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Investigations in regard to entire homogeneous functions of *n* differentials

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If one constructs a function from a given number of independently-varying quantities and their first differentials that is rational, entire, and homogeneous in the latter, but has arbitrary behavior relative to the variables then it will be clear that when one introduces an arbitrary system of new variable quantities in place of the original ones, the function will be converted into a new function of the new variables and their first differentials that is likewise entire, rational, homogeneous, and of the same degree as the original function relative to those differentials. One can regard a function of that type as an algebraic form in regard to the differentials that appear, and those expressions that seem to multiply the powers and products of powers of the differentials can be regarded as the coefficients. Accordingly, the analytical expression for the square of the line element of a given surface in two independent variables that *Gauss* had based his *Disquisitions generales circa superficies curvas* upon is a quadratic form in two differentials. *Gauss*'s research into that subject has found noteworthy application in the treatise *Über die Hypothesen, welche der Geometrie zu Grunde liegen* that was published by *Riemann*'s estate, and that is why interest in the cited class of forms, and in particular, the ones of degree two and with arbitrarily many differentials, has been growing significantly. A crucial point in *Riemann*'s investigations is that of ascertaining the criterion for whether a form of that type could be transformed by the introduction of a new system of variables into a form that is the sum of the squares of the differentials in the variables. In the case of a form in two differentials, the criterion is known to consist of the vanishing of the *Gauss*ian curvature that is associated with the line element in question. As soon as the number of differentials exceeds two, the criterion that *Riemann* posed (if I have grasped it correctly) assumes that one has found the solution of a problem in the calculus of variations that is formally close to it and has an indirect character because of that fact.

In the closed domain of forms for which the number of differentials and the degree have given values, one can consider two forms to belong to the same class or different ones according to whether or not there is a substitution with a non-vanishing functional determinant by which the one form can be transformed into the other form, and one also finds the counterpart of the usual basic concepts that one uses to construct the theory of transformations of entire homogeneous functions. Within that cited domain, however, those forms define a narrow subdomain whose coefficients are independent of the variables, or what amounts to the same thing, constant. Under any substitution by which the new variables are linear functions of the original ones, a form with constant coefficients will obviously go to a form that once more has constant coefficients. The theory of transformations of entire homogeneous functions by linear substitutions then finds a direct application here. Now, that raises the question of the conditions that will allow one to decide whether a given form in n differentials whose coefficients depend upon the variables can be transformed into a form with constant coefficients. As soon as the degree of the form exceeds unity, I will assume that a determinant whose elements are the second partial differential quotients of the form with respect to its differentials does not vanish identically.

For the forms of degree one, it is clear that a form with constant coefficients is equal to the differential of a linear function of the variables. In that case, the question that was raised then coincides with the question of the conditions under which a given form in n differentials can be equal to a differential. Those conditions are established and allow the following conception: When the expression:

$$a_1\, dx_1 + a_2\, dx_2\,, \ldots + a_n\, dx_n$$

means a form of degree one whose coefficients $a_1, a_2, \ldots, a_n$ depend upon the variables $x_1, x_2, \ldots, x_n$ arbitrarily, the form:

$$\sum_{\mathfrak{a},\mathfrak{b}=1}^{n} \left(\frac{\partial a_{\mathfrak{a}}}{\partial x_{\mathfrak{b}}} - \frac{\partial a_{\mathfrak{b}}}{\partial x_{\mathfrak{a}}} \right) dx_{\mathfrak{a}}\, \delta x_{\mathfrak{b}}\,,$$

which is bilinear in the differentials $dx_1, \ldots, dx_n$ and the variations $\delta x_1, \ldots, \delta x_n$, will have the property that when the given form of degree one is transformed by a substitution of new variables, it will change the given form in such a way that the relationship of the latter to the former will remain unchanged. The bilinear form is then the analogue of a covariant. In order for the given form to be equal to a form with constant coefficients, it is necessary and sufficient that the form thus-constructed, which is bilinear in the quantities $dx_{\mathfrak{a}}$ and $\delta x_{\mathfrak{a}}$ and linear in the combinations $dx_{\mathfrak{a}}\, \delta x_{\mathfrak{a}} - \delta x_{\mathfrak{a}}\, dx_{\mathfrak{a}}$, must vanish identically.

For forms of degree two, the content of the question that was posed will agree with the aforementioned problem in *Riemann*'s treatise. That is because every quadratic form in n differentials and with constant coefficients can be converted by a linear substitution into a form that is the sum of the squares of the differentials of the new variables, and in fact by a real substitution, provided that one deals with essentially positive forms, as *Riemann* did. The present work primarily pursues the goal of directly answering the question that was posed in regard to the forms of degree two in n differentials whose determinant does not vanish identically as a result of the restriction that one encounters, and in that sense, it extends the theory of the *Gauss*ian curvature. The result that is found corresponds precisely to the theorem that summarized the integrability conditions for forms of degree one just now. Namely, it shows that every given form of the indicated type in the differentials $dx_1, dx_2, \ldots, dx_n$ belongs to a second form that is quadrilinear in the four systems of differentials of the variables $x_{\mathfrak{a}}$, which might be denoted by $dx_{\mathfrak{a}}$, $\delta x_{\mathfrak{b}}$, $du_{\mathfrak{a}}$, $\delta u_{\mathfrak{b}}$, bilinear and symmetric in the combinations $dx_{\mathfrak{a}}\, \delta x_{\mathfrak{b}} - \delta x_{\mathfrak{a}}\, dx_{\mathfrak{b}}$ and $du_{\mathfrak{a}}\, \delta u_{\mathfrak{b}} - \delta u_{\mathfrak{a}}\, du_{\mathfrak{b}}$, and whose coefficients are formed with the help of single and double partial differentiations of the coefficients of the given form with respect to the variables. That second form has the property that when the given form and the former one are transformed by introducing new variables, they will be altered in such a way that the relationship of the latter to the former will remain unchanged. The necessary and sufficient condition for the given form to be transformable into a form with constant coefficients consists of the identical vanishing of the quadrilinear form that is associated with the one that we speak of (*). For forms in two differentials, the criterion for the quadrilinear form and the criterion for the curvature go over to each other.

The viewpoint from which the forms in *n* differentials are considered here allow one to regard the bilinear forms that refer to the integrability conditions and the quadrilinear form that is interpreted as the curvature as the first terms in a series. The desire still remains to continue that series of direct methods for the solution of the problem that was posed. By contrast, that problem will generally be addressed in what follows by an indirect method by which a corresponding problem in the calculus of variations will be assumed to be solved for each form.

1.

A form in *n* differentials $dx_1, dx_2, \ldots, dx_n$ whose degree is *p* and whose coefficients depend upon the *n* variables $x_1, x_2, \ldots, x_n$ arbitrarily, but in such a way that they can be partially differentiated with respect to the individual variables, will be denoted by $f(dx)$. A symbol from the lower-case German alphabet $\mathfrak{a}, \mathfrak{b}, \ldots$ always means an index that should come from the sequence of numbers 1 to *n*. The *n* variables $x_\mathfrak{a}$ will be regarded as independent functions of *n* new variables $y_\mathfrak{k}$, such that the *n* equations:

$$dx_\mathfrak{a} = c_{\mathfrak{a},1}\, dy_1 + c_{\mathfrak{a},2}\, dy_2 + \ldots + c_{\mathfrak{a},n}\, dy_n \tag{1}$$

will exist between the differentials $dx_\mathfrak{a}$ and $dy_\mathfrak{k}$. Here, one has set:

$$\frac{\partial x_\mathfrak{a}}{\partial y_\mathfrak{k}} = c_{\mathfrak{a},\mathfrak{k}}\,, \tag{2}$$

and the functional determinant:

$$C = \sum \pm c_{1,1}\, c_{2,2} \cdots c_{n,n}$$

should not vanish. The representation of the quantities $dy_\mathfrak{k}$ in terms of the quantities $dx_\mathfrak{a}$ is determined completely then. That yields the *n* equations:

$$dy_\mathfrak{k} = h_{\mathfrak{k},1}\, dx_1 + h_{\mathfrak{k},2}\, dx_2 + \ldots + h_{\mathfrak{k},n}\, dx_n\,. \tag{1*}$$

(*) It is doubtful to me that the result of *Riemann*'s treatise can be understood in such a way that it agrees with that theorem. However, I have some concerns about presently subjecting the methods that were applied in that treatise to a critical consideration, since the treatise had the goal of recording an oral presentation, so it included no analytical explanations and was not developed for the sake of publication by the author himself.

Now, if the variables $y_{\mathfrak{t}}$ are considered to be functions of the variables $x_{\mathfrak{a}}$ then one will have the equations:

(2*) $$\frac{\partial y_{\mathfrak{t}}}{\partial x_{\mathfrak{a}}} = h_{\mathfrak{t},\mathfrak{a}} .$$

Substituting the variables $y_{\mathfrak{t}}$ in the form $f(dx)$ might produce the equation:

(3) $$f(dx) = g(dy) ,$$

in which $g(dy)$ means a form of degree p in the n differentials $dy_{\mathfrak{t}}$ whose coefficients depend upon the variables $y_{\mathfrak{t}}$. However, one establishes the restriction that for $p \geq 2$ the determinant:

(3.a) $$\Delta = \sum \pm \frac{\partial^2 f(dx)}{\partial dx_1 \, \partial dx_1} \cdots \frac{\partial^2 f(dx)}{\partial dx_n \, \partial dx_n}$$

does not vanish identically, and the determinant:

(3.a*) $$E = \sum \pm \frac{\partial^2 g(dy)}{\partial dy_1 \, \partial dy_1} \cdots \frac{\partial^2 g(dy)}{\partial dy_n \, \partial dy_n}$$

will fulfill the same condition as a result of the equation:

$$E = C^2 \Delta .$$

In the study of the form $f(dx)$, it seems to me to be desirable, above all, to learn about those functions with the property that when they are transformed along with the form $f(dx)$, they will both be converted in such a way that their mutual relationship remains unchanged. I believe that one source for the derivation of such functions can be found in the theorems of the calculus of variations that *Lagrange* used in *Mécanique analytique*, part two, section IV, in order to transform the differential equations of mechanics. Let δ be the characteristic of the variation, let d denote complete differentiation, and let ∂ denote the partial differentiation, and in the latter, the form $f(dx)$ will be considered to depend upon $2n$ independent variables, namely, the n quantities $x_{\mathfrak{a}}$ and the n quantities $dx_{\mathfrak{a}}$, and likewise, the form $g(dy)$ is considered to depend upon the n quantities $y_{\mathfrak{t}}$ and the n quantities $dy_{\mathfrak{t}}$. The *Lagrange* equations are the following ones:

(4)
$$\left\{\begin{aligned}\delta f(dx) &= \sum_{\mathfrak{a}} \frac{\partial f(dx)}{\partial x_{\mathfrak{a}}}\delta x_{\mathfrak{a}} + \sum_{\mathfrak{a}} \frac{\partial f(dx)}{\partial\, dx_{\mathfrak{a}}}\delta dx_{\mathfrak{a}} \\ &= \sum_{\mathfrak{a}}\left(\frac{\partial f(dx)}{\partial x_{\mathfrak{a}}} - d\,\frac{\partial f(dx)}{\partial\, dx_{\mathfrak{a}}}\right)\delta x_{\mathfrak{a}} + d\sum_{\mathfrak{a}} \frac{\partial f(dx)}{\partial\, dx_{\mathfrak{a}}}\delta x_{\mathfrak{a}}\,,\end{aligned}\right.$$

(4*)
$$\left\{\begin{aligned}\delta g(dy) &= \sum_{\mathfrak{k}} \frac{\partial g(dy)}{\partial y_{\mathfrak{k}}}\delta y_{\mathfrak{k}} + \sum_{\mathfrak{k}} \frac{\partial g(dy)}{\partial\, dy_{\mathfrak{k}}}\delta dy_{\mathfrak{k}} \\ &= \sum_{\mathfrak{k}}\left(\frac{\partial g(dy)}{\partial y_{\mathfrak{k}}} - d\,\frac{\partial g(dy)}{\partial\, dy_{\mathfrak{k}}}\right)\delta y_{\mathfrak{k}} + d\sum_{\mathfrak{k}} \frac{\partial g(dy)}{\partial\, dy_{\mathfrak{k}}}\delta y_{\mathfrak{k}}\,.\end{aligned}\right.$$

Furthermore:

(5)
$$\delta f(dx) = \delta g(dy),$$

(6)
$$\sum_{\mathfrak{a}} \frac{\partial f(dx)}{\partial\, dx_{\mathfrak{a}}}\delta x_{\mathfrak{a}} = \sum_{\mathfrak{k}} \frac{\partial g(dy)}{\partial\, dy_{\mathfrak{k}}}\delta y_{\mathfrak{k}}\,,$$

and it follows from this that:

(7)
$$\sum_{\mathfrak{a}}\left(d\,\frac{\partial f(dx)}{\partial\, dx_{\mathfrak{a}}} - \frac{\partial f(dx)}{\partial x_{\mathfrak{a}}}\right)\delta x_{\mathfrak{a}} = \sum_{\mathfrak{k}}\left(d\,\frac{\partial g(dy)}{\partial\, dy_{\mathfrak{k}}} - \frac{\partial g(dy)}{\partial y_{\mathfrak{k}}}\right)\delta y_{\mathfrak{k}}\,.$$

The variations $\delta x_{\mathfrak{a}}$ and $\delta y_{\mathfrak{k}}$ are coupled by the equations that arise from (1) or (1*) when one replaces the characteristic d with the characteristic δ.

In order to separate the terms in equation (7) that include the differentials $d^2 x_{\mathfrak{a}}$ and $d^2 y_{\mathfrak{k}}$ from the remaining ones for the case in which the degree p exceeds unity, I shall form the equations:

(8)
$$\left\{\begin{aligned} d\,\frac{\partial f(dx)}{\partial\, dx_{\mathfrak{a}}} &= \sum_{\mathfrak{b}} \frac{\partial^2 f(dx)}{\partial\, dx_{\mathfrak{a}}\,\partial\, dx_{\mathfrak{b}}} d^2 x_{\mathfrak{b}} + \sum_{\mathfrak{c}} \frac{\partial f(dx)}{\partial\, dx_{\mathfrak{a}}\,\partial\, dx_{\mathfrak{c}}} dx_{\mathfrak{c}}\,, \\ d\,\frac{\partial g(dy)}{\partial\, dy_{\mathfrak{k}}} &= \sum_{\mathfrak{l}} \frac{\partial^2 g(dy)}{\partial\, dy_{\mathfrak{k}}\,\partial\, dy_{\mathfrak{l}}} d^2 x_{\mathfrak{l}} + \sum_{\mathfrak{m}} \frac{\partial g(dy)}{\partial\, dy_{\mathfrak{k}}\,\partial\, dy_{\mathfrak{m}}} dy_{\mathfrak{m}}\,,\end{aligned}\right.$$

and set:

(9)
$$\left\{\begin{aligned} f_{\mathfrak{a}}(dx) &= \sum_{\mathfrak{c}} \frac{\partial^2 f(dx)}{\partial dx_{\mathfrak{a}}\,\partial dx_{\mathfrak{c}}} dx_{\mathfrak{c}} - \frac{\partial f(dx)}{\partial x_{\mathfrak{a}}}\,, \\ g_{\mathfrak{k}}(dy) &= \sum_{\mathfrak{m}} \frac{\partial^2 g(dy)}{\partial dy_{\mathfrak{k}}\,\partial dy_{\mathfrak{m}}} dy_{\mathfrak{m}} - \frac{\partial g(dy)}{\partial y_{\mathfrak{k}}}\,.\end{aligned}\right.$$

(7) will then imply the equation:

(10) $$\sum_{\mathfrak{a}}\left(\sum_{\mathfrak{b}}\frac{\partial^2 f(dx)}{\partial\, dx_{\mathfrak{a}}\,\partial\, dx_{\mathfrak{b}}}\,d^2x_{\mathfrak{b}}+f_{\mathfrak{a}}(dx)\right)\delta x_{\mathfrak{a}} = \sum_{\mathfrak{k}}\left(\sum_{\mathfrak{l}}\frac{\partial^2 g(dy)}{\partial\, dy_{\mathfrak{k}}\,\partial\, dy_{\mathfrak{l}}}\,d^2y_{\mathfrak{l}}+g_{\mathfrak{k}}(dy)\right)\delta y_{\mathfrak{k}}\,.$$

At this point, a property of the forms $f_{\mathfrak{a}}(dx)$ of degree p that are defined by (9) might be mentioned that will find multiple uses later on. Since the partial differentiation with respect to $dx_{\mathfrak{a}}$ will yield the equation:

$$\frac{\partial f_{\mathfrak{a}}(dx)}{\partial\, dx_{\mathfrak{b}}}=\sum_{\mathfrak{c}}\frac{\partial^3 f(dx)}{\partial\, dx_{\mathfrak{a}}\,\partial\, dx_{\mathfrak{b}}\,\partial x_{\mathfrak{c}}}+\frac{\partial^2 f(dx)}{\partial\, dx_{\mathfrak{a}}\,\partial x_{\mathfrak{b}}}-\frac{\partial^2 f(dx)}{\partial x_{\mathfrak{a}}\,\partial\, dx_{\mathfrak{b}}}\,,$$

so when one switches the indices $\mathfrak{a}$ and $\mathfrak{b}$, one will get the relation:

(11) $$\frac{\partial f_{\mathfrak{a}}(dx)}{\partial\, dx_{\mathfrak{b}}}+\frac{\partial f_{\mathfrak{b}}(dx)}{\partial\, dx_{\mathfrak{a}}}= 2\sum_{\mathfrak{c}}\frac{\partial^3 f(dx)}{\partial\, dx_{\mathfrak{a}}\,\partial\, dx_{\mathfrak{b}}\,\partial x_{\mathfrak{c}}}\,dx_{\mathfrak{c}}\,.$$

In the case where the degree p is equal to unity, equations (6) and (7) go to the two following ones, respectively:

(6.a) $$f(dx) = g(dy)\,,$$

(7.a) $$\delta f(dx) - df(\delta x) = \delta g(dy) - dg(\delta y)\,.$$

If one sets:

(12) $$\begin{cases} f(dx) = a_1\,dx_1 + a_2\,dx_2 + \cdots + a_n\,dx_n\,, \\ g(dy) = e_1\,dy_1 + e_2\,dy_2 + \cdots + e_n\,dy_n \end{cases}$$

here then (7.a) will assume the form:

(7.b) $$\sum_{\mathfrak{a},\mathfrak{b}}\left(\frac{\partial a_{\mathfrak{a}}}{\partial x_{\mathfrak{b}}}-\frac{\partial a_{\mathfrak{b}}}{\partial x_{\mathfrak{a}}}\right)dx_{\mathfrak{a}}\,\delta x_{\mathfrak{b}} = \sum_{\mathfrak{k},\mathfrak{l}}\left(\frac{\partial e_{\mathfrak{k}}}{\partial y_{\mathfrak{l}}}-\frac{\partial e_{\mathfrak{l}}}{\partial y_{\mathfrak{k}}}\right)dy_{\mathfrak{k}}\,\delta y_{\mathfrak{l}}\,.$$

They express the property of the bilinear form:

(13) $$\delta f(dx) - df(\delta x) = \sum_{\mathfrak{a},\mathfrak{b}}\left(\frac{\partial a_{\mathfrak{a}}}{\partial x_{\mathfrak{b}}}-\frac{\partial a_{\mathfrak{b}}}{\partial x_{\mathfrak{a}}}\right)dx_{\mathfrak{a}}\,\delta x_{\mathfrak{b}}$$

that was suggested in the introduction that its relationship to the form $f(dx)$ will remain unchanged under a substitution of new variables. The fact that the identical vanishing of (13) includes the necessary and sufficient condition for the integrability of the form $f(dx)$ is well-known.

In the case where the number $p = 2$, the representation of curvature that *Gauss* had developed in art. 11 of the cited treatise defines a solid basis for the search for functions with the desired behavior. The elements of that representation and the elements of equation (10) shall then be carefully compared with each other. With the assumption that $p = 2$, the forms $f(dx)$ and $g(dy)$ will have the following forms:

$$(14)\qquad \begin{cases} f(dx) = \frac{1}{2}\sum_{\mathfrak{a},\mathfrak{b}} a_{\mathfrak{a},\mathfrak{b}}\, dx_{\mathfrak{a}}\, dx_{\mathfrak{b}}\,, \\ g(dy) = \frac{1}{2}\sum_{\mathfrak{k},\mathfrak{l}} e_{\mathfrak{k},\mathfrak{l}}\, dy_{\mathfrak{k}}\, dy_{\mathfrak{l}}\,. \end{cases}$$

Here, one has:

$$a_{\mathfrak{a},\mathfrak{b}} = a_{\mathfrak{b},\mathfrak{a}}\,, \qquad e_{\mathfrak{k},\mathfrak{l}} = e_{\mathfrak{l},\mathfrak{k}}\,,$$

and the determinant:

$$(15)\qquad \Delta = \sum \pm a_{1,1}\, a_{2,2} \cdots a_{n,n}$$

cannot vanish identically. From an earlier remark, the same thing will then be true of the determinant:

$$(15^*)\qquad E = \sum \pm e_{1,1}\, e_{2,2} \cdots e_{n,n}\,.$$

Moreover, let:

$$(16)\qquad \begin{cases} f_{\mathfrak{a}}(dx) = \frac{1}{2}\sum_{\mathfrak{g},\mathfrak{b}} f_{\mathfrak{a},\mathfrak{g},\mathfrak{b}}\, dx_{\mathfrak{g}}\, dx_{\mathfrak{b}}\,, \\ g_{\mathfrak{k}}(dy) = \frac{1}{2}\sum_{\mathfrak{p},\mathfrak{q}} g_{\mathfrak{k},\mathfrak{p},\mathfrak{q}}\, dy_{\mathfrak{p}}\, dy_{\mathfrak{q}}\,, \end{cases}$$

in which the expressions $f_{\mathfrak{a},\mathfrak{g},\mathfrak{b}}$ and $g_{\mathfrak{k},\mathfrak{p},\mathfrak{q}}$ have the following meaning:

$$(17)\qquad \begin{cases} f_{\mathfrak{a},\mathfrak{g},\mathfrak{b}}(dx) = \dfrac{\partial a_{\mathfrak{a},\mathfrak{g}}}{\partial x_{\mathfrak{b}}} + \dfrac{\partial a_{\mathfrak{a},\mathfrak{b}}}{\partial x_{\mathfrak{g}}} - \dfrac{\partial a_{\mathfrak{g},\mathfrak{b}}}{\partial x_{\mathfrak{a}}}\,, \\ g_{\mathfrak{k},\mathfrak{p},\mathfrak{q}}(dy) = \dfrac{\partial e_{\mathfrak{k},\mathfrak{p}}}{\partial y_{\mathfrak{q}}} + \dfrac{\partial e_{\mathfrak{k},\mathfrak{q}}}{\partial y_{\mathfrak{p}}} - \dfrac{\partial e_{\mathfrak{p},\mathfrak{q}}}{\partial y_{\mathfrak{k}}}\,, \end{cases}$$

due to (9), and as a result, one will have the equations:

$$f_{\mathfrak{a},\mathfrak{g},\mathfrak{b}} = f_{\mathfrak{a},\mathfrak{b},\mathfrak{g}}\,, \qquad g_{\mathfrak{k},\mathfrak{p},\mathfrak{q}} = g_{\mathfrak{k},\mathfrak{q},\mathfrak{p}}\,.$$

Gauss's study referred to curved surfaces for which the rectangular coordinates of a point were expressed in terms of the independent variables p and q. The square of the line element ds is set equal to the expression:

$$E\, dp^2 + 2F\, dp\, dq + G\, dq^2,$$

and the curvature will be called *k*. Under the substitutions:

$$p = x_1 , \qquad q = x_2 ,$$

$$E = a_{1,1} , \qquad F = a_{1,2} , \qquad G = a_{2,2} ,$$

those notations will go to the notations of this article for $p = 2$, $n = 2$, and one will have:

$$ds^2 = 2f\,(dx) .$$

Moreover, for the combinations that *Gauss* called Δ, m, m', m'', n, n', n'', one obtains the equations:

$$EG - FF = \Delta ,$$

$$\frac{\partial E}{\partial p} = 2m = f_{1,1,1} , \qquad \frac{\partial E}{\partial q} = 2m' = f_{1,1,2} = f_{1,2,1} , \qquad 2\frac{\partial F}{\partial q} - \frac{\partial G}{\partial p} = 2m'' = f_{1,2,2} ,$$

$$2\frac{\partial F}{\partial p} - \frac{\partial E}{\partial q} = 2n = f_{2,1,1} , \qquad \frac{\partial G}{\partial p} = 2n' = f_{2,1,2} = f_{2,2,1} , \qquad \frac{\partial G}{\partial q} = 2n'' = f_{2,2,2} .$$

The expression for the curvature that we spoke of can be deduced from these combinations by means of the following formula:

$$k = \frac{\dfrac{\partial m''}{\partial p} - \dfrac{\partial m'}{\partial q}}{\Delta} + \frac{E(n'n' - nn'') + F(nm'' - 2m'n' + mn'') + G(m'm' - mm'')}{\Delta\Delta} .$$

In the other notation, that reads:

$$k = \frac{1}{2\Delta}\left(\frac{\partial f_{1,2,2}}{\partial x_1} - \frac{\partial f_{1,2,1}}{\partial x_2}\right)$$
$$+ \frac{1}{4\Delta\Delta}[a_{11}(f_{2,1,2}^2 - f_{2,1,1}\, f_{2,2,2}) + a_{1,2}(f_{2,1,2}\, f_{1,2,2} - 2 f_{1,1,2}\, f_{2,1,2} + f_{1,1,1}\, f_{2,2,2}) + a_{2,2}(f_{1,1,2}^2 - f_{1,1,1}\, f_{1,2,2})] .$$

One finds all of the components of that representation on the left-hand side of equation (10) under the assumption that $p = 2$, $n = 2$. If one forms an expression *m* from the corresponding components on the right-hand side of that equation then it will represent the curvature for a line element whose square is equal to $2g\,(dy)$ with the notations that were introduced, in which y_1, y_2 will be the two independent coordinates. However, since the geometric meaning of the curvature does not depend upon the choice of independent coordinates, one must have the equation:

$$k = m\ .$$

The curvature k then has a relationship to the form $f\,(dx)$ that will remain unchanged under the substitution of new variables.

2.

If one wishes to employ equation (10) under the assumption that $p = 2$, $n = 2$ in order to give a direct proof of the property of the curvature that is expressed in the equation $k = m$ then that will be closely related to the fact that one begins by eliminating the second-order differentials $d^2 x_{\mathfrak{a}}$ and $d^2 y_{\mathfrak{k}}$. I will now perform an elimination by which the quantities $c_{\mathfrak{a},\mathfrak{k}}$ and their differentials $dc_{\mathfrak{a},\mathfrak{k}}$ will then enter into the equation under the general assumption that $p \geq 2$. It follows upon differentiating (1) that:

$$d^2 x_{\mathfrak{a}} = \sum_{\mathfrak{r}} c_{\mathfrak{a},\mathfrak{r}}\, d^2 y_{\mathfrak{r}} + \sum_{\mathfrak{r}} dc_{\mathfrak{a},\mathfrak{r}}\, dy_{\mathfrak{r}}\ , \tag{18}$$

and by converting d into δ:

$$\delta x_{\mathfrak{a}} = \sum_{\mathfrak{q}} c_{\mathfrak{a},\mathfrak{q}}\, \delta y_{\mathfrak{q}}\ .$$

Furthermore, as a result of (3), one has the equation:

$$\left\{\begin{aligned} \sum_{\mathfrak{a},\mathfrak{b}} \frac{\partial^2 f\,(dx)}{\partial\, dx_{\mathfrak{a}}\, \partial\, dx_{\mathfrak{b}}}\, \delta x_{\mathfrak{a}} \sum_{\mathfrak{r}} c_{\mathfrak{b},\mathfrak{r}}\, d^2 y_{\mathfrak{r}} &= \sum_{\mathfrak{a},\mathfrak{b}} \frac{\partial^2 f\,(dx)}{\partial\, dx_{\mathfrak{a}}\, \partial\, dx_{\mathfrak{b}}} \sum_{\mathfrak{q}} c_{\mathfrak{a},\mathfrak{q}}\, \delta y_{\mathfrak{q}} \sum_{\mathfrak{r}} c_{\mathfrak{b},\mathfrak{r}}\, d^2 y_{\mathfrak{r}} \\ &= \sum_{\mathfrak{k},\mathfrak{l}} \frac{\partial^2 g\,(dy)}{\partial\, dy_{\mathfrak{k}}\, \partial\, dy_{\mathfrak{l}}}\, \delta y_{\mathfrak{k}}\, d^2 y_{\mathfrak{l}} . \end{aligned}\right. \tag{19}$$

When one then expresses $d^2 x_{\mathfrak{a}}$ in (10) according to equation (18) and considers (19), that will give the new equation:

$$\sum_{\mathfrak{a},\mathfrak{b}} \frac{\partial^2 f\,(dx)}{\partial\, dx_{\mathfrak{a}}\, \partial\, dx_{\mathfrak{b}}}\, \delta x_{\mathfrak{a}} \sum_{\mathfrak{k}} dc_{\mathfrak{k},\mathfrak{l}}\, dy_{\mathfrak{l}} + \sum_{\mathfrak{a}} f_{\mathfrak{a}}\,(dx)\, \delta x_{\mathfrak{a}} = \sum_{\mathfrak{k}} g_{\mathfrak{k}}\,(dy)\, \delta y_{\mathfrak{k}}\ , \tag{20}$$

which is free of the quantities $d^2 x_{\mathfrak{b}}$, $d^2 y_{\mathfrak{l}}$.

It will now happen that consequences can be inferred from equation (20) for which the lost symmetry in the functions $f\,(dx)$ and $g\,(dy)$ will again occur, and in which it will be permissible to eliminate the differentials $dc_{\mathfrak{b},\mathfrak{l}}$. As long as the number $p = 2$, one can achieve the former goal by a simple means and approach the objective of this investigation in that way. For that reason, I shall introduce the assumption that $p = 2$ into equation (20), and obtain the following equation form (14):

(21) $$\sum_{\mathfrak{a},\mathfrak{b}} a_{\mathfrak{a},\mathfrak{b}}\,\delta x_{\mathfrak{a}} \sum_{\mathfrak{k}} dc_{\mathfrak{k},\mathfrak{l}}\,dy_{\mathfrak{l}} + \sum_{\mathfrak{a}} f_{\mathfrak{a}}(dx)\,\delta x_{\mathfrak{a}} = \sum_{\mathfrak{k}} g_{\mathfrak{k}}(dy)\,\delta y_{\mathfrak{k}} .$$

It will then suffice to take the partial differential quotients of both sides of it with respect to $dy_{\mathfrak{l}}$ and $\delta x_{\mathfrak{a}}$, where $\mathfrak{l}$ and $\mathfrak{a}$ have well-defined values. As a result of the defining equation (2), one has the relation:

$$\sum_{\mathfrak{l}} dc_{\mathfrak{k},\mathfrak{l}}\,dy_{\mathfrak{l}} = \sum_{\mathfrak{k},\mathfrak{l}} \frac{\partial^2 x_{\mathfrak{b}}}{\partial y_{\mathfrak{k}}\,\partial y_{\mathfrak{l}}}\,dy_{\mathfrak{k}}\,dy_{\mathfrak{l}} ,$$

and for that reason:

$$\frac{\partial \sum_{\mathfrak{l}} dc_{\mathfrak{k},\mathfrak{l}}\,dy_{\mathfrak{l}}}{\partial\, dy_{\mathfrak{q}}} = 2\,dc_{\mathfrak{b},\mathfrak{q}} .$$

Moreover, one has:

$$\frac{\partial f_{\mathfrak{a}}(dx)}{\partial\, dy_{\mathfrak{q}}} = \sum_{\mathfrak{b}} \frac{\partial f_{\mathfrak{a}}(dx)}{\partial\, dx_{\mathfrak{b}}}\,c_{\mathfrak{b},\mathfrak{q}} .$$

When one then partially-differentiates both sides of equation (21) with respect to a particular $dy_{\mathfrak{l}}$ that will give:

(22) $$2\sum_{\mathfrak{a},\mathfrak{b}} a_{\mathfrak{a},\mathfrak{b}}\,dc_{\mathfrak{b},\mathfrak{l}}\,\delta x_{\mathfrak{a}} + \sum_{\mathfrak{a},\mathfrak{b}} \frac{\partial f_{\mathfrak{a}}(dx)}{\partial\, dx_{\mathfrak{b}}}\,c_{\mathfrak{b},\mathfrak{l}}\,\delta x_{\mathfrak{a}} = \sum_{\mathfrak{k}} \frac{\partial g_{\mathfrak{k}}(dy)}{\partial\, dy_{\mathfrak{l}}}\,\delta y_{\mathfrak{k}} ,$$

and when one performs the partial differentiation with respect to a particular $\delta x_{\mathfrak{a}}$, because of (1^*), that will imply the relation:

(23) $$2\sum_{\mathfrak{b}} a_{\mathfrak{a},\mathfrak{b}}\,dc_{\mathfrak{b},\mathfrak{l}} + \sum_{\mathfrak{b}} \frac{\partial f_{\mathfrak{a}}(dx)}{\partial\, dx_{\mathfrak{b}}}\,c_{\mathfrak{b},\mathfrak{l}} = \sum_{\mathfrak{k}} \frac{\partial g_{\mathfrak{k}}(dy)}{\partial\, dy_{\mathfrak{l}}}\,h_{\mathfrak{k},\mathfrak{a}} .$$

The previously-derived equation (11) can be useful here. With the assumption that $p = 2$, it will take on the form:

(11.a) $$\frac{\partial f_{\mathfrak{a}}(dx)}{\partial\, dx_{\mathfrak{b}}} + \frac{\partial f_{\mathfrak{b}}(dx)}{\partial\, dx_{\mathfrak{a}}} = 2\sum_{\mathfrak{c}} \frac{\partial a_{\mathfrak{a},b}}{\partial x_{\mathfrak{c}}}\,dx_{\mathfrak{c}} ,$$

or, in summary:

(11.b) $$\frac{\partial f_{\mathfrak{a}}(dx)}{\partial\, dx_{\mathfrak{b}}} + \frac{\partial f_{\mathfrak{b}}(dx)}{\partial\, dx_{\mathfrak{a}}} = 2\,da_{\mathfrak{a},\mathfrak{b}} .$$

If one multiplies it by $c_{\mathfrak{b},\mathfrak{l}}$ and then subtracts it from (23), that will give the result that:

(24) $$2d\sum_{\mathfrak{b}} a_{\mathfrak{a},\mathfrak{b}}\,dc_{\mathfrak{b},\mathfrak{l}} = \sum_{\mathfrak{b}} \frac{\partial f_{\mathfrak{a}}(dx)}{\partial\, dx_{\mathfrak{b}}}\,c_{\mathfrak{b},\mathfrak{l}} + \sum_{\mathfrak{k}} \frac{\partial g_{\mathfrak{k}}(dy)}{\partial\, dy_{\mathfrak{l}}}\,h_{\mathfrak{k},\mathfrak{a}} .$$

Here, one finds the sum of two sums on the right-hand side, one of which has the the same relationship to the form $f(dx)$ and the variables $x_{\mathfrak{a}}$ that the other one has to the form $g(dy)$ and the variables $y_{\mathfrak{k}}$. In regard to the left-hand side, it should be remarked that from a known equation in the theory of quadratic forms, one must have:

$$\sum_{\mathfrak{b}} a_{\mathfrak{a},\mathfrak{b}}\, c_{\mathfrak{b},\mathfrak{l}} = \sum_{\mathfrak{k}} e_{\mathfrak{l},\mathfrak{k}}\, h_{\mathfrak{k},\mathfrak{a}} .$$

One then obtains the equations that correspond to (23) and (24), respectively:

$$(23^*) \qquad 2\sum_{\mathfrak{k}} e_{\mathfrak{l},\mathfrak{k}}\, dh_{\mathfrak{k},\mathfrak{a}} + \sum_{\mathfrak{k}} \frac{\partial g_{\mathfrak{l}}(dy)}{\partial\, dy_{\mathfrak{k}}} dh_{\mathfrak{k},\mathfrak{a}} = \sum_{\mathfrak{b}} \frac{\partial f_{\mathfrak{b}}(dx)}{\partial\, dx_{\mathfrak{a}}} c_{\mathfrak{b},\mathfrak{l}} ,$$

$$(24^*) \qquad 2d\sum_{\mathfrak{k}} e_{\mathfrak{l},\mathfrak{k}}\, h_{\mathfrak{k},\mathfrak{a}} = \sum_{\mathfrak{b}} \frac{\partial f_{\mathfrak{b}}(dx)}{\partial\, dx_{\mathfrak{a}}} c_{\mathfrak{b},\mathfrak{l}} + \sum_{\mathfrak{k}} \frac{\partial g_{\mathfrak{k}}(dy)}{\partial\, dy_{\mathfrak{l}}} h_{\mathfrak{k},\mathfrak{a}} .$$

Moreover, one succeeds in eliminating the differentials $dc_{\mathfrak{b},\mathfrak{l}}$ from equation (24) due to the fact that its left-hand side is an exact differential. On the same grounds, its right-hand side must fulfill the integrability conditions. In other words, when one constructs the bilinear forms that correspond to the first-degree form $\sum_{\mathfrak{b}} \frac{\partial f_{\mathfrak{b}}(dx)}{\partial\, dx_{\mathfrak{a}}} c_{\mathfrak{b},\mathfrak{l}}$ in the differentials $dx_{\mathfrak{g}}$ and the first-degree form $\sum_{\mathfrak{k}} \frac{\partial g_{\mathfrak{k}}(dy)}{\partial\, dy_{\mathfrak{l}}} h_{\mathfrak{k},\mathfrak{a}}$ in the differentials $dy_{\mathfrak{p}}$ according to the paradigm (13) and adds them, that sum must vanish identically. That principle implies the equation:

$$(25) \qquad \left\{ \begin{aligned} &\sum_{\mathfrak{b}} \left(\delta \frac{\partial f_{\mathfrak{b}}(dx)}{\partial\, dx_{\mathfrak{a}}} - d\, \frac{\partial f_{\mathfrak{b}}(\delta x)}{\partial\, \delta x_{\mathfrak{a}}} \right) c_{\mathfrak{b},\mathfrak{l}} + \sum_{\mathfrak{b}} \left(\frac{\partial f_{\mathfrak{b}}(dx)}{\partial\, dx_{\mathfrak{a}}} \delta c_{\mathfrak{b},\mathfrak{l}} - \frac{\partial f_{\mathfrak{b}}(\delta x)}{\partial\, \delta x_{\mathfrak{a}}} dc_{\mathfrak{b},\mathfrak{l}} \right) \\ +&\sum_{\mathfrak{k}} \left(\delta \frac{\partial g_{\mathfrak{k}}(dy)}{\partial\, dy_{\mathfrak{l}}} - d\, \frac{\partial g_{\mathfrak{k}}(\delta y)}{\partial\, \delta y_{\mathfrak{l}}} \right) h_{\mathfrak{k},\mathfrak{a}} + \sum_{\mathfrak{k}} \left(\frac{\partial g_{\mathfrak{k}}(dy)}{\partial\, dy_{\mathfrak{l}}} \delta h_{\mathfrak{k},\mathfrak{a}} - \frac{\partial g_{\mathfrak{k}}(\delta y)}{\partial\, \delta y_{\mathfrak{l}}} dh_{\mathfrak{k},\mathfrak{a}} \right) = 0. \end{aligned} \right.$$

However, under the prevailing assumption that the determinants Δ and E do not vanish, since the quantities $dc_{\mathfrak{b},\mathfrak{l}}$, $\delta c_{\mathfrak{b},\mathfrak{l}}$, $dh_{\mathfrak{k},\mathfrak{a}}$, $\delta h_{\mathfrak{k},\mathfrak{a}}$ can be represented as linear combinations of the quantities $c_{\mathfrak{g},\mathfrak{b}}$ and $h_{\mathfrak{p},\mathfrak{q}}$, a result will emerge from equation (25) that no longer includes the differentials and variations that were spoken of.

3.

One might use the following notations for the adjoint elements of the forms $f(dx)$ and $g(dy)$:

(26) $$A_{\mathfrak{a},\mathfrak{b}} = \frac{\partial \Delta}{\partial a_{\mathfrak{a},\mathfrak{b}}}, \qquad E_{\mathfrak{k},\mathfrak{l}} = \frac{\partial E}{\partial e_{\mathfrak{k},\mathfrak{l}}}.$$

One then obtains the representations:

(27) $$\begin{cases} 2\,dc_{\mathfrak{b},\mathfrak{l}} = \sum_{\mathfrak{c}} \dfrac{A_{\mathfrak{c},\mathfrak{b}}}{\Delta}\left(-\sum_{\mathfrak{b}} \dfrac{\partial f_{\mathfrak{c}}(dx)}{\partial\, dx_{\mathfrak{b}}} c_{\mathfrak{b},\mathfrak{l}} + \sum_{\mathfrak{k}} \dfrac{\partial g_{\mathfrak{k}}(dy)}{\partial\, dy_{\mathfrak{l}}} h_{\mathfrak{k},\mathfrak{c}}\right), \\ 2\,\delta c_{\mathfrak{b},\mathfrak{l}} = \sum_{\mathfrak{c}} \dfrac{A_{\mathfrak{c},\mathfrak{b}}}{\Delta}\left(-\sum_{\mathfrak{b}} \dfrac{\partial f_{\mathfrak{c}}(\delta x)}{\partial\, \delta x_{\mathfrak{b}}} c_{\mathfrak{b},\mathfrak{l}} + \sum_{\mathfrak{k}} \dfrac{\partial g_{\mathfrak{k}}(\delta y)}{\partial\, \delta y_{\mathfrak{l}}} h_{\mathfrak{k},\mathfrak{c}}\right), \\ 2\,dh_{\mathfrak{k},\mathfrak{a}} = \sum_{\mathfrak{m}} \dfrac{E_{\mathfrak{m},\mathfrak{k}}}{\Delta}\left(-\sum_{\mathfrak{n}} \dfrac{\partial g_{\mathfrak{m}}(dx)}{\partial\, dy_{\mathfrak{n}}} h_{\mathfrak{n},\mathfrak{a}} + \sum_{\mathfrak{b}} \dfrac{\partial f_{\mathfrak{b}}(dx)}{\partial\, dx_{\mathfrak{a}}} c_{\mathfrak{b},\mathfrak{m}}\right), \\ 2\,\delta h_{\mathfrak{k},\mathfrak{a}} = \sum_{\mathfrak{m}} \dfrac{E_{\mathfrak{m},\mathfrak{k}}}{\Delta}\left(-\sum_{\mathfrak{n}} \dfrac{\partial g_{\mathfrak{m}}(\delta x)}{\partial\, dy_{\mathfrak{n}}} h_{\mathfrak{n},\mathfrak{a}} + \sum_{\mathfrak{b}} \dfrac{\partial f_{\mathfrak{b}}(\delta x)}{\partial\, dx_{\mathfrak{a}}} c_{\mathfrak{b},\mathfrak{m}}\right). \end{cases}$$

When one multiplies these equations by the factors:

$$-\frac{\partial f_{\mathfrak{b}}(\delta x)}{\partial\, \delta x_{\mathfrak{a}}}, \quad -\frac{\partial f_{\mathfrak{b}}(dx)}{\partial\, dx_{\mathfrak{a}}}, \quad -\frac{\partial g_{\mathfrak{k}}(\delta y)}{\partial\, dy_{\mathfrak{a}}}, \quad -\frac{\partial g_{\mathfrak{k}}(dy)}{\partial\, dy_{\mathfrak{l}}},$$

in succession, adds them, and sums the result in the manner that was prescribed by (25), the first term in brackets on the right-hand side of (27) will yield the sum:

(28) $$\begin{cases} \tfrac{1}{2}\sum_{\mathfrak{b},\mathfrak{c},\mathfrak{d}} \dfrac{A_{\mathfrak{c},\mathfrak{d}}}{\Delta}\left(\dfrac{\partial f_{\mathfrak{c}}(dx)}{\partial\, dx_{\mathfrak{b}}}\dfrac{\partial f_{\mathfrak{b}}(\delta x)}{\partial\, \delta x_{\mathfrak{a}}} - \dfrac{\partial f_{\mathfrak{c}}(\delta x)}{\partial\, \delta x_{\mathfrak{b}}}\dfrac{\partial f_{\mathfrak{b}}(dx)}{\partial\, dx_{\mathfrak{a}}}\right) c_{\mathfrak{d},\mathfrak{l}} \\ +\tfrac{1}{2}\sum_{\mathfrak{k},\mathfrak{m},\mathfrak{n}} \dfrac{E_{\mathfrak{m},\mathfrak{k}}}{E}\left(\dfrac{\partial g_{\mathfrak{m}}(dy)}{\partial\, dy_{\mathfrak{n}}}\dfrac{\partial g_{\mathfrak{k}}(\delta y)}{\partial\, \delta y_{\mathfrak{l}}} - \dfrac{\partial g_{\mathfrak{m}}(\delta y)}{\partial\, \delta y_{\mathfrak{n}}}\dfrac{\partial g_{\mathfrak{k}}(dy)}{\partial\, dy_{\mathfrak{l}}}\right) h_{\mathfrak{n},\mathfrak{a}}. \end{cases}$$

Furthermore, the second term in the brackets on the right-hand side of (27) will yield the sum:

(29) $$\begin{cases} \tfrac{1}{2}\sum_{\mathfrak{b},\mathfrak{c},\mathfrak{k}} \dfrac{A_{\mathfrak{c},\mathfrak{b}}}{\Delta}\left(-\dfrac{\partial f_{\mathfrak{b}}(\delta x)}{\partial\, \delta x_{\mathfrak{a}}}\dfrac{\partial g_{\mathfrak{k}}(dy)}{\partial\, dy_{\mathfrak{l}}} + \dfrac{\partial f_{\mathfrak{b}}(dx)}{\partial\, dx_{\mathfrak{a}}}\dfrac{\partial g_{\mathfrak{k}}(\delta y)}{\partial\, \delta y_{\mathfrak{l}}}\right) h_{\mathfrak{k},\mathfrak{c}} \\ +\tfrac{1}{2}\sum_{\mathfrak{k},\mathfrak{m},\mathfrak{b}} \dfrac{E_{\mathfrak{m},\mathfrak{k}}}{E}\left(-\dfrac{\partial f_{\mathfrak{b}}(dx)}{\partial\, dx_{\mathfrak{a}}}\dfrac{\partial g_{\mathfrak{k}}(\delta y)}{\partial\, \delta y_{\mathfrak{l}}} + \dfrac{\partial f_{\mathfrak{b}}(\delta x)}{\partial\, \delta x_{\mathfrak{a}}}\dfrac{\partial g_{\mathfrak{k}}(dy)}{\partial\, dy_{\mathfrak{l}}}\right) c_{\mathfrak{b},\mathfrak{m}}. \end{cases}$$

Now, that leads to the fact (which is conducive to the simplicity of the result) that due to the equation:

$$\sum_{\mathfrak{c}} \frac{A_{\mathfrak{c},\mathfrak{b}}}{\Delta} h_{\mathfrak{k},\mathfrak{c}} = \sum_{\mathfrak{m}} \frac{E_{\mathfrak{m},\mathfrak{k}}}{E} c_{\mathfrak{b},\mathfrak{m}} ,$$

which expresses a property of the forms that are adjoint to the forms $f(dx)$ and $g(dy)$, respectively, the total sum (29) will have the value zero. The summation indices $\mathfrak{b}$ and $\mathfrak{d}$ in the sum (28) might be switched with each other, so its introduction into (25) will imply the desired equation:

$$(30)\quad \begin{cases} \sum_{\mathfrak{b}} \left[\delta \frac{\partial f_{\mathfrak{b}}(dx)}{\partial dx_{\mathfrak{a}}} - d \frac{\partial f_{\mathfrak{b}}(\delta x)}{\partial \delta x_{\mathfrak{a}}} + \frac{1}{2} \sum_{\mathfrak{c},\mathfrak{d}} \frac{A_{\mathfrak{c},\mathfrak{d}}}{\Delta} \left(\frac{\partial f_{\mathfrak{c}}(dx)}{\partial dx_{\mathfrak{b}}} \frac{\partial f_{\mathfrak{d}}(\delta x)}{\partial \delta x_{\mathfrak{a}}} - \frac{\partial f_{\mathfrak{c}}(\delta x)}{\partial \delta x_{\mathfrak{b}}} \frac{\partial f_{\mathfrak{d}}(dx)}{\partial dx_{\mathfrak{a}}} \right) \right] c_{\mathfrak{b},\mathfrak{l}} \\ \sum_{\mathfrak{k}} \left[\delta \frac{\partial g_{\mathfrak{k}}(dy)}{\partial dy_{\mathfrak{l}}} - d \frac{\partial g_{\mathfrak{k}}(\delta y)}{\partial \delta y_{\mathfrak{l}}} + \frac{1}{2} \sum_{\mathfrak{m},\mathfrak{n}} \frac{E_{\mathfrak{m},\mathfrak{n}}}{E} \left(\frac{\partial g_{\mathfrak{m}}(dy)}{\partial dy_{\mathfrak{k}}} \frac{\partial g_{\mathfrak{n}}(\delta y)}{\partial \delta y_{\mathfrak{l}}} - \frac{\partial g_{\mathfrak{m}}(\delta y)}{\partial \delta y_{\mathfrak{k}}} \frac{\partial g_{\mathfrak{n}}(dy)}{\partial dy_{\mathfrak{l}}} \right) \right] h_{\mathfrak{k},\mathfrak{a}} \\ = 0 . \end{cases}$$

Here, one can remark that the factor of the quantity $c_{\mathfrak{b},\mathfrak{l}}$ has the property that it will go to the opposite value when one switches the indices $\mathfrak{a}$ and $\mathfrak{b}$ with each other in it. Namely, equation (11.b) implies the relation:

$$\delta \left(\frac{\partial f_{\mathfrak{a}}(dx)}{\partial dx_{\mathfrak{b}}} + \frac{\partial f_{\mathfrak{b}}(dx)}{\partial dx_{\mathfrak{a}}} \right) = d \left(\frac{\partial f_{\mathfrak{a}}(\delta x)}{\partial \delta x_{\mathfrak{b}}} + \frac{\partial f_{\mathfrak{b}}(\delta x)}{\partial \delta x_{\mathfrak{a}}} \right),$$

which explains the statement immediately. The relations that are included in equation (30) can be combined in such a way that one introduces two new systems of differentials $du_{\mathfrak{a}}$ and $\delta u_{\mathfrak{a}}$ of the variables $x_{\mathfrak{a}}$, which might corresponds to the differentials $dv_{\mathfrak{k}}$ and $\delta v_{\mathfrak{k}}$ of the variables $y_{\mathfrak{k}}$, respectively. As a result of (1) and (1^*), one will have the following equations between them:

$$(31)\quad \begin{cases} du_{\mathfrak{a}} = c_{\mathfrak{a},1}\, dv_1 + c_{\mathfrak{a},2}\, dv_2 + \cdots + c_{\mathfrak{a},n}\, dv_n , \\ \delta u_{\mathfrak{a}} = c_{\mathfrak{a},1}\, \delta v_1 + c_{\mathfrak{a},2}\, \gamma v_2 + \cdots + c_{\mathfrak{a},n}\, \delta v_n , \\ dv_{\mathfrak{k}} = h_{\mathfrak{k},1}\, du_1 + h_{\mathfrak{k},2}\, du_2 + \cdots + h_{\mathfrak{k},n}\, du_n , \\ \delta v_{\mathfrak{k}} = h_{\mathfrak{k},1}\, \delta u_1 + h_{\mathfrak{k},2}\, \delta u_2 + \cdots + h_{\mathfrak{k},n}\, \delta u_n . \end{cases}$$

Now, if the left-hand side of (30) is multiplied by the factor $du_{\mathfrak{a}}\, \delta v_{\mathfrak{l}}$, and when the summation over the indices $\mathfrak{a}$ and $\mathfrak{l}$ is complete, a new equation will arise in which one has:

$$\sum_{\mathfrak{l}} du_{\mathfrak{a}}\, c_{\mathfrak{b},\mathfrak{l}}\, \delta v_{\mathfrak{l}} = du_{\mathfrak{a}}\, \delta u_{\mathfrak{b}} , \qquad \sum_{\mathfrak{a}} \delta v_{\mathfrak{l}}\, h_{\mathfrak{k},\mathfrak{a}}\, du_{\mathfrak{a}} = dv_{\mathfrak{k}}\, \delta v_{\mathfrak{l}} ,$$

as a result of (31). When I now define the two functions:

(32)
$$\begin{cases} \Psi = \sum_{\mathfrak{a},\mathfrak{b}} \left[\delta \frac{\partial f_{\mathfrak{a}}(dx)}{\partial\, dx_{\mathfrak{b}}} - d\, \frac{\partial f_{\mathfrak{a}}(\delta x)}{\partial\, \delta x_{\mathfrak{b}}} + \frac{1}{2} \sum_{\mathfrak{c},\mathfrak{d}} \frac{A_{\mathfrak{c},\mathfrak{d}}}{\Delta} \left(\frac{\partial f_{\mathfrak{c}}(dx)}{\partial\, dx_{\mathfrak{a}}} \frac{\partial f_{\mathfrak{d}}(\delta x)}{\partial\, \delta x_{\mathfrak{b}}} - \frac{\partial f_{\mathfrak{c}}(\delta x)}{\partial\, \delta x_{\mathfrak{a}}} \frac{\partial f_{\mathfrak{d}}(dx)}{\partial\, dx_{\mathfrak{b}}} \right) \right] du_{\mathfrak{a}}\, \delta u_{\mathfrak{b}}, \\ \Omega = \sum_{\mathfrak{k},\mathfrak{l}} \left[\delta \frac{\partial g_{\mathfrak{k}}(dy)}{\partial\, dy_{\mathfrak{l}}} - d\, \frac{\partial g_{\mathfrak{k}}(\delta y)}{\partial\, \delta y_{\mathfrak{l}}} + \frac{1}{2} \sum_{\mathfrak{m},\mathfrak{n}} \frac{E_{\mathfrak{m},\mathfrak{n}}}{E} \left(\frac{\partial g_{\mathfrak{m}}(dy)}{\partial\, dy_{\mathfrak{k}}} \frac{\partial f_{\mathfrak{n}}(\delta y)}{\partial\, \delta y_{\mathfrak{l}}} - \frac{\partial g_{\mathfrak{m}}(\delta y)}{\partial\, \delta y_{\mathfrak{k}}} \frac{\partial f_{\mathfrak{n}}(dy)}{\partial\, dy_{\mathfrak{l}}} \right) \right] dv_{\mathfrak{k}}\, \delta v_{\mathfrak{l}} \end{cases}$$

that equation will go to the following result:

(33.)
$$\Psi = \Omega,$$

due to the property that concerns the permutation of the indices $\mathfrak{a}$ and $\mathfrak{b}$. The function Ψ is a quadrilinear form in the four systems of differentials $du_{\mathfrak{a}}$, $\delta u_{\mathfrak{b}}$, $dx_{\mathfrak{g}}$, $\delta x_{\mathfrak{h}}$, and its coefficients are derived from the coefficients of the quadratic form $f(dx)$ with the help of single and double partial differentiations with respect to the variables $x_{\mathfrak{a}}$. The function Ω depends upon the four systems of differentials $dv_{\mathfrak{k}}$, $\delta v_{\mathfrak{l}}$, $dy_{\mathfrak{p}}$, $\delta y_{\mathfrak{q}}$, and the form $g(dy)$ in precisely the same way. Thus, equation (33) expresses the fact that the relationship of the quadrilinear form Ψ to the given quadratic form $f(dx)$ will not change when one introduces a new system of variables. However, this investigation was initially directed towards finding functions with that behavior.

4.

With the notations that were introduced in (16) and (17), the form Ψ, which shall now be focused on more closely, assumes the following form:

(34)
$$\Psi = \sum_{\mathfrak{a},\mathfrak{b},\mathfrak{g},\mathfrak{h}} \left[\frac{\partial f_{\mathfrak{a},\mathfrak{b},\mathfrak{g}}}{\partial x_{\mathfrak{h}}} - \frac{\partial f_{\mathfrak{a},\mathfrak{b},\mathfrak{h}}}{\partial x_{\mathfrak{g}}} + \frac{1}{2} \sum_{\mathfrak{c},\mathfrak{b}} (f_{\mathfrak{c},\mathfrak{a},\mathfrak{g}}\; f_{\mathfrak{b},\mathfrak{b},\mathfrak{h}} - f_{\mathfrak{c},\mathfrak{a},\mathfrak{h}}\; f_{\mathfrak{b},\mathfrak{b},\mathfrak{g}}) \right] du_{\mathfrak{a}}\; \delta u_{\mathfrak{b}}\; dx_{\mathfrak{g}}\; \delta x_{\mathfrak{h}} ,$$

and indeed one has:

(35)
$$\frac{\partial f_{\mathfrak{a},\mathfrak{b},\mathfrak{g}}}{\partial x_{\mathfrak{h}}} - \frac{\partial f_{\mathfrak{a},\mathfrak{b},\mathfrak{h}}}{\partial x_{\mathfrak{g}}} = \frac{\partial^2 a_{\mathfrak{a},\mathfrak{g}}}{\partial x_{\mathfrak{b}}\, \partial x_{\mathfrak{h}}} + \frac{\partial^2 a_{\mathfrak{b},\mathfrak{h}}}{\partial x_{\mathfrak{a}}\, \partial x_{\mathfrak{g}}} - \frac{\partial^2 a_{\mathfrak{a},\mathfrak{h}}}{\partial x_{\mathfrak{b}}\, \partial x_{\mathfrak{g}}} - \frac{\partial^2 a_{\mathfrak{b},\mathfrak{g}}}{\partial x_{\mathfrak{a}}\, \partial x_{\mathfrak{h}}} .$$

One sees from this that the coefficient of $du_{\mathfrak{a}}\; \delta u_{\mathfrak{b}}\; dx_{\mathfrak{g}}\; \delta x_{\mathfrak{h}}$ will be converted into the opposite value when one either switches $\mathfrak{a}$ and $\mathfrak{b}$ or $\mathfrak{g}$ and $\mathfrak{h}$, but it will go back to itself when one simultaneously switches $\mathfrak{a}$ with $\mathfrak{g}$ and $\mathfrak{b}$ with $\mathfrak{h}$. As a result, Ψ will be a bilinear form in the two systems of $n(n - 1) / 2$ combinations $du_{\mathfrak{a}}\; \delta u_{\mathfrak{b}} - \delta u_{\mathfrak{a}}\; du_{\mathfrak{b}}$ and $dx_{\mathfrak{g}}\; \delta x_{\mathfrak{h}} - \delta x_{\mathfrak{g}}\; dx_{\mathfrak{h}}$ that behaves symmetrically with respect to those two systems.

When the number of variables is two, Ψ will take on the following form:

(36)
$$\Psi = \left[\frac{\partial f_{1,1,2}}{\partial x_2} - \frac{\partial f_{1,2,2}}{\partial x_1} + \frac{1}{2} \sum_{\mathfrak{c},\mathfrak{b}} \frac{A_{\mathfrak{c},\mathfrak{b}}}{\Delta} (f_{\mathfrak{c},1,1}\; f_{\mathfrak{b},2,2} - f_{\mathfrak{c},1,2}\; f_{\mathfrak{b},2,1}) \right] (du_1\, \delta u_2 - \delta u_1\, du_2)(dx_1\, \delta x_2 - \delta x_1\, dx_2),$$

in which one has the equations:

$$A_{1,1} = a_{2,2}\,, \qquad A_{1,2} = -\,a_{1,2}\,, \quad A_{2,2} = a_{1,1}\,, \qquad \Delta = a_{1,1}\,a_{2,2} - a_{1,2}^2\,.$$

The consideration of the *Gauss*ian expression for the curvature k that belongs to the form $f(dx)$ that was presented above will then exhibit the following relationship between the curvature k and the form Ψ:

(37) $$\Psi = -\,2k\,\Delta\,(du_1\,\delta u_2 - \delta u_1\,du_2)\,(dx_1\,\delta x_2 - \delta x_1\,dx_2)\,.$$

For the curvature m that is associated with form $g(dy)$, one must have the equation:

(37*) $$\Omega = -\,2m\,E\,(dv_1\,\delta v_2 - \delta v_1\,dv_2)\,(dy_1\,\delta y_2 - \delta y_1\,dy_2)\,,$$

on the same grounds. It will then follow from the equation $\Psi = \Omega$ and the known equation:

$$\Delta\,(du_1\,\delta u_2 - \delta u_1\,du_2)\,(dx_1\,\delta x_2 - \delta x_1\,dx_2) = E\,(dv_1\,\delta v_2 - \delta v_1\,dv_2)\,(dy_1\,\delta y_2 - \delta y_1\,dy_2)$$

that one has the property of the curvature that was mentioned before and is expressed in the equation:

$$k = m\,.$$

There is some interest in summarizing all of the coefficients in the form Ψ for which the group of four indices $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{g}$, $\mathfrak{h}$ remains the same, except for their order. It is initially clear that no more than two of those four indices can be equal to each other if the associated coefficient is to not vanish. Therefore, either only two pairs of equal indices can be present or only one pair, or all four of them must be different. If the form Ψ is regarded as bilinear in the given way then that would imply only one coefficient in the first case of the group $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{a}$, $\mathfrak{b}$, two coefficients in the second case of the group $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{a}$, $\mathfrak{c}$, which are equal to each other on the cited symmetry grounds, and six coefficients in the third case of the group $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{g}$, $\mathfrak{h}$, which correspond to the six arrangements:

$$\begin{array}{ll} \mathfrak{a},\ \mathfrak{b},\ \mathfrak{g},\ \mathfrak{h}; & \mathfrak{g},\ \mathfrak{h},\ \mathfrak{a},\ \mathfrak{b}, \\ \mathfrak{a},\ \mathfrak{b},\ \mathfrak{h},\ \mathfrak{g}; & \mathfrak{h},\ \mathfrak{b},\ \mathfrak{a},\ \mathfrak{g}, \\ \mathfrak{a},\ \mathfrak{h},\ \mathfrak{b},\ \mathfrak{g}; & \mathfrak{b},\ \mathfrak{g},\ \mathfrak{a},\ \mathfrak{h}, \end{array}$$

and every two of which are equal to each other by symmetry. The number n of variables is significant in the exhibition of those three categories of groups. The form Ψ, which is thought to be bilinear, has only one term for $n = 2$, and in fact it belongs to the first category. For $n = 3$, it has six different terms, three of which belong to the first category and three of which belong to the second one. Finally, for $n \geq 4$, the number of different terms is $n\,(n-1)/2\,[n\,(n-1)/2 + 1]/2$,

$n\,(n-1)/2$ of which belong to the first category, $n\,(n-1)(n-2)/2$ belong to the second, and $n\,(n-1)(n-2)(n-3)/8$ belong to the third. There exists the relation between the terms in the last category that the sum of the three coefficients that correspond to the three arrangements:

$$\mathfrak{a},\ \mathfrak{b},\ \mathfrak{g},\ \mathfrak{h},$$
$$\mathfrak{a},\ \mathfrak{g},\ \mathfrak{h},\ \mathfrak{b},$$
$$\mathfrak{a},\ \mathfrak{h},\ \mathfrak{b},\ \mathfrak{g}$$

will always have the value zero, because one has the equations:

$$\frac{\partial f_{\mathfrak{a},\mathfrak{b},\mathfrak{g}}}{\partial x_{\mathfrak{h}}} - \frac{\partial f_{\mathfrak{a},\mathfrak{b},\mathfrak{h}}}{\partial x_{\mathfrak{g}}} + \frac{\partial f_{\mathfrak{a},\mathfrak{g},\mathfrak{h}}}{\partial x_{\mathfrak{b}}} - \frac{\partial f_{\mathfrak{a},\mathfrak{g},\mathfrak{b}}}{\partial x_{\mathfrak{h}}} + \frac{\partial f_{\mathfrak{a},\mathfrak{h},\mathfrak{b}}}{\partial x_{\mathfrak{g}}} - \frac{\partial f_{\mathfrak{a},\mathfrak{h},\mathfrak{g}}}{\partial x_{\mathfrak{b}}} = 0$$

and

$$\begin{aligned} & f_{\mathfrak{c},\mathfrak{a},\mathfrak{g}}\, f_{\mathfrak{d},\mathfrak{b},\mathfrak{h}} - f_{\mathfrak{c},\mathfrak{a},\mathfrak{h}}\, f_{\mathfrak{d},\mathfrak{b},\mathfrak{g}} \\ + & f_{\mathfrak{c},\mathfrak{a},\mathfrak{h}}\, f_{\mathfrak{d},\mathfrak{g},\mathfrak{b}} - f_{\mathfrak{c},\mathfrak{a},\mathfrak{h}}\, f_{\mathfrak{d},\mathfrak{g},\mathfrak{h}} \\ + & f_{\mathfrak{c},\mathfrak{a},\mathfrak{h}}\, f_{\mathfrak{d},\mathfrak{h},\mathfrak{g}} - f_{\mathfrak{c},\mathfrak{a},\mathfrak{g}}\, f_{\mathfrak{d},\mathfrak{h},\mathfrak{b}} = 0\ . \end{aligned}$$

On this occasion, one can observe that when the substitution:

$$du_{\mathfrak{a}} = dx_{\mathfrak{a}}\ , \qquad \delta u_{\mathfrak{b}} = \delta x_{\mathfrak{b}}\ ,$$

is introduced into the form Ψ, entirely the same relations will exist between the products $(dx_{\mathfrak{a}}\,\delta x_{\mathfrak{b}} - \delta x_{\mathfrak{a}}\,dx_{\mathfrak{b}})(dx_{\mathfrak{g}}\,\delta x_{\mathfrak{h}} - \delta x_{\mathfrak{g}}\,dx_{\mathfrak{h}})$ that are verified by the same associated coefficients of the form Ψ.

5.

The theory of forms in *n* differentials belongs, in part, to algebra, and in part, to the infinitesimal calculus. Up to now, the discussion of those forms has required no other operations than rational algebraic operations and the operation of differentiation. Other properties of those forms will come to light when one bases them upon the consideration of integral calculus.

The *n* variables $x_{\mathfrak{a}}$ might depend upon a new independent variable *t*, so the *n* variables $y_{\mathfrak{k}}$ will also depend upon *t*. Differentiation with respect to that variable *t* shall be denoted in the manner of *Lagrange*. I shall also establish that when other quantities $\xi_{\mathfrak{a}}$ are substituted for the differentials $dx_{\mathfrak{a}}$, respectively, in a form $f\,(dx)$, the result will be denoted by $f\,(\xi)$. Furthermore, when the special values $x_{\mathfrak{a}}(0)$ are substituted for the variables $x_{\mathfrak{a}}$, respectively, in the coefficients of that form, the result shall be called $(f\,(\xi))_0$, and the corresponding statement shall be true for the form $g\,(dy)$. Let the degree *p* again be equal to or greater than two. Now, The essence of the form $f\,(dx)$ is most closely connected with the problem of the calculus of variations that consists of determining the variables $x_\mathfrak{a}$ as functions of t in such a way that the first variation of the integral $\int f(x')dt$ vanishes, when taken between fixed limits. It is known that this problem leads to the integration of those systems of differential equations that arise from the left-hand side of equation (10) above when one divides it by the quantity $(dt)^p$ and sets the factor of each variation $dx_\mathfrak{a}$ equal to zero. This system of isoperimetric differential equations that is associated with the form $f(dx)$ reads as follows:

$$\sum_\mathfrak{b} \frac{\partial^2 f(x')}{\partial x'_\mathfrak{a}\,\partial x'_\mathfrak{b}}\, x''_\mathfrak{b} + f_\mathfrak{a}(x') = 0. \tag{38}$$

Upon introducing the variables $y_\mathfrak{k}$ in place of the variables $x_\mathfrak{a}$, the cited problem will be converted into the analogous problem that belongs to the form $g(dy)$, and that will imply the following system of differential equations for determining the variables $y_\mathfrak{k}$ as functions of t :

$$\sum_\mathfrak{l} \frac{\partial^2 g(y')}{\partial y'_\mathfrak{k}\,\partial y'_\mathfrak{l}}\, y''_\mathfrak{l} + g_\mathfrak{k}(y') = 0. \tag{38*}$$

The fact that this system emerges from the system (38) upon substituting the variables $y_\mathfrak{k}$ can be proved directly by means of equation (10).

It has already been remarked on another occasion that as long as the determinant that is denoted by Δ does not vanish identically in (3.a), the n quantities $x''_\mathfrak{b}$ can be represented as functions of the quantities $x_\mathfrak{a}$ and $x'_\mathfrak{a}$ using the system (38), and one can conclude from this that under certain continuity conditions that are developed at each location, a complete integration of that system of differential equations can be carried out in which the variables $x_\mathfrak{a}$ are determined by the demand that the equations:

$$x_\mathfrak{a} = x_\mathfrak{a}(0)\,, \qquad x'_\mathfrak{a} = x'_\mathfrak{a}(0)$$

must be fulfilled for a certain value $t = t_0$ (*). The fixed values $x_\mathfrak{a}(0)$ and $x'_\mathfrak{a}(0)$ must be chosen in such a way that upon substituting them for $x_\mathfrak{a}$ and $x'_\mathfrak{a}$, the determinant will not be equal to zero and the aforementioned expressions for the quantities $x''_\mathfrak{b}$ will remain entirely finite under the corresponding substitution. The values of the quantities $x_\mathfrak{a}$ that are obtained by integration will then be finite and continuous for a certain interval of the variable t that extends from the value $t = t_0$. Such an integration of the system (38) will be currently assumed to be given. Since entirely the

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(*) The integration of the system (38) that is appropriate to transforming the second variation of the integral $\int f(x')dt$ according to the method that is given in volume 65, pp. 26 of this journal has the values of the quantities $x_\mathfrak{a}$ and $\partial f(x') / \partial x'_\mathfrak{a} = F_{\mathfrak{a},1}$ at $t = t_0$ for its integration constants. When one introduces $x_\mathfrak{a}$ and $F_{\mathfrak{a},1}$ as dependent variables in the system (38), those constants will become initial values of the variables, and from *Jacobi*, the system will assume the following form:

$$\frac{dx_\mathfrak{a}}{dt} = \frac{\partial H}{\partial F_{\mathfrak{a},1}}, \quad \frac{dF_{\mathfrak{a},1}}{dt} = -\frac{\partial H}{\partial x_\mathfrak{a}}.$$

same considerations are true for the system (38[*]), it will be assumed that the quantities $y_\mathfrak{k}$ are determined by that and the demand that the equations:

$$y_\mathfrak{k} = y_\mathfrak{k}(0), \qquad y'_\mathfrak{k} = y'_\mathfrak{k}(0)$$

must be satisfied for the value $t = t_0$, where the constants $y_\mathfrak{k}(0)$ and $y'_\mathfrak{k}(0)$ are coupled with the constants $x_\mathfrak{a}(0)$ and $x'_\mathfrak{a}(0)$ by equations that correspond to the relationship between the variables $y_\mathfrak{k}$ and $x_\mathfrak{a}$. If one considers the quantities $c_{\mathfrak{a},\mathfrak{k}}$ that enter into equation (1) to be functions of the variables $y_\mathfrak{k}$ and suggests the result of the substitution $y_\mathfrak{k} = y_\mathfrak{k}(0)$ by $c_{\mathfrak{a},\mathfrak{k}}(0)$ then the equations between the constants $x'_\mathfrak{a}(0)$ and $y'_\mathfrak{k}(0)$ will be these:

$$x'_\mathfrak{a}(0) = c_{\mathfrak{a},1}(0)\, y'_1(0) + c_{\mathfrak{a},2}(0)\, y'_2(0) + \cdots + c_{\mathfrak{a},n}(0)\, y'_n(0). \tag{39}$$

A peculiarity of the integration value $x_\mathfrak{a}$ that we speak of consists of the fact that the variable t does not enter into it except in certain couplings with the constants $x'_\mathfrak{a}(0)$. In order to see that, one singles out one of the variables $x_\mathfrak{a}$ – say, x_1 – and deduces $n - 1$ expressions from equations (38) that are equal to:

$$\frac{d^2 x_\mathfrak{a}}{dx_1^2} = \frac{\dfrac{d^2 x_\mathfrak{a}}{dt^2}\dfrac{dx_1}{dt} - \dfrac{d^2 x_1}{dt^2}\dfrac{dx_\mathfrak{a}}{dt}}{\left(\dfrac{dx_1}{dt}\right)^2}$$

and contain only the quantities $x_\mathfrak{c}$ and $dx_\mathfrak{c}$, but no longer contain the differential dt. One thinks of the system of $n - 1$ second-order differential equations in the variables $x_2, x_3, \ldots, x_n$, and x_1 that is defined in that way as being integrated in such a way that for $x_1 = x_1(0)$, the equations:

$$x_2 = x_2(0), \qquad \ldots, \qquad x_n = x_n(0),$$

$$\frac{dx_2}{dx_1} = \left(\frac{dx_2}{dx_1}\right)_0, \quad \ldots, \quad \frac{dx_n}{dx_1} = \left(\frac{dx_n}{dx_1}\right)_0$$

will be fulfilled. Now, the system (38) yields the known integral:

$$f(x') = [f(x'(0))]_0 . \tag{40}$$

If one replaces the $x_2, x_3, \ldots, x_n$ in the left-hand side of this equation with their expressions that one finds for them in terms of x_1 and the $2n - 2$ constants $x_2(0), \ldots, x_n(0)$; $\left(\frac{dx_2}{dx_1}\right)_0, \ldots, \left(\frac{dx_n}{dx_1}\right)_0$ then due to the homogeneity of $f(x')$, one will have:

$$f(x') = f^{(1)}\left(x_1, x_1(0), x_2(0), \ldots, x_n(0), \left(\frac{dx_2}{dx_1}\right)_0, \ldots, \left(\frac{dx_n}{dx_1}\right)_0\right)\left(\frac{dx_1}{dt}\right)^p, \tag{41}$$

or, more briefly:

$$f(x') = f^{(1)}\left(\frac{dx_1}{dt}\right)^p .$$

Equation (40) then implies the relation:

$$\sqrt[p]{f^{(1)}}\,dt = \sqrt[p]{(f^{(1)}(x'(0))_0}\,dt, \tag{42}$$

and when one integrates the equation in x_1 and t over x_1 :

$$\int_{x_1(0)}^{x_1} \sqrt[p]{f^{(1)}}\,dt = \sqrt[p]{(f^{(1)}(x'(0))_0}\,(t - t_0) . \tag{43}$$

When one ponders the fact that the equations:

$$\left(\frac{dx_{\mathfrak{b}}}{dx_1}\right)_0 = \frac{(t - t_0)\,x'_{\mathfrak{b}}(0)}{(t - t_0)\,x'_1(0)}$$

are true for the expression $\left(\frac{dx_{\mathfrak{b}}}{dx_1}\right)_0$, and that due to the homogeneity of $f(x')$, one has the equation:

$$(f(x'(0)))_0\,(t - t0)^p = (f((t - t_0)\,x'(0)))_0 ,$$

the result will emerge from equation (43) that the value x_1 is a function of the n quantities $x_{\mathfrak{b}}(0)$ and the n quantities $(t - t_0)\,x'_{\mathfrak{b}}(0)$, and the variable t is not involved in any way. Equality is obviously true for any other value $x_{\mathfrak{a}}$, and since the system of differential equations (38*) has the same property as the system (38), the integration values $y_{\mathfrak{k}}$ in question are also purely functions of the n quantities $y_{\mathfrak{l}}(0)$ and the n quantities $(t - t_0)\,y'_{\mathfrak{l}}(0)$.

I shall consider the n quantities $x_{\mathfrak{b}}(0)$ in the representation of $x_{\mathfrak{a}}$ to be fixed and the n quantities $(t - t_0)\,x'_{\mathfrak{b}}(0)$ to be variable, and likewise consider the n quantities $y_{\mathfrak{l}}(0)$ in the representation of $y_{\mathfrak{k}}$ to be fixed and the n quantities $(t - t_0)\,y'_{\mathfrak{l}}(0)$ to be variable. I shall further assume that the $x_{\mathfrak{a}}$ can be partially differentiated with respect to the new variables $(t - t_0)\,x'_{\mathfrak{b}}(0)$ and that the $y_{\mathfrak{k}}$ can be partially differentiated with respect to the new variables $(t - t_0)\, y'_{\mathfrak{l}}(0)$ (*). From that standpoint, the introduction of the values of $x_{\mathfrak{a}}$ into the form $f(\delta x)$ and the introduction of the values $y_{\mathfrak{k}}$ into the form $g(\delta y)$ will yield the two transformations:

(44)
$$\begin{cases} f(\delta x) = \varphi(\delta\overline{(t-t_0)\, x'(0)}), \\ g(\delta y) = \chi(\delta\overline{(t-t_0)\, y'(0)}). \end{cases}$$

However, since (39) implies that the variables $(t - t_0)\, x'_{\mathfrak{b}}(0)$ and $(t - t_0)\, y'_{\mathfrak{l}}(0)$ are coupled to each other by equations that do not contain those variables themselves, corresponding equations:

(45)
$$\delta(\overline{(t-t_0)\; x'_{\mathfrak{a}}(0)}) = \sum_{\mathfrak{k}} c_{\mathfrak{a},\mathfrak{l}}\; \delta(\overline{(t-t_0)\; y'_{\mathfrak{k}}(0)})$$

will exist between the variations of those two systems of variables, and when one applies equations (39) and (45), the form $\varphi(\delta\,\overline{(t-t_0)x'(0)})$ must go to the form $\chi(\delta\,\overline{(t-t_0)y'(0)})$; that is, the equation:

(46)
$$\varphi(\delta\,\overline{(t-t_0)x'(0)})_0 = \chi(\delta\,\overline{(t-t_0)y'(0)}\,)_0$$

must be fulfilled.

It is obvious that the equation:

$$f(x') = g(y')$$

will remain correct when one sets $x_{\mathfrak{a}} = x_{\mathfrak{a}}(0)$ and $x'_{\mathfrak{a}} = x'_{\mathfrak{a}}(0)$ on the left-hand side and sets $y_{\mathfrak{k}} = y_{\mathfrak{k}}(0)$ and $y'_{\mathfrak{k}} = y'_{\mathfrak{k}}(0)$ and multiplies both sides by the factor $(t - t_0)^p$. The equation then arises:

(47)
$$(f((t-t_0)\, x'(0)))_0 = (g((t-t_0)\, y'(0)))_0\,.$$

The left-hand side of that is a form of degree p in the n variables $(t - t_0)\, x'_{\mathfrak{a}}(0)$ whose coefficients depend upon only the quantities $x_{\mathfrak{a}}(0)$, while the right-hand side is a form in the n variables $(t - t_0)\, y'_{\mathfrak{k}}(0)$ whose coefficients depend upon only the quantities $y_{\mathfrak{k}}(0)$. Now, since the equations:

(39.a)
$$(t-t_0)\, x'_{\mathfrak{a}}(0) = \sum_{\mathfrak{k}} c_{\mathfrak{a},\mathfrak{k}}(0)\, \delta(t-t_0)\, y'_{\mathfrak{k}}(0))$$

exist between the two systems of variables in question, it would be justified to replace the variables considered with their differentials in equation (47). Hence, the equation:

(48)
$$(f(\delta\,\overline{(t-t_0)x'(0)}))_0 = (g(\delta\,\overline{(t-t_0)y'(0)}\,)_0$$

(*) With the help of the cited article, one can exhibit continuity conditions for the functions that appear in the system (38), and therefore continuity conditions for the coefficients in the form $f(\delta x)$ and its differential quotients, into which that assumption must enter.

will be created by the substitution (39.a) and (45), as well as equation (46).

6.

The considerations that were presented in the previous section include the means for finding the general answer to the question:

When it is possible to transform a form f (dx) into a form g (dy) whose coefficients are constant?

If one sets the coefficients of the form g (dy) into which f (dx) goes upon introducing the variables $y_{\mathfrak{k}}$ equal to constants then that will imply the following consequences: Due to equations (17), all of the quantities $g_{\mathfrak{k},\mathfrak{p},\mathfrak{q}}$ will vanish, and for that reason, the forms $g_{\mathfrak{k}}$ (dy), as well. Thus, the system (38*) will assume the form:

$$\sum_{\mathfrak{k}} \frac{\partial^2 g(y')}{\partial y'_{\mathfrak{k}}\, \partial y'_{\mathfrak{l}}}\, y''_{\mathfrak{l}} = 0. \tag{49}$$

Now, from the foregoing, the values $y'_{\mathfrak{k}}(0)$ are thought of as being chosen in such a way that the determinant of the elements $\frac{\partial^2 g(y')}{\partial y'_{\mathfrak{k}}\, \partial y'_{\mathfrak{l}}}$ does not vanish under the substitution $y'_{\mathfrak{k}} = y'_{\mathfrak{k}}(0)$. That is the basis for the conclusion that the system (49) cannot be fulfilled by anything but the n equations:

$$y''_{\mathfrak{k}} = 0 . \tag{50}$$

However, integrating that will yield the result that:

$$y_{\mathfrak{k}} = y_{\mathfrak{k}}(0) + (t - t_0)\, y'_{\mathfrak{k}}(0) . \tag{51}$$

Now, as long as the n quantities $y_{\mathfrak{k}}$ can be considered to depend upon the n new variables $(t - t_0)$ $y'_{\mathfrak{k}}(0)$, while the n quantities $y_{\mathfrak{k}}(0)$ are fixed, equation (51) will imply the n relations between the differentials in question:

$$\delta y_{\mathfrak{k}} = \delta\overline{(t - t_0)\, y'_{\mathfrak{k}}(0)}. \tag{52}$$

Therefore, when the new variables are introduced into the form g (δy), whose coefficients do not include the quantities $y_{\mathfrak{k}}$, by assumption, that will give:

$$g(\delta y) = g\left(\delta\overline{(t - t_0)\, y'(0)}\right), \tag{53}$$

and with the notation that was used in (44), one will have:

(54) $$\chi\left(\delta\overline{(t-t_0)\,y'(0)}\right) = g\left(\delta\overline{(t-t_0)\,y'(0)}\right).$$

It is likewise clear that the substitution $y_{\mathfrak{k}} = y_{\mathfrak{k}}(0)$ will not alter the form $g(\delta y)$, or in symbols, that one will have the equation:

(55) $$\left(g\left(\delta\overline{(t-t_0)\,y'(0)}\right)\right)_0 = g\left(\delta\overline{(t-t_0)\,y'(0)}\right).$$

On those grounds, and under the current assumptions, equations (46) and (48) of the previous section will be converted into the following ones:

(46.a) $$\varphi\left(\delta\overline{(t-t_0)\,x'(0)}\right) = g\left(\delta\overline{(t-t_0)\,y'(0)}\right),$$

(48.a) $$\left(f\left(\delta\overline{(t-t_0)\,x'(0)}\right)\right)_0 = g\left(\delta\overline{(t-t_0)\,y'(0)}\right),$$

respectively. However, combining them will produce the identity:

(56) $$\varphi\left(\delta\overline{(t-t_0)\,x'(0)}\right) = \left(f\left(\delta\overline{(t-t_0)\,x'(0)}\right)\right)_0 .$$

When one admits that equation, the question that was posed can be answered in the following way:

Let the system of isoperimetric differential equations that is associated with the form $f(dx)$ *be integrated in such a way that the variables* $x_{\mathfrak{a}}$ *are represented in terms of the n quantities* $x_{\mathfrak{b}}(0)$ *and the n quantities* $(t-t_0)\,x'_{\mathfrak{b}}(0)$. *One regards the variables* $x_{\mathfrak{a}}$ *as functions of the n new variables* $(t-t_0)\,x'_{\mathfrak{b}}(0)$, *and the n quantities* $x_{\mathfrak{b}}(0)$ *as constant and transforms the form* $f(dx)$ *by introducing those new variables into the form* $\varphi(\delta\,\overline{(t-t_0)x'(0)}$. *As long as it is possible to transform the form* $f(dx)$ *into another form with constant coefficients, such a transformation will, in fact, be achieved by the given process, and indeed the form* $\varphi(\delta\,\overline{(t-t_0)x'(0)}$ *will emerge from the form* $f(dx)$ *when one replaces the differentials* $\delta x_{\mathfrak{a}}$ *with the differentials* $\delta\,\overline{(t-t_0)x'(0)}$ *and substitutes the constants* $x_{\mathfrak{b}}(0)$ *for the variables* $x_{\mathfrak{b}}$ *in the coefficients.*

One can give the criterion that was just described yet another form. As long as the form $f(\delta x)$ can be transformed into a form with constant coefficients, the coefficients of the form $\varphi(\delta\,\overline{(t-t_0)x'(0)}$, which are expressed in terms of the quantities $x_{\mathfrak{a}}(0)$ and $(t-t_0)\,x'_{\mathfrak{a}}(0)$ in each case, will be independent of the quantities $(t-t_0)\,x'_{\mathfrak{a}}(0)$, and for that reason, they will be independent of the quantity t. Now, one can also show the converse, namely, that when the coefficients of the form $\varphi(\delta\,\overline{(t-t_0)x'(0)}$ are independent of t, equation (56) will be true, and therefore the form $f(\delta x)$ can be transformed into a form with constant coefficients. In order to convince oneself of that fact, one notes that the construction of the form $\varphi(\delta\,\overline{(t-t_0)x'(0)}$ from the form $f(\delta x)$ will come about by first substituting the values for $x_{\mathfrak{a}}$ that are obtained by integrating the system (38) in the coefficients of the form $f(\delta x)$ and then replace the differentials $\delta x_{\mathfrak{a}}$ with the following expressions:

(57) $$\delta x_{\mathfrak{a}} = \frac{\partial x_{\mathfrak{a}}}{(t-t_0)\,\partial x_1'(0)}\,\delta\overline{(t-t_0)\,x_1'(0)} + \cdots + \frac{\partial x_{\mathfrak{a}}}{(t-t_0)\,\partial x_n'(0)}\,\delta\overline{(t-t_0)\,x_n'(0)}.$$

As long as the coefficients of the form that is produced by those operations are independent of t, one can determine its value for the variable t when it is known for $t = t_0$ because destroying the continuity is always excluded from those considerations. That explains the fact that the values of $x_{\mathfrak{a}}$ for $t = t_0$ that are introduced into the coefficients of the form $f(\delta x)$ are precisely the values $x_{\mathfrak{a}}(0)$. In the partial differential quotients $\frac{\partial x_{\mathfrak{a}}}{(t-t_0)\,\partial x'_{\mathfrak{b}}}$, the quantities $x_{\mathfrak{a}}$ are regarded as depending upon the n quantities $x_{\mathfrak{b}}(0)$, the n quantities $x'_{\mathfrak{b}}(0)$, and the quantity t. Since the quantities $x_{\mathfrak{a}} = x_{\mathfrak{a}}(0)$ for $t = t_0$ now, $\frac{\partial x_{\mathfrak{a}}}{\partial x'_{\mathfrak{b}}(0)}$ will be equal to zero for $t = t_0$, and therefore:

$$\lim \frac{\partial x'_{\mathfrak{a}}}{(t-t_0)\,\partial x'_{\mathfrak{b}}(0)} = \left(\frac{d}{dt}\frac{\partial x_{\mathfrak{a}}}{\partial x'_{\mathfrak{b}}(0)}\right)_{t=t_0} .$$

However, since differentiations with respect to $x'_{\mathfrak{b}}(0)$ and t are independent of each other, one will have:

$$\frac{d}{dt}\frac{\partial x_{\mathfrak{a}}}{\partial x'_{\mathfrak{b}}(0)} = \frac{\partial}{\partial x'_{\mathfrak{b}}(0)}\left(\frac{dx_{\mathfrak{a}}}{dt}\right) = \frac{\partial x'_{\mathfrak{a}}}{\partial x'_{\mathfrak{b}}(0)} .$$

The quantities $x'_{\mathfrak{a}}$ go to $x'_{\mathfrak{a}}(0)$ for $t = t_0$, so the expression $\frac{\partial x'_{\mathfrak{a}}}{\partial x'_{\mathfrak{b}}(0)}$ will be equal to zero for $t = t_0$ when $\mathfrak{b}$ is different from $\mathfrak{a}$ and equal to unity when $\mathfrak{b} = \mathfrak{a}$. Therefore, $\lim \frac{\partial x_{\mathfrak{a}}}{(t-t_0)\,\partial x'_{\mathfrak{b}}(0)}$ is also equal to zero in the former case and unity in the latter. Hence, the system (57) has the property that it will go to the system:

(58) $$\delta x_{\mathfrak{a}} = \delta\overline{(t-t_0)\,\partial x'_{\mathfrak{a}}(0)}$$

under the assumption that $t = t_0$. If one combines everything up to now then that will explain the fact that the form $f(\delta x)$ will be converted by the given operations into a form that coincides with the form $(f(\delta\,\overline{(t-t_0)x'_{\mathfrak{a}}(0)}))_0$ for $t = t_0$, and since its coefficients do not change with t, by hypothesis, that transformation will also exist for variable t, as was asserted.

When we denote differentiation with respect to the variable t, while $x_{\mathfrak{a}}(0)$ and $x'_{\mathfrak{a}}(0)$ are regarded as constant, as we have up to now, and likewise establishes that this differentiation should not affect the quantities $\delta\,\overline{(t-t_0)x'_{\mathfrak{a}}(0)}$, we can express the result that was obtained as follows:

The form f (dx) can or cannot be transformed into a form with constant coefficients according to whether the equation:

(59)
$$\frac{d\varphi(\delta\overline{(t-t_0)\,\partial x'_{\mathfrak{a}}(0)})}{dt} = 0$$

is or is not fulfilled, respectively.

7.

When the form *f* (*dx*) has degree two and is denoted as it was in equation (14) of section **1**, the form $\varphi\,(\delta\,\overline{(t-t_0)x'_{\mathfrak{a}}(0)})$ will have the following form:

(60)
$$\begin{cases} \varphi(\delta\overline{(t-t_0)\,\partial x'(0)}) = \frac{1}{2}(t-t_0)^{-2}\sum_{\mathfrak{k},\mathfrak{l}}\varphi_{\mathfrak{k},\mathfrak{l}}\,\delta\overline{(t-t_0)\,\partial x'_{\mathfrak{k}}(0)}\,\delta\overline{(t-t_0)\,\partial x'_{\mathfrak{l}}(0)}\,, \\ \varphi_{\mathfrak{k},\mathfrak{l}} = \sum_{\mathfrak{a},\mathfrak{b}} a_{\mathfrak{a},\mathfrak{b}}\,\frac{\partial x_{\mathfrak{a}}}{\partial x'_{\mathfrak{k}}(0)}\,\frac{\partial x_{\mathfrak{a}}}{\partial x'_{\mathfrak{k}}(0)}\,, \end{cases}$$

and the criterion that was presented will consist of satisfying the *n* (*n* + 1) / 2 equations:

(61)
$$\frac{d((t-t_0)^{-2}\varphi_{\mathfrak{k},\mathfrak{l}})}{dt} = 0\,.$$
,

Based upon the properties of the quadrilinear form Ψ that was defined in (32), this criterion, which depends upon integrating the system (38), can be converted into a direct criterion, and therein lies the characteristic significance of the form Ψ in regard to the quadratic form *f* (*dx*). The direct criterion that we spoke of is contained in the theorem:

The necessary and sufficient condition for the quadratic form f (*dx*) *to be convertible into a form with constant coefficients consists of the identical vanishing of the quadrilinear form* Ψ *that belongs to the form f* (*dx*).

It was already noted before that when the form *f* (*dx*) goes to the form *g* (*dy*) upon introducing the variables $y_{\mathfrak{k}}$, and when the latter has constant coefficients, the *n* forms $g_{\mathfrak{k}}$ (*dy*) will vanish identically. The identical vanishing of the form Ω that was defined in (32) will follow from that, and therefore the identical vanishing of the form Ψ, as well. Therefore, the criterion that was posed is shown to be necessary. However, in order to also show that the criterion is also sufficient, it will be shown that the identical vanishing of the form Ψ implies that the equations that are contained in (61) will be fulfilled.

Completing that proof will depend upon knowing the connection between the quantities $\varphi_{\mathfrak{k},\mathfrak{l}}$ and the form Ψ. In order to gain that knowledge, I shall turn to the relations that were the key to finding the form Ψ, namely, the relations (23), and see what follows from them when the form *g* (*dy*) has constant coefficients. The aforementioned vanishing of the *n* forms $g_{\mathfrak{k}}$ (*dy*) will also effect a vanishing of the right-hand side of (23) then, and the left-hand side must likewise be zero, independently of the *n* differentials $dx_{\mathfrak{c}}$. Now, as long as one regards the quantities $c_{\mathfrak{b},\mathfrak{l}}$ as functions of the *n* variables $x_{\mathfrak{c}}$, the aforementioned relation:

$$\sum_{\mathfrak{b}} a_{\mathfrak{a},\mathfrak{b}}\, dc_{\mathfrak{b},\mathfrak{l}} + \tfrac{1}{2}\sum_{\mathfrak{b}} \frac{\partial f_{\mathfrak{a}}(dx)}{\partial x_{\mathfrak{b}}} c_{\mathfrak{b},\mathfrak{l}} = 0 \tag{23.a}$$

will be the symbolic expression of a system of partial differential equations that the quantities $c_{\mathfrak{b},\mathfrak{l}}$ must necessarily satisfy. Moreover, it can be proved that when the form $f(dx)$ can be transformed into a form with constant coefficients, such a transformation can be actually performed by the process that was developed in the preceding section. The quantities $(t - t_0)\, x'_{\mathfrak{l}}(0)$ are the new variables by whose substitution the transformed form will take on constant coefficients. If one replaces the variables $y_{\mathfrak{k}}$ with them in equation (1) then, from (57), the quantities $c_{\mathfrak{b},\mathfrak{l}}$ in question will take the values:

$$c_{\mathfrak{b},\mathfrak{l}} = (t - t_0)^{-1}\frac{\partial x_{\mathfrak{a}}}{\partial x'_{\mathfrak{l}}(0)} . \tag{62}$$

Therefore, provided that those values are expressed in terms of the original variables $x_{\mathfrak{a}}$, they must fulfill the system of partial differential equations that was indicated in (23.a).

That system will still be fulfilled when the quantities $x_{\mathfrak{a}}$ are taken to be functions of *t* that integrate the system (38), by assumption, and when the differentials $dx_{\mathfrak{c}}$ are replaced with the differential quotients $dx_{\mathfrak{c}} / dt$, which is consistent with the same assumption. Since the quantities $x_{\mathfrak{a}}(0)$ are always constant, the differential quotients of the quantities $c_{\mathfrak{b},\mathfrak{l}} = (t - t_0)^{-1}\dfrac{\partial x_{\mathfrak{a}}}{\partial x'_{\mathfrak{l}}(0)}$ with respect to *t* will take on the same values when one imagines that they are expressed in terms of the *n* variables $x_{\mathfrak{a}}$, and when one expresses them in terms of the *n* variables $(t - t_0)^{-1} x'_{\mathfrak{a}}(0)$. Thus, as long as one ascribes the cited meaning to the quantities $x_{\mathfrak{a}}$ and $dx_{\mathfrak{a}} / dt$, when the relation (23.a) is divided by *dt*, it will represent a system of *n* ordinary first-order differential equations that is satisfied by the quantities:

$$c_{\mathfrak{b},\mathfrak{l}} = (t - t_0)^{-1}\frac{\partial x_{\mathfrak{a}}}{\partial x'_{\mathfrak{l}}(0)} .$$

I will now suggest that one can derive the existence of the aforementioned relation from the identical vanishing of the quadrilinear form Ψ and then derive the validity of equations (51) from that, and I will then confirm that suspicion completely.

8.

To begin with the second part of the assertion, set:

$$x'_{\mathfrak{k}}(0) = \kappa, \quad x'_{\mathfrak{l}}(0) = \lambda .$$

When one divides the left-hand side of (23.a) by dt, substitutes the values of $c_{\mathfrak{b},\mathfrak{l}}$ in (62), and multiplies by $(t - t_0)^2$, that will give:

$$\text{(63)} \qquad \Phi_{\mathfrak{a},\lambda} = (t-t_0)^{-2}\left(\sum_{\mathfrak{b}} a_{\mathfrak{a},\mathfrak{b}} \frac{d(t-t_0)^{-1}\dfrac{\partial x_{\mathfrak{b}}}{\partial \lambda}}{dt} + \tfrac{1}{2}\sum_{\mathfrak{b}} \frac{\partial f_{\mathfrak{a}}(x')}{\partial x'_{\mathfrak{b}}}(t-t_0)^{-1}\frac{\partial x_{\mathfrak{b}}}{\partial \lambda}\right),$$

$$\text{(64)} \qquad \Psi_{\mathfrak{a},\lambda} = \sum_{\mathfrak{b}} a_{\mathfrak{a},\mathfrak{b}} \frac{d\dfrac{\partial x_{\mathfrak{b}}}{\partial \lambda}}{dt} + \tfrac{1}{2}\sum_{\mathfrak{b}} \frac{\partial f_{\mathfrak{a}}(x')}{\partial x'_{\mathfrak{b}}}\frac{\partial x_{\mathfrak{b}}}{\partial \lambda}.$$

One will then have:

$$\text{(65)} \qquad \Phi_{\mathfrak{a},\lambda} = (t-t_0)\Psi_{\mathfrak{a},\lambda} - \sum_{\mathfrak{b}} a_{\mathfrak{a},\mathfrak{b}}\frac{\partial x_{\mathfrak{b}}}{\partial \lambda},$$

and upon switching $\mathfrak{b}$ with $\mathfrak{a}$ and λ with κ, that will give:

$$\text{(65.b)} \qquad \Phi_{\mathfrak{b},\kappa} = (t-t_0)\Psi_{\mathfrak{b},\kappa} - \sum_{\mathfrak{a}} a_{\mathfrak{b},\mathfrak{a}}\frac{\partial x_{\mathfrak{b}}}{\partial \kappa}.$$

It then follows from this upon summation that:

$$\text{(66)} \qquad \sum_{\mathfrak{a}} \Phi_{\mathfrak{a},\lambda}\frac{\partial x_{\mathfrak{a}}}{\partial \kappa} + \sum_{\mathfrak{b}} \Phi_{\mathfrak{b},\kappa}\frac{\partial x_{\mathfrak{b}}}{\partial \lambda} = (t-t_0)\left(\sum_{\mathfrak{a}} \Psi_{\mathfrak{a},\lambda}\frac{\partial x_{\mathfrak{a}}}{\partial \kappa} + \sum_{\mathfrak{b}} \Psi_{\mathfrak{b},\kappa}\frac{\partial x_{\mathfrak{b}}}{\partial \lambda}\right) - 2\sum_{\mathfrak{a},\mathfrak{b}} a_{\mathfrak{b},\mathfrak{a}}\frac{\partial x_{\mathfrak{a}}}{\partial \kappa}\frac{\partial x_{\mathfrak{b}}}{\partial \lambda}.$$

Now, from equation (11.b), one has:

$$\text{(67)} \qquad \sum_{\mathfrak{a}} \Psi_{\mathfrak{a},\lambda}\frac{\partial x_{\mathfrak{a}}}{\partial \kappa} + \sum_{\mathfrak{b}} \Psi_{\mathfrak{b},\kappa}\frac{\partial x_{\mathfrak{b}}}{\partial \lambda} = \frac{d}{dt}\sum_{\mathfrak{a},\mathfrak{b}} a_{\mathfrak{b},\mathfrak{a}}\frac{\partial x_{\mathfrak{a}}}{\partial \kappa}\frac{\partial x_{\mathfrak{b}}}{\partial \lambda},$$

so when one introduces the quantities $\varphi_{\mathfrak{k},\mathfrak{l}}$, that will give the result that:

$$\text{(68)} \qquad \sum_{\mathfrak{a}} \Phi_{\mathfrak{a},\lambda}\frac{\partial x_{\mathfrak{a}}}{\partial \kappa} + \sum_{\mathfrak{b}} \Phi_{\mathfrak{b},\kappa}\frac{\partial x_{\mathfrak{b}}}{\partial \lambda} = (t-t_0)\frac{d\varphi_{\mathfrak{k},\mathfrak{l}}}{dt} - 2\varphi_{\mathfrak{k},\mathfrak{l}},$$

or

$$\text{(68.a)} \qquad \sum_{\mathfrak{a}} \Phi_{\mathfrak{a},\lambda}\frac{\partial x_{\mathfrak{a}}}{\partial \kappa} + \sum_{\mathfrak{b}} \Phi_{\mathfrak{b},\kappa}\frac{\partial x_{\mathfrak{b}}}{\partial \lambda} = (t-t_0)^3\frac{d((t-t_0)^{-2}\varphi_{\mathfrak{k},\mathfrak{l}})}{dt}.$$

When the functions $\Phi_{\mathfrak{a},\lambda}$ and $\Phi_{\mathfrak{b},\kappa}$, which differ by only their indices, all vanish, it will follow from those equations that equations (61) are always fulfilled. With that, the second part of the assertion that was made is resolved, and I shall now turn to the still-remaining first part, according to which the identical vanishing of the form Ψ will require the validity of the relations:

$$\Phi_{\mathfrak{b},\kappa} = 0 .$$

One can verify that fact when one shows that the functions $\Phi_{\mathfrak{b},\kappa}$ once more satisfy a well-defined system of ordinary differential equations that are satisfied by nothing but vanishing values under the prevailing relationships. The system of isoperimetric equations (38) must be employed in the derivation of that system of differential equations, since the quantities $x_{\mathfrak{a}}$, $x'_{\mathfrak{a}}$, and $x'_{\mathfrak{k}}(0) = \kappa$ will first take on their special significance because of them. Upon introducing the expressions (14) and switching the indices $\mathfrak{a}$ and $\mathfrak{b}$, the left-hand side of (38) will take the form:

(69) $$F_{\mathfrak{b}} = \sum_{\mathfrak{a}} a_{\mathfrak{b},\mathfrak{a}} x''_{\mathfrak{a}} + f_{\mathfrak{b}} ,$$

and the system will become:

(69.a) $$F_{\mathfrak{b}} = 0 .$$

Now, the algebraic nature of the current considerations will become more transparent when one develops an identity between the functions $\Psi_{\mathfrak{b},\kappa}$, $F_{\mathfrak{b}}$, and the form Ψ, from which, the aforementioned system of differential equation in the functions $\Phi_{\mathfrak{b},\kappa}$ will emerge from the assumptions that $F_{\mathfrak{b}} = 0$ and that Ψ vanishes identically. In order to do that, I shall give the functions $F_{\mathfrak{b}}$ and $\Psi_{\mathfrak{b},\kappa}$ the forms:

(70) $$\left\{ \begin{aligned} \Psi_{\mathfrak{b},\kappa} &= \sum_{\mathfrak{a}} a_{\mathfrak{b},\mathfrak{a}} \frac{d}{dt}\left(\frac{\partial x_{\mathfrak{a}}}{\partial \kappa}\right) + \tfrac{1}{2} \sum_{\mathfrak{a}} \frac{\partial f_{\mathfrak{b}}(x')}{\partial x'_{\mathfrak{a}}} \frac{\partial x_{\mathfrak{a}}}{\partial \kappa}, \\ F_{\mathfrak{b}} &= \sum_{\mathfrak{a}} a_{\mathfrak{b},\mathfrak{a}} \frac{d^2 x_{\mathfrak{a}}}{dt^2} + \tfrac{1}{2} \sum_{\mathfrak{a}} \frac{\partial f_{\mathfrak{b}}(x')}{\partial x'_{\mathfrak{a}}} \frac{dx_{\mathfrak{a}}}{dt}, \end{aligned} \right.$$

in which κ now means any quantity that the variables $x_{\mathfrak{a}}$ depend upon, and which is independent of the quantity t, and one has $dx_{\mathfrak{a}} / dt = x'_{\mathfrak{a}}$, as before. From a known property of homogeneous functions of degree two, one has:

$$\sum_{\mathfrak{a}} \frac{\partial f_{\mathfrak{b}}(x')}{\partial x'_{\mathfrak{a}}} \frac{\partial x_{\mathfrak{a}}}{\partial \kappa} = \sum_{\mathfrak{a}} \frac{\partial f_{\mathfrak{b}}\left(\dfrac{\partial x}{\partial \kappa}\right)}{\partial\left(\dfrac{\partial x_{\mathfrak{a}}}{\partial \kappa}\right)} x'_{\mathfrak{a}}$$

here, and as a result of (11.b):

$$\frac{\partial f_{\mathfrak{a}}\left(\frac{\partial x}{\partial \kappa}\right)}{\partial\left(\frac{\partial x_{\mathfrak{b}}}{\partial \kappa}\right)}+\frac{\partial f_{\mathfrak{b}}\left(\frac{\partial x}{\partial \kappa}\right)}{\partial\left(\frac{\partial x_{\mathfrak{a}}}{\partial \kappa}\right)}=2\,\frac{\partial a_{\mathfrak{a},\mathfrak{b}}}{\partial \kappa},$$

$$\frac{\partial f_{\mathfrak{a}}(x')}{\partial x'_{\mathfrak{b}}}+\frac{\partial f_{\mathfrak{b}}(x')}{\partial x'_{\mathfrak{a}}}=2a'_{\mathfrak{a},\mathfrak{b}}\,.$$

In that way, equations (70) will go to the following ones:

(71)
$$\begin{cases}\Psi_{\mathfrak{b},\kappa}=\sum_{\mathfrak{a}}\frac{\partial(a_{\mathfrak{b},\mathfrak{a}}\,x'_{\mathfrak{a}})}{\partial\kappa}-\frac{1}{2}\sum_{\mathfrak{a}}\frac{\partial f_{\mathfrak{a}}\left(\frac{\partial x}{\partial \kappa}\right)}{\partial\left(\frac{\partial x_{\mathfrak{b}}}{\partial \kappa}\right)}\,x'_{\mathfrak{a}}\,,\\[2ex] F_{\mathfrak{b}}=\sum_{\mathfrak{a}}\frac{d(a_{\mathfrak{b},\mathfrak{a}}\,x'_{\mathfrak{a}})}{dt}-\frac{1}{2}\sum_{\mathfrak{a}}\frac{\partial f_{\mathfrak{a}}(x')}{\partial x'_{\mathfrak{b}}}\,x'_{\mathfrak{a}}\,.\end{cases}$$

Due to the independence of the differentiations with respect to κ and t that must be done, that will imply that:

(72)
$$\frac{d\Psi_{\mathfrak{b},\kappa}}{dt}-\frac{\partial F_{\mathfrak{b}}}{\partial\kappa}=\frac{1}{2}\sum_{\mathfrak{a}}\frac{\partial f_{\mathfrak{a}}(x')}{\partial x'_{\mathfrak{b}}}\,x'_{\mathfrak{a}}+\frac{1}{2}\sum_{\mathfrak{d}}\frac{\partial f_{\mathfrak{b}}(x')}{\partial x'_{\mathfrak{b}}}\frac{\partial x'_{\mathfrak{d}}}{\partial\kappa}_{\mathfrak{a}}-\frac{1}{2}\sum_{\mathfrak{a}}\frac{d}{dt}\frac{\partial f_{\mathfrak{a}}\left(\frac{\partial x}{\partial \kappa}\right)}{\partial\left(\frac{\partial x_{\mathfrak{b}}}{\partial \kappa}\right)}\,x'_{\mathfrak{a}}-\frac{1}{2}\sum_{\mathfrak{d}}\frac{d}{dt}\frac{\partial f_{\mathfrak{d}}\left(\frac{\partial x}{\partial \kappa}\right)}{\partial\left(\frac{\partial x_{\mathfrak{b}}}{\partial \kappa}\right)}\,x'_{\mathfrak{b}}\,.$$

Equations (70) will then yield the following ways of determining $\partial x'_{\mathfrak{b}}/\partial\kappa$ and $x''_{\mathfrak{b}}$:

$$\Psi_{\mathfrak{b},\kappa}-\frac{1}{2}\sum_{\mathfrak{a}}\frac{\partial f_{\mathfrak{c}}(x')}{\partial x'_{\mathfrak{a}}}\frac{\partial x_{\mathfrak{a}}}{\partial\kappa}=\sum_{\mathfrak{a}}a_{\mathfrak{c},\mathfrak{a}}\,\frac{\partial x'_{\mathfrak{a}}}{\partial\kappa}\,,$$

$$F_{\mathfrak{c}}-\frac{1}{2}\sum_{\mathfrak{a}}\frac{\partial f_{\mathfrak{c}}(x')}{\partial x'_{\mathfrak{a}}}\,x'_{\mathfrak{a}}\quad=\sum_{\mathfrak{a}}a_{\mathfrak{c},\mathfrak{a}}\,x''_{\mathfrak{a}}\,,$$

so

(73)
$$\begin{cases}\frac{\partial x'_{\mathfrak{b}}}{\partial\kappa}=\sum_{\mathfrak{c}}\frac{A_{\mathfrak{c},\mathfrak{b}}}{\Delta}\,\Psi_{\mathfrak{c},\kappa}-\frac{1}{2}\sum_{\mathfrak{c},\mathfrak{a}}\frac{A_{\mathfrak{c},\mathfrak{b}}}{\Delta}\,F_{\mathfrak{c}}\,\frac{\partial f_{\mathfrak{c}}\left(\frac{\partial x}{\partial \kappa}\right)}{\partial\left(\frac{\partial x_{\mathfrak{a}}}{\partial \kappa}\right)}\,x'_{\mathfrak{a}}\,,\\[2ex] x''_{\mathfrak{b}}=\sum_{\mathfrak{c}}\frac{A_{\mathfrak{c},\mathfrak{b}}}{\Delta}\,F_{\mathfrak{c}}-\frac{1}{2}\sum_{\mathfrak{c},\mathfrak{a}}\frac{A_{\mathfrak{c},\mathfrak{b}}}{\Delta}\,\frac{\partial f_{\mathfrak{c}}(x')}{\partial x'_{\mathfrak{a}}}\,x'_{\mathfrak{a}}\,.\end{cases}$$

Substituting these expressions in (72) will then produce the desired relation:

$$(74)\quad \begin{cases} \dfrac{d\Psi_{\mathfrak{b},\kappa}}{dx} - \tfrac{1}{2}\sum\limits_{\mathfrak{c},\mathfrak{d}} \dfrac{A_{\mathfrak{c},\mathfrak{d}}}{\Delta}\,\Psi_{\mathfrak{c},\kappa}\,\dfrac{\partial f_{\mathfrak{d}}(x')}{\partial x'_{\mathfrak{b}}} - \dfrac{\partial F_{\mathfrak{b}}}{\partial\kappa} + \tfrac{1}{2}\sum\limits_{\mathfrak{c},\mathfrak{d}} \dfrac{A_{\mathfrak{c},\mathfrak{d}}}{\Delta}\,F_{\mathfrak{c}}\,\dfrac{\partial f_{\mathfrak{d}}\left(\dfrac{\partial x}{\partial\kappa}\right)}{\partial\left(\dfrac{\partial x_{\mathfrak{b}}}{\partial\kappa}\right)} \\[2ex] = \tfrac{1}{2}\sum\limits_{\mathfrak{a}}\left[\dfrac{\partial}{\partial\kappa}\left(\dfrac{\partial f_{\mathfrak{a}}(x')}{\partial x'_{\mathfrak{b}}}\right) - \dfrac{d}{dt}\left(\dfrac{\partial f_{\mathfrak{a}}\left(\dfrac{\partial x}{\partial\kappa}\right)}{\partial\left(\dfrac{\partial x_{\mathfrak{b}}}{\partial\kappa}\right)}\right) + \tfrac{1}{2}\sum\limits_{\mathfrak{c},\mathfrak{d}} \dfrac{A_{\mathfrak{c},\mathfrak{d}}}{\Delta}\left(\dfrac{\partial f_{\mathfrak{c}}(x')}{\partial x'_{\mathfrak{a}}}\,\dfrac{\partial f_{\mathfrak{d}}\left(\dfrac{\partial x}{\partial\kappa}\right)}{\partial\left(\dfrac{\partial x_{\mathfrak{b}}}{\partial\kappa}\right)} - \dfrac{\partial f_{\mathfrak{c}}\left(\dfrac{\partial x}{\partial\kappa}\right)}{\partial\left(\dfrac{\partial x_{\mathfrak{a}}}{\partial\kappa}\right)}\,\dfrac{\partial f_{\mathfrak{d}}(x')}{\partial x'_{\mathfrak{b}}}\right)\right] x'_{\mathfrak{a}}\,. \end{cases}$$

In fact, if one compares the right-hand side of (74) with the defining equation of the form Ψ in (32) then that will explain the fact that when the quantities $dx_{\mathfrak{g}} / dt$ are substituted for $dx_{\mathfrak{g}}$ and the quantities $dx_{\mathfrak{a}} / dt$ are substituted for $\delta x_{\mathfrak{h}}$ in Ψ, one-half of the partial differential quotient of the resulting function $\Psi(x', \delta u, x', \partial x / \partial\kappa)$ with respect to the quantities $\delta u_{\mathfrak{g}}$ will be equal to the right-hand side of (74), and that relation will take the following form:

$$(74.a)\qquad \frac{d\Psi_{\mathfrak{c},\kappa}}{dt} - \tfrac{1}{2}\sum_{\mathfrak{c},\mathfrak{d}} \frac{A_{\mathfrak{c},\mathfrak{d}}}{\Delta}\,\Psi_{\mathfrak{c},\kappa}\,\frac{\partial f_{\mathfrak{d}}(x')}{\partial x'_{\mathfrak{b}}} - \frac{\partial F_{\mathfrak{b}}}{\partial\kappa} + \tfrac{1}{2}\sum_{\mathfrak{c},\mathfrak{d}} \frac{A_{\mathfrak{c},\mathfrak{d}}}{\Delta}\,F_{\mathfrak{c}}\,\frac{\partial f_{\mathfrak{d}}\left(\dfrac{\partial x}{\partial\kappa}\right)}{\partial\left(\dfrac{\partial x_{\mathfrak{b}}}{\partial\kappa}\right)} = \tfrac{1}{2}\,\frac{\partial\Psi\left(x', \delta u, x', \dfrac{\partial x}{\partial\kappa}\right)}{\partial(\delta u_{\mathfrak{b}})}\,.$$

The assumption that the form Ψ vanishes identically might now enter in, such that the quantities $x_{\mathfrak{a}}$ will satisfy the system of equations $F_{\mathfrak{b}} = 0$ when one lets κ be an integration constant of that system, which is why the equations $\partial F_{\mathfrak{b}} / \partial\kappa = 0$ will also remain valid. The relation (74.a) will then go to this system of differential equations for the functions $\Psi_{\mathfrak{b},\kappa}$:

$$(75)\qquad \frac{d\Psi_{\mathfrak{c},\kappa}}{dt} - \tfrac{1}{2}\sum_{\mathfrak{c},\mathfrak{d}} \frac{A_{\mathfrak{c},\mathfrak{d}}}{\Delta}\,\Psi_{\mathfrak{c},\kappa}\,\frac{\partial f_{\mathfrak{d}}(x')}{\partial x'_{\mathfrak{b}}} = 0\,.$$

The functions $\Phi_{\mathfrak{b},\kappa}$ and $\Psi_{\mathfrak{b},\kappa}$ are connected with each other by equation (65.b). Differentiating it will give:

$$\frac{d\Phi_{\mathfrak{c},\kappa}}{dt} = (t - t_0)\,\frac{d\Psi_{\mathfrak{c},\kappa}}{dt} + \Psi_{\mathfrak{b},\kappa} - \sum_{\mathfrak{a}} a'_{\mathfrak{b},\mathfrak{a}}\,\frac{\partial x_{\mathfrak{a}}}{\partial\kappa} - \sum_{\mathfrak{a}} a_{\mathfrak{b},\mathfrak{a}}\,\frac{\partial x'_{\mathfrak{a}}}{\partial\kappa}\,,$$

and from formula (11.b) and upon introducing the values of $\Psi_{\mathfrak{b},\kappa}$ in (70), one will have:

$$\frac{d\Phi_{\mathfrak{b},\kappa}}{dt} = (t - t_0)\,\frac{d\Psi_{\mathfrak{b},\kappa}}{dt} - \sum_{\mathfrak{a}} \frac{\partial f_{\mathfrak{a}}(x')}{\partial x'_{\mathfrak{b}}}\,\frac{\partial x_{\mathfrak{a}}}{\partial\kappa}\,.$$

Moreover, one has:

$$\tfrac{1}{2}\sum_{\mathfrak{c},\mathfrak{d}} \frac{A_{\mathfrak{c},\mathfrak{d}}}{\Delta} \Phi_{\mathfrak{c},\kappa} \frac{\partial f_{\mathfrak{d}}(x')}{\partial x'_{\mathfrak{b}}} = (t-t_0)\tfrac{1}{2}\sum_{\mathfrak{c},\mathfrak{d}} \frac{A_{\mathfrak{c},\mathfrak{d}}}{\Delta} \Psi_{\mathfrak{c},\kappa} \frac{\partial f_{\mathfrak{d}}(x')}{\partial x'_{\mathfrak{b}}} - \tfrac{1}{2}\sum_{\mathfrak{c},\mathfrak{d}} \frac{A_{\mathfrak{c},\mathfrak{d}}}{\Delta} \sum_{\mathfrak{a}} a_{\mathfrak{c},\mathfrak{a}} \frac{\partial x_{\mathfrak{a}}}{\partial \kappa} \frac{\partial f_{\mathfrak{d}}(x')}{\partial x'_{\mathfrak{b}}},$$

or, as a result of the relations that exist between the quantities $a_{\mathfrak{c},\mathfrak{a}}$ and $A_{\mathfrak{c},\mathfrak{d}}$:

$$\tfrac{1}{2}\sum_{\mathfrak{c},\mathfrak{d}} \frac{A_{\mathfrak{c},\mathfrak{d}}}{\Delta} \Phi_{\mathfrak{c},\kappa} \frac{\partial f_{\mathfrak{d}}(x')}{\partial x'_{\mathfrak{b}}} = (t-t_0)\tfrac{1}{2}\sum_{\mathfrak{c},\mathfrak{d}} \frac{A_{\mathfrak{c},\mathfrak{d}}}{\Delta} \Psi_{\mathfrak{c},\kappa} \frac{\partial f_{\mathfrak{d}}(x')}{\partial x'_{\mathfrak{b}}} - \tfrac{1}{2}\sum_{\mathfrak{c},\mathfrak{d}} \frac{\partial f_{\mathfrak{a}}(x')}{\partial x'_{\mathfrak{b}}} \frac{\partial x_{\mathfrak{a}}}{\partial \kappa},$$

and therefore the system (75) has the system of differential equations:

(75) $$\frac{d\Phi_{\mathfrak{c},\kappa}}{dt} - \tfrac{1}{2}\sum_{\mathfrak{c},\mathfrak{d}} \frac{A_{\mathfrak{c},\mathfrak{d}}}{\Delta} \Phi_{\mathfrak{c},\kappa} \frac{\partial f_{\mathfrak{d}}(x')}{\partial x'_{\mathfrak{b}}} = 0$$

as a consequence, which refer to the functions $\Phi_{\mathfrak{b},\kappa}$ and are the ones that were sought.

Up to now, the determination applied to the case when κ was understood to mean any integration constant of the system by which the equations $F_{\mathfrak{b}} = 0$ were integrated. We shall now once more make the assumption in regard to the integration that $x_{\mathfrak{a}}$ is represented in terms of the quantities t, $x_{\mathfrak{l}}(0)$, and $x'_{\mathfrak{k}}(0)$, as well as the assumption that:

$$\kappa = x'_{\mathfrak{k}}(0) .$$

Since all breaks in continuity for the integral that extends from $t = t_0$ have been excluded for the integration in question, the function $\Psi_{\mathfrak{b},\mathfrak{k}}$ is always a finite quantity by means of equation (70), and therefore for $t = t_0$. Equation (65.b):

$$\Phi_{\mathfrak{c},\kappa} = (t-t_0)\Psi_{\mathfrak{b},\mathfrak{k}} - \sum_{\mathfrak{a}} a_{\mathfrak{b},\mathfrak{a}} \frac{\partial x'_{\mathfrak{a}}}{\partial \kappa}$$

then says that for $t = t_0$ the function $\Phi_{\mathfrak{b},\kappa}$ will be equal to the expression $-\sum_{\mathfrak{a}} a_{\mathfrak{b},\mathfrak{a}} \frac{\partial x_{\mathfrak{a}}}{\partial \kappa}$. The quantities $a_{\mathfrak{b},\mathfrak{a}}$ remain likewise finite under the substitution $x_{\mathfrak{a}} = x_{\mathfrak{a}}(0)$, but the partial differential quotients $\frac{\partial x_{\mathfrak{a}}}{\partial \kappa} = \frac{\partial x_{\mathfrak{a}}}{\partial x'_{\mathfrak{k}}(0)}$ must go to zero for $t = t_0$, from a remark that was already applied before. Hence the functions $\Phi_{\mathfrak{b},\kappa}$ have the property that they vanish for $t = t_0$. Now, since the system of linear differential equations (76), as long as the functions that appear in them as coefficients do not exhibit any breaks in continuity, can be integrated in only one way when the values of $\Phi_{\mathfrak{b},\kappa}$ that correspond to the assumption that $t = t_0$ are given (*), and since the determination $\Phi_{\mathfrak{b},\kappa} = 0$ satisfies the system for a moving t, moreover, that determination will also be the only one that is valid (**).

(*) Cf., the treatise that was cited above: *Erörterung der Möglichkeit*, etc.

(**) If is known to *Jacobi* that the system $\partial F_{\mathfrak{b}} / \partial\kappa = 0$, when regarded as a system of linear differential equations for the quantities $\partial x_{\mathfrak{a}} / \partial\kappa$, can be integrated completely in terms of the $2n$ systems of solutions $\frac{\partial x_{\mathfrak{a}}}{\partial x_{\mathfrak{k}}(0)}$ and $\frac{\partial x_{\mathfrak{a}}}{\partial x'_{\mathfrak{k}}(0)}$. It will then follow from the above that the system $\Phi_{\mathfrak{b},\kappa} = 0$, when regarded as a system of linear differential equations

However, that was to be proved, and in that way, the assertion that was made is in regard to the form Ψ is established in full generality.

9.

The identical vanishing of the form Ψ – that is, the vanishing of all coefficients of that form for all systems of values $x_{\mathfrak{a}}$ – yields equations between the coefficients of the form $f(dx)$ and their first and second partial differential quotients to which the remarks in section **4** will apply. The coefficient of the product:

$$(du_{\mathfrak{a}}\,\delta u_{\mathfrak{b}} - \delta u_{\mathfrak{a}}\,du_{\mathfrak{b}})\,(dx_{\mathfrak{g}}\,\delta x_{\mathfrak{h}} - \delta x_{\mathfrak{g}}\,dx_{\mathfrak{h}})$$

is the sum of the two components:

$$[\mathfrak{a},\mathfrak{b},\mathfrak{g},\mathfrak{h}] = \frac{\partial^2 a_{\mathfrak{a},\mathfrak{g}}}{\partial x_{\mathfrak{b}}\,\partial x_{\mathfrak{h}}} + \frac{\partial^2 a_{\mathfrak{b},\mathfrak{h}}}{\partial x_{\mathfrak{a}}\,\partial x_{\mathfrak{g}}} - \frac{\partial^2 a_{\mathfrak{a},\mathfrak{h}}}{\partial x_{\mathfrak{b}}\,\partial x_{\mathfrak{g}}} - \frac{\partial^2 a_{\mathfrak{b},\mathfrak{g}}}{\partial x_{\mathfrak{a}}\,\partial x_{\mathfrak{h}}}$$

and

$$(\mathfrak{a},\mathfrak{b},\mathfrak{g},\mathfrak{h}) = \tfrac{1}{2}\sum_{\mathfrak{c},\mathfrak{d}} \frac{A_{\mathfrak{c},\mathfrak{d}}}{\Delta}(f_{\mathfrak{c},\mathfrak{d},\mathfrak{g}}\; f_{\mathfrak{d},\mathfrak{b},\mathfrak{h}} - f_{\mathfrak{c},\mathfrak{a},\mathfrak{h}}\; f_{\mathfrak{d},\mathfrak{b},\mathfrak{g}})\,.$$

As one sees, $[\mathfrak{a}, \mathfrak{b}, \mathfrak{g}, \mathfrak{h}]$ is a linear function of the second partial differential quotients of the quantities $a_{\mathfrak{c},\mathfrak{d}}$, and $(\mathfrak{a}, \mathfrak{b}, \mathfrak{g}, \mathfrak{h})$ is a function of degree two in the first partial differential quotients of the same quantities $a_{\mathfrak{c},\mathfrak{d}}$. As long as the group of four indices $\mathfrak{a}, \mathfrak{b}, \mathfrak{g}, \mathfrak{h}$ and the group of four indices $\mathfrak{a}', \mathfrak{b}', \mathfrak{g}', \mathfrak{h}'$ are not identical, the corresponding expressions $[\mathfrak{a}, \mathfrak{b}, \mathfrak{g}, \mathfrak{h}]$ and $[\mathfrak{a}', \mathfrak{b}', \mathfrak{g}', \mathfrak{h}']$ will contain only differing second partial differential quotients of the quantities $a_{\mathfrak{c},\mathfrak{d}}$. The orderings of those groups of four indices $\mathfrak{a}, \mathfrak{b}, \mathfrak{g}, \mathfrak{h}$ are considered at the aforementioned place and exhibit three different cases: As long as two or one pairs of equal indices occur among the four indices the different orderings will produce one non-vanishing coefficient of the form Ψ. However, as long as the four indices $\mathfrak{a}, \mathfrak{b}, \mathfrak{g}, \mathfrak{h}$ are all distinct from each other, the different orderings will produce three different coefficients of the form Ψ, whose sum always has the value zero. With the current notations, one then has:

$$[\mathfrak{a}, \mathfrak{b}, \mathfrak{g}, \mathfrak{h}] + [\mathfrak{a}, \mathfrak{g}, \mathfrak{h}, \mathfrak{b}] + [\mathfrak{a}, \mathfrak{h}, \mathfrak{b}, \mathfrak{g}] = 0$$

and

$$(\mathfrak{a}, \mathfrak{b}, \mathfrak{g}, \mathfrak{h}) + (\mathfrak{a}, \mathfrak{g}, \mathfrak{h}, \mathfrak{b}) + (\mathfrak{a}, \mathfrak{h}, \mathfrak{b}, \mathfrak{g}) = 0\,.$$

Now, if all of the coefficients of the form Ψ are to be equal to zero then the linear function of the second differential quotients $[\mathfrak{a}, \mathfrak{b}, \mathfrak{g}, \mathfrak{h}]$ must be equal to the quadratic function of the first

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for the quantities $\partial x_{\mathfrak{a}} / \partial \kappa$, can be integrated completely in terms of the *n* systems of solutions $\dfrac{\partial x_{\mathfrak{a}}}{\partial x'_{\mathfrak{k}}(0)}$ as long as the form Ψ vanishes identically.

differential quotients – (𝔞, 𝔟, 𝔤, 𝔥) . The number of those equations, in which the corresponding complex [𝔞, 𝔟, 𝔤, 𝔥] are not coupled to each other by any first-degree equation, can then be determined as follows: As one has seen, among the different coefficients of the form Ψ, $n(n-1)/2$ of them belong to the first category 𝔞, 𝔟, 𝔞, 𝔟, and then $n(n-1)(n-2)/2$ belong to the second category 𝔞, 𝔟, 𝔞, 𝔠, and $n(n-1)(n-2)(n-3)/8$ will belong to the third category 𝔞, 𝔟, 𝔤, 𝔥. For the first and second category, the vanishing of each coefficient will imply one equation for each of them. However, for the third category, the vanishing of each two coefficients from three associated ones will also imply the vanishing of the third one, so the vanishing of each three coefficients will always correspond to only two equations. The number of equations in question is then equal to the sum of the numbers:

$$\frac{n(n-1)}{2}+\frac{n(n-1)(n-2)}{2}+\frac{n(n-1)(n-2)(n-3)}{12}$$

or to the number $(n^4 - n^2)/12$. For $n = 2, 3, 4$, it takes the values 1, 6, 20, respectively.

Bonn, 4 January 1869.