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Quantum-mechanical interpretation of Weyl’s theory ([1])

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Chapter I – Weyl’s theory.

The idea of a “purely-local geometry,” which was first conceived by **Riemann**, is known to have recently experienced an extraordinarily beautiful and simple completion by **Weyl**. One can consider the **Riemannian** conception of space to be the elimination of the prejudice that the *curvature* behavior at *one* location in space must imply the curvature at *all others*. In order to give some sense to **Riemann**’s statement, it was then necessary that the yardstick that enabled one to determine the coefficients g_{ik} of the fundamental metric form:

$$ds^2 = g_{ik}\, dx^i dx^k$$

would have to be a “rigid” yardstick.

By contrast, **Weyl** made it legitimate to assume that such a rigid yardstick would be contrary to a radical local geometry in which only the *ratios* of the g_{ik} at *one* location, and not their absolute values, could be reasonably fixed, and he correspondingly set the change dl in a gauge yardstick of length l under an infinitesimal displacement dx^i equal to:

$$dl = l\,\varphi_i\, dx^i , \tag{1}$$

in which the proportionality factors φ_i are functions of position that are characteristic of the metric behavior of space, similar to the g_{ik} , or when one integrates (1):

([1]) Presented, in part, at the session of the Gauvereins Württemberg of the D. Phys. Ges. Stuttgart on 18 December 1926; cf., also a tentative survey report in Naturwiss. **15** (1927), pp. 187.

$$l = l_0 \, e^{\int \varphi_i \, dx^i} \tag{2}$$

($l_0 = l$ at the beginning of the displacement). The gauge is generally path-dependent (i.e., non-integrable), unless the quantities:

$$f_{ik} = \frac{\partial \varphi_i}{\partial x^k} - \frac{\partial \varphi_k}{\partial x^i} \tag{3}$$

vanish. One can say nothing about those quantities f_{ik}, except that their definition (3) (let the dimension of the manifold be four) will imply that:

$$\frac{\partial f_{ik}}{\partial x^l} + \frac{\partial f_{kl}}{\partial x^i} + \frac{\partial f_{li}}{\partial x^k} = 0 \qquad i \neq k \neq l, \qquad i, k, l = 1, 2, 3, 4. \tag{4}$$

The formal agreement between those four equations and the one system of **Maxwell** equations:

$$\operatorname{rot} \mathfrak{E} + \frac{1}{c} \dot{\mathfrak{H}} = 0 \,,$$
$$\operatorname{div} \mathfrak{H} = 0 \,,$$

along with some further formal analogies, led **Weyl** to the conclusion that φ_i could be identified (up to a constant proportionality factor) with the components Φ_i of the electromagnetic four-potentials, while the f_{ik} corresponded to the electromagnetic field strengths $\mathfrak{E}$, $\mathfrak{H}$. In a logical extension of the geometric interpretation of *gravitation* in terms of the *variable curvature* of **Riemannian** space, **Weyl** then imagined that the remaining part of physical effects – viz., the *electromagnetic field* – would likewise be a property of the metric behavior of space that was characterized by the *variability of the gauge*. One would then write:

$$l = l_0 \, e^{\alpha \int \Phi_i \, dx^i} \quad (\alpha = \text{proportionality factor}). \tag{2.a}$$

One must admire the sheer boldness that led **Weyl** to his study of the gauge-geometric interpretation of electromagnetism on the basis of nothing but that purely-formal association: In the theory of gravitation, it was a *physical fact*, namely, the principle of the equivalence of inertial and gravitational mass, that inspired **Einstein**'s geometric interpretation. By contrast, in the theory of electromagnetism, no such fact was known: That did not give one the right to imagine that the electromagnetic field might have a universal influence on the so-called rigid yardsticks (clocks, resp.). Quite the contrary, e.g., atomic clocks represent yardsticks whose *independence* of their *prior history* is confirmed by the sharpness of the spectral lines, which contradicts the non-integrable metric (2.a) that **Weyl** assumed in a magnetic field. It would probably require an uncommonly-clear metaphysical conviction for **Weyl** to believe that despite such elementary experimental facts, one should not give up on the idea that nature *must* make use of those beautiful geometric possibilities as being necessary. He held fast to his viewpoint and kept the contradiction that was just depicted out of the discussion by way of a somewhat-vague reinterpretation of the concept of "real measurements," which generally gave his theory its concise physical meaning, but in that way surrendered much of its power to convince.

I shall not need to go into the details of that abstract formulation of the theory. Rather, I will show that it is precisely the concise original conception of **Weyl**'s theory that inherently gives it a much greater resilience than its creator has already made use of, namely, that one can glimpse in it nothing less than a *logical path to undulatory mechanics*, and it is from that viewpoint that one will first arrive at an immediately-understandable physical meaning.

Chapter II – De Broglie's undulatory mechanics and Weyl's theory.

What I am calling "**de Broglie**'s theory" is that still-incomplete precursor to undulatory mechanics in which the wave function for the motion of *one* electron (to which we shall confine ourselves here):

$$\psi = e^{2\pi i W(x_i)/h}, \qquad i = 1, 2, 3, 4 \tag{5}$$

emerges from a complete solution W of the **Hamilton-Jacobi** partial differential equation:

$$\left(\frac{\partial W}{\partial x^i} - \frac{e}{c}\Phi_i\right)\left(\frac{\partial W}{\partial x_i} - \frac{e}{c}\Phi^i\right) = -\, m_0^2\, c^2\,, \tag{6}$$

in which the integration constants are determined in a known way such that ψ will be a single-valued function of space, i.e., W will be additively-periodic with a whole-number multiple of the **Planck** constant as its period.

When one gets serious about the radical continuum conception of matter, with the solution of the discontinuously-bounded electron in field quantities that vary continuously in space and time, as was suggested by **de Broglie**'s theory and more consistently by the theory that **Schrödinger** considered later ([1]), one will arrive at an especially-definitive complication when one examines the sense that one might assign to metric statements inside of the undulatory continuum, if at all. That is because in that oscillating and fluctuating infinitely-broad medium that enters in place of the bounded electron, one finds no discontinuities that cannot be understood nor rigid bodies that might permit one to establish a metric as a reproducible yardstick.

I do not at all agree with the opinion that in order to speak of geometry in the atomic domain, one must give a prescription for measurements that *can be performed*. Indeed, one cannot speak of such a thing in the theory of the electron either. However, if one would like to give *any well-*

([1]) It is known to lead to compelling reasons for reinterpreting the entire undulatory formalism statistically, which was proposed by, above all, **Born** and his collaborators. To the extent that the charge density can be reinterpreted as a statistical weighting function, it is not difficult to see that this indeterminacy in regard to the applicability of the law of identity to which we will refer here must be translated accordingly. However, since that conception initially rejects *any* interpretation in space and time, there is little of interest in its relation to **Weyl**'s theory of space.

defined meaning to the specification of a metric then it seems to me that at the very least one can demand is the specification of *some real object (as a "prototype")* to which the metric statements will already relate, such as an electron diameter or distance, etc., although such a statement might still have a very problematic connection with a performable measurement.

However, such a real object does not exist in the undulatory continuum. The law of identity is not applicable to the *πάντα ῥεῖ* (†) of standing and travelling waves, since there are no features of a continuum that would be suitable for defining a reproducible measurement. The main position that one is placed in here would be completely hopeless if it were not for the fact that **Weyl**, in his generalization of the **Riemannian** conception of space, had already addressed a type of space in which it is precisely that non-reproducibility of the gauge unit that is intended to be a logical postulate in a radical local geometry. If that theory had once been a superfluous burden in the world-view of the theory of discontinuous electrons, since one indeed believed that the electrons should possess reproducible measurable quantities, then the situation has changed fundamentally now. One is almost compelled to go back to the general **Weyl** conception of space and attempt to apply it to the **Schrödinger** continuum. A simpler connection now reveals itself.

§ 1. – We shall first assume that we already possess a yardstick l that varies according to the **Weyl** prescription (2.a) and move it around in the ψ-field, and indeed it will move with the flow velocity of matter, which is group four-velocity:

$$u^i = \frac{dx^i}{d\tau} = \frac{1}{m_0}\left(\frac{\partial W}{\partial x_i} - \frac{e}{c}\,\Phi^i\right) . \tag{7}$$

I assert that with that obvious prescription on the path, ***Weyl**'s scalar l will be numerically identical to the de Broglie field scalar ψ.* There are two clarifications that must be made:

A factor of α was left undetermined in **Weyl**'s gauge. I shall make the hypothesis for it that it equals $2\pi\, i\, e / h\, c$. Thus:

$$l = l_0 \exp\frac{2\pi i}{h}\int \frac{e}{c}\,\Phi_i\,dx^i . \tag{2.a}$$

Ultimately: I shall not employ precisely the ψ in equation (5), but the five-dimensional ψ that is equipped with the factor $e^{2\pi i\, m_0 c^2 \tau / h}$, which corresponds to the suggestions of **Klein**, **Fock**, and **Kudar**, in which τ is understood to mean the proper time ([1]). One now has:

$$\psi = e^{2\pi i (W + m_0 c^2 \tau)/h} \tag{5.a}$$

(†) Translator: "Everything flows." This is a reference to the philosophy of Heraclitus that the universe exists in an eternal state of flux.

([1]) This conception of *t*, which goes back to **Kudar**, Ann. Phys. (Leipzig) **81** (1926), pp. 632, agrees completely with the recently-discussed interpretation as the angle coordinate of the proper rotational motion of the electron [Naturwissenschaften **15** (1927), pp. 15], because that angle of rotation is regarded as a clock that moves with the electron. It therefore transforms like proper time.

or

$$= \exp \frac{2\pi i}{h} \left\{ \int \frac{\partial W}{\partial x^i} dx^i + m_0 c^2 \tau \right\}.$$

That quantity ψ should be compared with the **Weyl** gauge (2.a) that gets carried along the current of the continuum. One will get:

$$\frac{\psi}{l} = \frac{1}{l_0} \exp \frac{2\pi i}{h} \left\{ \int \left(\frac{\partial W}{\partial x^i} - \frac{e}{c} \Phi_i \right) dx^i + m_0 c^2 \tau \right\},$$

in which the dx^i are carried along the flow that is given by (7)

$$= \frac{1}{l_0} \cdot \exp \frac{2\pi i}{h} \left\{ \int \left(\frac{\partial W}{\partial x^i} - \frac{e}{c} \Phi_i \right) \left(\frac{\partial W}{\partial x_i} - \frac{e}{c} \Phi^i \right) \frac{d\tau}{m_0} + m_0 c^2 \tau \right\}.$$

As a result of the **Hamilton-Jacobi** differential equation (6), the integrand is equal to $-m_0 c^2$, so one will get:

$$\frac{\psi}{l} = \frac{1}{l_0} \exp \left(\frac{2\pi i}{h} \cdot \text{const.} \right) = \text{const.} \tag{8}$$

The physical object that behaves like **Weyl**'s metric has then been found, viz., *the complex amplitude of the de Broglie wave*. It will then have the same effect in an electromagnetic field that **Weyl** had postulated for his gauge, and to which he had to assign a purely metaphysical existence (as an unused term in the physics of his era). It is, so to speak, the prototype of the **Weyl** metric. Just as in the theory of gravitation we are free to speak of *deflected* light rays and masses or their *geodetic* motion in a **Riemannian** space, (8) will make it possible for us to geometrically interpret the **de Broglie** oscillation process for matter and the influence of the electromagnetic potential in terms of a **Weyl** space that is filled *homogeneously* with matter, but whose metric connection is not integrable.

According to (2.a), the gauge will be constant in the absence of an electromagnetic field. One must then obtain a constant value for the **de Broglie** wave function when one follows it along the associated current, i.e., *group velocity* (v, which is always $< c$). That would seem to contradict **de Broglie**'s fundamental result that the phases of his waves advance with a much larger *phase velocity* ($u = c^2 / v$). However, that is not applicable here, since it was not precisely the **de Broglie** ψ that was employed above, but the extended five-dimensional one, which is dispersion-less, and the distinction between group and phase velocity will go away here. One also easily convinces oneself immediately that the plane wave:

$$\psi = \exp\left[-\frac{2\pi i}{h}\left(\frac{m_0 c^2}{\sqrt{1-\beta^2}}t - \frac{m_0 v}{\sqrt{1-\beta^2}}x - m_0 c^2 \tau\right)\right] \qquad (\beta = v / c)$$

does in fact exhibit constant phase when one follows it with the velocity v.

A further objection that we are comparing ψ to a density with a length l here likewise seems to present no difficulty. One must compare ψ to l^{-3} from the outset, which would mean only a change in the choice of the undetermined factor α. It would probably be more natural to infer from the connection that was revealed here that **Weyl**'s gauge l is assigned the same dimension to begin with as that of **de Broglie**'s ψ. No such statement can be made within the context of **Weyl**'s theory, since nothing is known about the "nature" of l in it.

The failure to understand the *complex form* for the transportation of line segments seems to present a grave difficulty. In that way, it is not permissible to, say, restrict oneself to the real part. One can see a counterpart for that in the fact that the wave function itself ψ is regarded as *essentially* complex, or better yet, it represents a combination of *two* physical state quantities, namely, $\psi\bar{\psi}$ and the real part of $(h / 2\pi i) \ln \psi$. In that sense, one must also understand that in the variational problem of wave mechanics, ψ and $\bar{\psi}$ are varied *independently* of each other. However, I would not like to pause to discuss here whatever it should mean that every line segment is regarded as a complex quantity and what it should mean that the entire **Weyl** variability of the measure of the line segment is presented as a *change in only the phase* while preserving the absolute value.

§ 2. – Nonetheless, the objection that we alluded to above still exists, namely that experiments *contradict* the non-integrability of the gauge. One can now foresee already how that difficulty must be resolved: Quantum theory allows matter to have only a discrete series of equations of motion, and one suspects that those distinguished motions will allow one transport the gauge only in such a way that the phase will have made precisely a whole number of circuits upon returning to the starting point, such that *despite the non-integrability* of the transport of line segments, the gauge *will always be realized in a single-valued way* at every location. In fact, one recalls the resonance property of the **de Broglie** waves, which is the same way that the older **Sommerfeld-Epstein** quantum condition first reinterpreted the **de Broglie** condition so successfully. That is generally coupled with the phase velocity, but as a result of the five-dimensional extension of the wave function, the oscillation process will be dispersion-less, and as a result, our current velocity will be identical to the phase velocity. In that way, and as a result of the identity of the wave function ψ with **Weyl**'s metric, it would then seem to have been already proved ([1]) that **Weyl**'s metric will also take part in the resonance of the **de Broglie** waves when I follow it along the quantum-theoretically possible matter current, and despite the non-integrability of the differential expression (2.a) in the electromagnetic field, it will still lead to a single-valued measurement at every location.

([1]) That terminology is imprecise and will be soon rectified.

Had one included the single-valuedness of the metric as a generally-known fact of experiment in **Weyl**'s theory axiomatically, then one would be logically led to the system of discrete states of motion of "classical" quantum theory and their **de Broglie** waves.

I would not like to leave this subject without making note of the fact that the resonance property of the **Weyl** line metric that we proposed as a characteristic law of undulatory mechanics here was already alluded to by **Schrödinger** ([1]) in 1922 as a "remarkable property of quantum orbits" and he proved it in a number of examples without knowing what it meant at the time. He also drew attention to the possibility that $\alpha = 2\pi i \cdot e / h c$, but without giving preference to a different choice of α. Thus, **Schrödinger** already had the characteristic wave-mechanical periodicity available to him at the time that he later encountered once more from a completely different viewpoint.

For that reason, it would perhaps not be superfluous for me to also prove **Schrödinger**'s suggestion independently of the wave-mechanical connections as a law of "classical" quantum theory, as it was originally intended. I then assert: When the line exponent in the **Weyl** metric is led around a closed quantum orbit, it will be a whole-number multiple of **Planck**'s constant:

$$\oint \frac{e}{c} \Phi_i \, dx^i = n \, h \, . \tag{9}$$

In order to prove that, one utilizes the relation that was employed in **§ 1** already:

$$\int \left(\frac{\partial W}{\partial x^i} - \frac{e}{c} \Phi_i \right) dx^i = - \int m_0 \, c^2 \, d\tau = - \int m_0 \, c^2 \sqrt{1 - \left(\frac{v}{c} \right)^2} \, dt \, .$$

As a result of the quantum condition:

$$\sum_{i=1}^{3} \oint \frac{\partial W}{\partial x^i} dx^i = n \, h \, ,$$

one will then get:

$$\oint \left(\frac{\partial W}{\partial x_4} dx_4 - \frac{e}{c} \Phi_i \, dx^i \right) = - n \, h - \int m_0 \, c^2 \sqrt{1 - \left(\frac{v}{c} \right)^2} \, dt \, .$$

Assuming that an energy integral exists, one will have:

$$\frac{\partial W}{\partial x_4} dx_4 = - (E_{\text{kin}} + E_{\text{pot}}) \, dt \, ,$$

so

$$- \oint \frac{e}{c} \Phi_i \, dx^i = - n \, h + \int \left(- m_0 \, c^2 \sqrt{1 - \left(\frac{v}{c} \right)^2} + E_{\text{kin}} + E_{\text{pot}} \right) dt \, .$$

([1]) **E. Schrödinger**, Zeit. Phys. **12** (1922), pp. 13.

The integral on the right-hand side vanishes here as a result of the relativistic generalization of the virial theorem ([1]) under the *assumption* that the potential is homogeneous of degree – 1 in the x^i, from which the assertion (9) will follow immediately.

One sees from that derivation that one can prove the single-valuedness of **Weyl**'s metric only under *two assumptions*. Those assumptions (in particular, the first one) are obviously very essential, and one can certainly not get around them completely. It guarantees certain stationary relationships in space that *one is first permitted to speak of spatially-closed orbits in the* ***Minkowski*** *world*, which is a statement that generally depends entirely upon the choice of reference system. That is why one must refer to those assumptions as *conditions for the possibility of applying the law of identity to space.*

Most of the time, trajectories will not be exactly periodic, but only quasi-periodic. Under suitable continuity assumptions, one can then prove that the **Weyl** metric will coincide with its original value at the starting point to a sufficiently good approximation and up to an arbitrarily-small given amount. One does not need to demand any more than that.

The fact that the transport of the gauge element with the velocity (7) will always have matter as a consequence in that way seems especially satisfying, because a transport with any other velocity would not be possible quantum-mechanically (mechanically, resp.) However, I would like to defer a more rigorous justification of those connections and their inclusion into an epistemologically-based theory of measurement for later, since an essentially different viewpoint will have to be singled out. If we have also seen how **Weyl**'s ideas have found an unanticipated incorporation into the present physical intuitions then I nonetheless do not believe that one should be over-satisfied with that success. I have placed the continuum picture of quantum mechanics in the foreground here in a *biased way* that does not correspond to my beliefs. Anyway, it would

([1]) I do not know of any proof of the relativistic generalization of the virial theorem in the literature, so I would like to present one here. One has:

$$\oint\left(-m_0\,c^2\sqrt{1-\left(\frac{v}{c}\right)^2}+\frac{m_0\,c^2}{\sqrt{1-\left(\frac{v}{c}\right)^2}}+E_{\text{pot}}\right)dt=\oint\left(\frac{m_0\,v^2}{\sqrt{1-\left(\frac{v}{c}\right)^2}}+E_{\text{pot}}\right)dt=\oint\left(\sum_{i=1}^{3}p_i\,\frac{dx^i}{dt}+E_{\text{pot}}\right)dt.$$

Under product integration, while observing the periodicity condition, that will be:

$$=\oint\left(-\sum_{i=1}^{3}x^i\,\frac{dp_i}{dt}+E_{\text{pot}}\right)dt.$$

As a result of the equations of motion, one will have $\frac{dp_i}{dt}=-\frac{\partial E_{\text{pot}}}{\partial x^i}$, so one will then get:

$$=\oint\left(\sum_{i=1}^{3}x^i\,\frac{\partial E_{\text{pot}}}{\partial x^i}+E_{\text{pot}}\right)dt.$$

The integrand will vanish here as a result of **Euler**'s theorem on homogeneous functions.

seem desirable to me to next pursue those ideas to their logical conclusions. In that spirit, what will be developed in the following chapter should be considered to be largely provisional. I hope to return to the whole connection with more general physical viewpoints soon.

Chapter III – Quantum-mechanical reinterpretation of Weyl's theory.

The investigations in the previous chapter were expressly concerned with the early stage of quantum mechanics that is characterized as the "**de Broglie** theory." They will then become false when one would like to adapt them to the **Schrödinger** theory directly, at least in the domain where the two theories overlap. However, one can already say that in any event our results must remain *asymptotically* correct in the limit of large quantum numbers, since both theories will merge into each other then.

One can then characterize the advance of the **Schrödinger** form of wave mechanics by the fact that it involved the "incorporation" of the trajectories of classical mechanics, which **de Broglie** had initially attributed to a wave only superficially by way of (5), in the calculations for a connected wave continuum. In geometric optics, the consideration of the isolated individual trajectories is physically equivalent to that of the wave fronts. By contrast, in wave optics, an isolated wave-ray will experience a certain influence due to its neighbor when it is *incorporated* into a front of rays. The fact that this influence is expressed is the characteristic statement of **Schrödinger**'s theory when it describes the wave function ψ by a wave equation instead of a **Jacobi** differential equation (6). Upon splitting into imaginary and real components, the **Schrödinger** wave equation for $\psi = |\psi| e^{2\pi i W/h}$ (W real) will read:

$$\left.\begin{aligned}\left(\frac{h}{2\pi i}\right)^2 \frac{\Box|\psi|}{|\psi|} + \left(\frac{\partial W}{\partial x^i} - \frac{e}{c}\Phi_i\right)\left(\frac{\partial W}{\partial x_i} - \frac{e}{c}\Phi^i\right) + m^2c^2 = 0, \\ \frac{\partial}{\partial x_k}\left\{|\psi|^2 \frac{e}{m}\left(\frac{\partial W}{\partial x_k} - \frac{e}{c}\Phi^k\right)\right\} = 0.\end{aligned}\right\} \tag{10}$$

In that representation, one recognizes the counterpart to the **de Broglie** theory in the appearance of the term $\Box|\psi|/|\psi|$. At the same time, it will also be obvious here that one is dealing with a problem with two unknown real functions. The second equation is the continuity equation of the current whose four components are included in the curly brackets.

There is *no question* that we presently give preference to the ideas of **Schrödinger**'s theory over those of **de Broglie** without reservation, due to the better agreement with experiments in the former. We must see that the discrepancy between **Schrödinger**'s theory and that of **Weyl** is no oversight in **Schrödinger**'s theory.

If one observes that the deviations will characteristically break down for small quantum numbers then there can be no doubt as to where the difficulty arises: All of the competency of **Weyl**'s theory is, so to speak, tailored to classical mechanics, and therefore to **de Broglie**'s theory, which is associated with it. As a result, one would not at all expect or demand that it should already be suited to **Schrödinger**'s theory. Rather, the problem must be to complete the step in **Weyl**'s now-dated theory that would correspond to the one that led from **de Broglie** to **Schrödinger**. It must be modified corresponding to the quantum-mechanical correction to classical laws in its own right.

One can foresee the direction that the correction to **Weyl**'s metric might take. Up to now, it was assumed that the four potentials Φ_i that provide a complete description of the electromagnetic field are the only thing that is definitive of the displacement of line segments (2.a). The situation has changed now in that the **Schrödinger** ψ is added to the four state quantities of the field Φ_i as a fifth one, which presents the field quantities Φ_i symmetrically in many respects [above all, in its representation by a variational problem (1)]. Matter, which is banished from the field behind impenetrable boundary surfaces or relegated to its singularities in its electron-theoretic conception, is now spread out over all of space, and whereas in **Weyl**'s theory, one has every right to imagine that a yardstick in "empty" space is influenced by only the electromagnetic potentials that prevail there, one will now have to account for the fact that the older separation between *impenetrable* matter and the $\kappa\varepsilon\nu\grave{o}\nu$ (†) has vanished, and one will *always* find oneself in the interior, so to speak, of the everywhere-penetrable (2) new substance $|\psi|$.

One should then expect that in addition to the external electromagnetic field quantities, one will also have to consider internal ones that depend upon only $|\psi|$. **Madelung** (3) gave the "potential" for that internal action of the ψ-field on itself. I would like to propose the relativistic generalization of it as:

$$e\,\Phi_5 = m_0\,c^2\left(1-\sqrt{1+\left(\frac{h}{2\pi i}\right)^2\frac{1}{m_0^2\,c^2}\frac{\Box|\psi|}{|\psi|}}\right). \tag{11}$$

The word "potential" must be used with care. Φ_5 does not correspond to the "scalar" potential Φ_4 that figures relativistically as the *temporal component* of a four-vector since is it also a scalar invariant relativistically. Correspondingly, Φ_5 cannot govern the change in a line segment along a certain world-direction either. If one wishes to assume that it has any influence on the gauge at all then it can depend upon *only the magnitude* of the four-dimensional displacement of the line segment, but *not on its direction*. If one correspondingly introduces a fifth coordinate by way of the world-line element $dx_5 = c\,d\tau$ (τ = proper time) that is not independent of the remaining dx_i, but is coupled to them by the condition (4):

(1) **E. Schrödinger**, Ann. Phys. (Leipzig) **82** (1927), pp. 265.

(†) Translator: "empty, void, vacuum."

(2) That is because ψ satisfies a linear differential equation, and therefore the superposition principle. However, the property of impenetrability seems to find its quantum-mechanical expression in the form of the Pauli exclusion principle. [**P. Ehrenfest**, Naturwissenschaften **15** (1927), pp. 161]

(3) **E. Madelung**, Zeit. Phys. **40** (1940), pp. 322.

(4) The appearance of that five-dimensional quadratic form is entirely reasonable in the spirit of **Weyl**'s demand of gauge invariance. The world-line element $d\tau$ (dx_5, resp.) is indeed a *relativistic* invariant, but not a gauge invariant (the transition to a different gauge will change $d\tau$), but the *vanishing* of the quadratic form (12) is probably gauge invariant. Obviously, one must understand **Kaluza**'s five-dimensional Ansätze in that sense.

$$dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2 + dx_5^2 = 0 \tag{12}$$

then one might suspect that:

$$l = l_0 \exp\left[\frac{2\pi i}{h}\int \sum_{i=1}^{5} \frac{e}{c}\,\Phi_i\,dx^i\right] \tag{13}$$

represents the quantum-mechanical generalization of **Weyl**'s line metric.

In order to verify the identity of (12) with **Schrödinger**'s wave function, we must next give the path along which the generalized line metric (13) is translated. We would once more like to prescribe that the transport should have the current velocity. However, in so doing, we must observe that the components u^i of the four-velocity are no longer given by (7), although the representation of the current in the second equation (10) relates to the splitting off of the factor $e\,\psi\,\bar{\psi}$ as a rest charge density. Due to (10.1), the velocity components that are split off in that way would not, in fact, fulfill the identity of the four-velocity ([1]):

$$u_k\,u^k = \frac{dx_k}{d\tau}\frac{dx^k}{d\tau} = -c^2 . \tag{12′}$$

Rather, we must write:

$$\frac{dx_k}{dx_5} = \frac{u_k}{c} = \frac{\psi\,\bar{\psi}}{\rho}\cdot\frac{e}{m_0\,c}\left(\frac{\partial W}{\partial x} - \frac{e}{c}\,\Phi_k\right), \tag{7.a}$$

in which the factor:

$$\rho = e\psi\,\bar{\psi}\sqrt{1 + \left(\frac{h}{2\pi\,i}\right)^2 \frac{1}{m_0^2\,c^2}\frac{\Box|\psi|}{|\psi|}} = e\psi\,\bar{\psi}\left(1 - \frac{1}{m_0\,c^2}\,\Phi_5\right) \tag{14}$$

has been split off as the "rest charge density."

In that notation, one gets:

$$e\,\Phi_5 = m_0\,c^2\left(1 - \frac{\rho}{e\psi\,\bar{\psi}}\right), \tag{11.a}$$

and the five-dimensional form ([2]) of the first **Schrödinger** equation will read:

$$\sum_{i=1}^{5}\left(\frac{\partial W}{\partial x_i} - \frac{e}{c}\,\Phi^i\right)\left(\frac{\partial W}{\partial x^i} - \frac{e}{c}\,\Phi_i\right) = 0 . \tag{10.a}$$

([1]) Unless otherwise stated to the contrary, the summation over equal indices from 1 to 4 will alwaysbe understood in what follows, as it was up to now.

([2]) One should observe in this that Φ_5 is initially an undetermined unknown in its own right. It is known that it is still a source of wonder why the same thing is not true for the potentials Φ_1, Φ_2, Φ_3, Φ_4, as one might expect. [**E. Schrödinger**, Ann. Phys. (Leipzig) **82** (1927), pp. 265.] One has $\partial W / \partial x_5 = m_0\,c$ [cf., (5.a)].

We shall now compare the segment l (13) with the **Schrödinger** scalar ψ along the current (7.a). We will then get for ψ / l that:

$$\frac{\psi}{l} = \frac{|\psi|}{l_0}\exp\left[\frac{2\pi i}{h}\int\sum_{i=1}^{5}\left(\frac{\partial W}{\partial x_i} - \frac{e}{c}\,\Phi_i\right)dx^i\right].$$

(7.a) will then imply that this:

$$= \frac{|\psi|}{l_0}\exp\left[\frac{2\pi i}{h}\int\sum_{i=1}^{4}\frac{\psi\bar{\psi}}{\rho}\,\frac{e}{mc}\left(\frac{\partial W}{\partial x_i} - \frac{e}{c}\,\Phi^i\right)\left(\frac{\partial W}{\partial x^i} - \frac{e}{c}\,\Phi_i\right)dx_5 + \left(\frac{\partial W}{\partial x_5} - \frac{e}{c}\,\Phi_5\right)dx_5\right]$$

and (11.a) will give:

$$= \frac{|\psi|}{l_0}\exp\left[\frac{2\pi i}{h}\int\frac{\psi\bar{\psi}}{\rho}\,\frac{e}{mc}\sum_{i=1}^{5}\left(\frac{\partial W}{\partial x_i} - \frac{e}{c}\,\Phi^i\right)\left(\frac{\partial W}{\partial x^i} - \frac{e}{c}\,\Phi_i\right)dx_5\right]$$

$$= \frac{|\psi|}{l_0}.$$

The last step is true due to (10.a). One will not get ψ / l = const. next, but:

$$\frac{\psi}{l} = \frac{|\psi|}{l_0}, \tag{8.a}$$

which is a single-valued function of position ([1]). However, the potentials Φ_k are established physically only up to an additive gradient. If I introduce:

([1]) One can express that method of proof more logically in the spirit of five-dimensional geometry as follows:

$\left(\frac{\partial W}{\partial x^i} - \frac{e}{c}\,\Phi_i\right)$ is parallel to the five-current $j_i = \frac{e}{m}\psi\bar{\psi}\left(\frac{\partial W}{\partial x^i} - \frac{e}{c}\,\Phi_i\right)$,

dx^i shall be chosen to be parallel to the five-current j^i.

The five-current is orthogonal to itself $\left(\sum_{i=1}^{5} j_i\, j^i = 0\right)$. Thus, j_i is also orthogonal to dx^i, and therefore $\sum_{i=1}^{5}\left(\frac{\partial W}{\partial x^i} - \frac{e}{c}\,\Phi_i\right)dx^i = 0$.

I would like to thank Herrn **A. Landé** for communicating this beautiful formulation to me. In that way, the fifth component of the five-current will be $j_5 = \rho c$.

$$\Phi_k^* = \Phi_k - \frac{hc}{2\pi i e} \frac{\partial}{\partial x^k} \ln |\psi|$$

in place of them, which will leave the electromagnetic field strengths untouched, then it will follow that ψ / l = const.

The single-valuedness of the gauge that moves along with the current, which goes back to the resonance of the waves, now carries over from **de Broglie**'s theory to **Schrödinger**'s with no further discussion, such that we do not have to add anything to the arguments in Chapter 2 here.

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