"Sulle soluzioni stazionarie dei sistemi pfaffiani," Rend. Acc. Lincei (6) **19** (1934), Nota I, pp. 261-267; Nota II, pp. 470-477.

On the stationary solutions of Pfaffian systems

Note I ([*])

GENERALITIES AND REVIEW

By **T. Levi-Civita**

Translated by D. H. Delphenich

At the end of 1901 ([1]), I indicated a rule that would permit one to construct particular solutions of the canonical systems when one knows an integral, as well as possibly some invariant relations. The guiding criterion would also be true for arbitrary differential systems, as will be shown in what follows ([2]). However, for the ultimate applications of the rule, one will be led to solutions that are endowed with less generality (i.e., ones that contain a smaller number of arbitrary constants) of the kind that one might deal with in the case of canonical systems with some ulterior specification. Conversely, one might indeed presume that the systems that one calls *Pfaffian*, into which the canonical systems will transform, continue to enjoy more noteworthy properties than the one that concerns the greater generality of the solutions that are provided by the constructive rule. That is what I propose to clarify in the present communications, and more precisely in Note II. In this Note I, I must recall some known things in a simplified form or in a summary form that is appropriate to the applications being discussed.

The fact that the Pfaffian systems behave like canonical ones in regard to the aforementioned, truly requires a justification, since (as opposed to the notions of integrals or invariant relations) the rule that was alluded to above for the construction of stationary solutions to a canonical system *does not* have an invariant aspect under the passage from the canonical variables:

$$t\,;\quad p_h\,,\quad q_h \qquad\qquad (h = 1, 2, \ldots, n)$$

to $2n + 1$ arbitrary, but independent, combinations of them:

$$x^0,\ x^1,\ x^2,\ \ldots,\ x^{2n}.$$

([*]) Presented at the session on 4 March 1934.

([1]) These "Rendiconti," vol. X, pp. 3-9, 35-41. [In these *Opere:* Volume Two, II, pp. 87-100]

([2]) See, e.g., **T. Levi-Civita** and **U. Amaldi**, *Lezioni do meccanica razionale*, vol. II, Bologna, Zanichelli, 1927, pp. 339-348.

We shall treat the transformation of the formalism in precisely such a way as to give it an invariant character.

1. – Stationary solutions that can be inferred from a known integral in the general case.

Consider a general differential system of order $2n$ in $2n + 1$ arguments x^i $(i = 0, 1, \ldots, 2n)$. One can assume that it takes the usual form:

$$\frac{dx^0}{X^0} = \frac{dx^1}{X^1} = \ldots = \frac{dx^{2n}}{X^{2n}},$$

in which the X denote functions of the x (which are not all zero and are regular in the region considered). If one would like to also avoid the appearance of divisors, some of which might become zero, then one can also write:

$$dx^i = \varepsilon X^i \qquad (i = 0, 1, \ldots, 2n), \tag{1}$$

in which ε denotes an (infinitesimal) proportionality factor that is undetermined *a priori*. It would help (or at least, it would not hurt) to single out one of the variables. For example, suppose that X_0 is non-zero, and when one eliminates ε, one will have the normal system of order $2n$:

$$\frac{dx^i}{dx^0} = \frac{X^i}{X^0} \qquad (i = 1, 2, \ldots, 2n). \tag{1′}$$

However, one can more symmetrically introduce an auxiliary variable ξ, take $\varepsilon = d\xi$, and refer to the normal system:

$$\frac{dx^i}{d\xi} = X^i \qquad (i = 0, 1, \ldots, 2n), \tag{1″}$$

which nonetheless has order $2n + 1$, but it will offer the advantage that it does not contain the independent variable ξ explicitly.

The theorem that was stated on pp. 244 of the cited *Lezioni di meccanica razionale* of Prof. **Amaldi** and myself referred to the form (1′) specifically, in which one of the arguments was treated differently from the others. One knows that the passage to the form (1) or (1″), in which all of the variables are on the same footing, is elementary. However, instead of performing that transformation, we shall recall the question directly and refer to the symmetric form (1) without the introduction of an extrinsic parameter.

We start from the hypothesis that the system (1) admits a known integral:

$$f(x^0, x^1, \ldots, x^{2n}) = \text{const.}$$

That is equivalent to the formal situation in which:

$$df = \sum_{i=0}^{2n} \frac{\partial f}{\partial x^i} dx^i$$

is annulled for all systems of differentials that satisfy (1), or that one has:

$$\sum_{i=0}^{2n} X^i \frac{\partial f}{\partial x^i} = 0 . \tag{2}$$

Say that S is the $(2n + 1)$-dimensional space that represents the arguments $x^0, x^1, \ldots, x^{2n}$. The points P in that space in which f has a stationary value are, by definition, the ones such that df will be annulled when one starts from them, no matter how one chooses the increments dx^i, i.e., the points at which the $2n + 1$ conditions:

$$\frac{\partial f}{\partial x^i} = 0 \qquad (i = 0, 1, \ldots, 2n) \tag{3}$$

are satisfied.

That constitutes $2n + 1$ equations in finite terms between the x. However, due to (2), at most $2n$ of the aforementioned $2n + 1$ equations will be independent, so one of them must necessarily be a consequence of the other. Thus, in the general case (i.e., in the absence of incompatibility or special relations that are derived from the special nature of the function f), (3) will represent $2n$ independent relations between the $2n + 1$ unknowns and will then define one or more curves in S. If one limits the region conveniently then one can refer to a well-defined curve, which shall be denoted by σ.

The fundamental property is that *each σ constitutes a particular solution to the system* (1). One should also recall that, more generally, even when (3) reduces to less than $2n$ independent equations and then represents, not a line, but a manifold τ of two or more dimensions, *that manifold τ will be invariant with respect to the system* (1), in the sense that *when one starts from an arbitrary point P of τ, any displacement dx^i that satisfies equations* (1) *will belong to τ*, i.e., one will pass to a neighboring point that is also situated on τ. The proof is immediate. Indeed, it will follow from the identity (2) that, firstly (when one denotes the summation index by j and) differentiates with respect to x^i:

$$\sum_{j=0}^{2n} X^j \frac{\partial}{\partial x^i}\left(\frac{\partial f}{\partial x^j}\right) = -\sum_{j=0}^{2n} \frac{\partial X^j}{\partial x^i} \cdot \frac{\partial f}{\partial x^j},$$

from which it will appear that when one annuls the right-hand sides, by virtue of (3), the left-hand sides will also be annulled. If one multiplies by ε and keeps (1) in mind then one can infer that:

$$d\left(\frac{\partial f}{\partial x^j}\right) = 0 \qquad (i = 0, 1, \ldots, 2n),$$

in which d is the symbol for the total differential when one refers to the system (1). That amounts to the same thing as saying that equations (3) admit the infinitesimal displacement (1). Q.E.D.

Observation. – Let us now direct our attention to what must follow. In general case in which the stationary manifold τ reduces to a curve σ, a particular solution to the system (1) will follow with no further discussion (which is σ itself). However, when one is given a particular situation in which the number of independent equations in (3) is less than $2n$, and therefore define a manifold τ in the space S with a number of dimensions that is equal to $1 + \nu$ ($\nu > 0$), from the preceding theorem, that manifold must then be invariant or composed entirely of integral curves of the system (1). Moreover, determining them will require the integration of the system that is subordinate to (1) on τ or a differential operation of order ν that introduces just as many arbitrary constants. That is because when τ has a dimension greater than unity, there will no longer be enough operations in finite terms for one to assign those particular solutions, as one would have in the general case. However, to compensate for that, one will find a more ample class of them: namely, ∞^ν of them, if $\nu + 1$ is the dimension of τ.

2. – The hypothesis in which one knows *m* invariant relations in addition to an integral.

Let:

$$f_r = 0 \qquad (r = 1, 2, \ldots, m) \tag{4}$$

be $m < n$ relations that are invariant with respect to our differential system (1), in which the functions $f_r(x^0, x^1, \ldots, x^{2n})$ must be assumed to be independent (in the region considered), with which (4) will be soluble for just as many arguments x. The relations (4) will then define a $(2n + 1 - m)$-dimensional manifold Σ in the space S. The formal specification that corresponds to the qualitative hypothesis that the f must be independent is that one lets m be the characteristic of the functional matrix of the $f_1, f_2, \ldots, f_m$ with respect to the $2n + 1$ arguments x on Σ (and therefore in the immediate neighborhood of Σ, as well). We shall also introduce a slightly less restrictive condition and consider the matrix with $m + 1$ rows and $2n + 1$ columns:

$$M \equiv \left\| \begin{matrix} X^0 & X^1 & \cdots & X^{2n} \\ \dfrac{\partial f_1}{\partial x^0} & \dfrac{\partial f_1}{\partial x^1} & \cdots & \dfrac{\partial f_1}{\partial x^{2n}} \\ \dfrac{\partial f_2}{\partial x^0} & \dfrac{\partial f_2}{\partial x^1} & \cdots & \dfrac{\partial f_2}{\partial x^{2n}} \\ \vdots & \vdots & \cdots & \vdots \\ \dfrac{\partial f_m}{\partial x^0} & \dfrac{\partial f_m}{\partial x^1} & \cdots & \dfrac{\partial f_m}{\partial x^{2n}} \end{matrix} \right\| ,$$

and suppose, in addition, that one of the functional determinants of the f with respect to just as many x, say:

$$B = \begin{pmatrix} f_1 & f_2 & \cdots & f_m \\ x^{\beta_1} & x^{\beta_2} & \cdots & x^{\beta_m} \end{pmatrix},$$

is ultimately non-zero for one of the X – say, X^α – *which has an index α that is distinct from each of the β.*

One knows that this condition is certainly satisfied when the matrix M has characteristic m + 1, but it can also be satisfied when the characteristic is simply m. In order to account for that, it is enough to consider the case of a second-order determinant:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix}.$$

If its characteristic is 2 then at least one of the products ad, bc will be non-zero. However, it might very well be the case that both ad and bc are non-zero, and the determinant $ad - bc$ is still annulled.

The essential hypothesis that (4) constitutes a system of relations that is invariant with respect to the differential system (1) is formally expressed by saying that one has:

$$(5) \qquad \sum_{i=0}^{2n} X^i \frac{\partial f_r}{\partial x^i} = 0 \qquad (r = 1, 2, \ldots, m)$$

by virtue of (4), i.e., on Σ.

Otherwise, let f = const. be an integral of (1). It will still be the case that:

$$(2) \qquad \sum_{i=0}^{2n} X^i \frac{\partial f}{\partial x^i} = 0$$

identically. Having done that, one can show how to generalize the procedure in the preceding section that led to particular solutions from the stationarity of f : It is no longer absolute, but only subordinate to the system (4) of invariant relations or relative to Σ.

Now, to make f stationary on Σ means to annul the differentials that are compatible with (4). Introduce m Lagrange multipliers λ_r and set:

$$(6) \qquad F = f + \sum_{r=1}^{m} \lambda_r f_r .$$

The stationarity condition that is subordinate to (4) translates into:

$$(7) \qquad dF = 0$$

for *any* system of increments dx^i, and that will then require that all of the derivatives of F must be annulled. Since it follows, in addition, from (2), (5), and (6) that by virtue of (4), or on Σ, one has:

$$\sum_{i=0}^{2n} X^i \frac{\partial F}{\partial x^i} = 0 , \tag{2′}$$

it will be sufficient to take $2n$ of the $2n + 1$ derivatives of F equal to zero, since it will then result that the $(2n + 1)^{\text{th}}$ one is also zero by virtue of (2′). If one supposes that $X^\alpha \cdot B \neq 0$ then one must assume that:

$$\frac{\partial F}{\partial x^i} = 0 \qquad (i \neq \alpha). \tag{8}$$

With that, (2′) will reduce to $X^\alpha (\partial F / \partial x^\alpha) = 0$, and that will show that all of the $\partial F / \partial x^\alpha$ must also be annulled as a consequence.

On the other hand, if one replaces F with its expression in (6) then the m equations (8) that correspond to the values $\beta_1, \beta_2, \ldots, \beta_m$ of the index i will become linear in the λ, and they will be soluble for those λ since $B \neq 0$. When one makes use of that fact in order to eliminate the λ from the $2n - m$ remaining equations (8), what will result are $2n - m$ equations between only the x that must be associated with (4). One will then obtain the manifold τ of stationarity relative to the integral f, i.e., the one that is situated on Σ. That manifold will be merely a linear one, *in general*, because it will be defined, on the whole, in the space S of the $2n + 1$ arguments $x^0, x^1, \ldots, x^n$ by:

$$(2n - m) + m = 2n$$

equations. However, special circumstances can occur in which τ can have a number of dimensions $1 + \nu$ that is greater than unity (as in the elementary case in the preceding section). However, from the considerations that were developed in no. **1**, it will obviously follow that if τ is a linear simplex then it will certainly define a particular solution σ of the differential system (1). Nonetheless, if it has a dimension $1 + \nu$ that is greater than unity then it will not just characterize one solution, but ∞^ν of them, so one will have to integrate a differential system of order ν.

3. – Case of canonical systems and involutory relations.

An important application of the generalities that were just presented is to the canonical systems:

$$\frac{dp_h}{dt} = -\frac{\partial H}{\partial q_h}, \qquad \frac{dq_h}{dt} = \frac{\partial H}{\partial p_h} \qquad (h = 1, 2, \ldots, n), \tag{9}$$

in which the characteristic function $H(p \mid q)$ is independent of t. In addition, since f = const., one has that the (generalized) energy integral H = const. If one finally supposes that the system admits $m < n$ invariant relations:

$$f_r = 0 \qquad\qquad (r = 1, 2, \ldots, m)$$

that are also free of t and in involution, while H and the f_r are $m + 1$ independent functions, then one can state [3] that ν *is equal to exactly m*, or that one arrives at a category of particular solutions that depend upon m arbitrary constants if m is the rank of the involutory group.

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[3] Page 351, vol. II_2 of the cited *Lezioni di meccanica razionale*, or also **E. T. Whittaker**, *Analytical Dynamics*, Cambridge University Press, 3rd ed., 1927, § 145.

Note II (*)

THE MOST SIGNIFICANT CASE

(*Ibidem*, pp. 369-375.)

The general premises of Note I ([4]) (which bears the same main title as the present one) were essentially aimed at making it possible extend the favorable circumstances that present themselves in the search for stationary solutions to the canonical systems to Pfaffian systems, when they are opportunely specified. By leaning on the aforementioned Note, we can now truly enter into that subject, while also continuing the numbering of the sections, formulas, and footnotes directly.

4. – Review of some facts about Pfaffian systems.

With **Morera** and **Birkhoff** ([5]), we shall say *Pfaffians* to mean those systems of ordinary differential equations that are produced by canonical ones under an arbitrary change of variables, i.e., by replacing t, p_h, q_h with $2n + 1$ independent combinations of them x^0, x^1, …, x^{2n}.

One can say, autonomously, that Pfaffian systems are the ones in the aforementioned arguments that one can infer from a Pfaffian form:

$$\omega_d = \sum_{i=0}^{2n} u_i \, dx^i \qquad (10)$$

(u_i are assigned functions of the x)

when one imposes the variational condition:

$$\delta \int_s \omega_d = 0 \qquad (11)$$

on *any* closed line s in S or on arc s between fixed extremes.

Now introduce the bilinear covariant of (10):

$$\omega' = \sum_{i,j=0}^{2n} v_{ij} \, dx^j \, \delta x^i , \qquad (12)$$

in which:

$$v_{ij} = \frac{\partial u_j}{\partial x^i} - \frac{\partial u_i}{\partial x^j} \qquad (i, j = 0, 1, \dots, 2n), \qquad (13)$$

(*) Presented at the session on 4 March 1934.

([4]) Page 261 of this same volume of the "Rendiconti." [In this volume of the *Opere matematiche*, pp. 463]

([5]) *Dynamical Systems*, New York, 1927, American Math. Society Publications, vol. IV, Chap. II.

so (11) will be equivalent to $\omega' = 0$ for any system of increments δx^i, or to the Pfaffian system:

$$\sum_{j=0}^{2n} v_{ij}\, dx^j = 0 \qquad (i = 0, 1, \ldots, 2n), \tag{14}$$

which consists of $2n + 1$ equations. Moreover, due to the skew symmetry of the v, no more than $2n$ can be independent.

The case that one calls *ordinary* is the one in which the determinant of order $2n + 1$:

$$\Delta = \| v_{ij} \|,$$

which is always annulled by the skew symmetry of the v, nonetheless has characteristic $2n$. That situation will present itself if and only if ω_d is produced by transforming the canonical form $\sum_{h=1}^{n} p_h\, dq_h - H\, dt$. No matter what variables it refers to, from the fact that the determinant Δ has characteristic $2n$, one can state that the algebraic complements V^{ij} of the elements v_{ij} that belong to two parallel rows (or columns) are proportional to each other. When one then takes into account the symmetry of the V^{ij} (viz., $V^{ij} = V^{ji}$, which follows from the skew symmetry of the v_{ij}), one can even set:

$$V^{ij} = X^i X^j \tag{15}$$

then, in which the X^i are well-defined functions of the v, and therefore of the x, *but they are not all zero*.

One immediately sees that one can attribute the explicit form:

$$X f \equiv \sum_{i=0}^{2n} X^i \frac{\partial f}{\partial x^i} = 0 \tag{16}$$

to the condition that $f(x^0, x^1, \ldots, x^{2n})$ = const. must be an integral of (14).

On the other hand, as in no. **2**, let:

$$f_r = 0 \qquad (r = 1, 2, \ldots, m) \tag{4}$$

be m independent equations in the variables x that collectively constitute a set of relations that are invariant with respect to the Pfaffian system (14). Formally, that is equivalent to the existence of the m equations:

$$X f_r = 0 \qquad (r = 1, 2, \ldots, m) \tag{17}$$

on the manifold Σ that is defined by (4).

With **Cartan** ([6]), whose fundamental research has conferred a high degree of elegance and simplicity on the general theory of **Pfaff**ian forms and systems, let us introduce the successive derivatives ω', ω'', and in general:

$$\omega^{(k)} \qquad\qquad (k = 1, 2, \ldots, 2n\ ;\ \omega^{(2n+1)} = 0),$$

which are all exterior (or alternating) forms, and ω' coincides with the bilinear covariant (12). With corresponding symbolism, (16) and (17) can be written ([7]):

$$\omega^{(2n-1)} \cdot df = 0$$

identically, and:

$$\omega^{(2n-1)} \cdot df_r = 0 \qquad\qquad (r = 1, 2, \ldots, m)$$

on Σ.

Further recall the extension of the **Poisson** parentheses of two generic functions *f* and *g* that is subordinate to a given Pfaffian ω_d.

One sets:

$$(f, g)_\omega = \frac{\omega^{(2n-2)} \cdot df \cdot dg}{\omega^{(2n)}},$$

in which an alternating bilinear form in the derivatives of the two functions remains to be defined ([8]). Naturally, when ω_d has the canonical expression:

$$\sum_{h=1}^{n} p_h\, dq_h - H\, dt,$$

those parentheses will reduce to their classical definition:

$$(f, g) = \sum_{h=1}^{n} \left(\frac{\partial f}{\partial p_h} \frac{\partial g}{\partial q_h} - \frac{\partial f}{\partial q_h} \frac{\partial g}{\partial p_h} \right).$$

In any case, when the parenthesis is zero, one says that the functions are *in involution*.

Given all of that, *suppose that the invariant relations* (4) *are in involution with each other and with the integral f with respect to the Pfaffian system* (14). That translates into formulas that say that their $m\,(m + 1) / 2$ parentheses are annulled *on* Σ, or that:

$$(18) \qquad\qquad (f_r, g_s)_\omega = 0 \qquad\qquad (r, s = 1, 2, \ldots, m),$$

([6]) Cf., in addition to his *Leçons sur les invariants intégraux*, Paris, Hermann, 1922, the beautiful volume by **Goursat**, *Leçons sur le problème de Pfaff*, Paris, Hermann, 1922, pp. 154.

([7]) **Goursat**, *loc. cit.*, pp. 161-162, and 34.

([8]) *Ibidem*, pp. 165.

(19) $$(f_r, f)_\omega = 0 \qquad (r = 1, 2, \ldots, m).$$

The last m equations are, like (16), linear and homogeneous in the partial derivatives of the integral f. With notations that are analogous to the ones in (16), one can also write the r^{th} equation (19) in the form:

(19′) $$X_r f \equiv \sum_{i=0}^{2n} X_r^i \frac{\partial f}{\partial x^i} = 0 \qquad (r = 1, 2, \ldots, m),$$

in which the X_r^i are, in turn, linear forms in the derivatives of the functions f_r . Based upon the definition of the parentheses, those forms must be considered to be provided directly from facts of the matter.

5. – A noteworthy class of stationary solutions to Pfaffian systems.

In order to apply the general rule of no. **2** to Pfaffian systems of the special kind in the preceding section, one must add a qualitative restriction (that is analogous to the one that was introduced in the aforementioned no. **2**) that implies the independence of the functions $f_1, f_2, \ldots, f_m$, but requires a complement. That restriction is stated most conveniently by considering the matrix:

$$M' \equiv \left\| \begin{matrix} X^0 & X^1 & \cdots & X^{2n} \\ X_1^0 & X_1^1 & \cdots & X_1^{2n} \\ \vdots & \vdots & \cdots & \vdots \\ X_m^0 & X_m^1 & \cdots & X_m^{2n} \\ \dfrac{\partial f_1}{\partial x^0} & \dfrac{\partial f_1}{\partial x^1} & \cdots & \dfrac{\partial f_1}{\partial x^{2n}} \\ \dfrac{\partial f_2}{\partial x^0} & \dfrac{\partial f_2}{\partial x^1} & \cdots & \dfrac{\partial f_2}{\partial x^{2n}} \\ \vdots & \vdots & \cdots & \vdots \\ \dfrac{\partial f_m}{\partial x^0} & \dfrac{\partial f_m}{\partial x^1} & \cdots & \dfrac{\partial f_m}{\partial x^{2n}} \end{matrix} \right\|$$

with $2m + 1$ rows and $2n + 1$ columns, which includes the M in no. **2**, but contains the m rows of coefficients X_r^i, in addition. We suppose that at least one product of the type AB is non-zero, in which A is intended to mean a determinant of order $m + 1$ that is extracted from the first $m + 1$ rows by keeping the columns $\alpha_0, \alpha_1, \ldots, \alpha_m$, and B means a determinant of order m that is extracted from the *complementary* matrix to A in M', i.e., to the (functional) matrix that is composed of the last m rows of M' and the $2n + 1 - (m + 1) = 2n - m$ columns that exclude the indices $\alpha_0, \alpha_1, \ldots, \alpha_m$. As in no. **2**, the determinant B is then a functional determinant:

$$B = \begin{pmatrix} f_1 & f_2 & \cdots & f_m \\ x^{\beta_1} & x^{\beta_2} & \cdots & x^{\beta_m} \end{pmatrix},$$

whose indices β are all distinct form the α, by definition.

Here as well, one should note that if the characteristic of M' is the maximum possible, viz., $2m + 1$, then there will certainly be some product $AB \neq 0$. However, the latter situation can also occur without the aforementioned characteristic being exactly $2m + 1$.

We can now finally recall the considerations of no. **2**, in the context of the Pfaffian system (14), under the circumstances that were specified in the preceding section and with the additional condition that $AB \neq 0$. Everything comes down to showing that at most $2(n - m) + 1$ of the equations for the stationarity of f will be independent on the manifold Σ that is defined by (4).

If one introduces the m auxiliary variables λ_r by way of:

$$F = f + \sum_{r=1}^{m} \lambda_r f_r \tag{6}$$

then the conditions for stationarity will be provided, as in no. **2**, by the association of:

$$f_r = 0 \tag{4}$$

with

$$\frac{\partial F}{\partial x^i} = 0 \qquad (i = 0, 1, \ldots, 2n). \tag{20}$$

If one introduces F in place of f and takes (17), (18), and (4) into account then (16) and (19) will give:

$$X F = 0 \tag{21}$$

and

$$(f_r, F)_\omega = 0 \qquad (r = 1, 2, \ldots, m),$$

the last of which can also be written:

$$X_r F = 0, \tag{22}$$

when one defines the operators X_r .

Due to the fact that $A \neq 0$, (21), (22), when taken together, can be solved for $\partial F / \partial x^{\alpha_0}$, $\partial F / \partial x^{\alpha_1}$, …, $\partial F / \partial x^{\alpha_m}$, and that will define those functions as ones that are linear and homogeneous in the remaining derivatives. That is because of the equations (20), it is enough to take into consideration the ones for which the index i is different from $\alpha_0, \alpha_1, \ldots, \alpha_m$. The remaining ones can then be satisfied automatically. Let us say, generically, that j are the indices that are different from the α. From the definition of the manifold of stationarity τ, other than the original equation (4), we will then have the $2n + 1 - (m + 1) = 2n - m$ equations:

$$\frac{\partial F}{\partial x^j} = 0 \,, \tag{23}$$

which also involve the multipliers λ, due to (6).

There are certainly m of them (which correspond to the indices $\beta_1, \beta_2, \ldots, \beta_m$) that are soluble for the λ since the determinant B of their coefficients is non-zero. Meanwhile, one can (linearly) eliminate the λ from (23), and what will remain are then $2n - 2m$ equations that are free of the λ and involve only the coordinates x. As a result, an (at least) $(m + 1)$-dimensional manifold τ will then be defined in the space S of the $2n + 1$ variables x. Q.E.D.

6. – Return to the canonical form.

The verification that was just achieved in reference to a generic Pfaffian system will be valid, in particular, for a canonical system under the hypotheses that were imposed in no. **3**, as long as one recognizes that those hypotheses also imply that some product of the type AB must not be annulled. In order to see that, fix one's attention on the matrix M', while keeping in mind the form that it takes (with the usual notations for canonical systems) under the indicated circumstances. The first $m + 1$ rows, whose elements are the X in the schema of the preceding section, can then be written (recalling that one always supposes that $m < n$):

$$\left\|\begin{matrix} 1 & -\dfrac{\partial H}{\partial q_1} & \cdots & -\dfrac{\partial H}{\partial q_n} & \dfrac{\partial H}{\partial p_1} & \cdots & \dfrac{\partial H}{\partial p_m} & \cdots & \dfrac{\partial H}{\partial p_n} \\ \dfrac{\partial f_1}{\partial t}(=0) & -\dfrac{\partial f_1}{\partial q_1} & \cdots & -\dfrac{\partial f_1}{\partial q_n} & \dfrac{\partial f_1}{\partial p_1} & \cdots & \dfrac{\partial f_1}{\partial p_m} & \cdots & \dfrac{\partial f_1}{\partial p_m} \\ \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ \dfrac{\partial f_n}{\partial t}(=0) & -\dfrac{\partial f_m}{\partial q_1} & \cdots & -\dfrac{\partial f_m}{\partial q_n} & \dfrac{\partial f_m}{\partial p_1} & \cdots & \dfrac{\partial f_m}{\partial p_m} & \cdots & \dfrac{\partial f_m}{\partial p_n} \end{matrix}\right\| .$$

Recall that the functional determinant of the f_r (at least one of which is independent of those functions, by assumption) is non-zero, say:

$$D = \begin{pmatrix} f_1 & f_2 & \cdots & f_m \\ x^1 & x^2 & \cdots & x^m \end{pmatrix} .$$

Assume that A is the determinant of order $m + 1$ that one obtains from the matrix that was written above by taking the first column and the m columns that follow after the $(m + 1)^{\text{th}}$ one. One will certainly have:

$$A = D \neq 0$$

then. In the functional matrix of the m functions f with respect to t, p_h, q_h :

$$\left\| \begin{matrix} \frac{\partial f_1}{\partial t}(=0) & \frac{\partial f_1}{\partial p_1} & \cdots & \frac{\partial f_1}{\partial p_n} & \frac{\partial f_1}{\partial q_n} & \cdots & \frac{\partial f_1}{\partial q_n} \\ \frac{\partial f_2}{\partial t}(=0) & \frac{\partial f_2}{\partial p_1} & \cdots & \frac{\partial f_2}{\partial p_n} & \frac{\partial f_2}{\partial q_n} & \cdots & \frac{\partial f_2}{\partial q_n} \\ \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ \frac{\partial f_n}{\partial t}(=0) & \frac{\partial f_m}{\partial p_1} & \cdots & \frac{\partial f_m}{\partial p_n} & \frac{\partial f_m}{\partial q_1} & \cdots & \frac{\partial f_m}{\partial q_n} \end{matrix} \right\|,$$

a group of columns is given by the m columns that follow the first, and they are certainly all distinct from the ones that D is composed of. The corresponding determinant B is once more D. Hence, the product $AB = D^2$ is non-zero, and therefore one can certainly apply the proof that was given for general Pfaffian systems. One then has a way of deducing the stationary solutions for canonical systems under the various hypotheses that were given before that avoids not only having to develop calculations, but also purely-conceptual transformations, by exploiting the premises in a direct and synthetic manner.