would be: How must the n functions be arranged in order for:

$$v_1 \sum_{k=1}^{n} P_{1k}(u_k) + v_2 \sum_{k=1}^{n} P_{2k}(u_k) + \cdots + v_n \sum_{k=1}^{n} P_{nk}(u_k) \tag{2}$$

to be representable by an aggregate of m exact differential quotients with respect to the individual arguments?

The property that is required of the expression (2) shall be denoted by ~ 0 in what follows, to abbreviate. In order to also express the demand:

$$\sum_{h=1}^{n}\sum_{k=1}^{n} v_h\, P_{hk}(u_k) \sim 0, \tag{3}$$

we shall bring into consideration the fact that the differential expression that is *adjoint* to $P_{hk}(u)$:

$$\text{adj. } P_{hk}(u) = u \cdot \alpha_{hk} - \frac{d}{dx_1}(u \cdot \beta_{hk}) + \cdots + (-1)^\nu \frac{d^\nu}{dx_{i_1}\,dx_{i_2}\cdots dx_{i_\nu}}(u \cdot \mu_{hk}) + \cdots$$

is known (*) to be characterized completely by the property:

(*) **G. Frobenius**, “Über adjungierte lineare Differentialausdrücke,” J. reine angew. Math. **85** (1878), pp. 207.

$$v \cdot P_{hk}(u) - u \cdot \text{adj.}\, P_{hk}(v) \sim 0 .$$

Thus, our requirement above can also be written:

$$\sum_{h=1}^{n}\sum_{k=1}^{n} u_k \cdot \text{adj.} P_{hk}(v_h) \sim 0 . \tag{4}$$

Since the u_k are undetermined here, the individual coefficients:

$$\sum_{h=1}^{n} \text{adj.}\, P_{hk}(v_h) \sim 0$$

must vanish.

We shall call the sums:

$$\sum_{h=1}^{n} \text{adj.}\, P_{h1}(v_h) ,\ \sum_{h=1}^{n} \text{adj.}\, P_{h2}(v_h) ,\ \ldots,\ \sum_{h=1}^{n} \text{adj.}\, P_{hn}(v_h) , \tag{5}$$

which then define the left-hand sides of some remarkable differential equations, the system of linear differential expressions that is *adjoint* to the system (1).

If the two systems (1) *and* (5) *coincide once we identify the symbols u_k and v_k then we will call the system* (1) ***self-adjoint****.*

The necessary and infallible criterion for the occurrence of that case is expressed by the equations:

$$\text{adj.}\, P_{hk}(u) = P_{kh}(u) \qquad (h, k = 1, 2, \ldots, n),$$

which we can also write in the form:

$$u \cdot P_{kh}(v) - v \cdot P_{hk}(u) \sim 0 \qquad (h, k = 1, 2, \ldots, n). \tag{6}$$

2. – The kinetic potential:

$$f\,(x_1, \ldots, x_m ; y_1, \ldots, y_n ; y_1^{(1)}, \ldots, y_h^{(i)}, \ldots, y_h^{(i_1 i_2 \cdots i_\nu)}, \ldots)$$

leads to the derived system:

$$F_1 = V_1(f) , \quad F_2 = V_2(f) , \quad \ldots, \quad F_n = V_n(f) .$$

That is defined by the property that is known from the calculus of variations that the difference of $u_k \cdot F_h$ and:

$$\delta_{u_h} f = \frac{\partial f}{\partial y_h} u_h + \frac{\partial f}{\partial y_h^{(1)}} u_h^{(1)} + \cdots + \frac{\partial f}{\partial y_h^{(i_1 i_2 \cdots i_\nu)}} u_h^{(i_1 i_2 \cdots i_\nu)} + \cdots$$

can be represented by an aggregate of *m* exact differential quotients with respect to the individual arguments $x_1, x_2, \ldots, x_m$. That can be expressed with the help of the symbol ~ in the form:

$$\delta_{u_h} f \sim u_k \cdot F_h \qquad (h = 1, 2, \ldots, n).$$

If we apply those relations to the process:

$$\delta_{v_k} f = \frac{\partial f}{\partial y_k} v_k + \frac{\partial f}{\partial y_k^{(1)}} u_k^{(1)} + \cdots + \frac{\partial f}{\partial y_k^{(i_1 i_2 \cdots i_\nu)}} v_k^{(i_1 i_2 \cdots i_\nu)} + \cdots$$

then since $\frac{d}{dx_i}$ and δ_{v_k} commute, that will give:

$$\delta_{v_k}(\delta_{u_h} f) \sim u_h \cdot \delta_{v_k} F_h \ .$$

One will likewise have:

$$\delta_{u_h}(\delta_{v_k} f) \sim v_k \cdot \delta_{u_h} F_k \ .$$

As a result of the commutability of the two processes δ_{u_h} and δ_{v_k}, it will follow from this that:

(7) $$u_h \cdot \delta_{v_k} F_h \sim v_k \cdot \delta_{u_h} F_k \qquad (h, k = 1, 2, \ldots, n).$$

If we introduce the symbol $P_{hk}(u_k)$ for $\delta_{u_h} F_k$ then those formulas will go to the system (6). Their content can then be expressed by saying:

I. *The system of linear differential expressions that is derived from the functions:*

$$F_1 = V_1(f), \quad F_2 = V_2(f), \quad \ldots, \quad F_n = V_n(f),$$

namely:

(8) $$\delta F_h = \delta_{u_1} F_h + \delta_{u_2} F_h + \cdots + \delta_{u_n} F_h \qquad (h = 1, 2, \ldots, n),$$

is self-adjoint.

3. – The theorem that was just proved can be inverted. That is almost self-explanatory when the derivatives of $y_1, y_2, \ldots, y_n$ are not included in F_h , but in addition to the:

$$x_1, x_2, \ldots, x_m ,$$

only:

$$y_1, y_2, \ldots, y_n$$

will appear. Namely, in that case, the system that is adjoint to the system:

$$\delta F_h = \frac{\partial F_h}{\partial y_1} u_1 + \frac{\partial F_h}{\partial y_2} u_2 + \cdots + \frac{\partial F_h}{\partial y_n} u_n \qquad (h = 1, 2, \ldots, n)$$

takes the following form:

$$\frac{\partial F_1}{\partial y_h} v_1 + \frac{\partial F_2}{\partial y_h} v_2 + \cdots + \frac{\partial F_n}{\partial y_h} v_n \qquad (h = 1, 2, \ldots, n).$$

If those two systems coincide as soon as we identify the symbols u_k and v_k then:

$$\frac{\partial F_h}{\partial y_k} = \frac{\partial F_k}{\partial y_h} \qquad (h, k = 1, 2, \ldots, n).$$

If that condition is satisfied in the neighborhood of a location:

$$x_1 = a_1, \quad x_2 = a_2, \quad \ldots, \quad x_m = a_m, \quad y_1 = b_1, \quad y_2 = b_2, \quad \ldots, \quad y_m = b_m$$

then one will have that:

(9) $$F_1\, dy_1 + F_2\, dy_2 + \ldots + F_2\, dy_2$$

is a complete differential there, when one considers $x_1, x_2, \ldots, x_m$ to be only parameters. $F_1, F_2, \ldots, F_n$, can then be represented in the form:

$$F_1 = V_1(f) = \frac{\partial f}{\partial y_1}, \qquad F_2 = V_2(f) = \frac{\partial f}{\partial y_2}, \quad \ldots, \qquad F_n = V_n(f) = \frac{\partial f}{\partial y_n}$$

in the stated neighborhood. If we choose f to be the rectilinear integral of the differential (9) of $(b_1, b_2, \ldots, b_n)$ to $(y_1, y_2, \ldots, y_n)$ then we will have:

(10) $$f = \sum_{h=1}^{n} \int_0^1 (y_h - b_h)\, F_h(x_1, \ldots, x_m; t(y_1 - b_1) + b_1, \ldots, t(y_n - b_n) + b_n)\, dt .$$

4. – The generalization of the known formula (10) leads to the following general converse to Theorem I in a simple way:

II. – *Let $F_1, F_2, \ldots, F_n$ be given functions of the quantities:*

$$x_1, x_2, \ldots, x_m; \; y_1, y_2, \ldots, y_n; \; \ldots; \; y_k^{(i_1 i_2 \cdots i_\nu)}, \ldots,$$

and let:

$$x_1 = a_1, \quad x_2 = a_2, \ldots, x_m = a_m \,; \ y_1 = b_1, \quad y_2 = b_2, \ldots, y_n = b_n \,; \ldots ; \ y_k^{(i_1 i_2 \cdots i_\nu)} = b_k^{(i_1 i_2 \cdots i_\nu)}, \ldots$$

mean a location (S) *such that the coefficients that appear in the system:*

$$\delta F_h = \delta_{u_1} F_h + \delta_{u_2} F_h + \cdots + \delta_{u_n} F_h \qquad (h = 1, 2, \ldots, n) \tag{8}$$

and the system adjoint to it exist at not only that location, but also in a certain neighborhood (C) *of it. In addition, let the system* (8) *be self-adjoint at* (C). *One can then construct a function:*

$$f\,(x_1, \ldots, x_m \,; y_1, \ldots, y_n \,; \ldots, y_h^{(i_1 i_2 \cdots i_\nu)}, \ldots)$$

with the help of a quadrature by means of which the functions F_h *can be represented in the form:*

$$F_1 = V_1\,(f)\,, \quad F_2 = V_2\,(f)\,, \quad \ldots, \quad F_n = V_n\,(f) \qquad (h = 1, 2, \ldots, n),$$

in a certain neighborhood (C') *of* (S).

If we again denote by the symbol $P_{hk}\,(u_k)$ then our assumption that the system (6) is self-adjoint can be expressed by the following equations:

$$\sum_{k=1}^{n} P_{hk}(u_k) = \sum_{k=1}^{n} \text{adj.}\, P_{kh}(u_k) \qquad (h = 1, 2, \ldots, n). \tag{11}$$

By our assumption, those equations are satisfied identically. Thus, they will still be true when we replace the y_k, u_k, and their derivatives with other quantities. Above all, we would like to replace:

$$y_1, y_2, \ldots, y_n \,,$$

and its derivatives with the functions:

$$t\,(y_1 - \varphi_1) + \varphi_1\,, \qquad t\,(y_2 - \varphi_2) + \varphi_2\,, \qquad \ldots, \qquad t\,(y_n - \varphi_n) + \varphi_n\,,$$

and their derivatives. Here, the φ_k mean arbitrary, but given, functions of $x_1, x_2, \ldots, x_m$. The symbol t denotes a new parameter that does not appear in the φ_k. With the stated substitution, for which we introduce the symbol [ ], equations (11) will go to:

$$\sum_{k=1}^{n} [P_{hk}(u_k)] = \sum_{k=1}^{n} [\text{adj.}\, P_{kh}(u_k)] \qquad (h = 1, 2, \ldots, n). \tag{12}$$

Here, we have:

(13) $$[P_{hk}(u)] = \left[\frac{\partial F_h}{\partial y_k}\right] u + \left[\frac{\partial F_h}{\partial y_k^{(1)}}\right] u^{(1)} + \cdots + \left[\frac{\partial F_h}{\partial y_k^{(i_1 i_2 \cdots i_\nu)}}\right] u^{(i_1 i_2 \cdots i_\nu)} + \cdots$$

and

$$[\text{adj. } P_{kh}(u)] = \left[u \cdot \frac{\partial F_k}{\partial y_h}\right] + \left[\frac{d}{dx_1}\left(u \cdot \frac{\partial F_h}{\partial y_k^{(1)}}\right)\right] - \cdots + (-1)^\nu \left[\frac{d^\nu}{dx_{i_1}\, dx_{i_2} \cdots dx_{i_\nu}}\left(u \cdot \frac{\partial F_h}{\partial y_k^{(i_1 i_2 \cdots i_\nu)}}\right)\right] + \cdots$$

If we bring under consideration the fact that for every function Φ:

$$\frac{\partial[\Phi]}{\partial y_h^{(i_1 i_2 \cdots i_\nu)}} = \left[\frac{\partial[\Phi]}{\partial y_h^{(i_1 i_2 \cdots i_\nu)}}\right] t$$

and

$$\frac{d\,[\Phi]}{dx_i} = \left[\frac{d\,\Phi}{dx_i}\right]$$

then we can also write:

(14) $$[\text{adj. } P_{kh}(u)] = u \cdot \frac{\partial[F_k]}{\partial y_h} - \frac{d}{dx_1}\left(u \cdot \frac{\partial[F_h]}{\partial y_k^{(1)}}\right) - \cdots + (-1)^\nu \frac{d^\nu}{dx_{i_1}\, dx_{i_2} \cdots dx_{i_\nu}}\left(u \cdot \frac{\partial[F_h]}{\partial y_k^{(i_1 i_2 \cdots i_\nu)}}\right) + \cdots$$

If we now replace $u_1, u_2, \ldots, u_n$, and their derivatives with the functions:

$$y_1 - \varphi_1, \qquad y_2 - \varphi_2, \qquad \ldots, \qquad y_n - \varphi_n,$$

and their derivatives then the sum $\sum_{k=1}^{n} [P_{hk}(u_k)]$ will go to:

$$\frac{\partial[F_h]}{\partial t},$$

and the product t [adj. $P_{kh}(u)$] will go to:

$$V_h((y_k - \varphi_k)[F_k]) - \delta_{hk}[F_h],$$

in which δ_{hk} means one or zero according to whether h and k are equal or different, resp. Equation (12) will ultimately go to the following one then:

$$t\,\frac{\partial[F_h]}{\partial t} = \sum_{k=1}^{n} V_h((y_k - \varphi_k)[F_k]) - [F_h] \qquad (h = 1, 2, \ldots, n),$$

i.e., to:

$$\frac{\partial}{\partial t}(t[F_h]) = \sum_{k=1}^{n} V_h((y_k - \varphi_k)[F_k]) \qquad (h = 1, 2, \ldots, n).$$

If we now integrate over t from $t = 0$ to $t = 1$. and in so doing consider that this integration commutes with the operation V_h, then we will, in fact, get:

$$F_h = V_h(f) ,$$

in which:

(15)
$$f = \sum_{k=1}^{n} \int_0^1 (y_k - \varphi_k)[F_k]\,dt \; .$$

That is the desired generalization of (10).

Naturally, the formulas that led to this result can be applied only within a certain domain of validity. We must then establish a domain in which our result is valid unconditionally.

Let the neighborhood (C) of the location (S) in which equations (11) have been assumed to be satisfied identically be defined by:

$$|x_i - a_i| < \delta_i , \qquad |y_h - b_h| < \varepsilon_h , \qquad \left| y_h^{(i_1 i_2 \cdots i_\nu)} - b_h^{(i_1 i_2 \cdots i_\nu)} \right| < \varepsilon_h^{(i_1 i_2 \cdots i_\nu)} .$$

We would like to choose $\varphi_1, \varphi_2, \ldots, \varphi_n$, to be entire rational functions such that for $x_1 = a_1, \ldots, x_m = a_m$, the derivative:

$$\frac{d^\nu}{dx_{i_1}\, dx_{i_2} \cdots dx_{i_\nu}} \varphi_h(x_1, x_2, \ldots, x_m)$$

is equal to $b_h^{(i_1 i_2 \cdots i_\nu)}$ or zero according to whether $y_h^{(i_1 i_2 \cdots i_\nu)}$ does or does not appear in $F_1, F_2, \ldots, F_n$, resp. Let the initial value of the φ_h be:

$$\varphi_h(a_1, a_2, \ldots, a_m) = b_h ,$$

which would be natural. We now determine $\delta_1', \delta_2', \ldots, \delta_n'$ such that for:

$$|x_i - a_i| < \delta_i' , \qquad |y_h - b_h| < \frac{\varepsilon_h}{2} , \qquad \ldots, \qquad |x_m - a_m| < \delta_m' ,$$

we have the inequalities:

$$|\varphi_h - b_h| < \frac{\varepsilon_h}{2} ,$$

$$\left| \varphi_h^{(i_1 i_2 \cdots i_\nu)} - b_h^{(i_1 i_2 \cdots i_\nu)} \right| < \frac{\varepsilon_h^{(i_1 i_2 \cdots i_\nu)}}{2} .$$

If we now define the neighborhood (C') of (S) by the inequalities:

$$|x_i - a_i| < \delta_i' , \qquad |y_h - b_h| < \frac{\varepsilon_h}{2} , \qquad \left| y_h^{(i_1 i_2 \cdots i_\nu)} - b_h^{(i_1 i_2 \cdots i_\nu)} \right| < \frac{\varepsilon_h^{(i_1 i_2 \cdots i_\nu)}}{2}$$

then all of our formulas will be true in that domain for $0 \leq t \leq 1$. Therefore, the system:

$$F_1, F_2, \ldots, F_n$$

will actually possess the generalized kinetic potential (15) in (C').

Budapest, 8 June 1905.