"Sulla integrazione della equazione di Hamilton-Jacobi per separazione di variabili," Math. Ann. **59** (1904), 383-397.

# On the integration of the Hamilton-Jacobi equation by separation of variables.

By

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[Extract from a letter to Sig. Prof. **P. Stäckel** in Kiel]

Translated by D. H. Delphenich

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In recent days, I have had occasion to revisit your beautiful research (*) on the integration of the Hamilton-Jacobi equation:

$$H\,(p_1, p_2, \ldots, p_n\,;\, x_1, x_2, \ldots, x_n) = h$$

($p_1 = dW / dx_1$, $p_2 = dW / dx_2$, …; $h$ is an arbitrary constant) by separation of variables.

It is known that one can easily assign (in the explicit form of partial differential equations with respect to the arguments $p$ and $x$) the necessary and sufficient conditions that an $H$ must satisfy in order for the equation:

$$H = h$$

to admit a complete integral of the form:

$$\sum_{i=1}^{n} W_i$$

($W_i$ is a function of only $x_i$).

One can derive some consequences of a general nature from those conditions that seem rather interesting to me, and from which I will deduce the complete solution to the problem, which appears quite laborious, and for which (dare I say it) there is not even much hope of finding essentially new types other than the ones that you had discovered.

However, I do not think that it would be unwelcome of me to communicate what little that I have done by way of argument since it has been elegantly discussed in the case of two variables, which was already treated exhaustively by Prof. Morera (**) and yourself (***), among others, but perhaps not as simply.

Therefore, here are my observations:

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(*) *Habilitationsschrift*, Halle, 1891, as well as Math. Ann., Bd. **42**, pps. 546-549.

(**) Atti della R. Accademia di Torino **16** (1881).

(***) Math. Ann., Bd. **35**.

1. – The hypothesis that $W$ must have the form $\sum_{i=1}^{n} W_i$ is equivalent to (*):

(1) $$\frac{dp_i}{dx_j} = 0 \qquad (i \neq j).$$

On the other hand, upon differentiating the equation $H = h$ with respect to a generic $x_i$ and observing that $H$ depends upon $x_i$ both directly and by way of the $p$, one will have:

$$\frac{\partial H}{\partial x_i} + \sum_{j=1}^{n} \frac{\partial H}{\partial p_j} \frac{\partial p_j}{\partial x_i} = 0 ,$$

and consequently, upon recalling (1):

$$\frac{\partial H}{\partial x_i} + \frac{\partial H}{\partial p_i} \frac{\partial p_i}{\partial x_i} = 0 .$$

Now, in the cases that actually correspond to a Hamilton-Jacobi equation, none of the $\partial H / \partial p_i$ can be identically zero, and therefore those equations are equivalent to:

(2) $$\frac{\partial p_i}{\partial x_i} = -\frac{\dfrac{\partial H}{\partial x_i}}{\dfrac{\partial H}{\partial p_i}} .$$

(1) and (2) then define all of the derivatives of the $p$. In order for functions $p$ to actually exist that have those derivatives [and therefore, by virtue of (1), for $W$ to also exist], the integrability conditions must be satisfied, and as soon as they appear, they will reduce to:

$$\frac{d}{dx_j} \left\{ \frac{\dfrac{\partial H}{\partial x_i}}{\dfrac{\partial H}{\partial p_i}} \right\} = 0 \qquad (i \neq j),$$

or when one develops the operator symbol $\dfrac{d}{dx_j}$ into $\dfrac{\partial}{\partial x_j} + \dfrac{\partial p_j}{\partial x_j} \dfrac{\partial}{\partial p_j}$ and takes (2) into account, to:

(3) $$\frac{\partial H}{\partial p_i} \frac{\partial H}{\partial p_j} \frac{\partial^2 H}{\partial x_i \partial x_j} - \frac{\partial H}{\partial p_i} \frac{\partial H}{\partial x_j} \frac{\partial^2 H}{\partial x_i \partial p_j} - \frac{\partial H}{\partial x_i} \frac{\partial H}{\partial p_j} \frac{\partial^2 H}{\partial p_i \partial x_j} + \frac{\partial H}{\partial x_i} \frac{\partial H}{\partial x_j} \frac{\partial^2 H}{\partial p_i \partial p_j} = 0 \qquad (i \neq j).$$

(*) It is intended that the indices $i$, $j$, and therefore all of the ones that appear in the following, must be attributed all values from 1 to $n$ that are compatible with any possible restrictions that might be stated explicitly.

Since all of the derivatives of the $p$ are well-defined, all that remains arbitrary is at most their initial values. Those initial values must effectively remain arbitrary when we treat (as we are assuming) a complete integral $W$ to the Hamilton-Jacobi equation (in which we should not forget that the constant $h$ in the right-hand side is arbitrary *a priori* and is meant to be determined by the initial values).

Since the complete arbitrariness of the initial values of the $p$ implies that it is necessary and sufficient that the integrability conditions (3) are satisfied *identically* with respect to all the $2n$ symbols $p$ and $x$, one can conclude, with no further discussion, that:

*The Hamilton-Jacobi equation:*

$$H = h$$

(in which it is intended that $H$ contains *all* of the $p$ explicitly) *is integrable by separation of variables if and only if the characteristic function $H$ satisfies the* $n$ $(n - 1) / 2$ *second-order equations* (3).

**2.** – Suppose that $H$ corresponds to a dynamical problem with constraints that are independent of time and let:

$$T = \tfrac{1}{2} \sum_{r,s=1}^{n} a_{rs}\, x'_r\, x'_s \tag{4}$$

be the *vis viva* of the system, with the usual notations, while $U$ is the potential of the force that acts upon it.

If $a^{(rs)}$ are the coefficients of the form that is reciprocal to $T$, and one sets:

$$K = \tfrac{1}{2} \sum_{r,s=1}^{n} a^{(rs)}\, p_r\, p_s \tag{5}$$

then one will have:

$$H = K - U\,. \tag{6}$$

In addition, as is well known:

$$x'_i = \frac{\partial K}{\partial p_i}\,, \tag{7}$$

$$\frac{\partial K}{\partial x_i} = -\frac{\partial T}{\partial x_i}\,, \tag{8}$$

when one regards the $x$ and $p$ in the left-hand side of (8) and the $x$ and $x'$ in the right-hand side as independent variables.

Having said that, replace $H$ with its value $K - U$ in the left-hand sides of (3) and note that they include a part that has degree four in the $p$, one of degree two, and one of degree zero.

Since we are dealing with an identity, the coefficients of those polynomials must all be zero. If we confine ourselves, for the moment, to expressing that the terms of various degrees are annulled separately then (3) will split into three groups:

$$\text{(I)} \qquad \frac{\partial K}{\partial p_i}\frac{\partial K}{\partial p_j}\frac{\partial^2 K}{\partial x_i\,\partial x_j} - \frac{\partial K}{\partial p_i}\frac{\partial K}{\partial x_j}\frac{\partial^2 K}{\partial x_i\,\partial p_j} - \frac{\partial K}{\partial x_i}\frac{\partial K}{\partial p_j}\frac{\partial^2 K}{\partial p_i\,\partial x_j} + \frac{\partial K}{\partial x_i}\frac{\partial K}{\partial x_j}\frac{\partial^2 K}{\partial p_i\,\partial p_j} = 0\,,$$

$$\text{(II)} \qquad \frac{\partial K}{\partial p_i}\frac{\partial K}{\partial p_j}\frac{\partial^2 U}{\partial x_i\,\partial x_j} - \frac{\partial K}{\partial p_i}\frac{\partial^2 K}{\partial x_i\,\partial p_j}\frac{\partial U}{\partial x_j} - \frac{\partial K}{\partial p_j}\frac{\partial^2 K}{\partial x_i\,\partial p_j}\frac{\partial K}{\partial x_i} + \frac{\partial K}{\partial p_j}\frac{\partial^2 K}{\partial x_j\,\partial p_i}\frac{\partial U}{\partial x_i}$$
$$+ \frac{\partial^2 K}{\partial p_i\,\partial p_j}\left\{\frac{\partial K}{\partial x_i}\frac{\partial U}{\partial x_j} + \frac{\partial K}{\partial x_j}\frac{\partial U}{\partial x_i}\right\} = 0\,,$$

$$\text{(III)} \qquad \frac{\partial^2 K}{\partial p_i\,\partial p_j}\frac{\partial U}{\partial x_i}\frac{\partial U}{\partial x_j} = a^{(ij)}\frac{\partial U}{\partial x_i}\frac{\partial U}{\partial x_j} = 0 \qquad (i \neq j).$$

Equations (I) differ from (3) only by the fact that $K$ appears in place of $H$. From that, one has the proposition:

*If a dynamical problem with the characteristic function $H = K - U$ is integrable by separation of variables then the same property will be shared by the equation $K = h$, which defines the geodesics.*

Equations (II) and (III) constitute additional conditions (that involve both $K$ and $U$) under which the separation of variables will be possible for non-zero forces, as well. In particular, (III) will prove to be satisfied identically in the cases that you studied.

**3.** – The observation that was just made shows that in order to classify the dynamical problems that admit separation of variables, one must:

1. Characterize those material systems, or if one prefers, those $ds^2 = \sum_{r,s=1}^{n} a_{rs}\,dx_r\,dx_s$ for which the form $K$ verifies (I).

2. Examine what sort of forces can be applied to each of those systems that are compatible with (II), (III). This second part would certainly be easy when one can solve the first one, especially

since one can benefit from the known result concerning the analytical expression for the potential $U$ that you established (*) without any restricting hypotheses on the nature of the system.

Therefore, we shall confine ourselves to the geodetic case, as would be natural.

The explicit conditions on the coefficients $a^{(rs)}$ of $K$ that are given by (I) are developed materially and their individual coefficients are equated to zero. However, one can point out a criterion that serves to simplify the calculations.

**4.** – Fix a generic index $i$ and set:

$$\sigma_{ij} = -\frac{\partial K}{\partial p_j}\frac{\partial^2 K}{\partial p_i\,\partial x_j} + a^{(ij)}\frac{\partial K}{\partial x_j}.$$

Keeping in mind that $\frac{\partial^2 K}{\partial p_i\,\partial x_j}$ is not $a^{(ij)}$, one will see that the $n$ – 1 equations (I) relative to the fixed index $i$ (and to the $n$ – 1 indices $j$ that are different from $i$) can be written:

$$\frac{\partial K}{\partial p_i}\left\{\frac{\partial K}{\partial p_j}\frac{\partial^2 K}{\partial x_i\,\partial x_j} - \frac{\partial K}{\partial x_j}\frac{\partial^2 K}{\partial x_i\,\partial p_j}\right\} + \frac{\partial K}{\partial x_i}\sigma_{ij} = 0\,.$$

Now, $\frac{\partial K}{\partial p_i}$ is a form that is homogeneous of degree one in the $p$, while $\frac{\partial K}{\partial x_i}$ and $\sigma_{ij}$ are ones of degree two. One or the other of them must be divisible by $\frac{\partial K}{\partial p_i}$.

In order to specify the divisibility conditions, it helps to imagine introducing the $x'$ in place of the $p$, and the former are linear combinations of the latter, according to (7). One has precisely $\frac{\partial K}{\partial p_i} = x_i'$, and from (8):

$$\frac{\partial K}{\partial x_i} = -\frac{\partial T}{\partial x_i} = -\sum_{r,s=1}^{n}\frac{\partial a_{rs}}{\partial x_i}x_r'\,x_s'\,.$$

It will then happen that the conditions for $\frac{\partial K}{\partial x_i}$ to be divisible by $\frac{\partial K}{\partial p_i}$ are:

$$\text{(9)}\qquad \frac{\partial a_{rs}}{\partial x_i} = 0 \qquad (r, s \neq i).$$

(*) *Habilitationsschrift*, pp. 8; Math. Ann., Bd. **42**, pp. 548.

If one also expresses the $\sigma_{ij}$ in terms of the $x'$ then one will have:

$$\sigma_{ij} = x'_j \sum_{l=1}^{n} a^{(il)} \frac{\partial^2 T}{\partial x'_l \, \partial x'_j} - a^{(ij)} \frac{\partial T}{\partial x_j},$$

so the conditions for divisibility by $x'_i$ will be:

$$(10) \qquad \left\{ \begin{aligned} & a^{(ij)} \frac{\partial a_{rs}}{\partial x_j} = 0, \\ & \sum_{l=1}^{n} a^{(il)} \frac{\partial a_{rl}}{\partial x_j} - a^{(ij)} \frac{\partial a_{rj}}{\partial x_j} = 0, \qquad (j, r, s \neq i \, ; \, r, s \neq j). \\ & \sum_{l=1}^{n} a^{(il)} \frac{\partial a_{jl}}{\partial x_j} - \tfrac{1}{2} a^{(ij)} \frac{\partial a_{jj}}{\partial x_j} = 0 \end{aligned} \right.$$

Either (9) or (10) will be satisfied for any value of *i*. The last one, in turn, gives rise to distinct subcases according to the hypotheses that are made on orthogonality (i.e., whether various $a^{(ij)}$ are or are not annulled).

If one takes (9) [or (10), respectively] into account then for all values of *i*, the left-hand sides of (I) will be divisible by $\frac{\partial K}{\partial p_i} \frac{\partial K}{\partial p_j}$ and will then reduce to polynomials of degree two in the *p* (or, if one prefers, in the $x'$). If one expresses the idea that the individual coefficients are annulled then one can infer the final condition that they have order two in the *a*.

Despite those relative simplifications, if one does not find some synthetic artifice then one will need to review all of the eventualities that are possible *a priori*, while supposing that (9) is satisfied for a certain number of values of the *i* and (10) is satisfied for the remaining values, with the aforementioned subcases.

For $n = 2$, things work quite well, which is easy to predict in light of the foregoing. However, even for $n = 3$, one needs to undertake a detailed discussion that I shall not seek to elaborate upon.

The calculations will be simple for arbitrary *n when one fixes the case beforehand, as well as the subcase* (I have attempted a few of them by way of experiment). However, I must only add that I did not encounter any truly interesting types in those experiments.

**5.** – As an example, take the case in which *all* of the $\frac{\partial K}{\partial x_i}$ are divisible by the corresponding $\frac{\partial K}{\partial p_i}$.

If one introduces the $a_{ij,\,r}$ (viz., the Christoffel symbols of the first kind), which are defined by:

(11) $$2\,a_{ij,r} = \frac{\partial a_{ri}}{\partial x_j} + \frac{\partial a_{rj}}{\partial x_i} - \frac{\partial a_{ij}}{\partial x_r}\,,$$

then it will result immediately from (9) that:

(9′) $$a_{ij,r} = 0 \qquad (i \neq j).$$

Indeed, if neither of the two indices $i$ and $j$ is equal to $r$ then all of the terms in the right-hand side of (11) will be annulled. However, if one has, e.g., $j = r$, and therefore $i \neq j$, then $\frac{\partial a_{ri}}{\partial x_j}$ will drop out with $\frac{\partial a_{ij}}{\partial x_r}$, and what will remain is $\frac{\partial a_{rj}}{\partial x_i}$, which is zero by virtue of (9).

Since (9′) is a consequence of (9), (9) is conversely a consequence of (9′). Indeed, from the identity:

$$\frac{\partial a_{rs}}{\partial x_i} = a_{ri,s} + a_{si,r}\,,$$

one will then have that (9′) will give $\frac{\partial a_{rs}}{\partial x_i} = 0$ whenever $i$ is different from both $r$ and $s$.

Recall, as well, that the Christoffel symbols of the second kind are defined by:

$$\begin{Bmatrix} i\ j \\ s \end{Bmatrix} = \sum_{s=1}^{n} a^{(rs)}\, a_{ij,r}\,,$$

and one will see that (9′) can be written in the new equivalent form:

(9″) $$\begin{Bmatrix} i\ j \\ s \end{Bmatrix} = 0 \qquad (i \neq j).$$

Having said that, observe that the ratios:

$$\frac{\partial K}{\partial x_i} : \frac{\partial K}{\partial p_i}\,, \qquad \text{i.e.,} \qquad -\frac{\partial T}{\partial x_i} : x_i'\,,$$

will reduce to:

$$\frac{1}{2}\frac{\partial a_{ii}}{\partial x_i}\,x_i' - \sum_{r=1}^{n} \frac{\partial a_{ir}}{\partial x_i}\,x_r'$$

by virtue of (9).

Since $\partial a_{ii} / \partial x_r = 0$ for $r \neq i$ [as always, from (9)], those expressions can also be presented in the form:

$$- \sum_{r=1}^{n} \left( \frac{\partial a_{ir}}{\partial x_i} - \frac{1}{2} \frac{\partial a_{ii}}{\partial x_r} \right) x'_r \ ,$$

or from (11):

$$- \sum_{r=1}^{n} a_{ii,r} \, x'_r \ .$$

If one replaces the $x'$ with their values in (7) and takes (12) into account then one will finally have:

$$- \frac{\dfrac{\partial K}{\partial x_i}}{\dfrac{\partial K}{\partial p_i}} = \sum_{s=1}^{n} \begin{Bmatrix} i\ i \\ s \end{Bmatrix} p_s \ .$$

In the present case ($H = K$), (2) will become:

$$\frac{dp_i}{dx_i} = - \frac{\dfrac{\partial K}{\partial x_i}}{\dfrac{\partial K}{\partial p_i}} ,$$

and therefore, the system that must be completed will consist of:

$$\text{(1)} \qquad \frac{dp_i}{dx_i} = 0 \qquad (i \neq j),$$

and:

$$\text{(13)} \qquad \frac{dp_i}{dx_i} = \sum_{s=1}^{n} \begin{Bmatrix} i\ i \\ s \end{Bmatrix} p_s \ .$$

At this point, there is no need to recall the general formulas (I). The conditions that must associated with (9) are clearly:

$$\frac{d}{dx_j} \sum_{s=1}^{n} \begin{Bmatrix} i\ i \\ s \end{Bmatrix} p_s = 0 \qquad (i \neq j),$$

and when that is developed, according to (1), (13), it will become:

$$\sum_{s=1}^{n} \left[ \frac{\partial}{\partial x_j} \begin{Bmatrix} i\ i \\ s \end{Bmatrix} + \begin{Bmatrix} i\ i \\ j \end{Bmatrix} \begin{Bmatrix} j\ j \\ s \end{Bmatrix} \right] p_s = 0 \qquad (i \neq j),$$

and if that is true identically then that must imply:

(14) $$\frac{\partial}{\partial x_j}\begin{Bmatrix} i\ i \\ s \end{Bmatrix} + \begin{Bmatrix} i\ i \\ j \end{Bmatrix}\begin{Bmatrix} j\ j \\ s \end{Bmatrix} = 0 \qquad (i \neq j).$$

We shall now address the specification of $K$, or what amounts to the same thing, the corresponding $ds^2$, by means of (9″) and (14).

The integration of those equations can be performed without calculation on the basis of known principles in differential geometry.

In the first place, by the definition of the Riemann symbols, one has:

$$a_{rs,ij} = \frac{\partial}{\partial x_j}\begin{Bmatrix} r\ i \\ s \end{Bmatrix} - \frac{\partial}{\partial x_i}\begin{Bmatrix} r\ j \\ s \end{Bmatrix} + \sum_{l=1}^{n}\left[\begin{Bmatrix} r\ i \\ l \end{Bmatrix}\begin{Bmatrix} l\ j \\ s \end{Bmatrix} - \begin{Bmatrix} r\ j \\ l \end{Bmatrix}\begin{Bmatrix} l\ i \\ s \end{Bmatrix}\right].$$

Let us distinguish four cases:

a) $r \neq i, \qquad r \neq j,$

b) $r = i \neq j,$

c) $r = j \neq i,$

d) $r = i = j.$

Any symbol $a_{rs,ij}$ will obviously belong to one of the four categories. The ones in category a) will be zero, by virtue of (9″).

For the ones in b), one will have $\begin{Bmatrix} r\ j \\ s \end{Bmatrix} = 0$, $\begin{Bmatrix} r\ j \\ l \end{Bmatrix} = 0$, but not that $\begin{Bmatrix} l\ j \\ s \end{Bmatrix} = 0$ $(l \neq j)$, from (9″) itself. What will then remain are:

$$a_{is,ij} = \frac{\partial}{\partial x_j}\begin{Bmatrix} i\ i \\ s \end{Bmatrix} + \begin{Bmatrix} i\ i \\ j \end{Bmatrix}\begin{Bmatrix} j\ j \\ s \end{Bmatrix},$$

which is zero by virtue of (14).

One sees that the symbols in category c) will be annulled in the same way. The ones in d) are then zero identically.

One then has:

(14′) $$a_{is,ij} = 0$$

for all values of the four indices $r, s, i, j$.

Conversely, it is important to note that from (14′), one will come back to (14) when one takes (9″) into account.

*By definition, one can then regard:*

(14′) $$a_{is,ij} = 0,$$

$$(9'') \qquad \begin{Bmatrix} i\ j \\ s \end{Bmatrix} = 0 \qquad (i \neq j)$$

*as the equations of condition for our* $ds^2$ *. The first of them says that one is dealing with a Euclidian manifold.*

It still remains for us to characterize the surface coordinate $x_i$ = const.

A better way of achieving the goal is to look for expressions for the Cartesian coordinates $y$ as a function of $x$.

In any event, the Cartesian coordinates (when considered to be functions of the arbitrary coordinates $x$) must satisfy the Ricci equations (*):

$$y_{r\,|\,ij} = 0\ ,$$

or when one replaces the covariant derivatives $y_{r\,|\,ij}$ with their actual expressions:

$$(15) \qquad \frac{\partial^2 y_r}{\partial x_i\, \partial x_j} - \sum_{s=1}^{m} \begin{Bmatrix} i\ j \\ s \end{Bmatrix} \frac{\partial y_r}{\partial x_s} = 0\ .$$

In the present case, if one appeals to (9) then one can infer, in particular:

$$(16) \qquad \frac{\partial^2 y_r}{\partial x_i\, \partial x_j} = 0 \qquad (i \neq j),$$

which can be integrated by inspection to give:

$$(16') \qquad y_r = \sum_{i=1}^{n} X_i^{(r)}(x_i)\ ,$$

in which the $X_i^{(r)}(x_i)$ are arbitrary functions of the indicated argument that are constrained by only the restriction that the determinant of their first derivatives must be non-zero. [That restriction is necessary because (16′) must define a non-degenerate transformation between the $y$ and $x$.]

Those values (16) of the $y$ actually verify all of the required conditions because when one substitutes them in:

$$ds^2 = \sum_{r=1}^{n} dy_r^2\ ,$$

that will give rise to an expression for $ds^2$ in terms of the variables $x$ that looks the same as before and satisfies (14′) and (9″).

(*) *Lezioni sulla teoria delle superficie*, Padua, 1898, published by Drucker; Chap. V.

That is obvious from (14′) (which characterizes the possibility of transforming to the Euclidian type $\sum_{r=1}^{n} dy_r^2$ ). From (9″), that will result from a comparison of (16) with (15).

If we observe (16′) then we will now be in a position to characterize *the manifolds* $x_i$ = const. very simply by saying that *they are hypersurfaces of translation* (for $n$ = 2, 3, they are curves of surfaces, respectively).

**6. Complete discussion for** $n = 2$. – The cases to distinguish are:

1. The two ratios:

$$\frac{\dfrac{\partial K}{\partial x_1}}{\dfrac{\partial K}{\partial p_1}}, \quad \frac{\dfrac{\partial K}{\partial x_2}}{\dfrac{\partial K}{\partial p_2}}$$

are both integers (of course, with respect to the $p$).

2. One only one of them is an integer.

3. Neither of them is an integer.

*Case 1:* From what we have seen, in general, this case corresponds to a planar $ds^2$ when referred to lines of translation. With an opportune choice of parameters for the coordinate lines, we will immediately have your three types (*):

$$ds^2 = dx_1^2 + 2\cos(X_1 + X_2)\,dx_1\,dx_2 + dx_2^2,$$

in which $X_1$ is an arbitrary function of $x_1$, $X_2$ of $x_2$ .

*Case 2:* Let:

$$\tau = \frac{\dfrac{\partial K}{\partial x_2}}{\dfrac{\partial K}{\partial p_2}} \tag{17}$$

be the integer ratio, so one must naturally exclude the case in which the other one $\dfrac{\partial K}{\partial x_1} : \dfrac{\partial K}{\partial p_1}$ is, as well.

(*) Math. Ann. (Leipzig), Bd. XXXV, pp. 94.

(I) will then consist of only one equation that one can write as:

$$\frac{\partial K}{\partial p_1}\frac{\partial \tau}{\partial x_1} - \frac{\partial K}{\partial x_1}\frac{\partial \tau}{\partial p_1} = 0$$

upon dividing by $\left(\frac{\partial K}{\partial p_1}\right)^2$.

From that, one sees that the product $\frac{\partial K}{\partial x_1}\frac{\partial \tau}{\partial p_1}$ is divisible by $\frac{\partial K}{\partial p_1}$. That is not true of the first factor, so it must be true of $\frac{\partial \tau}{\partial p_1}$. However, if one lets $\tau$ be a linear function of the $p$ then $\frac{\partial \tau}{\partial p_1}$ will depend upon only the $x$, and it can then be divisible by the linear function $\frac{\partial K}{\partial p_1}$ only on the condition that it is annulled identically. Therefore, the preceding equation will split into two:

$$\frac{\partial \tau}{\partial p_1} = 0 , \qquad \frac{\partial \tau}{\partial x_1} = 0 ,$$

and $\tau$ will then reduce to the product of $p_2$ with a function of only $x_2$. If one denotes it by $\frac{d \log X_2}{dx_2}$, as is always permissible, and sets:

(18) $$f = X_2\, p_2$$

then one can obviously give $\tau$ the expression:

$$\tau = \frac{\dfrac{\partial f}{\partial x_2}}{\dfrac{\partial f}{\partial p_2}} ,$$

and with that, (17) will assume the form:

(17′) $$\frac{\partial K}{\partial p_2}\frac{\partial f}{\partial x_2} - \frac{\partial K}{\partial x_2}\frac{\partial f}{\partial p_2} = 0 .$$

Since $f$, like $\tau$, is independent of $x_1$ and $p_1$, the left-hand side of (17′) can be considered to be the Poisson parentheses ($K$, $f$). As you have taught me, the annulling of those parentheses expresses the idea that the geodetics of $K$ admit the integral:

$$f = \text{const.}$$

The existence of a linear integral, which has the form $X_2\,p_2$ = const., for the geodetic lines permits one to assert that *the corresponding manifold can be mapped to a surface of revolution whose coordinate lines* $x_1$ = const. *represent the parallels.*

That is your type (II″).

*Case 3:* We shall appeal to (10) in no. **4** for $i = 1, 2$. There cannot be more than two distinct indices, so the first two groups will be missing, and the third one, in which we set $i = 1$, $j = 2$; $i = 2$, $j = 1$ in succession will give:

$$(19)\qquad \begin{cases} a^{(11)}\dfrac{\partial a_{12}}{\partial x_2}+\dfrac{1}{2}a^{(12)}\dfrac{\partial a_{22}}{\partial x_2}=0, \\[2ex] \dfrac{1}{2}a^{(12)}\dfrac{\partial a_{11}}{\partial x_1}+\quad a^{(22)}\dfrac{\partial a_{12}}{\partial x_1}=0. \end{cases}$$

If the coordinates are orthogonal then $a_{12}$, and therefore $a^{(12)}$, will be annulled, and (19) will still be satisfied identically.

If one writes $e^\alpha$ for $a^{(11)}$ and $e^\beta$ for $a^{(22)}$ then one will have:

$$K = \tfrac{1}{2}\,(e^\alpha p_1^2 + e^\beta p_2^2)\ ,$$

and with that:

$$\frac{\partial^2 K}{\partial p_1\,\partial p_2} = 0\ ,$$

$$\frac{\dfrac{\partial^2 K}{\partial p_1\,\partial x_2}}{\dfrac{\partial K}{\partial p_1}} = \frac{\partial\alpha}{\partial x_1}\,p_1\,,$$

$$\frac{\dfrac{\partial^2 K}{\partial x_1\,\partial p_2}}{\dfrac{\partial K}{\partial p_2}} = \frac{\partial\beta}{\partial x_1}\,p_2\,,$$

and more explicitly:

$$(20)\qquad \begin{cases} \dfrac{\partial^2\alpha}{\partial x_1\,\partial x_2}-\dfrac{\partial\beta}{\partial x_1}\dfrac{\partial\alpha}{\partial x_2}=0, \\[2ex] \dfrac{\partial^2\beta}{\partial x_1\,\partial x_2}-\dfrac{\partial\beta}{\partial x_1}\dfrac{\partial\alpha}{\partial x_2}=0, \end{cases}$$

which agrees with (I). In particular, it will follow from (20) that:

$$\frac{\partial^2 (\alpha - \beta)}{\partial x_1 \, \partial x_2} = 0 ,$$

and when that is integrated, it will give:

$$\alpha - \beta = X_1 - X_2 ,$$

in which $X_1$, $X_2$ denote arbitrary functions of the arguments $x_1$, $x_2$, respectively, as usual.

Set:

$$-\log \lambda = \alpha - X_1 = \beta - X_2 ,$$

so (20) will reduce to:

$$(20') \qquad \frac{\partial^2 \lambda}{\partial x_1 \, \partial x_2} = 0 ,$$

and one will get the expression for $K$ :

$$\frac{1}{2} \frac{X_1 \, p_1^2 + X_2 \, p_2^2}{U - V} ,$$

which obviously reduces to:

$$\frac{1}{2} \frac{p_1^2 + p_2^2}{U - V} ,$$

in which $U$ is a function of only $x_1$, $V$, and $x_2$ .

*The corresponding* $ds^2$ *will have the Liouville form:*

$$(U - V)(dx_1^2 + dx_2^2) .$$

It is known that when one is give the criteria that follow from our classification, $U$ and $V$ must actually be considered to be functions of $x_1$, $x_2$ . If one or the other of them does, in fact, reduce to a constant then $\frac{\partial K}{\partial x_1}$ or $\frac{\partial K}{\partial x_2}$ will prove to be divisible by $\frac{\partial K}{\partial p_1}$ or $\frac{\partial K}{\partial p_2}$, respectively, and one would come back to one of the two preceding cases. Moreover, even in those cases, as was observed before by Prof. Morera and yourself, the $ds^2$ will be reducible to the Liouville form, except that one of the two functions $U$, $V$ (*Case 2*) or both of them (*Case 1*) will prove to be constant.

With that, we have exhausted the types that you enumerated, and no others can actually exist. It then remains for us to complete the discussion of why it is not obvious *a priori* that the coordinate lines must be orthogonal.

Therefore, let us suppose that $a_{12} \neq 0$ and try to show that it would not be possible to satisfy all of the required conditions then.

In the first place, when one replaces the reciprocal elements $a^{(11)}$, $a^{(12)}$, $a^{(22)}$ with their values $a_{22} / a$, $- a_{12} / a$, $a_{11} / a$ ($a = a_{11} \, a_{22} - a_{12}^2$), resp., in (19) and takes into account the fact that $a_{12}$ is non-zero, it can be written:

$$\frac{\partial}{\partial x_2}\frac{a_{22}}{a_{12}^2} = 0\,, \quad \frac{\partial}{\partial x_1}\frac{a_{11}}{a_{12}^2} = 0\,,$$

in which $X_1$, $X_2$ mean:

$$a_{22} = a_{12}^2\, X_1\,, \quad a_{11} = a_{12}^2\, X_2\ .$$

Thus, we will have a $ds^2$ of the form:

$$a_{12}^2\, X_1\, X_2 \left\{\frac{dx_1^2}{X_1} + \frac{2}{a_{12}\sqrt{X_1\, X_2}}\frac{dx_1}{\sqrt{X_1}}\frac{dx_2}{\sqrt{X_2}} + \frac{dx_2^2}{X_2}\right\}\,,$$

or more simply, if one switches $x_1$, $x_2$ in $\int \sqrt{X_1}\, dx_1$, $\int \sqrt{X_2}\, dx_2$ (in the real domain, $X_1$, $X_2$ cannot be zero, so the transformation is certainly legitimate) and if one writes $1 / \lambda$ for $a_{12}\sqrt{X_1\, X_2}$ then:

$$ds^2 = \frac{1}{\lambda^2}\{dx_1^2 + 2\lambda\, dx_1\, dx_2 + dx_2^2\}\ .$$

That gives:

$$K = \frac{1}{2}\left(\frac{1}{1-\lambda^2} - 1\right)\{p_1^2 - 2\,\lambda\, p_1 p_2 + p_2^2\}\,,$$

which will make $K$ depend upon the variables $x_1$, $x_2$ only by way of the argument $\lambda$, and consequently:

$$\frac{\partial K}{\partial x_1} = \frac{\partial K}{\partial \lambda}\frac{\partial \lambda}{\partial x_1}\,, \qquad \frac{\partial K}{\partial x_2} = \frac{\partial K}{\partial \lambda}\frac{\partial \lambda}{\partial x_2}\,,$$

$$\frac{\partial^2 K}{\partial x_1\,\partial x_2} = \frac{\partial K}{\partial \lambda}\frac{\partial^2 \lambda}{\partial x_1\,\partial x_2} + \frac{\partial^2 K}{\partial \lambda^2}\frac{\partial \lambda}{\partial x_1}\frac{\partial \lambda}{\partial x_2}\,.$$

The first derivatives of $\lambda$ cannot be annulled identically since that would imply that $\partial K / \partial x_1$ or $\partial K / \partial x_2$ would be zero in that case (and therefore divisible by the corresponding $\partial K / \partial p$).

Having said that, imagine that one substitutes the values for $\frac{\partial K}{\partial x_1}$, $\frac{\partial K}{\partial x_2}$, $\frac{\partial^2 K}{\partial x_1\,\partial x_2}$ that were just written out in the fundamental equation:

$$\frac{\partial K}{\partial p_1}\frac{\partial K}{\partial p_2}\frac{\partial^2 K}{\partial x_1\,\partial x_2} - \frac{\partial K}{\partial p_1}\frac{\partial K}{\partial x_2}\frac{\partial^2 K}{\partial x_1\,\partial p_2} - \frac{\partial K}{\partial x_1}\frac{\partial K}{\partial p_2}\frac{\partial^2 K}{\partial p_1\,\partial x_2} + \frac{\partial K}{\partial x_1}\frac{\partial K}{\partial x_2}\frac{\partial^2 K}{\partial p_1\,\partial p_2} = 0$$

and isolates the term:

$$\frac{\partial K}{\partial p_1}\frac{\partial K}{\partial p_2}\frac{\partial^2 K}{\partial \lambda^2}\frac{\partial \lambda}{\partial x_1}\frac{\partial \lambda}{\partial x_2}$$

in the left-hand side. All of the other terms contain the factor $\partial K / \partial \lambda$, which can be taken out of them, so one can attribute the following form to the preceding equation (which must be true identically with respect to the $p$ and $x$):

$$\frac{\partial K}{\partial p_1}\frac{\partial K}{\partial p_2}\frac{\partial^2 K}{\partial \lambda^2}\frac{\partial \lambda}{\partial x_1}\frac{\partial \lambda}{\partial x_2} = \frac{\partial K}{\partial \lambda}\Omega ,$$

in which $\Omega$ is an expression that is quadratic in the $p$.

Such an identity implies, in particular, that the right-hand side must be divisible by $\frac{\partial K}{\partial p_1}\frac{\partial K}{\partial p_2}$. Neither of those two factors divides $\frac{\partial K}{\partial \lambda}$ (since it would also have to divide $\frac{\partial K}{\partial x_1}$, $\frac{\partial K}{\partial x_2}$ then). Therefore, $\Omega$ will be divisible by $\frac{\partial K}{\partial p_1}\frac{\partial K}{\partial p_2}$, and one can get:

(21) $$\frac{\partial^2 K}{\partial \lambda^2}\frac{\partial \lambda}{\partial x_1}\frac{\partial \lambda}{\partial x_2} = \frac{\partial K}{\partial \lambda}\cdot \text{ functions of only the } x .$$

Now:

$$\frac{\partial K}{\partial \lambda} = \frac{1}{(1-\lambda^2)^2}\{p_1^2 - 2\lambda\, p_1\, p_2 + p_2^2\} - \left(\frac{1}{1-\lambda^2} - 1\right) p_1\, p_2 ,$$

or when one sets:

$$A = \frac{1}{(1-\lambda^2)^2}, \qquad B = \frac{\lambda^2(\lambda^2-3)}{(1-\lambda^2)^2},$$

$$\frac{\partial K}{\partial \lambda} = A(p_1^2 + p_2^2) + B\, p_1\, p_2 ,$$

so

$$\frac{\partial^2 K}{\partial \lambda^2} = \frac{dA}{d\lambda}(p_1^2 + p_2^2) + \frac{dB}{d\lambda}\, p_1\, p_2 ,$$

and the determinant:

$$\begin{vmatrix} A & B \\ \dfrac{dA}{d\lambda} & \dfrac{dB}{d\lambda} \end{vmatrix}$$

is not identically zero since under that hypothesis, the ratio $B / A = \lambda(\lambda^2 - 3)$ would have to prove to be independent of $\lambda$, but that is not true.

Therefore, $\frac{\partial K}{\partial \lambda}$ and $\frac{\partial^2 K}{\partial \lambda^2}$ are independent linear forms in the arguments $p_1^2 + p_2^2, p_1 p_2$.

Suppose that the values of $x_1$, $x_2$ are fixed, and therefore $\lambda$ is fixed by the single restriction that:

$$\begin{vmatrix} A & B \\ \frac{dA}{d\lambda} & \frac{dB}{d\lambda} \end{vmatrix} \frac{\partial \lambda}{\partial x_1} \frac{\partial \lambda}{\partial x_2} \neq 0 ,$$

and the function of the $x$ that appears in the right-hand side of (21) will remain regular.

One can then (and in an infinitude of ways) attribute values to the arguments $p_1^2 + p_2^2, p_1 p_2$ (i.e., ultimately to the $p$) for which $\frac{\partial K}{\partial \lambda}$, and therefore the right-hand side of (21), are annulled, while $\frac{\partial^2 K}{\partial \lambda^2}$ is not annulled, and therefore the product $\frac{\partial^2 K}{\partial \lambda^2} \frac{\partial \lambda}{\partial x_1} \frac{\partial \lambda}{\partial x_2}$ would not be either, but it constitutes the left-hand side of (21) in its own right. However, that equation must be satisfied for any choice of initial values.

The hypothesis that $a_{12} \neq 0$ must then be excluded, which was to be proved.

**Padua**, 3 February 1904.