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Einstein's theory and Fermat's principle

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1. – Preliminary.

Let:

$$ds^2 = \sum_{i,k=0}^{3} g_{ik}\, dx_i\, dx_k \tag{1}$$

be the quadratic form that encompasses the measure of space and time, according to Einstein.

Suppose that the variable $x_0 = t$ can be physically interpreted as time, and that x_1, x_2, x_3 represent the spatial coordinates, so $-(ds^2)_{dx_0=0}$ or:

$$dl^2 = -\sum_{i,k=1}^{3} g_{ik}\, dx_i\, dx_k = \sum_{i,k=1}^{3} a_{ik}\, dx_i\, dx_k \tag{2}$$

will be the (positive-definite) form that specifies the metric on the ambient space.

For the physical interpretation, it is convenient to separate the terms in dx_0 (i.e., dt) in ds^2 and write:

$$ds^2 = g_{00}\, dt^2 + 2dt \sum_{i=1}^{3} g_{0i}\, dx_i - dl^2. \tag{1′}$$

The quaternary form on the right-hand side is indefinite ([1]) and can assume positive, negative, or zero values according to the circumstances.

A system of differentials $dx_0 = ct$, dx_1, dx_2, dx_3 are known to define a direction (d) in the quadri-dimensional variety. One says that (d) is *temporal, spatial,* or of *null length* according to whether ds^2 proves to be positive, negative, or zero, respectively. The hypothesis that when t varies by itself, it will yield a measure of time implies, in particular, that the direction:

$$(dt \neq 0,\ dx_1 = 0,\ dx_2 = 0,\ dx_3 = 0)$$

([1]) Even more precisely, consider it to be an algebraic form in dt, dx_1, dx_2, dx_3 (in the neighborhood of a general point) and to be such that when it is reduced to its canonical form (by real linear transformations), *the index of inertia* (i.e., the number of negative coefficients) *will be three.*

is temporal. It will then follow that $g_{00}\, dt^2 > 0$, which justifies the position that:

$$g_{00} = V^2, \tag{3}$$

in which V is a real quantity that has the dimension of a velocity.

In the kinematic phenomenon of the motion of a point, x_1, x_2, x_3 are well-defined functions of t and, any dt will remain uniquely subordinate to the differentials of the other three variables. It is well-known that the postulate of elementary relativity that any material motion will proceed with a velocity that is less than that of light can be generalized by assuming that $ds^2 > 0$ for any material point in motion, while one has $ds^2 = 0$ for the propagation of light. When one eliminates t, the equation of motion:

$$x_i = x_i\,(t) \qquad (i = 1, 2, 3)$$

will define a line in the ambient space – i.e., the trajectory of motion. However, when interpreted in four-dimensional space, it will define the so-called *time-lines* (*linea oraria*) ([1]).

If one takes into account the qualitative specification that $ds^2 > 0$ then the quantitative law that governs the motion of a material point will be included in Einstein's variational principle:

$$\delta \int ds = 0, \tag{4}$$

for variations (of the coordinates and t) that are zero at the extremes. With expressive geometric imagery, one can say: *The time-line of a material point is a temporal geodetic of the quadri-dimensional metric* [(1) *or* (1′)].

2. – Light rays. Relativistic principle. Fermat's principle. Coincidence under stationary conditions.

The formal application of (4) to the case of a time-line of null length cannot be materially performed (i.e., putting δ inside the sign and proceeding with only the algorithm of the calculus of variations) due to the singularity that results from annulling ds. However, it is known that it is enough to make a simple change of parameter in the differential equations that is equivalent to (4) in order to remove any trace of the singularity. In that sense, the notion of null-length geodesic is perfectly legitimate, and one can assume that it is a fundamental postulate of geometric optics in Einstein's theory, as stated by Hilbert ([2]), that: *Light rays are null-length geodesics of the quadri-dimensional metric* [(1) *or* (1′)].

([1]) For Weyl, they are "world-lines," i.e. *linea universale*. [Cf., the Italian translation of the article "Spazio Tempo" that is due to Prof. Gianfrancheschi in vol. XVIII, 1909, of this journal, page 336.] Granted, the terminology that Weyl introduced has been generally accepted. To me, it seems entirely preferable to appeal to the common usage in elementary kinematics, in which one calls the diagram in the plane (s, t) of a motion (with an arbitrary trajectory) that is defined by the equation $s = s\,(t)$ the "time-line." If the motion is defined by the three equations $x_i = x_i\,(t)$ then one will have an analogous quadri-dimensional diagram, and there is no reason to refer to it by a different terminology.

([2]) "Die Grundlagen der Physik," part II, Göttingen Nachrichten, 1917.

Another inductive criterion for the definition of the course of light rays that is plausible *a priori* is to *associate the equation* $ds^2 = 0$ *with Fermat's principle* of the minimum time to travel between two generic points. That is, to assume that:

$$\delta \int dt = 0, \tag{5}$$

in which it is intended that dt is coupled with t and the spatial coordinates, and their differentials will give $ds^2 = 0$. In regard to that, one should note that whereas in the quadri-dimensional geometric principle (4), one must keep not only the δx_i , but also δt, zero at the extremes of the interval of integration, nevertheless, one must obviously suppress that last condition in (5), since otherwise it would reduce (5) to merely an identity. Moreover, it is enough to consider the meaning of the principle in order to infer its precise analytical formulation. Meanwhile, one must refer to the three-dimensional space x_1, x_2, x_3 (whose line element is dl) and regard the starting and stopping points as fixed, as well as the instant when the light signal started. One can then address the search for the path that minimizes the duration of the trip $\int dt$ when it is subordinate to the differential constraint $ds^2 = 0$. That will imply that $\delta\, ds^2 = 0$, which can be considered to be a (first-order differential) relation between δt, the variations (which are arbitrary, but zero at the extremes), dx^i, and their differentials $d\,\delta t$, $d\,\delta x_i$, respectively. When one imposes the condition on δt that it must be zero at the beginning, it will remain determined uniquely as a function of the δx_i along the curve that joins the extreme positions. One can say that (5) expresses the idea that such a curve must be determined in such a way that δt is also zero at the end no matter how one chooses the δx_i (at the intermediate points). The differential translation of (5) is obtained as follows:

If one imagines that the equation $ds^2 = 0$ has been solved for dt then one will have:

$$dt = f\,(t\,;\, x_1, x_2, x_3\,;\, dx_1, dx_2, dx_3)\ ,$$

in which f is a function that is homogeneous of degree one with respect to dx_1, dx_2, dx_3 . Under stationary conditions (i.e., when the coefficients of ds^2 do not depend upon t), even f is exempt from that condition, and the principle (5), when one replaces dt with f, will assume a purely-geometric character, which will lead to the equations:

$$d\left(\frac{\partial f}{\partial\, dx_i}\right) - \frac{\partial f}{\partial x_i} = 0 \qquad (i = 1, 2, 3).$$

In general, it also necessary to introduce the identity:

$$\delta\, dt - \delta f = 0$$

and to recall the usual method of multipliers, so one must replace (5) with the equivalent condition:

$$\int [\delta\, dt + \lambda\,(\delta\, dt - \delta f)] = 0,$$

in which δt is zero at only one of the extremes.

One will get the differential equations:

$$\begin{cases} d\lambda + \lambda \dfrac{\partial f}{\partial t} = 0, \\ d\left(\lambda \dfrac{\partial f}{\partial\, dx_i}\right) - \lambda \dfrac{\partial f}{\partial x_i} = 0 \quad (i = 1, 2, 3), \end{cases}$$

with the condition that $\lambda + 1 = 0$, along with the one that δt must remain arbitrary at the two extremes. For $\partial f / \partial t = 0$, the first differential equation, when combined with the limit conditions, will yield $\lambda = -1$, so one will recover the equations in the stationary case, as one must.

I have already had occasion to avail myself of Fermat's principle under static conditions ([1]) – i.e., when not only are all of the coefficients of ds^2 independent of t, but one also annuls the cross terms in dt (viz., $g_{0i} = 0$). As a result, Weyl ([2]) has noted that there is an essential equivalence between the two criteria (viz., Hilbert's and Fermat's) under such conditions. Weyl's proof is not really complicated, but it still requires some formal steps. *I propose to establish* (more generally and also with more simplicity) *the equivalence of the two principles of geometric optics – viz., (quadri-dimensional) geodetics and minimum time – for any stationary metric.*

3. – Proof.

As I have already pointed out, one should consider the null-length geodetics to be something that is derived from temporal geodetics ($ds > 0$) in the limit. In order to do that, when one lets c denote an arbitrary constant that is identified with the speed of light in the absence of any perturbing circumstances, one usually sets:

(6) $$\dot{x}_i = \frac{dx_i}{dt} \quad (i = 1, 2, 3), \qquad v^2 = \frac{dl^2}{dt^2} = \sum_{i,k=1}^{3} a_{ik}\dot{x}_i\,\dot{x}_k,$$

(7) $$L = c\frac{dx_i}{dt} = c\sqrt{V^2 + \sum_{i,k=1}^{3} g_{0i}\dot{x}_i - v^2},$$

in which the function L admits finite partial derivatives (because one excludes the possibility that it annuls the ds, and therefore the quantity under the radical).

(4) can be written:

(4′) $$\delta \int L\, dt = 0\,.$$

([1]) "Statica einsteiniana," Rend. della R. Acc. dei Lincei **26** (1st sem. 1917), pp. 470.

([2]) "Zur Gravitationstheorie," Ann. Phys. (Leipzig) **54** (1917), 127-128.

The variation that is performed on the coordinates x_1, x_2, x_3 classically leads to the Lagrange equations:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} - \frac{\partial L}{\partial x_i} = 0 \qquad (i = 1, 2, 3). \tag{8}$$

Given the origin of (4′) from (4), one needs to treat t like the spatial coordinates, and therefore subject it to variation (which is zero at the extremes), as well. However, that will not lead to any new conditions. Indeed, after integrating by parts, one will have:

$$\frac{d}{dt}\left(\sum_{i=1}^{3}\frac{\partial L}{\partial \dot{x}_i}\dot{x}_i - L\right) + \frac{\partial L}{\partial t} = 0,$$

which is a necessary consequence of (8).

Under the hypothesis that characterizes the stationary case that L does not contain t explicitly, one will get:

$$L - \sum_{i=1}^{3}\frac{\partial L}{\partial \dot{x}_i}\dot{x}_i = E, \tag{9}$$

in which the constant E can be interpreted as the total energy of the moving point ([1]).

When one multiplies that by L, the left-hand side can be written:

$$\tfrac{1}{2}L^2 + \tfrac{1}{2}\left(L^2 - \sum_{i=1}^{3}\frac{\partial L^2}{\partial \dot{x}_i}\dot{x}_i\right).$$

By virtue of (7), L^2 is a polynomial of degree two in $\dot{x}_1$, $\dot{x}_2$, $\dot{x}_3$. It already takes a form that is split into three addends that are homogeneous of degrees 0, 1, 2, respectively. From Euler's theorem on homogeneous functions, the linear terms in the difference $L^2 - \sum_{i=1}^{3}\frac{\partial L^2}{\partial \dot{x}_i}\dot{x}_i$ will disappear, since they reduce to $c^2\,(V^2 + v^2)$. With that, when (9) is multiplied by L, that will give:

$$\tfrac{1}{2}L^2 + \tfrac{1}{2}c^2(V^2 + v^2) = E\,L. \tag{9′}$$

The left-hand side is essentially positive, as well as $\geq \tfrac{1}{2}c^2\,V^2$ (which is considered to be endowed with a non-zero lower limit in the present context). The product $E\,L$ *can be regarded as a function of* x *and* $\dot{x}_i$ *that remains regular and non-zero, even when* L *converges to zero.* Under those hypotheses, the constant E will clearly tend to infinity.

([1]) Cf., in addition to the previously-cited "Statica einsteiniana" (pages 465-468), the article "ds^2 einsteiniani in campi newtoniana. I. Generalità e prima approssimazione," *ibidem*, 2nd semester 1917, pp. 309.

On the other hand, as was observed before in the cited note on Einsteinian statics, for any motions that have the same total energy E, one can replace (4′), in which one supposes that δt is annulled at the extremes of the integration interval, with an analogous principle that presents the major advantage that it no longer requires that condition. To that end, it is enough to observe that $\delta \int dt = 0$ when δt is zero at the extremes, and consequently, (4′) will be equivalent to $\delta \int (L - E)$ $dt = 0$, or even to $\delta \int \left(1 - \frac{L}{E}\right) dt = 0$ for $E \neq 0$. Finally, in the latter, one can drop the constraint that δt is annulled at the extremes, because when one moves δ inside the sign and applies it to dt (in which it appears explicitly in terms of the $\dot{x}_i$), one will have, materially:

$$\frac{1}{E} \int \delta\, dt \left(L - E - \sum_{i=1}^{3} \frac{\partial L}{\partial \dot{x}_i} \dot{x}_i \right),$$

which is annulled by virtue of (9).

We have therefore learned that for an assigned (non-zero) value of E, the equation of motion can be summarized in the formula:

$$\delta \int \left(1 - \frac{L}{E}\right) dt = 0 . \tag{10}$$

The function under the sign can be written $1 - \frac{L^2}{E\,L}$, in which it appears that it will stay regular and tend to unity under the hypothesis that L converges to zero when one keeps in mind the observed behavior of the denominator $E\,L$. Now, it is exactly that hypothesis that makes one pass from the material motion to the limiting case of light propagation. If one expects regularity then the passage to the limit can be exchanged with the operator $\delta \int$, and (10) will then give rise to Fermat's principle:

$$\delta \int dt = 0 .$$

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4. – Geometric complements.

One can obviously make any direction (d) in quadri-dimensional space (t, x_1, x_2, x_3) – i.e., to any system of increments (dt, dx_1, dx_2, dx_3) – correspond to a (velocity) vector **v** in the physical space with the line element dl, by which, one means, more precisely, in the tangent Euclidian space (at a generic point from which one considers the aforementioned increments).

Assume that the contravariant system of that vector with respect to the metric (2) consists of the ratios:

$$\frac{dx_i}{dt} = \dot{x}_i \qquad (i = 1, 2, 3).$$

If one gives them the form:

$$\frac{dx_i}{dl} \frac{dl}{dt}$$

then one will exhibit the direction parameters dx_i / dl, and therefore the positive factor dl / dt will measure the length of the vector. When referred to the position (6), one will have:

$$v^2 = \frac{dl^2}{dt^2} = \sum_{i,k=1}^{3} a_{ik}\, \dot{x}_i\, \dot{x}_k$$

for the square of that length.

Another vector **w** that is a function of position and time exclusively (merely position under stationary conditions, resp.) can be made to correspond with the triad g_{0i} (which is covariant with respect to arbitrary transformations of only the spatial coordinates) when one assumes that the triad is the covariant system for the vector. Therefore, when one lets $a^{(ik)}$ denote the coefficients of the form that is reciprocal to (2) and sets:

(11) $$w^2 = \sum_{i,k=1}^{3} a^{(ik)}\, g_{0i}\, g_{0k},$$

one will have the length of that vector in w and (for $w > 0$) the moments (i.e., the reciprocal system to the parameters) of its direction in the ratios g_{0i} / w. One should note that if the spatial coordinates x have the dimensions of length then the coefficients a_{ik} of dl^2, and therefore their reciprocals $a^{(ik)}$, will be pure numbers, while the coefficients g_{0i} of the cross terms in dt will have the dimensions of a velocity. Therefore, the vector **w** can be interpreted as a velocity, just like **v**. That conclusion (which should be obvious) will remain valid even when one leaves the dimensions of the coordinates x_1, x_2, x_3 undetermined.

If one given φ as the angle between **v** and **w** (both of which are supposed to be non-zero, for the moment) then based upon the metric (2), one will have:

$$\cos\varphi = \sum_{i,k=1}^{3} \frac{g_{0i}}{w} \frac{\dot{x}_i}{v},$$

and therefore, the identity:

(12) $$v\, w \cos\varphi = \sum_{i,k=1}^{3} g_{0i}\, \dot{x}_i,$$

Assuming all of that, on the basis of (3), (6), and (12), the expression (1′) for ds^2 can be written:

$$ds^2 = dt^2\,(V^2 + 2v\, w \cos\varphi - v^2)\,.$$

That makes it obvious that the condition $ds^2 = 0$, which characterizes the propagation of light, defines the velocity v as a function of position and the direction of the ray, as well as time, in the general case in which the coefficients of ds^2, along with those of V, w, and φ, depend upon t.

Let β and p represent the ratios (which are both positive and pure numbers) v / V and w / V, resp., so one will have the following second-degree equation for β:

(13) $$\beta^2 - 2p \cos\varphi\, \beta - 1 = 0,$$

whose roots have – 1 for their product, and therefore one of them is positive, while the other is negative. From its meaning, v must be positive, so that (13) will define it *uniquely*.

When all of the cross terms in dt are annulled (viz., the static case), one will have $w = 0$, so $\beta = 1$, and v will coincide with V. In general, one will have $p > 0$, and the discrepancy in V (at a fixed position and instant) will depend upon the direction of the ray, or rather, the angle φ that it forms with **w**. One also has $v = V$ for any ray that is perpendicular to **w**. Of course, (13) then shows that the maximum and minimum values of β will correspond with $\varphi = 0$ and $\varphi = \pi$, resp. That is to say, the maximum velocity of propagation:

$$V(\sqrt{1+p^2}+p)$$

will be found along **w**; the minimum:

$$V(\sqrt{1+p^2}-p)$$

will be found in the same direction, but in the opposite sense.

As one sees, except for the static case, the propagation of light in physical space has a behavior that is not only anisotropic, but even irreversible.

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