“Complementi al teorema di Malus-Dupin. Nota I” Rend. R. Accad. dei Lincei **9** (1900), 185-189.

Complements to the Malus-Dupin theorem. I

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It is well-known that if one subjects a normal congruence of rays to an arbitrary number of refractions (or reflections, in particular) then one will get another normal congruence. The normality is then a character of the rectilinear congruences that is invariant under as many refractions as one pleases. We shall also see that it is the only invariant property. I propose to show that, in fact, two congruences of lines (which are both normal or not) are always deducible from each other by a finite number of refractions. More precisely, one refraction is sufficient for normal congruences, while two are needed for the others, in general.

The indices of refraction are assumed to be arbitrary, in particular, they are equal to – 1, which corresponds to reflections. Refringent surfaces must satisfy certain differential conditions. The existence of such surfaces and the degree of generality are inferred from the fundamental theorems of the theory of equations.

Thus, e.g., the transition surface between two normal congruences will remain well-defined when one fixes a point of it or (what amounts to the same thing) the continuation of an incident ray.

For the non-normal congruences, one can arrange the two refringent surfaces in such a way that ∞^1 rays of the first congruence are transformed into ∞^1 rays that are chosen at will from the second one, which then corresponds to a ruled surface and the individual rays in it.

1. – Let x_1, x_2, x_3 denote Cartesian coordinates, while X_1, X_2, X_3 are functions of those variables that are coupled by the identity:

(1) $$\sum_{j=1}^{3} X_j^2 = 1 .$$

The congruence:

(2) $$\frac{dx_i}{ds} = X_i \qquad (i = 1, 2, 3)$$

will be rectilinear, provided that the direction cosines X_i keep constant values along the individual curve (2), i.e., if one has:

$$\frac{dX_i}{ds} = \sum_{j=1}^{3} \frac{\partial X_i}{\partial x_j} X_j = 0 \qquad (i = 1, 2, 3).$$

On the other hand, (1) implies that:

$$\sum_{j=1}^{3} \frac{\partial X_j}{\partial x_i} X_j = 0 \qquad (i = 1, 2, 3),$$

in any case, so upon subtraction, one will have:

$$\sum_{j=1}^{3} \left(\frac{\partial X_i}{\partial x_j} - \frac{\partial X_j}{\partial x_i} \right) X_j = 0 \qquad (i = 1, 2, 3),$$

which expresses the idea that the differences:

$$\frac{\partial X_2}{\partial x_3} - \frac{\partial X_3}{\partial x_2}, \quad \frac{\partial X_3}{\partial x_1} - \frac{\partial X_1}{\partial x_3}, \quad \frac{\partial X_1}{\partial x_2} - \frac{\partial X_2}{\partial x_1}$$

are proportional to X_1, X_2, X_3, respectively. If one lets A represent the proportionality factor (viz., the *abnormality* of the rectilinear congruence considered) then one can write:

(3)
$$\begin{aligned} \frac{\partial X_2}{\partial x_3} - \frac{\partial X_3}{\partial x_2} &= A\, X_1, \\ \frac{\partial X_3}{\partial x_1} - \frac{\partial X_1}{\partial x_3} &= A\, X_2, \\ \frac{\partial X_1}{\partial x_2} - \frac{\partial X_2}{\partial x_1} &= A\, X_3, \end{aligned}$$

and one can infer the expression for A from that:

(4)
$$A = X_1 \left(\frac{\partial X_2}{\partial x_3} - \frac{\partial X_3}{\partial x_2} \right) + X_2 \left(\frac{\partial X_3}{\partial x_1} - \frac{\partial X_1}{\partial x_3} \right) + X_3 \left(\frac{\partial X_1}{\partial x_2} - \frac{\partial X_2}{\partial x_1} \right).$$

Differentiate (3) with respect to x_1, x_2, x_3, respectively, and add them. That will give:

$$\frac{\partial (A\, X_1)}{\partial x_1} + \frac{\partial (A\, X_2)}{\partial x_2} + \frac{\partial (A\, X_3)}{\partial x_3} = 0,$$

which can be written ([1]):

$$\frac{dA}{ds} = -A\left(\frac{\partial X_1}{\partial x_1} + \frac{\partial X_2}{\partial x_2} + \frac{\partial X_3}{\partial x_3}\right).$$

That equation explains the obvious geometrical fact that a rectilinear congruence cannot be normal to a surface without all the surfaces in the family being parallel. It will then, in fact, result that if A is annulled at one point then it will remain zero along the entire ray that passes through that point.

Now, when a surface meets the rays of a congruence normally, one must have $A = 0$ on it, and therefore, from the observation that was made, A must be identically zero.

(3) then say that X_1, X_2, X_3 are the derivatives of the same function.

One might add that when that is the case, the congruence (2) will necessarily be rectilinear. Here, one has a known proposition by Hamilton ([2]):

A necessary and sufficient condition for a congruence (2) to be rectilinear and normal is that the expression $X_1\,dx_1 + X_2\,dx_2 + X_3\,dx_3$ *must constitute an exact differential*.

2. – Consider the separation surface σ between two optical media. If $-X_1$, $-X_2$, $-X_3$ represent the direction cosines (in the sense of the propagation of light) of an incident ray on σ, while Y_1, Y_2, Y_3 represent those of the corresponding refracted one (in the sense of propagation, as always), and n represents the relative index of the two media considered then the cosines of normal to the surface σ will be proportional to $X_1 - n\,Y_1$, $X_2 - n\,Y_2$, $X_3 - n\,Y_3$. That is equivalent to saying that for any displacement dx_1, dx_2, dx_3 that belongs to σ, one must have:

(5)
$$\sum_{i=1}^{3} X_i\,dx_i + n\sum_{i=1}^{3} Y_i\,dx_i = 0\,.$$

Having said that, if one is given two rectilinear congruences $[C]$ and $[C']$ with direction cosines X_i and Y_i , respectively, then one can regard $[C']$ as being produced from $[C]$ by the index of refraction n as long as there exists a surface σ on which (5) is valid.

If the two congruences $[C]$ and $[C']$ are both normal then $\sum_{i=1}^{3} X_i\,dx_i$, $\sum_{i=1}^{3} Y_i\,dx_i$ will be differentials of two particular functions U and U', and all of the surfaces of the family:

$$U + n\,U' = \text{const.}$$

will satisfy the desired condition.

It then follows that the separation surface of two media can be imagined to be something that goes through an arbitrary point of space.

([1]) More generally, the Ricci formulas lead to a relation of that type for geodetic congruences in an arbitrary space. See the recent note of A. Dall'Acqua: "Ricerche sulle congruenze di curve in una varietà qualunque a tre dimensioni," Atti del R. Istituto Veneto, 1900.

([2]) Darboux, *Leçons sur la théorie générale des surfaces*, t. II, page 275.

In order to see the one-to-one correspondence between the rays of [C] and those of [C'], one again needs to assume that the surface in question does not consist of the rays of one of the two congruences; i.e., one has neither:

$$-\sum_{i=1}^{3} X_i \frac{\partial}{\partial x_i}(U + nU') = 0 \qquad \text{nor} \qquad \sum_{i=1}^{3} Y_i \frac{\partial}{\partial x_i}(U + nU') = 0 .$$

That is why one must exclude any surface $U + n\,U' =$ const. for which one might have $-1 + n \cos\omega = 0$, $-\cos\omega + n = 0$ (where ω denotes the angle between the directions of propagation of the rays of the two congruences at a generic point).

It should be noted that from the optical viewpoint those two directions must form angles of the same type with the normal to the surface (i.e., both acute or both obtuse, according to the direction that is assumed to be positive along the normal). That demands that the two binomials $-1 + n\cos\omega$, $-\cos\omega + n$ must have the same sign, i.e., that ω is not greater than the *complement to the limiting angle*.

Suppose that the two congruences [C] and [C'] have a ray g in common (and opposite to the positive direction along it), so the aforementioned restriction will certainly be verified inside of g, because one has $\cos\omega = 1$ on g, and the two binomials $-1 + n\cos\omega$, $-\cos\omega + n$ will remain equal.

In the case of reflections, that will be true for any ω, and the only exception that will remain is the value $\omega = \pi$, from what was said.

Starting from the hypothesis that the two congruences [C] and [C'] are normal, the left-hand side of (5) cannot be an exact differential, in general. Nothing less than a family of surfaces σ will exist as long as the $X_i + n\,Y_i$ are proportional to the derivatives of the same function. That will lead to the condition:

$$\begin{aligned}
&(X_1 + nY_1)\left\{\frac{\partial(X_2 + nY_2)}{\partial x_3} - \frac{\partial(X_3 + nY_3)}{\partial x_2}\right\} \\
&+ (X_2 + nY_2)\left\{\frac{\partial(X_3 + nY_3)}{\partial x_1} - \frac{\partial(X_1 + nY_1)}{\partial x_3}\right\} \\
&+ (X_3 + nY_3)\left\{\frac{\partial(X_1 + nY_1)}{\partial x_2} - \frac{\partial(X_2 + nY_2)}{\partial x_1}\right\} = 0,
\end{aligned}$$

and when one introduces the abnormalities A, A' of the two congruences [C], [C'] and observes (3), that will simplify to:

$$(6) \qquad A + n^2 A' - n\,(A + A')\cos\omega = 0 .$$

That is not an identity, since there can exist at most one refringent surface σ, etc.

In (6), we have an indirect confirmation of the Malus-Dupin theorem. Indeed, it asserts the impossibility of passing from a normal congruence to another congruence that is not normal or *vice versa*. In truth, if one supposes that $[C]$ is normal, but not $[C']$, then the existence of a surface σ will demand that $n - \cos\omega = 0$, which will exclude the existence of a one-to-one correspondence between the rays of the two congruences.

“Complementi al teorema di Malus-Dupin. Nota II,” Rend. R. Accad. dei Lincei **9** (1900), 237-245.

Complements to the Malus-Dupin theorem. II ([1])

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3. – Let [C] and [C'] be two non-normal congruences. We wish to see if we can determine a third one [Γ] that is joined to both of them by refraction.

Say that σ is the transition surface between [C] and [Γ], while $1 / n$ is the index of refraction for that transition. Let σ' be the surface that joins [Γ] to [C'] and let n' be the relative index of refraction. x_1, x_2, x_3 will be coordinates of an arbitrary point P on σ.

The ray in [Γ] that passes through P will cut σ' at a certain point P', whose coordinates will be denoted by y_1, y_2, y_3.

Set:

$$\overline{PP'} = r = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2},$$

so one must have:

$$\sum_{i=1}^{3} \left(n\, X_i - \frac{\partial r}{\partial x_i} \right) dx_i = 0 \tag{7}$$

on σ, and:

$$\sum_{i=1}^{3} \left(n' Y_i - \frac{\partial r}{\partial y_i} \right) dy_i = 0 \tag{8}$$

on σ', in which the X_i are understood to mean functions of x_1, x_2, x_3, while the Y_i are functions of y_1, y_2, y_3.

Conversely, if one can determine six functions x_i, y_i of the two parameters u, v for which (7), (8) are verified, and neither of the determinants:

([1]) See these Rendiconti, pp. 185.

$$\Delta = \begin{vmatrix} X_1 & X_2 & X_3 \\ \dfrac{\partial x_1}{\partial u} & \dfrac{\partial x_2}{\partial u} & \dfrac{\partial x_3}{\partial u} \\ \dfrac{\partial x_1}{\partial v} & \dfrac{\partial x_2}{\partial v} & \dfrac{\partial x_3}{\partial v} \end{vmatrix},$$

$$\Delta' = \begin{vmatrix} Y_1 & Y_2 & Y_3 \\ \dfrac{\partial y_1}{\partial u} & \dfrac{\partial y_2}{\partial u} & \dfrac{\partial y_3}{\partial u} \\ \dfrac{\partial y_1}{\partial v} & \dfrac{\partial y_2}{\partial v} & \dfrac{\partial y_3}{\partial v} \end{vmatrix}$$

are annulled then one will have the means to pass from one $[C]$ to $[C']$ by way of two refractions.

In the first place, the set of all points x_i (u, v) will truly constitute a surface (as opposed to a curve) as long as not all of the minors of the matrix:

$$\begin{vmatrix} \dfrac{\partial x_1}{\partial u} & \dfrac{\partial x_2}{\partial u} & \dfrac{\partial x_3}{\partial u} \\ \dfrac{\partial x_1}{\partial v} & \dfrac{\partial x_2}{\partial v} & \dfrac{\partial x_3}{\partial v} \end{vmatrix}$$

are annulled at the same time when $\Delta \neq 0$. For the same reason, the surface σ effectively cuts the congruence $[C]$. Analogous statements are true for σ' and $[C']$.

The points P (x_1, x_2, x_3) and $P'(y_1, y_2, y_3)$ on the two surfaces that represent equal values of the parameters u, v will be coupled in that way.

Select a generic ray of $[C]$ and let P be precisely its intersection with σ. By virtue of (7), PP' will be a refracted ray. It is, in turn, refracted upon crossing σ', and from (8), its continuation is given by the ray of $[C']$ that passes through P'. That proves the assertion.

One must then establish that one can satisfy the simultaneous system (7), (8) with functions of the two parameters u, v for which neither Δ nor Δ' are annulled.

The system (7), (8) is equivalent to the following four partial differential equations:

$$\text{(9)} \qquad \begin{cases} H_1 \equiv \displaystyle\sum_{i=1}^{3} \left(n\, X_i - \frac{\partial r}{\partial x_i} \right) \frac{\partial x_i}{\partial u} = 0, \\ K_1 \equiv \displaystyle\sum_{i=1}^{3} \left(n' Y_i - \frac{\partial r}{\partial y_i} \right) \frac{\partial y_i}{\partial u} = 0, \end{cases}$$

(10)
$$\begin{cases} H_2 \equiv \sum_{i=1}^{3}\left(n\,X_i - \dfrac{\partial r}{\partial x_i}\right)\dfrac{\partial x_i}{\partial v} = 0, \\ K_2 \equiv \sum_{i=1}^{3}\left(n'\,Y_i - \dfrac{\partial r}{\partial y_i}\right)\dfrac{\partial y_i}{\partial v} = 0, \end{cases}$$

to which, one must add the integrability conditions:

$$H_3 \equiv \frac{\partial H_1}{\partial v} - \frac{\partial H_2}{\partial u} = 0,$$
$$K_3 \equiv \frac{\partial K_1}{\partial v} - \frac{\partial K_2}{\partial u} = 0.$$

Using the notation $\begin{pmatrix} \varphi & \psi \\ u & v \end{pmatrix}$ to denote the Jacobian determinant of the generic functions φ, ψ with respect to the variables u, v and recalling (3), one will have immediately:

(11)
$$\begin{cases} H_3 \equiv n\,A\Delta - \sum_{i,j=1}^{3}\dfrac{\partial^2 r}{\partial x_i\,\partial y_j}\begin{pmatrix} x_i & y_j \\ u & v \end{pmatrix} = 0, \\ K_3 \equiv n'\,A'\,\Delta' - \sum_{i,j=1}^{3}\dfrac{\partial^2 r}{\partial x_i\,\partial y_j}\begin{pmatrix} y_i & x_j \\ u & v \end{pmatrix} = 0, \end{cases}$$

which will give:

$$n\,A\,\Delta + n'A'\Delta' = 0$$

when summed.

That system of equations will obviously remain invariant under an arbitrary change of the two independent variables u, v. One can fix them by either identifying them with two of the unknowns x_i, y_i or, more generally, by adding two more equations to the system, namely:

(12)
$$H_4 = 0\,, \qquad K_4 = 0$$

[which are not invariant with respect to the transformations $u' = u'(u, v)$, $v' = v'(u, v)$].

However, suppose that we start with functions x_i, y_i *of just one variable* u that verify the two equations (9). In general, it will be possible to satisfy the remaining equations of the system, namely, (10), (11), (12), which are six in number, with some other functions of the two variables u, v that reduce to the given functions of just u for a certain value of $v = v_0$. We will prove that rigorously later on. For the moment, we assume that it is known that our system (9), (10), (11), (12) has been integrated completely, so (9) will also be (which is verified by construction only when v has the value v_0), which will persist for any value of v.

Indeed, when one takes (10) into account, (11) will be equivalent to:

$$\frac{\partial H_1}{\partial v} = 0, \quad \frac{\partial K_1}{\partial v} = 0.$$

Since H_1 and K_1 are annulled for $v = v_0$, the same thing will be true for any other value of v.

The six functions $x_i(u)$, $y_i(u)$, which will be integrals of (9) when one takes the aforementioned steps, can be chosen in such a way that ∞^1 rays of [C] are transformed, ray by ray, into ∞^1 rays that belong to [C'].

In order to prove that, suppose that:

$$x_i = x_i(u, \alpha) \qquad (i = 1, 2, 3), \tag{13}$$

$$y_i = y_i(u, \beta) \qquad (i = 1, 2, 3) \tag{14}$$

define the two ruled surfaces γ and γ' of [C] and [C'], respectively, and above all, u = const. represents a rectilinear generator, while α = const., β = const. represent orthogonal trajectories, and it is understood that the rays that represent the same value of u will correspond to each other. One can also assume that α and β represent lengths when measured along the rectilinear generators upon starting from one of the orthogonal trajectories, and that u measures the arc-length along that trajectory: With those hypotheses, one will have:

$$\frac{\partial x_i}{\partial \alpha} = X_i, \quad \frac{\partial y_i}{\partial \beta} = Y_i.$$

Furthermore, the $\partial x_i / \partial u$ for $\alpha = 0$ and the $\partial y_i / \partial u$ for $\beta = 0$ represent direction cosines.

Constrain the parameters α and β to be functions of u in such a way that (9) is satisfied, i.e., when written out explicitly:

$$\tag{9$'$} \left\{ \begin{aligned} \frac{d\alpha}{du} \sum_{i=1}^{3} \left(n X_i - \frac{\partial r}{\partial x_i} \right) X_i + \sum_{i=1}^{3} \left(n X_i - \frac{\partial r}{\partial x_i} \right) \frac{\partial x_i}{\partial u} = 0, \\ \frac{d\beta}{du} \sum_{i=1}^{3} \left(n' Y_i - \frac{\partial r}{\partial y_i} \right) Y_i + \sum_{i=1}^{3} \left(n' Y_i - \frac{\partial r}{\partial y_i} \right) \frac{\partial y_i}{\partial u} = 0, \end{aligned} \right.$$

conforming to what was assumed. That will be possible, at least within a certain domain of values for u, α, β, only as long as the initial values correspond to points P_0, P'_0 of the two ruled surfaces γ, γ' for which the coefficients:

$$\sum_{i=1}^{3} \left(n X_i - \frac{\partial r}{\partial x_i} \right) X_i, \quad \sum_{i=1}^{3} \left(n' Y_i - \frac{\partial r}{\partial y_i} \right) Y_i$$

of $d\alpha / du$, $d\beta / du$ are not annulled.

With that caveat, choose the initial values, so the six functions:

$$x_i(u, \alpha(u)), \qquad y_i(u, \beta(u)),$$

which are integrals of (9) that are still determined by (9′), and from them, the refringent surfaces σ and σ', in such a way that for $v = v_0$ they will correspond precisely to the two assumed systems of ∞^1 rays.

4. – The two congruences $[C]$ and $[C']$ have a common ray g, and $-X_1, -X_2, -X_3$ will coincide precisely with Y_1, Y_2, Y_3, respectively, along g.

The values of the abnormality will become non-zero at the points of g.

We want to show that all of the restricting inequalities that ensure the existence of the integral system and the one-to-one character of the correspondence between the rays of the two congruence will be effectively satisfied, at least within a sufficiently-small neighborhood of g. In other words, *if we are given an arbitrary ruled surface γ of $[C]$ and a ruled surface γ' of $[C']$ that have a ray g in common and a correspondence between their rectilinear generators for which g is the uniting ray then there will exist* (and they are specified uniquely by the indices of refraction and the condition that they must pass through given points P_0, P'_0 of g) *two refringent surfaces σ and σ' that are suitable for transforming $[C]$ into $[C']$* (more precisely, for transforming a sufficiently-small pencil of rays around g in $[C]$ into an analogous pencil in $[C']$).

For simplicity, take the line g to be the x_3-axis and choose two non-coincident points P_0, P'_0 along it arbitrarily, and assume that their coordinates are the initial values of our functions x_i, y_i.

Locate the origin of the coordinates at the midpoint of the segment P_0, P'_0 $(= 2l > 0)$, and choose $P_0\,P'_0$ to be the positive direction of the x_3-axis. One will then have:

$$\begin{aligned} &x_1^0 = 0, \quad x_2^0 = 0, \quad x_3^0 = -l, \\ &y_1^0 = 0, \quad y_2^0 = 0, \quad y_3^0 = l \end{aligned} \tag{15}$$

for the initial values, and in addition:

$$\left\{ \begin{gathered} X_1^0 = 0, \quad X_2^0 = 0, \quad X_3^0 = -1, \\ Y_1^0 = 0, \quad Y_2^0 = 0, \quad Y_3^0 = 1, \\ \left(\frac{\partial r}{\partial x_1}\right)_0 = -\left(\frac{\partial r}{\partial y_1}\right)_0 = 0, \quad \left(\frac{\partial r}{\partial x_2}\right)_0 = -\left(\frac{\partial r}{\partial y_2}\right)_0 = 0, \quad \left(\frac{\partial r}{\partial x_3}\right)_0 = -\left(\frac{\partial r}{\partial y_3}\right)_0 = -1, \\ r_0 = 2l > 0, \quad A_0 \neq 0, \quad A'_0 \neq 0. \end{gathered} \right. \tag{16}$$

No matter how one assigns the ruled surfaces γ, γ', and the generators that must correspond between them in order to have g be the uniting ray, the parametric representation (of the type just described) must make the ray g correspond to the same value u_0 of the parameter u in both of the ruled surfaces. We will assume that it specifies the arc of the trajectory on g that is orthogonal to the generators and passes through P_0 and the arc of the trajectory on γ' that passes through P'_0.

Since the tangents to those curves in P_0, P'_0 are both orthogonal to g, the directions of the bisectors, together with g, will constitute an orthogonal triad, and one can suppose that the x_1, x_2

axes are parallel to those bisectors. Let 2ϑ denote the angle between the two tangents in question. The values of $\partial x_i / \partial u$, $\partial y_i / \partial u$ for $u = u_0$ (viz., the direction cosines of those tangents) will then be of the type:

$$\frac{\partial x_1}{\partial u} = \cos\vartheta, \quad \frac{\partial x_2}{\partial u} = -\sin\vartheta, \quad \frac{\partial x_3}{\partial u} = 0,$$
$$\frac{\partial y_1}{\partial u} = \cos\vartheta, \quad \frac{\partial y_2}{\partial u} = \quad \sin\vartheta, \quad \frac{\partial y_3}{\partial u} = 0,$$

in which one can further assume that $0 < \vartheta < \pi / 2$.

We begin by asserting that the coefficients of $d\alpha / du$, $d\beta / du$ in (9) are not annulled with the initial values (16). In fact, they become $n - 1$, $n' - 1$, which are properly non-zero, because $n = 1$ or $n' = 1$ would imply the absence of refraction, and we are essentially supposing that refraction (or reflection) exists. We determine the functions α and β of u from (9), substitute them in (13), (14), and in that way infer six functions:

$$x_1(u),\ x_2(u),\ x_3(u);\ y_1(u),\ y_2(u),\ y_3(u) \tag{17}$$

that satisfy (9) and reduce to the values (15) for $u = u_0$.

In regard to the values of the derivatives of those functions for $u = u_0$, one has meanwhile has from (9) that:

$$\left(\frac{dx_3}{du}\right)_0 = 0, \quad \left(\frac{dy_3}{du}\right)_0 = 0.$$

On the other hand:

$$\frac{dx_1}{du} = \frac{\partial x_1}{\partial u} + \frac{\partial x_1}{\partial \alpha}\frac{d\alpha}{du} = \frac{\partial x_1}{\partial u} + X_1\frac{d\alpha}{du},$$

so

$$\left(\frac{dx_1}{du}\right)_0 = \left(\frac{\partial x_1}{\partial u}\right)_0 = \cos\vartheta.$$

Analogous statements are true for:

$$\left(\frac{dx_2}{du}\right)_0, \quad \left(\frac{dy_1}{du}\right)_0, \quad \left(\frac{dy_2}{du}\right)_0.$$

By definition, the initial values of the derivatives of the functions (17) are:

$$\begin{cases} \left(\dfrac{dx_1}{du}\right)_0 = \cos\vartheta, & \left(\dfrac{dx_2}{du}\right)_0 = -\sin\vartheta, & \left(\dfrac{dx_3}{du}\right)_0 = 0, \\ \left(\dfrac{dy_1}{du}\right)_0 = \cos\vartheta, & \left(\dfrac{dy_2}{du}\right)_0 = \quad \sin\vartheta, & \left(\dfrac{dy_3}{du}\right)_0 = 0. \end{cases} \tag{18}$$

Having said that, consider the system (10), (11), to which one adds the auxiliary equations (12), which are specialized, e.g., by:

(12) $$\begin{cases} H_4 \equiv \dfrac{\partial x_1}{\partial u}\dfrac{\partial x_1}{\partial v} - \dfrac{\partial x_2}{\partial u}\dfrac{\partial x_2}{\partial v} + l = 0, \\ K_4 \equiv \dfrac{\partial y_1}{\partial u}\dfrac{\partial y_1}{\partial v} - \dfrac{\partial y_2}{\partial u}\dfrac{\partial y_2}{\partial v} - l = 0. \end{cases}$$

Everything now comes down to showing that for $u = u_0$, $x_i = x_i^0$, $y_i = y_i^0$, the six equations (10), (11), (12) can be solved for the derivatives $\partial x_i / \partial v$, $\partial y_i / \partial v$, and that the values that one infers for those derivatives do not annul Δ or Δ'. Indeed, with that, the integral system of (10), (11), (12), which reduces to the functions (17) of just u for an arbitrary value v_0 of v, will satisfy all of the desired conditions.

In order to prove that, we note that in the first place, for the values (16) and (18), equations (10) and our auxiliary equations (12) will reduce to:

(10′) $$\left(\frac{\partial x_3}{\partial v}\right)_0 = 0, \quad \left(\frac{\partial y_3}{\partial v}\right)_0 = 0,$$

(12) $$\cos\vartheta\left(\frac{\partial x_1}{\partial v}\right)_0 + \sin\vartheta\left(\frac{\partial x_2}{\partial v}\right)_0 = -l, \quad \cos\vartheta\left(\frac{\partial y_1}{\partial v}\right)_0 - \sin\vartheta\left(\frac{\partial y_2}{\partial v}\right)_0 = l.$$

One then has:

$$-\Delta_0 = \sin\vartheta\left(\frac{\partial x_1}{\partial v}\right)_0 + \cos\vartheta\left(\frac{\partial x_2}{\partial v}\right)_0, \quad \Delta'_0 = -\sin\vartheta\left(\frac{\partial y_1}{\partial v}\right)_0 + \cos\vartheta\left(\frac{\partial y_2}{\partial v}\right)_0.$$

The identity:

$$\frac{\partial^2 r}{\partial x_i\,\partial y_j} = -\frac{1}{r}\left(\frac{\partial r}{\partial x_i}\frac{\partial r}{\partial y_j} + \varepsilon_{ij}\right) \qquad (i, j = 1, 2, 3)$$

(in which, as usual, ε_{ij} denotes zero or unity according to whether the indices i and j are distinct or coincident, resp.) gives:

$$-\sum_{i,j=1}^{3}\frac{\partial^2 r}{\partial x_i\,\partial y_j}\begin{pmatrix} x_i & y_j \\ u & v\end{pmatrix} = \frac{1}{r}\sum_{i,j=1}^{3}\frac{\partial r}{\partial x_i}\frac{\partial r}{\partial y_j}\begin{pmatrix} x_i & y_j \\ u & v\end{pmatrix} + \frac{1}{r}\sum_{i=1}^{3}\begin{pmatrix} x_i & y_j \\ u & v\end{pmatrix},$$

and when one introduces the values (16) and (18), the right-hand side will become:

$$\frac{\cos\vartheta\left\{\left(\dfrac{\partial y_1}{\partial v}\right)_0 - \left(\dfrac{\partial x_1}{\partial v}\right)_0\right\} - \sin\vartheta\left\{\left(\dfrac{\partial y_2}{\partial v}\right)_0 - \left(\dfrac{\partial x_2}{\partial v}\right)_0\right\}}{2l},$$

which will be equal to unity, by virtue of (12′).

With that, (11) will assume the form:

$$(11') \qquad \begin{cases} \Delta_0 = -\sin\vartheta\left(\dfrac{\partial x_1}{\partial v}\right)_0 - \cos\vartheta\left(\dfrac{\partial x_2}{\partial v}\right)_0 = -\dfrac{1}{n\,A_0}, \\ \Delta_0' = -\sin\vartheta\left(\dfrac{\partial y_1}{\partial v}\right)_0 + \cos\vartheta\left(\dfrac{\partial y_2}{\partial v}\right)_0 = \dfrac{1}{n'\,A_0'}. \end{cases}$$

One infers the four derivatives:

$$\left(\frac{\partial x_1}{\partial v}\right)_0, \left(\frac{\partial x_2}{\partial v}\right)_0, \left(\frac{\partial y_1}{\partial v}\right)_0, \left(\frac{\partial y_2}{\partial v}\right)_0$$

from the last set of equations and (12′), as long as the tangent planes to the two ruled surfaces γ and γ' at P_0, P'_0 do not meet at a right angle, i.e., $\cos 2\vartheta$ is non-zero.

From its geometric significance, that condition is independent of the way by which the auxiliary equations (12) were specialized. Moreover, one can see that directly by observing that since the auxiliary equations are arbitrary, if the system has to imply non-infinite values for:

$$\left(\frac{\partial x_1}{\partial v}\right)_0, \left(\frac{\partial x_2}{\partial v}\right)_0, \left(\frac{\partial y_1}{\partial v}\right)_0, \left(\frac{\partial y_2}{\partial v}\right)_0$$

then the same thing must be true for:

$$\Delta_0, \quad \Delta_0', \quad -\sum_{i,j=1}^{3}\left(\frac{\partial^2 r}{\partial x_i\,\partial y_j}\right)\begin{pmatrix} x_i & y_j \\ u & v \end{pmatrix}.$$

Now, the square of the matrix of coefficients of:

$$\left(\frac{\partial x_1}{\partial v}\right)_0, \left(\frac{\partial x_2}{\partial v}\right)_0, \left(\frac{\partial y_1}{\partial v}\right)_0, \left(\frac{\partial y_2}{\partial v}\right)_0$$

in those three expressions is equal to $\cos^2 2\vartheta / 2l^2$ and will be annulled when $\cos 2\vartheta = 0$. On the other hand, if one lets k denote the value of:

$$-\sum_{i,j=1}^{3}\left(\frac{\partial^2 r}{\partial x_i\,\partial y_j}\right)\begin{pmatrix} x_i & y_j \\ u & v \end{pmatrix}$$

then pursuant to (11), the following three equations must be valid:

$$-\sum_{i,j=1}^{3}\left(\frac{\partial^2 r}{\partial x_i\,\partial y_j}\right)\begin{pmatrix} x_i & y_j \\ u & v \end{pmatrix} = k\,,$$

$$-\Delta_0 = \frac{k}{n\,A_0}\,,$$

$$\Delta'_0 = \frac{k}{n'\,A'_0}$$

(for $\vartheta = \pi / 4$, $k \neq 0$), but as one soon recognizes, that would require that:

$$2l + \frac{1}{n\,A_0} + \frac{1}{n'\,A'_0} = 0\,,$$

which is not true, in general.

Therefore, if we exclude the case in which 2ϑ is a right angle then the equations of our system will be soluble with respect to the derivatives:

$$\left(\frac{\partial x_i}{\partial v}\right)_0, \quad \left(\frac{\partial y_i}{\partial v}\right)_0,$$

and

$$\Delta_0 = -\frac{1}{n\,A_0}, \quad \Delta'_0 = \frac{1}{n'\,A'_0}$$

will prove to be non-zero, as is necessary.

It is useful to add, in regard to an observation that was made at the end of § **2**, that in the domain under consideration *the solution to the problem is possible, not just geometrically, but also physically*.