"Über den Hilbertschen Unabhängigkeitssatz in der Theorie des Maximums und Minimums der einfachen Integrale," Math. Ann. **58** (1904), 235-248.

# On Hilbert's independence theorem in the theory of the maximum and minimum of a simple integral (\*)

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In his most-interesting "Mathematischen Problemen," **Hilbert** had suggested, *inter alia*, a very promising new method for deriving the criteria for the maxima and minima of simple and multiple integrals (\*\*). At the basis for that new method is a theorem that, for the simplest problem in the calculus of variations, where one deals with the greatest or least value of a given integral:

$$J \equiv \int_{x_0}^{x_1} f(x, y, y') dx \qquad \left( y' \equiv \frac{dy}{dx} \right)$$

with fixed values of *x*, *y* at the limits, then reads:

If one has found the simply-infinite family of solutions:

y = Y(x, a) ,

which assumes the prescribed fixed value  $y_0$  for  $x = x_0$ , for the second-order differential equation of the problem, then one needs only to form the first-order differential equation:

$$y'=p\left(x,y\right),$$

whose complete solution will be represented by that family, in order to have arrived at the function p = p(x, y), and at the same time, a function that makes the expression:

$$f(x, y, p) + (y' - p) \frac{\partial f(x, y, p)}{\partial p}$$

<sup>(\*)</sup> Reproduced from the Leipziger Berichten on 4 May 1903.

<sup>(\*\*)</sup> Göttinger Nachrichten (1900), 291-296.

and therefore makes the integral:

$$J^* \equiv \int_{x_0}^{x_1} \left\{ f(x, y, p) + (y' - p) \frac{\partial f(x, y, p)}{\partial p} \right\} dx$$

independent of the choice of the function y of x as a result of the prescribed limit values (as long as the function under the integral sign remains continuous).

In what follows, we shall address the extension of **Hilbert**'s independence theorem to the more general problem:

Among all functions  $y_1, ..., y_n$  of x that satisfy r < n given condition equations:

$$f_{\rho}(x, y_1, \dots, y_n, y'_1, \dots, y'_n) = 0$$
  $\rho = 1, 2, \dots, r,$ 

which are mutually-independent relative to the differential quotients  $y'_1$ , ...,  $y'_n$ , possess fixed values at the two given limits  $x_0$  and  $x_1 > x_0$ , and remain continuous between those limits, along with their first differential quotients, find the ones that provide a greatest or least value to the given integral:

$$\int_{x_0}^{x_1} f(x, y_1, \dots, y_n, y_1', \dots, y_n') dx .$$

Indeed, it will be shown that the theorem in question will follow naturally when one has integrated the differential equations of the problem by means of its **Hamilton-Jacobi** partial differential equation using the method of **Clebsch** (\*), which has a very close relationship to the theorem. Originally, I was of the opinion that this kind of integration could be assumed to be known, which would then complete the proof of the theorem in a few sentences. However, in the foundations of the **Clebsch** method of integration that were known to me, there was one point that was not sufficiently clear to me and which will matter essentially here. For the sake of clarity, I have therefore ultimately preferred to preface the proof of the generalized independence theorem with the derivation of the canonical method of integrating the differential equations of the calculus of variations (\*\*).

<sup>(\*)</sup> Crelle's Journal, **55**, pp. 337-340.

<sup>(\*\*)</sup> The theorem is already given correctly for the case of n = 2, in general, but only an entirely-flawed proof of it was produced, although one does not at all need to appeal to the partial differential equation of the problem in that special case, and one can also show that the integrability conditions are fulfilled without it.

# Integrating the differential equation of the problem by a complete solution to its Hamilton-Jacobi partial differential equation.

If one sets:

(1) 
$$\Omega = f(x, y_1, \dots, y_n, y'_1, \dots, y'_n) + \sum_{\rho=1}^r \lambda_r f_\rho(x, y_1, \dots, y_n, y'_1, \dots, y'_n)$$

then the present problem will lead to the *n* second-order **Lagrange** differential equations between  $y_1, ..., y_n, \lambda_1, ..., \lambda_r$ , and *x*:

(2) 
$$\frac{d}{dx}\frac{\partial\Omega}{\partial y'_i} = \frac{\partial\Omega}{\partial y_i}.$$

In order for that to be possible at all and well-defined for fixed limiting values of x,  $y_1$ , ...,  $y_n$ , the very next thing to address is whether the n + r equations (2) and:

(3) 
$$f(x, y_1, \dots, y_n, y'_1, \dots, y'_n) = 0$$

define a system of differential equations of order 2n, i.e., their complete integration must include 2n arbitrary constants, and once more that is identical to demanding that the n + r unknowns (\*):

$$y'_1, \ldots, y'_n, \lambda_1, \ldots, \lambda_r$$

Substituting its solutions:

$$y'_{i} = \Psi_{i}\left(x, y_{1}, \dots, y_{n}, \frac{\partial V}{\partial y_{1}}, \dots, \frac{\partial V}{\partial y_{n}}\right),$$
$$\lambda_{\rho} = \Pi_{\rho}\left(x, y_{1}, \dots, y_{n}, \frac{\partial V}{\partial y_{1}}, \dots, \frac{\partial V}{\partial y_{n}}\right)$$

will then take the equation:

$$\frac{\partial V}{\partial x} = \Omega - \sum_{h=1}^{n} y'_{h} \frac{\partial \Omega}{\partial y'_{h}} = f - \sum_{h=1}^{n} \frac{\partial V}{\partial y_{h}} y'_{h}$$

to a first-order partial differential equation:

(4) 
$$\frac{\partial V}{\partial x} = H\left(x, y_1, \dots, y_n, \frac{\partial V}{\partial y_1}, \dots, \frac{\partial V}{\partial y_n}\right)$$

<sup>(\*)</sup> Cf., Leipziger Berichte (1895), pp. 138.

between the unknown function V and the independent variables  $x, y_1, ..., y_n$ .

Therefore, if V = W is any solution of that **Hamilton-Jacobi** partial differential equation then the values:

(5)  
$$\begin{cases} y_i' = \Psi_i \left( x, y_1, \dots, y_n, \frac{\partial W}{\partial y_1}, \dots, \frac{\partial W}{\partial y_n} \right), \\ \lambda_\rho = \prod_\rho \left( x, y_1, \dots, y_n, \frac{\partial W}{\partial y_1}, \dots, \frac{\partial W}{\partial y_n} \right) \end{cases}$$

will satisfy the n + r equations:

(6) 
$$\frac{\partial \Omega}{\partial y'_i} = \frac{\partial W}{\partial y_i}, \quad f_{\rho} = 0,$$

which they are solutions of, as well as the equation:

(7) 
$$\frac{\partial W}{\partial x} = \Omega - \sum_{h=1}^{n} y'_{h} \frac{\partial \Omega}{\partial y'_{h}},$$

identically for all values of the variables  $x, y_1, ..., y_n$ , as well as the possible arbitrary constants of the solution W.

Let *a* be any constant that does not merely enter into *W* additively.

If one then imagines setting the  $y'_i$  and  $\lambda_r$  equal to the values (5) and then differentiating the relations (7) partially with respect to  $y_i$  and *a* then one will get:

$$\frac{\partial^2 W}{\partial x \, \partial y_i} = \frac{\partial \Omega}{\partial y_i} + \sum_{\rho=1}^r f_\rho \frac{\partial \lambda_\rho}{\partial y_i} - \sum_{h=1}^n y'_h \frac{\partial}{\partial y_i} \frac{\partial \Omega}{\partial y'_h} ,$$
$$\frac{\partial^2 W}{\partial x \, \partial a} = \sum_{\rho=1}^r f_\rho \frac{\partial \lambda_\rho}{\partial a} - \sum_{h=1}^n y'_h \frac{\partial}{\partial a} \frac{\partial \Omega}{\partial y'_h} .$$

Therefore, their solutions will satisfy equations (6), as well as the equations:

(8)  
$$\begin{cases} \frac{\partial^2 W}{\partial y_i \partial x} = \frac{\partial \Omega}{\partial y_i} - \sum_{h=1}^n \frac{\partial^2 W}{\partial y_h \partial x_i} y'_h, \\ \frac{\partial^2 W}{\partial a \partial x} = -\sum_{h=1}^n \frac{\partial^2 W}{\partial y_h \partial a} y'_h, \end{cases}$$

identically for all values of x,  $y_1$ , ...,  $y_n$ . Now let:

(9) 
$$V = W(x, y_1, ..., y_n, a_1, ..., a_n) + \text{const.}$$

be any particular complete solution of the partial differential equation (4). The determinant:

(10) 
$$\sum \pm \frac{\partial^2 W}{\partial y_1 \partial a_1} \frac{\partial^2 W}{\partial y_2 \partial a_2} \cdots \frac{\partial^2 W}{\partial y_n \partial a_n}$$

will not be identically zero then. Therefore, the *n* equations:

(11) 
$$\frac{\partial W}{\partial a_i} = \alpha_i ,$$

in which  $\alpha_1, ..., \alpha_n$  shall also mean arbitrary constants, will always determine the *n* unknowns  $y_1, ..., y_n$ , and when their solutions:

(12) 
$$y_i = \varphi_i (x, a_1, \dots, a_n, \alpha_1, \dots, \alpha_n) \equiv [y_i]$$

are substituted in equations (5), the latter will go to:

(13) 
$$\begin{cases} y_i' = \psi_i (x, a_1, \dots, a_n, \alpha_1, \dots, \alpha_n) \equiv [y_i'], \\ \lambda_\rho = \pi_\rho (x, a_1, \dots, a_n, \alpha_1, \dots, \alpha_n) \equiv [\lambda_\rho], \end{cases}$$

so the values (12) and (13) will satisfy equations (11), (6), (7), and (8) identically for every x,  $a_1$ , ...,  $a_n$ ,  $\alpha_1$ , ...,  $\alpha_n$ . If one suggests the substitution of the values as in (12) and (13) by [] then one will have, on the one hand:

$$\begin{bmatrix} \frac{\partial \Omega}{\partial y'_i} \end{bmatrix} \equiv \begin{bmatrix} \frac{\partial W}{\partial y_i} \end{bmatrix}, \qquad \alpha_i \equiv \begin{bmatrix} \frac{\partial W}{\partial a_i} \end{bmatrix},$$

and partial differentiation with respect to x will then yield:

$$\frac{\partial}{\partial x} \left[ \frac{\partial \Omega}{\partial y'_i} \right] \equiv \left[ \frac{\partial^2 W}{\partial y_i \partial x} \right] + \sum_{h=1}^n \left[ \frac{\partial^2 W}{\partial y_i \partial y_h} \right] \frac{\partial [y_h]}{\partial x} ,$$
$$0 \equiv \left[ \frac{\partial^2 W}{\partial a_i \partial x} \right] + \sum_{h=1}^n \left[ \frac{\partial^2 W}{\partial a_i \partial y_h} \right] \frac{\partial [y_h]}{\partial x} .$$

On the other hand, from (8), one has:

$$\left[\frac{\partial^2 W}{\partial y_i \, \partial x}\right] \equiv \left[\frac{\partial \Omega}{\partial y_i}\right] - \sum_{h=1}^n \left[\frac{\partial^2 W}{\partial y_i \, \partial y_h}\right] [y'_h] ,$$

$$\left[\frac{\partial^2 W}{\partial a_i \,\partial x}\right] \equiv -\sum_{h=1}^n \left[\frac{\partial^2 W}{\partial y_h \,\partial a_i}\right] [y'_h] ,$$

and when one adds those identities to the foregoing ones, that will yield:

(14) 
$$\frac{\partial}{\partial x} \left[ \frac{\partial \Omega}{\partial y'_i} \right] = \left[ \frac{\partial \Omega}{\partial y_i} \right] + \sum_{h=1}^n \left[ \frac{\partial^2 W}{\partial y_i \partial y_h} \right] \left( \frac{\partial [y_h]}{\partial x} - [y'_h] \right),$$

(15) 
$$0 = \sum_{h=1}^{n} \left[ \frac{\partial^2 W}{\partial y_h \partial a_i} \right] \left( \frac{\partial [y_h]}{\partial x} - [y'_h] \right).$$

Now since the determinant (10) is non-zero, by its nature, and is entirely free of the undetermined quantities  $\alpha_1, ..., \alpha_n$ , it also cannot vanish identically as a result of equations (11). Therefore, the determinant of the *n* linear homogeneous relations (15) will not be identically zero either, and as a result those relations will demand that one must have:

$$\frac{\partial[y_h]}{\partial x} - [y'_h] \equiv 0$$

in each case, which will make the relations (14) reduce to:

$$\frac{\partial}{\partial x} \left[ \frac{\partial \Omega}{\partial y'_i} \right] \equiv \left[ \frac{\partial \Omega}{\partial y_i} \right],$$

i.e., since one also has that each  $[f_{\rho}] \equiv 0$ , one has, however:

The differential equations (2) and (3) will be integrated completely by way of the solutions (12) and (13) to equations (11) and (6), which are solutions that will, at the same time, fulfill the equation (7) identically for all arbitrary values of  $a_1, \ldots, a_n, \alpha_1, \ldots, \alpha_n$ , and in general one has:

(16) 
$$\frac{\partial [y_h]}{\partial x} = [y'_h] \; .$$

With that, we have achieved the canonical method of integrating the differential equations of our problem, and we can now move on to the derivation of **Hilbert**'s independence theorem.

### § 2.

# **Connection between the Hamilton-Jacobi partial differential equation** and the Hilbert independence theorem.

The problem in question prescribes fixed given values for the functions  $y_1, \ldots, y_n$  at the two fixed limits  $x_0$  and  $x_1$ . If one then lets  $y_{10}, \ldots, y_{n0}$  denote the given initial values of the y then in order for the problem to be soluble, the *n* equations (12), along with the *n* equations:

(17) 
$$y_{i0} = \varphi_i(x_0, a_1, ..., a_n, \alpha_1, ..., \alpha_n)$$

must also be soluble for their 2n arbitrary constants a and  $\alpha$ , or in other words, the last n equations must be soluble for *n* of those constants, and after substituting the solutions, the *n* equations (12) must determine the remaining n constants.

However, when one sets:

(18) 
$$W_0 \equiv W(x_0, y_{10}, ..., y_{n0}, a_1, ..., a_n, \alpha_1, ..., \alpha_n),$$

equations (11) will directly imply the *n* equations:

(19) 
$$\frac{\partial (W - W_0)}{\partial a_k} = 0,$$

and when the initial values  $x_0, y_{10}, \ldots, y_{n0}$  are considered to be given, those equations will determine the unknowns  $y_1, \ldots, y_n$  as functions of merely x and the n arbitrary constants  $a_1, \ldots, a_n$ . Let:

(20) 
$$y_i = Y_i (x, a_1, ..., a_n)$$

be the solutions of equations (19) for the *n* unknowns  $y_i$  that assume the values  $y_i = y_{i0}$  for  $x = x_0$ .

Hence, at the same time, equations (19) will always possess solutions:

(21) 
$$a_k = A_k (x, y_1, ..., y_n) \equiv \{a_k\}$$

for the *n* unknowns  $a_k$  that fulfill equations (20) identically, so they will also be once more solutions of those equations, and when one substitutes the values (20) in equations (5), from (16), one will get a system of values for the  $y'_i$  and  $\lambda_r$ :

(22) 
$$\begin{cases} y_i' = Y_i'(x, a_1, \dots, a_n) \equiv \frac{\partial Y_i}{\partial x}, \\ \lambda_\rho = L_\rho (x, a_1, \dots, a_n), \end{cases}$$

and together with the values (20) for the  $y_i$ , they satisfy equations (6) and (7) identically for every  $a_1, ..., a_n$ , whereby, at the same time, the values of the  $y_i$  and  $\lambda_\rho$  will define a system of particular solutions to the differential equations (2) and (3) with only *n* arbitrary constants  $a_1, ..., a_n$ .

Now, the values (21) for the  $a_k$ , which might be emphasized by { }, fulfill equations (20) identically when they are substituted, and that will convert the values (22) for the  $y'_i$  and  $\lambda_\rho$  into functions of just  $x, y_1, ..., y_n$ . If we denote those functions by  $p_i$  and  $\mu_\rho$ , so we define:

(23) 
$$p_i \equiv \left\{ \frac{\partial Y_i}{\partial x} \right\}, \qquad \mu_\rho \equiv \left\{ L_\rho \right\},$$

then the substitution { } will take equations (20) and (22) to the identities and equations:

(24) 
$$y_i \equiv y_i$$
 and  $y'_i = p_i$ ,  $\lambda_\rho = \mu_\rho$ ,

of which the *n* equations:

$$y'_i = p_i$$

define a system of first-order differential equations that will be integrated completely by equations (20) (\*).

Finally,  $\overline{f}$ ,  $\overline{f}_{\rho}$ , and  $\overline{\Omega}$  might denote the functions that arise from  $f, f_{\rho}$ , and  $\Omega$ , resp., by the substitutions (24) when one initially regards the  $p_i$  and  $\mu_{\rho}$  as arbitrary variables.

The values (20) and (22) satisfy equations (6) and (7) identically for all values of  $a_1, ..., a_n$ . Those identities can therefore not be eliminated by substituting the values (21) for the  $a_k$ . However, from (24) and the definitions that were just given, they will then go to:

$$\frac{\partial \overline{\Omega}}{\partial p_i} \equiv \left\{ \frac{\partial W}{\partial y_i} \right\} \,, \qquad \overline{f}_\rho \equiv 0 \,, \qquad \overline{\Omega} - \sum_{i=1}^n p_i \, \frac{\partial \overline{\Omega}}{\partial p_i} \equiv \left\{ \frac{\partial W}{\partial x} \right\} \,,$$

in which the  $p_i$  and  $\mu_\rho$  are naturally understood to mean the well-defined functions of x,  $y_1$ , ...,  $y_n$  that are defined by (23). Those identities show that for all arbitrary functions  $y_1$ , ...,  $y_n$  of x, one will have the relation:

$$\overline{f} + \sum_{i=1}^{n} (y'_i - p_i) \frac{\partial \overline{\Omega}}{\partial p_i} \equiv \left\{ \frac{\partial W}{\partial x} \right\} + \sum_{i=1}^{n} \left\{ \frac{\partial W}{\partial y_i} \right\} y''_i.$$

<sup>(\*)</sup> Equations (20) reduce to  $y_i = y_{i0}$  for  $x = x_0$ , and therefore they will no longer determine the unknowns  $a_k$  at all. Thus, their solutions (21), and with them the functions  $p_i$  and  $\mu_\rho$ , will necessarily assume the undefined form 0 / 0 at the location  $x = x_0$  where the  $y_i$  are prescribed the given initial values  $y_{i0}$ , i.e., the initial values of the functions  $\{a_k\}$ ,  $p_i$ , and  $\mu_\rho$  are still not determined by the initial values of the functions y alone, but will also still depend upon the initial values of the differential quotients of those functions. In particular, the definition of the  $p_i$  and the way that equations (21) arose from equations (20) shows that for any system of functions  $y_1, \ldots, y_n$  that possesses the given initial values, the initial values of the  $p_i$  will be nothing but the initial values of the differential quotients  $y'_i$ .

On the other hand, when one also converts the function  $W - W_0$  to a function of only  $x, y_1, ..., y_n$  by the substitution  $\{ \}$ , one will have:

$$\frac{d\{W-W_0\}}{dx} \equiv \left\{\frac{\partial W}{\partial x}\right\} + \sum_{i=1}^n \left\{\frac{\partial W}{\partial y_i}\right\} y'_i + \sum_{k=1}^n \left\{\frac{\partial (W-W_0)}{\partial a_k}\right\} \frac{d\{a_k\}}{dx} ,$$

and the values (21) of the  $a_k$  will also be solutions of equations (19), so one will have:

$$\left\{\frac{\partial \left(W-W_{0}\right)}{\partial a_{k}}\right\}\equiv0$$

for each of them. The following formula ensues directly from the two foregoing identities:

(26) 
$$\overline{f} + \sum_{i=1}^{n} (y_i' - p_i) \frac{\partial \overline{\Omega}}{\partial p_i} \equiv \frac{d\{W - W_0\}}{dx} ,$$

which is nothing but the canonical form of **Hilbert**'s independence theorem, and from it, for all functions  $y_1, ..., y_n$  that possess the fixed initial values  $y_{10}, ..., y_{n0}$  at the location  $x = x_0$  and keep the function  $\{W - W_0\}$  continuous between the limits  $x_0$  and x, it will follow that:

(27) 
$$\int_{x_0}^x \left(\overline{f} + \sum_{i=1}^n (y_i' - p_i) \frac{\partial \overline{\Omega}}{\partial p_i}\right) dx \equiv \{W - W_0\}.$$

Although that result already suffices to enable one to represent the desired extension of the independence theorem to full generality, let me make the following remarks, which are irrelevant to that extension itself, because the general relations between the **Hamilton-Jacobi** partial differential equation (4) and the problem of making the expression:

(28) 
$$\overline{\Omega} + \sum_{i=1}^{n} (y_i' - p_i) \frac{\partial \overline{\Omega}}{\partial p_i}$$

become a complete differential quotient, and along with that, also satisfying the r finite equations:

$$(29) f_{\rho} = 0,$$

on the one hand, while also shedding a brighter light upon the last problem and the integration of the differential equations (2) and (3), on the other.

Any system of functions  $p_1, ..., p_n, \mu_1, ..., \mu_r$  of  $x, y_1, ..., y_n$  for which the expression (28) becomes a complete differential quotient and which simultaneously satisfy equations (29)

identically, one can in fact, by means of merely a quadrature, add a function V of the same variable that will make the functions considered also satisfy the equation:

(30) 
$$\overline{\Omega} + \sum_{i=1}^{n} (y'_i - p_i) \frac{\partial \overline{\Omega}}{\partial p_i} = \frac{dV}{dx}$$

identically. However, those functions will then fulfill the equations:

$$\overline{\Omega} - \sum_{i=1}^{n} p_i \frac{\partial \overline{\Omega}}{\partial p_i} = \frac{\partial V}{\partial x}$$

and

(31) 
$$\frac{\partial \overline{\Omega}}{\partial p_i} = \frac{\partial V}{\partial y_i}, \quad \overline{f}_{\rho} = 0$$

identically, and therefore the function *V* will be a solution of the first-order partial differential equation that arises when one eliminates the n + r quantities  $p_i$  and  $\mu_\rho$  from the foregoing equation by means of the last n + r equations, i.e., it will even be a solution of the **Hamilton-Jacobi** partial differential equation (4). (Hence, our function  $\{W - W_0\}$  will also be such a thing then, which should also be obvious, moreover). Conversely, one will obtain a system of functions  $p_1, ..., p_n$ ,  $\mu_1, ..., \mu_r$  from any solution *V* of that equation that will fulfill equations (29) and (30) identically when one solves the n + r equations (31) for those n + r unknowns.

However, in order for the *p* and  $\mu$  to be functions of *x*, *y*<sub>1</sub>, ..., *y<sub>n</sub>* that make the expression:

$$\overline{\Omega} + \sum_{i=1}^{n} (y'_{i} - p_{i}) \frac{\partial \overline{\Omega}}{\partial p_{i}} \equiv B + \sum_{h=1}^{n} B_{h} y'_{h}$$

into a complete differential quotient, they must satisfy the n + n(n-1)/2 integrability conditions:

$$\frac{\partial B_i}{\partial x} - \frac{\partial B}{\partial y_i} = 0 , \qquad \frac{\partial B_i}{\partial y_h} - \frac{\partial B_h}{\partial y_i} = 0$$

identically. As a result of the last one, one can replace the first *n* of them with:

$$\frac{\partial B_i}{\partial x} - \frac{\partial B}{\partial y_i} + \sum_{h=1}^n p_i \left( \frac{\partial B_i}{\partial y_h} - \frac{\partial B_h}{\partial y_i} \right) = 0.$$

Moreover, since the partial differential quotients of the  $p_h$  drop out, one will have:

$$\frac{\partial B}{\partial y_i} \equiv \frac{\partial}{\partial y_i} \left( \overline{\Omega} - \sum_{h=1}^n p_h \frac{\partial \overline{\Omega}}{\partial p_h} \right) \equiv \frac{\partial \overline{\Omega}}{\partial y_i} + \sum_{\rho=1}^r \frac{\partial \mu_\rho}{\partial y_i} \overline{f_\rho} - \sum_{h=1}^n p_h \frac{\partial}{\partial y_i} \frac{\partial \overline{\Omega}}{\partial p_h}$$

Since:

$$B_h \equiv \frac{\partial \bar{\Omega}}{\partial p_h},$$

the integrability conditions can then be written:

(32) 
$$\frac{\partial}{\partial x}\frac{\partial\overline{\Omega}}{\partial p_i} + \sum_{h=1}^n p_h \frac{\partial}{\partial y_i}\frac{\partial\overline{\Omega}}{\partial p_h} - \frac{\partial\overline{\Omega}}{\partial y_i} = \sum_{\rho=1}^r \overline{f}_\rho \frac{\partial\mu_\rho}{\partial y_i}$$

(33) 
$$\frac{\partial}{\partial y_h} \frac{\partial \overline{\Omega}}{\partial p_i} - \frac{\partial}{\partial y_i} \frac{\partial \overline{\Omega}}{\partial p_h} = 0,$$

in which one first considers the dependency of the *p* and  $\mu$  on *x*, *y*<sub>1</sub>, ..., *y<sub>n</sub>* in the second partial differentiations.

From just those remarks, every solution V to the **Hamilton-Jacobi** partial differential equation (4) will produce a system of solutions  $p_1, ..., p_n, \mu_1, ..., \mu_r$  of those n + n (n - 1) / 2 first-order partial differential equations (29) that simultaneously satisfy the r finite equations (29), and the most general system of solutions of that type will be defined by equations (31) when one understands the V in them to mean the general solution of equation (4).

Finally, if the  $p_i$  and  $\mu_{\rho}$  are functions of x,  $y_1$ , ...,  $y_n$  that fulfill equations (29) and (32) identically, and one constructs the n first-order differential equations and r finite equations:

(34) 
$$y'_i = p_i(x, y_1, ..., y_n), \qquad \lambda_\rho = \mu_\rho(x, y_1, ..., y_n)$$

from those functions then one will see immediately from the form of equations (32) that the complete solutions of equations (34) are, at the same, solutions of the differential equations (2) and (3) as well. Hence, from the foregoing, along with our previous equations (24), there are also infinitely-many other systems of equations (34) that possess the double property that their complete integration will also produce solutions of the differential equations (2) and (3) and along their right-hand sides, the expression (28) will also be a complete differential quotient, which immediately includes the known theorem that any solution of their **Hamilton-Jacobi** partial differential equations (2) and (3) with *n* new arbitrary constants.

By contrast, the integrability conditions (33) are not merely consequences of the conditions (32) and equations (29). Therefore, it will no longer be true that the right-hand sides of any system of equations (34) whose complete solutions likewise satisfy the differential equations (2) will be functions that make the expression (28) into a complete differential quotient, as one would have in the case of n = 1.

#### § 3.

#### General statement for Hilbert's independence theorem.

With the result that was obtained in (26) that the expression:

$$\overline{f} + \sum_{i=1}^{n} (y_i' - p_i) \frac{\partial \Omega}{\partial p_i}$$

will be a complete differential quotient for the functions  $p_i$  and  $\mu_{\rho}$  that are defined by (23), the extension of **Hilbert**'s independence theorem to the present problem has been achieved and proved, although, of course, only under the assumption that one has introduced canonical constants  $a_1, ..., a_n, \alpha_1, ..., \alpha_n$  into the complete solutions:

$$y_i = \varphi_i (\alpha, a_1, ..., a_n, \alpha_1, ..., \alpha_n),$$
  
 $\lambda_{\rho} = \pi_{\rho} (\alpha, a_1, ..., a_n, \alpha_1, ..., \alpha_n)$ 

to the differential equations of the problem. However, since our functions:

$$Y_i$$
,  $\frac{\partial Y_i}{\partial x}$ , and  $L_{\rho}$ 

must emerge from the functions:

$$\varphi_i$$
,  $\frac{\partial \varphi_i}{\partial x}$ , and  $\pi_\rho$ ,

respectively, when one substitutes the solutions  $\alpha_1, ..., \alpha_n$  of the *n* equations (17) in them, one can briefly say that: One obtains the functions  $p_i$  and  $\lambda_\rho$  that were defined in (23) when one uses the 2n equations:

$$\varphi_i (x_0, a_1, ..., a_n, \alpha_1, ..., \alpha_n) = y_{i 0},$$
  
 $\varphi_i (x, a_1, ..., a_n, \alpha_1, ..., \alpha_n) = y_i$ 

to eliminate the 2*n* canonical constants  $a_1, ..., a_n, \alpha_1, ..., \alpha_n$  from the equations:

$$p_i = \frac{\partial \varphi_i}{\partial x}, \qquad \mu_{\rho} = \pi_{\rho}$$

Now, if the equations:

(35) 
$$y_i = \theta_i (x, c_1, ..., c_{2n}), \qquad \lambda_\rho = X_\rho (x, c_1, ..., c_{2n})$$

represent any system of complete differential solutions of the differential equations (2) and (3) then there must necessarily be 2n substitutions that are free of x and take the form:

(36) 
$$c_h = C_h (a_1, ..., a_n, \alpha_1, ..., \alpha_n)$$

that make:

and therefore also:

$$\theta_{i} \equiv \varphi_{i}, \qquad X_{\rho} \equiv \pi_{\rho},$$
$$\frac{\partial \theta_{i}}{\partial x} \equiv \frac{\partial \varphi_{i}}{\partial x},$$

and one can then obtain our functions  $p_i$  and  $\mu_{\rho}$  in such a way that one can use the 4n equations (36) and:

(37) 
$$\theta_i(x_0, c_1, ..., c_{2n}) = y_{i0}, \qquad \theta_i(x, c_1, ..., c_{2n}) = y_i$$

to eliminate the 4n constants:

 $c_1, ..., c_{2n}, a_1, ..., a_n, \alpha_1, ..., \alpha_n$ 

from the formulas:

(38) 
$$p_i = \frac{\partial \theta_i}{\partial x}, \qquad \mu_\rho = X_\rho.$$

However, the constants a and  $\alpha$  can be eliminated due to the fact that one can skip the transformation formulas (36) completely, and one will then need to eliminate only the 2n integration constants c from (38) by means of the 2n equations (37). If we do not perform that elimination all at once, but in two steps, then we can express our result in general in the form:

*If one uses the function:* 

$$\Omega \equiv f(x, y_1, \dots, y_n, y'_1, \dots, y'_n) + \sum_{\rho=1}^r \lambda_\rho f_\rho(x, y_1, \dots, y_n, y'_1, \dots, y'_n)$$

to construct the n + r differential equations:

$$\frac{d}{dx}\frac{\partial\Omega}{\partial y'_i} = \frac{\partial\Omega}{\partial y_i}, \qquad f_{\rho} = 0,$$

and one has found any system of complete solutions:

$$y_i = \theta_i (x, c_1, ..., c_{2n}), \qquad \lambda_\rho = X_\rho (x, c_1, ..., c_{2n})$$

to it then one can determine n of the 2n integration constants  $c_1, ..., c_{2n}$  in terms of the n remaining ones from the n equations:

$$\theta_i(x, c_1, \ldots, c_{2n}) = y_{i0}.$$

If the complete solutions go to the particular solutions (for fixed given  $x_0, y_{10}, ..., y_{n0}$ ) with only n constants  $c_1, ..., c_n$  that are still arbitrary when one substitutes the solutions:

$$y_i = Y_i (x, c_1, ..., c_{2n}), \qquad \lambda_\rho = L_\rho (x, c_1, ..., c_{2n})$$

then one needs only to appeal to the first n of the latter equations to eliminate those constants from the n + r equations:

$$y'_i = \frac{\partial Y_i}{\partial x}, \qquad \lambda_{\rho} = L_{\rho}$$

in order to obtain functions of  $x, y_1, ..., y_n$  in the right-hand sides of the equations that arise in that way:

( $\alpha$ )  $y'_i = p_i$ ,  $\lambda_{\rho} = \lambda_{\rho}$ 

that make the expression:

$$\overline{f} + \sum_{h=1}^{n} (y'_h - p_h) \frac{\partial \overline{\Omega}}{\partial p_h}$$

into a complete differential quotient and satisfy the r equations:

$$\overline{f}_{\rho} = 0$$

identically, moreover.

In those expressions,  $\overline{\Omega}$ ,  $\overline{f}$ ,  $\overline{f}_{\rho}$  mean the functions of  $x, y_1, ..., y_n, p_1, ..., p_n, \mu_1, ..., \mu_r$  that arise from  $\Omega$ ,  $f, f_{\rho}$ , resp., and the first n of those substitutions define a system of first-order differential equations that will be integrated completely by the n equations  $y_i = Y_i$ .

If one would like to prove that theorem independently of the **Hamilton-Jacobi** partial differential equation then one would have to show that the functions  $p_1, ..., p_n, \mu_1, ..., \mu_r$  that are obtained in the given way satisfy the n + n (n - 1) / 2 first-order partial differential equations (32) and (33). Indeed, that is immediately obvious for equations (32) and can also be verified in the case n = 2 without any laborious calculation for the one equation to which the system (33) will then reduce. However, producing such a direct proof for equations (33) would seem to present significant difficulties in the general case.

"Über den Hilbertschen Unabhängigkeitssatz in der Theorie des Maximums und Minimums der einfachen Integrale. II" Math. Ann. **62** (1906), 335-350.

# On Hilbert's independence theorem in the theory of the maximum and minimum of a simple integral (\*)

Part II.

By

## A. Mayer in Leipzig.

Translated by D. H. Delphenich

In my communication on **Hilbert**'s independence theorem (<sup>\*\*</sup>), I had only one goal in mind, that of arriving at the particular form of that theorem that would relate the Weierstrass *E*-function to the special extremal field for the problem in the calculus of variations that was posed that would lead directly to Jacobi's criterion for conjugate points (<sup>\*\*\*</sup>). It was only recently that I noticed that this special, but generally quite important, form of the theorem essentially emerges from the fact that one completely integrates Hamilton's partial differential equation for the problem by the Jacobi-Hamilton method. However, with that in mind, it was immediately clear to me from the outset that one must be able to arrive at the general solution of the Hilbert problem upon whose solution the independence theorem is based (<sup>†</sup>) when one employs just the general Cauchy method (<sup>††</sup>) instead of that special method in order to integrate the partial differential equation thus-obtained does, in fact, encompass all possible solutions of that problem, but is even more general than the independence theorem itself. That result shall be developed in what follows independently of the previous one and without assuming that the Cauchy method is known.

<sup>(\*)</sup> Reproduced from the Leipziger Berichten on 1 May 1905 in a somewhat-altered form.

<sup>(\*\*)</sup> These Annals, Bd. 58, pp. 235-248.

<sup>(\*\*\*)</sup> Cf., **Bolza**, *Lectures on the calculus of variations*, Chicago, 1904, pps. 91, 60, 82. The heading of § **3** in my first was not chosen correctly and might perhaps read: "Liberating Hilbert's independence theorem from Hamilton's partial differential equation."

<sup>(&</sup>lt;sup>†</sup>) Previously, I had considered that problem to be virtually identical to the independence theorem itself because the latter follows immediately from the former. However, it is much clearer, and also more correct, to treat both separately.

<sup>(&</sup>lt;sup>††</sup>) Cf., these Annals, Bd. 3, pp. 447-8.

### The connection between the problem in the calculus of variations and the Hilbert problem.

As before, we shall once more address the problem in the calculus of variations:

I. – Among all functions  $y_1, ..., y_n$  of x that satisfy r < n given first-order differential equations:

(1) 
$$f_{\rho}(x, y_1, ..., y_n, y'_1, ..., y'_n) = 0$$
  $(\rho = 1, 2, ..., r)$ 

that are soluble for r of the differential quotients  $y', ..., y'_n$ , possess fixed values at two given limits  $x_0$  and  $x_1 > x_0$ , and remain continuous between those two limits, find the ones for which the given integral:

$$\int_{x_0}^{x_1} f(x, y_1, ..., y_n, y_1', ..., y_n') dx$$

attains a greatest or least value.

That problem will be solved by the n + r differential equations:

(2) 
$$\frac{d}{dx}\frac{\partial\Omega}{\partial y'_i} = \frac{\partial\Omega}{\partial y_i}, \qquad f_{\rho} = 0,$$

in which one has:

(3) 
$$\Omega \equiv f + \sum_{\rho=1}^{r} \lambda_{\rho} f_{\rho} ,$$

and it can be possible and well-determined only when the n + r equations:

(4) 
$$\frac{\partial \Omega}{\partial y'_i} = v_i, \qquad f_\rho = 0$$

are soluble for the n + r unknowns:

 $y'_1, \ldots, y'_n, \lambda_1, \ldots, \lambda_r.$ 

Let:

(5) 
$$\begin{cases} y'_i = p_i(x, y_1, ..., y_n, v_1, ..., v_n), \\ \lambda_\rho = \mu_\rho(x, y_1, ..., y_n, v_1, ..., v_n) \end{cases}$$

~

be those solutions, and upon substituting them, one will have:

(6) 
$$\sum_{i=1}^{n} y_i' \frac{\partial \Omega}{\partial y_i'} - \Omega = H(x, y_1, \dots, y_n, v_1, \dots, v_n).$$

With that equation, the values (5) also satisfy the equations:

$$y'_i = \frac{\partial H}{\partial v_i}, \qquad -\frac{\partial \Omega}{\partial y_i} = \frac{\partial H}{\partial y_i}$$

identically. The introduction of the variables v in place of the differential quotients y' and the multipliers  $\lambda$  then converts the differential equations (2) into the 2n canonical differential equations:

(7) 
$$\frac{dy_i}{dx} = \frac{\partial H}{\partial v_i}, \qquad \frac{dv_i}{dx} = -\frac{\partial H}{\partial y_i}.$$

When one substitutes any system of solutions:

$$y_i = y_i(x)$$
,  $\lambda_\rho = \lambda_\rho(x)$ 

to the differential equations (2) in the equations:

$$v_i = rac{\partial \Omega}{\partial y'_i} \; ,$$

one will get a corresponding system of solutions:

$$y_i = y_i(x)$$
,  $v_i = v_i(x)$ 

to the differential equations (7), and conversely when one substitutes any system of solutions to the latter differential equations in the last r of equations (5), it will once more produce a system of solutions to the differential equations (2) *that likewise satisfy equations* (5) *identically for the solutions to the differential equations* (7) *under consideration*.

Assuming that, I will show that instead of:

$$y'_1, \ldots, y'_n, \lambda_1, \ldots, \lambda_r,$$

one should write:

$$p_1, \ldots, p_n, \mu_1, \ldots, \mu_r,$$

and indicate that by inclusion within [], and thus define:

(3') 
$$|\Omega| \equiv |f| + \sum_{\rho=1}^{r} \mu_{\rho} |f_{\rho}|,$$

and in general:

$$|f_{\sigma}| \equiv f_{\sigma}(x, y_1, \ldots, y_n, v_1, \ldots, v_n),$$

and I will now investigate the connection that exists between Problem I and the following Hilbert problem:

II. Determine the variables  $p_1, ..., p_n, \mu_1, ..., \mu_r$  as functions of  $x, y_1, ..., y_n$  such that the expression:

(8) 
$$|\Omega| + \sum_{i=1}^{n} (y'_i - p_i) \frac{\partial |\Omega|}{\partial p_i},$$

in which  $y_1, ..., y_n$  are considered to be undetermined functions of x, will be a complete differential quotient, and at the same time satisfy the r conditions:

$$(1') |f_{\rho}| = 0$$

*identically*.

If (8) is a complete differential quotient then there will exist a function of x,  $y_1$ , ...,  $y_n$ , for which one has:

(9) 
$$|\Omega| + \sum_{i=1}^{n} (y'_i - p_i) \frac{\partial |\Omega|}{\partial p_i} \equiv \frac{dV}{dx},$$

and which one will find with that Ansatz by a mere quadrature.

However, the requirement (9) decomposes into the 1 + n identically-fulfilled conditions:

(10) 
$$|\Omega| - \sum_{i=1}^{n} p_i \frac{\partial |\Omega|}{\partial p_i} \equiv \frac{\partial V}{\partial x},$$

(11) 
$$\frac{\partial |\Omega|}{\partial p_i} = \frac{\partial V}{\partial y_i} \ .$$

When combined with the r condition equations (1'), the n equations (11) now determine:

$$p_1, \ldots, p_n, \mu_1, \ldots, \mu_r$$

as functions of:

$$x, y_1, \ldots, y_n, \frac{\partial V}{\partial y_1}, \ldots, \frac{\partial V}{\partial y_n},$$

and indeed, they imply the meaning of equations (5) as a result:

(5')  
$$\begin{cases} p_i = p_i \left( x, y_1, \dots, y_n, \frac{\partial V}{\partial y_1}, \dots, \frac{\partial V}{\partial y_n} \right), \\ \mu_\rho = \mu_\rho \left( x, y_1, \dots, y_n, \frac{\partial V}{\partial y_1}, \dots, \frac{\partial V}{\partial y_n} \right). \end{cases}$$

From definition (6) of the function H, moreover, the substitution of those values will take equation (10) to the first-order partial differential equation between V and the n + 1 independent variables  $x, y_1, ..., y_n$ :

(12) 
$$\frac{\partial V}{\partial x} + H\left(x, y_1, \dots, y_n, \frac{\partial V}{\partial y_1}, \dots, \frac{\partial V}{\partial y_n}\right) = 0.$$

That immediately implies the theorem:

III. From every system of functions  $p_1, ..., p_n, \mu_1, ..., \mu_r$  that solve Problem II, one will obtain a solution V by a simple quadrature such that the partial differential equation (12) will be coupled with those functions by the relations (5'). Conversely, when any solution V to that partial differential equation is substituted in equations (5), that will yield a system of solutions to Problem II.

Furthermore, the expression (8) has the form:

$$|\Omega| + \sum_{i=1}^{n} (y'_i - p_i) \frac{\partial |\Omega|}{\partial p_i} \equiv B + \sum_{h=1}^{n} B_h y'_h .$$

Any system of functions p and  $\mu$  that satisfies the requirement (9) must then fulfill the n + n (n - 1) / 2 conditions:

$$\frac{\partial B_i}{\partial x} - \frac{\partial B}{\partial y_i} = 0 , \qquad \qquad \frac{\partial B_i}{\partial y_h} - \frac{\partial B_h}{\partial y_i} = 0$$

identically.

One can use the last of them to replace the first *n* of them with the following *n* :

$$\frac{\partial B_i}{\partial x} - \frac{\partial B}{\partial y_i} + \sum_{h=1}^n p_h \left( \frac{\partial B_i}{\partial y_h} - \frac{\partial B_h}{\partial y_i} \right) = 0.$$

However, one has:

$$B_h \equiv \frac{\partial |\Omega|}{\partial p_h},$$

and since the partial differential quotients of the functions  $p_h$  drop out by themselves:

$$\frac{\partial B}{\partial y_i} \equiv \frac{\partial}{\partial y_i} \left\{ |\Omega| - \sum_{h=1}^n p_h \frac{\partial |\Omega|}{\partial p_h} \right\} \equiv \frac{\partial |\Omega|}{\partial y_i} + \sum_{\rho=1}^r \frac{\partial \mu_{\rho}}{\partial p_h} |f_{\rho}| - \sum_{h=1}^n p_h \frac{\partial}{\partial y_i} \frac{\partial |\Omega|}{\partial p_h} + \sum_{\rho=1}^r \frac{\partial \mu_{\rho}}{\partial p_h} |f_{\rho}| + \sum_{h=1}^n p_h \frac{\partial}{\partial y_i} \frac{\partial |\Omega|}{\partial p_h} + \sum_{\rho=1}^r \frac{\partial \mu_{\rho}}{\partial p_h} |f_{\rho}| + \sum_{h=1}^n p_h \frac{\partial}{\partial y_i} \frac{\partial |\Omega|}{\partial p_h} + \sum_{\rho=1}^r \frac{\partial \mu_{\rho}}{\partial p_h} |f_{\rho}| + \sum_{h=1}^n p_h \frac{\partial}{\partial y_i} \frac{\partial |\Omega|}{\partial p_h} + \sum_{\rho=1}^r \frac{\partial \mu_{\rho}}{\partial p_h} |f_{\rho}| + \sum_{h=1}^n p_h \frac{\partial}{\partial y_i} \frac{\partial |\Omega|}{\partial p_h} + \sum_{\rho=1}^n \frac{\partial \mu_{\rho}}{\partial p_h} |f_{\rho}| + \sum_{h=1}^n p_h \frac{\partial \mu_{\rho}}{\partial p_h} + \sum_{h=1}^n \frac{\partial \mu_{\rho}}{\partial p_h} |f_{\rho}| + \sum_{h=1}^n p_h \frac{\partial \mu_{\rho}}{\partial p_h} + \sum_{h=1}^n \frac{\partial \mu_{\rho}}{\partial p_h} |f_{\rho}| + \sum_{h=1}^n p_h \frac{\partial \mu_{\rho}}{\partial p_h} + \sum_{h=1}^n \frac{\partial \mu_{\rho}}{\partial p_h} |f_{\rho}| + \sum_{h=1}^n p_h \frac{\partial \mu_{\rho}}{\partial p_h} + \sum_{h=1}^n \frac{\partial \mu_{\rho}}{\partial p_h} |f_{\rho}| + \sum_{h=1}^n p_h \frac{\partial \mu_{\rho}}{\partial p_h} + \sum_{h=1}^n \frac{\partial$$

As a result of the condition equations (1') that one further prescribes, the integrability conditions of the expression (8) can then be written:

(13) 
$$\frac{\partial}{\partial x}\frac{\partial|\Omega|}{\partial p_i} + \sum_{h=1}^n p_h \frac{\partial}{\partial y_h}\frac{\partial|\Omega|}{\partial p_i} = \frac{\partial|\Omega|}{\partial y_i},$$

(14) 
$$\frac{\partial}{\partial y_h} \frac{\partial |\Omega|}{\partial p_i} = \frac{\partial}{\partial y_i} \frac{\partial |\Omega|}{\partial p_h} ,$$

and upon performing the second partial differentiations, the first *n* of them will become:

$$(13') \begin{cases} \frac{\partial^2 |\Omega|}{\partial p_i \partial x} + \sum_{h=1}^n p_h \frac{\partial^2 |\Omega|}{\partial p_i \partial y_i} + \sum_{k=1}^n \frac{\partial^2 |\Omega|}{\partial p_i \partial p_k} \left( \frac{\partial p_k}{\partial x_i} + \sum_{h=1}^n p_h \frac{\partial p_k}{\partial x_h} \right) + \sum_{\rho=1}^r \frac{\partial |f_{\rho}|}{\partial p_i} \left( \frac{\partial \mu_{\rho}}{\partial x} + \sum_{h=1}^n p_h \frac{\partial \mu_{\rho}}{\partial y_h} \right) \\ = \frac{\partial |\Omega|}{\partial y_i}. \end{cases}$$

On the other hand, if one couples the functions  $y_1$ , ...,  $y_n$ ,  $\lambda_1$ , ...,  $\lambda_r$  with any system of solutions to Problem II by means of the n + r equations:

- $(15) y'_k = p_k$
- and
- (16)  $\lambda_{\rho} = \mu_{\rho}$

then differentiating those equations will imply that:

$$y_k'' = \frac{\partial p_k}{\partial x} + \sum_{h=1}^n \frac{\partial p_k}{\partial y_h} p_h ,$$
  
$$\lambda_{\rho}' = \frac{\partial \mu_{\rho}}{\partial x} + \sum_{h=1}^n \frac{\partial \mu_{\rho}}{\partial y_h} p_h .$$

Furthermore,  $|\Omega|$  and  $|f_{\rho}|$  once more go back to the original functions  $\Omega$  and  $f_{\rho}$ , and the conditions (13) will then become:

$$\frac{\partial^2 \Omega}{\partial y'_i \partial x} + \sum_{h=1}^n \frac{\partial^2 \Omega}{\partial y'_i \partial y_i} y'_h + \sum_{k=1}^n \frac{\partial^2 \Omega}{\partial y'_i \partial y'_k} y''_i + \sum_{\rho=1}^r \frac{\partial f_\rho}{\partial y_i} \lambda'_\rho = \frac{\partial \Omega}{\partial y_i},$$

i.e., those conditions and the conditions (1) will be converted into the equations:

(2) 
$$\frac{d}{dx}\frac{\partial\Omega}{\partial y'_i} = \frac{\partial\Omega}{\partial y_i}, \qquad f_{\rho} = 0$$

Hence, as long as  $p_1, ..., p_n, \mu_1, ..., \mu_r$  are functions of  $x, y_1, ..., y_n$  in equations (15) and (16) that fulfill the conditions (1') and (13) identically, the functions  $y_1, ..., y_n, \lambda_1, ..., \lambda_r$  of x that one obtains by completely integrating the n differential equations (15) and substituting the solutions in equations (16) will be solutions with n arbitrary constants of the differential equations of Problem I.

It will then further follow directly from Theorem III:

IV. Any solution V to the partial differential equation (12) will belong to a system of solutions  $y_1, ..., y_n, \lambda_1, ..., \lambda_r$  to the differential equations (2) with n arbitrary constants, relative to which the solutions  $y_1, ..., y_n$  are mutually independent, and one will get that system of solutions when one completely integrates the first n of them, which define a system of first-order differential equations between  $y_1, ..., y_n$  and x, and substitutes their solutions in the last r equations in (5').

#### § 2.

# Derivation of a certain system of solutions to the differential equations (2) with *n* arbitrary constants from the complete solutions to those equations.

However, let us return to the differential equations (2) and their canonical form! Let:

(17) 
$$y_i = \varphi_i(x, c_1, ..., c_{2n}), \qquad v_i = \psi_i(x, c_1, ..., c_{2n})$$

be the complete solutions to the differential equations (7) that are obtained with the help of the equations:

$$v_i = \frac{\partial \Omega}{\partial y'_i}$$

from any known system of complete solutions:

(18) 
$$y_i = \varphi_i (x, c_1, ..., c_{2n}), \qquad \lambda_\rho = \Theta_\rho (x, c_1, ..., c_{2n})$$

to the differential equations (2).

If *a* is a new arbitrary constant or also just any value of *x* that is determined in such a way that the 2*n* equations (17) will also remain soluble for their 2*n* integration constants  $c_1, ..., c_{2n}$  when x = a then one can introduce the initial values:

$$a_i = \varphi_i (a, c_1, ..., c_{2n}), \qquad b_i = \psi_i (a, c_1, ..., c_{2n})$$

for the variables  $y_i$  and  $v_i$  for x = a as new arbitrary constants instead of the latter and in that way obtain a new complete system of solutions for the differential equations (7) that have the form:

(19) 
$$\begin{cases} y_i = y_i(x, a_1, \dots, a_n, b_1, \dots, b_n), \\ v_i = v_i(x, a_1, \dots, a_n, b_1, \dots, b_n), \end{cases}$$

in which one has:

$$y_i(a, a_1, ..., a_n, b_1, ..., b_n) \equiv a_i,$$
  
 $v_i(a, a_1, ..., a_n, b_1, ..., b_n) \equiv b_i.$ 

When one then makes an arbitrary choice of the function:

$$A \equiv A (a, a_1, \ldots, a_n)$$

in order to set each:

$$b_h=rac{\partial A}{\partial a_h},$$

that new complete system of solutions will itself once more produce a new system of solutions to the differential equations (7):

(20) 
$$\begin{cases} y_i = y_i \left( x, a_1, \dots, a_n, \frac{\partial A}{\partial a_1}, \dots, \frac{\partial A}{\partial a_n} \right) \equiv \overline{y}_i (x, a_1, \dots, a_n, b_1, \dots, b_n), \\ v_i = v_i \left( x, a_1, \dots, a_n, \frac{\partial A}{\partial a_1}, \dots, \frac{\partial A}{\partial a_n} \right) \equiv \overline{v}_i (x, a_1, \dots, a_n, b_1, \dots, b_n) \end{cases}$$

that will yield:

$$y_i = a_i$$
,  $v_i = \frac{\partial A}{\partial a_i}$ ,

and that system corresponds to the system of solutions to the differential equations (2):

(21) 
$$y_i = \overline{y}_i(x, a_1, \dots, a_n), \qquad \lambda_\rho = \overline{\lambda}_\rho(x, a_1, \dots, a_n)$$

whose last *r* equations emerge from the last *r* equations in (5) by substituting the values (20). Its *n* arbitrary constants  $a_1, ..., a_n$  are the initial values of the solutions  $y_1, ..., y_n$  for x = a, and from pp. 3, the equations:

(5") 
$$\begin{cases} y'_i = p_i(x, \overline{y}_1, \dots, \overline{y}_n, \overline{v}_1, \dots, \overline{v}_n), \\ \lambda_\rho = \mu_\rho(x, \overline{y}_1, \dots, \overline{y}_n, \overline{v}_1, \dots, \overline{v}_n) \end{cases}$$

will be satisfied identically.

As a result of the way that equations (19) arise from equations (17), and they, in turn, from equations (18), one will obtain those generally particular solutions of the system (2) from its complete solutions (18) directly in such a way that one introduces the latter into the equations:

(22) 
$$\varphi_i(a, c_1, ..., c_{2n}) = a_i, \qquad \left\lfloor \frac{\partial \Omega}{\partial y'_i} \right\rfloor_{x=a} = \frac{\partial A}{\partial a_i}$$

and then eliminates the 2n integration constants  $c_1, ..., c_{2n}$  from them by means of those 2n equations.

§ 3.

# The system of solutions to the differential equations (2) that was obtained corresponds to a well-defined solution V to the partial differential equation (12), in the sense of Theorem IV.

If one indicates the substitution of the solutions (20) to the differential equations (7) that were just obtained by an overbar and defines *V* as a function of *x*,  $a_1, ..., a_n$  by the formula (\*):

(23) 
$$V \equiv A (a, a_1, \dots, a_n) + \int_a^x \left\{ \sum_{h=1}^n v_h \frac{\partial H}{\partial v_h} - H \right\} dx$$

then upon partial-differentiating with respect to  $a_k$ , one will get:

$$\frac{\partial V}{\partial a_k} \equiv \frac{\partial A}{\partial a_k} + \int_a^x \sum_{h=1}^n \left\{ \overline{v}_h \frac{\partial}{\partial a_k} \frac{\partial \overline{H}}{\partial v_h} - \frac{\partial \overline{H}}{\partial y_h} \frac{\partial \overline{y}_h}{\partial a_k} \right\} dx \,.$$

However, from (7), one has:

$$\frac{\partial \overline{H}}{\partial v_h} \equiv \frac{\partial \overline{y}_h}{\partial x}, \qquad \frac{\partial \overline{H}}{\partial y_h} \equiv -\frac{\partial \overline{v}_h}{\partial x},$$

so the sum under the integral sign possesses the value:

$$\int_{a}^{x} \Omega dx \quad \text{or} \quad \int_{a}^{x} f dx$$

upon substituting the solutions (21), while on the other hand, the following argument will itself likewise yield the Cauchy method of integrating the partial differential equation (12).

<sup>(\*)</sup> The integral in that formula is identical to the one that arises from:

$$\sum_{h=1}^{n} \left\{ \overline{v}_{h} \frac{\partial^{2} \overline{y}_{h}}{\partial x \partial a_{k}} + \frac{\partial \overline{v}_{h}}{\partial x} \frac{\partial \overline{y}_{h}}{\partial a_{k}} \right\} \equiv \frac{\partial}{\partial x} \sum_{h=1}^{n} \overline{v}_{h} \frac{\partial \overline{y}_{h}}{\partial a_{k}}$$

Moreover, one will have:

$$\overline{y}_h \equiv a_h , \qquad \overline{v}_h \equiv \frac{\partial A}{\partial a_h}$$

for x = a, and therefore:

$$\left[\sum_{h=1}^{n} \overline{v}_{h} \frac{\partial \overline{y}_{h}}{\partial a_{k}}\right]_{a}^{x} \equiv \sum_{h=1}^{n} \left(\overline{v}_{h} \frac{\partial \overline{y}_{h}}{\partial a_{k}} - \frac{\partial A}{\partial a_{h}} \frac{\partial a_{h}}{\partial a_{k}}\right) \equiv \sum_{h=1}^{n} \overline{v}_{h} \frac{\partial \overline{y}_{h}}{\partial a_{k}} - \frac{\partial A}{\partial a_{k}} ,$$

.

so all that will remain is:

(24) 
$$\frac{\partial V}{\partial a_k} \equiv \sum_{h=1}^n \overline{v}_h \frac{\partial \overline{y}_h}{\partial a_k}$$

Having established that, let:

(25) 
$$a_i = a_i (x, y_1, ..., y_n) \equiv (a_i)$$

be the solutions to the first n equations in (20), so the n equations:

(26) 
$$y_i = \overline{y}_i (x, a_1, ..., a_n),$$

in terms of their *n* arbitrary constants  $a_1, ..., a_n$ . The substitution of those solutions, which will be indicated by (), takes the function (23) to the function:

(27) 
$$(V) \equiv W(x, y_1, ..., y_n)$$
.

Now equations (26) reduce to  $y_i = a_i$  for x = a, so their solutions (25) must also yield  $a_i = y_i$  for x = a. However, from (23), one will have:

$$V = A (a, a_1, \ldots, a_n)$$

for x = a. The new function W will next possess the property that it assumes the value:

$$W = A (a, y_1, \ldots, y_n)$$

for x = a then. It follows further from the way that it came about that:

$$\frac{\partial W}{\partial y_i} \equiv \sum_{k=1}^n \left( \frac{\partial V}{\partial a_k} \right) \frac{\partial (a_k)}{\partial y_i},$$

so from (24):

$$\frac{\partial W}{\partial y_i} \equiv \sum_{h=1}^n (\overline{v}_h) \sum_{k=1}^n \left( \frac{\partial \overline{y}_h}{\partial a_k} \right) \frac{\partial (a_k)}{\partial y_i}.$$

However, since equations (25) are the solutions to equations (26), one will have:

$$y_h \equiv (\overline{y}_h)$$
,

and therefore, at the same time:

$$\frac{\partial y_h}{\partial y_i} \equiv \sum_{k=1}^n \left( \frac{\partial \overline{y}_h}{\partial a_k} \right) \frac{\partial (a_k)}{\partial y_i};$$

one will then get simply:

(28)  $\frac{\partial W}{\partial y_i} \equiv (\overline{y_i}) \ .$ 

Finally, it follows from the definition (24) that:

(29) 
$$\frac{dV}{dx} \equiv \sum_{h=1}^{n} \overline{v}_{h} \frac{\partial H}{\partial y_{h}} - \overline{H} .$$

Furthermore, equations (26) are, conversely, once more the solutions of their solutions (25). The substitution of the latter will then once more be inverted by the substitution of the values (26). From (27), one will then have:

$$(27') V \equiv W ,$$

and from (28):

(28') 
$$\frac{\overline{\partial W}}{\partial y_i} \equiv \overline{v_i} \ .$$

On the other hand, one will also have:

$$\frac{dV}{dx} \equiv \frac{\overline{\partial W}}{\partial x} + \sum_{h=1}^{n} \frac{\overline{\partial W}}{\partial y_h} \frac{\partial \overline{y}_h}{\partial x}$$

then, or from (28'), and since the overbar indicates the substitution of the solutions (20) to the system (7):

$$\frac{dV}{dx} \equiv \frac{\partial W}{\partial x} + \sum_{h=1}^{n} \overline{v}_{h} \frac{\partial H}{\partial v_{h}},$$

and from (29) and (28'), it will follow from this that:

$$\overline{\frac{\partial W}{\partial x}} + \overline{H} = \overline{\frac{\partial W}{\partial x}} + \overline{H} \left( x, \overline{y}_1, \dots, \overline{y}_n, \frac{\partial W}{\partial y_1}, \dots, \frac{\partial W}{\partial y_n} \right) \equiv 0.$$

However, one can once more invert the substitutions (26) in that identity with the substitutions of their solutions (25), and in that way, one will see that V = W is a solution of the partial differential equation:

(12) 
$$\frac{\partial V}{\partial x} + H\left(x, y_1, \dots, y_n, \frac{\partial H}{\partial y_1}, \dots, \frac{\partial H}{\partial y_n}\right) = 0.$$

Finally, if one observes that equations (5'), which our solutions (21) to the differential equations (2) satisfy identically, go to the identities:

$$y'_{i} = p_{i}\left(x, \overline{y}_{1}, \dots, \overline{y}_{n}, \frac{\overline{\partial W}}{\partial y_{1}}, \dots, \frac{\overline{\partial W}}{\partial y_{n}}\right),$$
$$\lambda_{\rho} = \mu_{\rho}\left(x, \overline{y}_{1}, \dots, \overline{y}_{n}, \frac{\overline{\partial W}}{\partial y_{1}}, \dots, \frac{\overline{\partial W}}{\partial y_{n}}\right)$$

by way of the identities (28') then one can now state the theorem:

V. If one has integrated the differential equations (3) completely and eliminated the 2n integration constants  $c_1, ..., c_{2n}$  from the complete solutions that were obtained:

$$y_i = \varphi_i (x, c_1, ..., c_{2n}), \qquad \lambda_\rho = \Theta_\rho (x, c_1, ..., c_{2n})$$

with the help of the 2n equations:

$$\varphi_i(x, c_1, \ldots, c_{2n}) = a_i, \qquad \left[\frac{\partial \Omega}{\partial y'_i}\right]_{x=a} = \frac{\partial A}{\partial a_i},$$

in which a is a new arbitrary constant or also a suitably-chosen well-defined value of x, then one will get a new system of solutions to those differential equations:

$$y_i = \overline{y}_i (x, a_1, \ldots, a_n), \qquad \lambda_\rho = \overline{\lambda}_\rho (x, a_1, \ldots, a_n),$$

whose n arbitrary constants  $a_1, ..., a_n$  are the initial values of the variables  $y_1, ..., y_n$  for x = a, and those new solutions to the differential equations (2) will belong to a certain solution V = W to the partial differential equation (12), and indeed a solution that assumes the value:

$$W = A (x, y_1, \ldots, y_n)$$

for x = a in such a way that it will satisfy the n + r equations (5') identically for V = W.

#### **§ 4.**

## Significance of Theorem V.

If we let the initial value *a* of *x* be entirely arbitrary in the foregoing then we can also partially differentiate (\*) formula (23) with respect to *a* and then get:

$$\frac{\partial V}{\partial a} \equiv \frac{\partial A}{\partial a} - \left[\sum_{h=1}^{n} \overline{v}_{h} \frac{\overline{\partial H}}{\partial v_{h}} - \overline{H}\right]_{x=a} + \sum_{h=1}^{n} \overline{v}_{h} \frac{\partial \overline{y}_{h}}{\partial a} - \left[\sum_{h=1}^{n} \overline{v}_{h} \frac{\partial \overline{y}_{h}}{\partial a}\right]_{x=a}$$

in the same way that led to formula (24), i.e., since:

$$\overline{y}_h = a_h$$
,  $\overline{v}_h = \frac{\partial A}{\partial a}$ ,  $\frac{\partial H}{\partial v_h} = \left[\frac{\partial \overline{y}_h}{\partial x}\right]_{x=a}$ 

for x = a, we will have:

$$\frac{\partial V}{\partial a} \equiv \frac{\partial A}{\partial a} + H\left(a, a_1, \dots, a_n, \frac{\partial A}{\partial a_1}, \dots, \frac{\partial A}{\partial a_n}\right) + \sum_{h=1}^n \overline{v}_h \frac{\partial \overline{y}_h}{\partial a} - \sum_{h=1}^n \frac{\partial A}{\partial a_h} \left[\frac{\partial \overline{y}_h}{\partial x} + \frac{\partial \overline{y}_h}{\partial a}\right]_{x=a}.$$

However,  $\overline{y}_h$  now has the form:

$$\overline{y}_h \equiv \overline{y}_h (x, a, a_1, ..., a_n)$$
.

It will then follow from  $a_h \equiv [\overline{y}_h]_{x=a}$  upon partially-differentiating with respect to *a* that:

$$0 \equiv \frac{\partial a_h}{\partial a} \equiv \left[\frac{\partial \overline{y}_h}{\partial x} + \frac{\partial \overline{y}_h}{\partial a}\right]_{x=a},$$

and therefore, all that will remain is:

$$\frac{\partial V}{\partial a} \equiv \frac{\partial A}{\partial a} + H\left(a, a_1, \dots, a_n, \frac{\partial A}{\partial a_1}, \dots, \frac{\partial A}{\partial a_n}\right) + \sum_{h=1}^n \overline{v}_h \frac{\partial \overline{y}_h}{\partial a}$$

On the other hand, one will indirectly obtain from the formula:

$$(27') V \equiv \overline{W}$$

<sup>(\*)</sup> The following calculation is essentially the same as the one that led to the second Hamilton partial differential equation in Problem I.

upon partially-differentiating with respect to *a* that:

$$\frac{\partial V}{\partial a} \equiv \overline{\frac{\partial W}{\partial a}} + \sum_{h=1}^{n} \overline{\frac{\partial W}{\partial y_h}} \frac{\partial \overline{y}_h}{\partial a},$$
$$\frac{\partial V}{\partial a} \equiv \overline{\frac{\partial W}{\partial a}} + \sum_{h=1}^{n} \overline{v}_h \frac{\partial \overline{y}_h}{\partial a},$$

and a comparison of those two values for  $\partial V / \partial a$  will immediately yield the formula:

(30) 
$$\frac{\overline{\partial W}}{\partial a} \equiv \frac{\partial A}{\partial a} + H\left(a, a_1, \dots, a_n, \frac{\partial A}{\partial a_1}, \dots, \frac{\partial A}{\partial a_n}\right).$$

Therefore, in particular, if the function:

or from (28'):

$$V = A(x, y_1, \ldots, y_n),$$

which is free of a, is itself a solution to the partial differential equation (12) then that will imply that:

$$\frac{\partial W}{\partial a} \equiv 0 ,$$

and thus, at the same time, when one again inverts the solutions (26) by substituting their solutions (25):

$$\frac{\partial W}{\partial a} \equiv 0 ,$$

i.e., the new solution V = W that is derived from (23) will then be free of *a*, as well. However, it will take on the value:

$$W = A (x, y_1, \ldots, y_n)$$

for x = a, and since every function  $F(x, y_1, ..., y_n)$  that is free of the arbitrary constant a is already given immediately by its value  $F(a, y_1, ..., y_n)$  for x = a, it will coincide entirely with the solution:

$$V = A(x, y_1, \ldots, y_n).$$

Hence, in order to obtain a system of solutions to the differential equations (2) that satisfies equations (5') for an arbitrary given solution:

$$V = F(x, y_1, \ldots, y_n)$$

to the partial differential equation (12), one needs only to let the constant a in Theorem V be completely arbitrary and take:

$$A(a, a_1, ..., a_n) \equiv F(a, a_1, ..., a_n).$$

However, from Theorem III, any system of solutions  $p_1, ..., p_n, \mu_1, ..., \mu_r$  to Problem II is coupled with a solution V to the partial differential equation (12) by equations (5'), and from Theorem IV, any such solution V will correspond to a system of solutions  $y_1, ..., y_n, \lambda_1, ..., \lambda_r$  to the differential equations (2) that satisfies equations (5'). One then sees that:

VI. After completely integrating the differential equations (2), Theorem V will allow one to obtain any system of solutions at all to those differential equations that are related to any system of solutions to Problem II by:

(
$$\alpha$$
)  $y'_i = p_i$ ,  $\lambda_{\rho} = \mu_{\rho}$ .

Indeed, the simultaneous discovery of such associated solutions to the differential equations (2) and Problem II from Theorem V and the identities (28) will then immediately produce the rule:

Once one has derived the new system of solutions:

(21) 
$$x_i = \overline{y}_i (x, a_1, ..., a_n), \qquad \lambda_\rho = \overline{\lambda}_\rho (x, a_1, ..., a_n)$$

from the complete solutions of the differential equations (2) according to the prescription in Theorem V, one substitutes those new solutions in the partial differential quotients of the function  $\Omega$  with respect to the  $y'_i$  and eliminates the *n* constants  $a_1, \ldots, a_n$  from the values thus-obtained:

$$\frac{\partial \Omega}{\partial y'_i} = \overline{v}_i \ (x, a_1, \dots, a_n)$$

by means of the first *n* of equations (21). If one obtains:

$$(\boldsymbol{\beta}) \qquad \qquad \frac{\partial \Omega}{\partial y'_i} = (\overline{v}_i) \equiv w_i (x, y_1, \dots, y_n)$$

in that way then one solves those n equations, together with the r given condition equations in Problem I:

(1) 
$$f_{\rho} = 0 ,$$

for the n + r unknowns  $y'_1, ..., y'_n, \lambda_1, ..., \lambda_r$ . The solutions ( $\alpha$ ) to those n + r equations then define a system of equations that the solutions (21) will satisfy identically and whose right-hand sides are, at the same time, solutions to Problem II.

However, calculating the values of the partial differential quotients of  $\Omega$  would be an unnecessary detour.

The solutions (21) do, in fact, satisfy equations (1) and ( $\beta$ ), on the one hand, but on the other hand, they also obviously satisfy the equations:

(
$$\gamma$$
)  $y'_i = \frac{\partial \overline{y}_i}{\partial x}, \qquad \lambda_{\rho} = \overline{\lambda}_{\rho}$ 

Along with the latter equations, they also therefore satisfy the ones that arise from those equations when one eliminates  $a_1, ..., a_n$  with the help of the first *n* equations (21), so the values of  $y'_i$  and  $\lambda_{\rho}$  that are obtained in that way, when expressed in the variables *x*,  $y_1, ..., y_n$ , must necessarily coincide with the solutions ( $\alpha$ ) to equations (1) and ( $\beta$ ).

One then obtains the solutions to Problem II that are coupled with the solutions (21) to the differential equations (2) by formula ( $\alpha$ ) simply by substituting the solutions to the first *n* equations in (21) for  $a_1, ..., a_n$  in the equations ( $\gamma$ ) that follow from (21).

#### § 5.

#### The limits of applicability of the independence theorem itself.

The special associated solutions to the differential equations (2) and Problem II that were obtained in my first communication, just like the Jacobi-Hamilton solution to the partial differential equation (12), exist only as long a solution to Problem I is actually possible and well-defined, so as long as the complete solutions:

$$y_i = \varphi_i (x, c_1, \ldots, c_{2n})$$

to its differential equations contain exactly 2n integration constants, in such a way that one can prescribe two fixed values for those solutions for two given values of x. By contrast, Theorem V assumes only that the n + r equations (4) are soluble for the n + r unknowns  $y'_1, ..., y'_n, \lambda_1, ..., \lambda_r$ or that the system of differential equations (2) actually has order 2n, and it will also still be true under that assumption when the complete solutions  $y_1, ..., y_n$  to those differential equations include less than 2n arbitrary constants. In particular, that will always be the case when the function f in Problem I is a complete differential quotient. However, it does not, unfortunately, follow from this that a *Hilbert independence theorem* also exists for the following problem of the calculus of variations:

VII. Among all continuous functions  $y_1, ..., y_n$  of x that satisfy r < n given first-order differential equations:

$$f_{\rho}(x, y_1, ..., y_n, y'_1, ..., y'_n) = 0$$
,

which are soluble for  $y'_1$ , ...,  $y'_n$  and the last n - 1 of which possess given values for two given values of  $x_0$  and  $x_1 > x_0$  of x, while only the value of  $y_1$  is prescribed for  $x = x_0$ , find the ones that are associated with a greatest or least value of the first function for  $x = x_1$ .

Namely, if one understands  $p_1, ..., p_n, \mu_1, ..., \mu_r$  to mean any well-defined system of solutions to Problem II then for all functions  $y_1, ..., y_n$  of x that are allowed by the conditions of Problem I that keep the expression (8) continuous over the integration interval, or what amounts to the same thing because of the conditions (1'), the expression:

$$|f| + \sum_{h=1}^{n} (y'_{h} - p_{h}) \frac{\partial |\Omega|}{\partial p_{h}},$$

then the integral:

(31) 
$$I^* \equiv \int_{x_0}^{x_1} \left\{ |f| + \sum_{h=1}^n (y'_h - p_h) \frac{\partial |\Omega|}{\partial p_h} \right\} dx$$

will unchangingly preserve the same value, since it depends upon only the limits  $x_0$ ,  $x_1$ , and the values of the functions  $y_1$ , ...,  $y_n$  at those two limits, and all of those limiting values are assumed to be fixed, and that result is the actual *Hilbert independence theorem*.

However, for a system of solutions:

$$y_i = \overline{y}_i, \qquad \lambda_{\rho} = \overline{\lambda}_{\rho}$$

to Problem I that satisfy the equations:

$$y'_i = p_i, \qquad \lambda_\rho = \mu_\rho,$$

that integral will take on the value:

$$I^* = \int_{x_0}^{x_1} \overline{f} \, dx \, .$$

If one then sets:

(32) 
$$E \equiv f - |f| - \sum_{h=1}^{n} (y'_h - p_h) \frac{\partial |\Omega|}{\partial p_h}$$

then one can (while always assuming that the continuity requirement has been fulfilled) express the change:

$$\Delta I = \int_{x_0}^{x_1} f \, dx - \int_{x_0}^{x_1} \overline{f} \, dx$$

that the given integral:

$$I \equiv \int_{x_0}^{x_1} f \, dx$$

experiences when it is first defined with a well-defined system of such solutions and then defined with any other functions  $y_1, ..., y_n$  that satisfy the conditions of Problem I as follows:

$$\Delta I = \int_{x_0}^{x_1} E \, dx \quad .$$

By contrast, although the superficial form of Problem VII seems to coincide with that of Problem I for  $f \equiv y'_1$ , they still differ quite essentially due to the fact that one can no longer prescribe the value of  $y_1$  for  $x = x_1$  in the former problem, but rather it must necessarily be left arbitrary. Hence, if one refers the integral (31) to Problem VII when one replaces f with  $y'_1$  then its value will no longer remain unvarying for all functions  $y_1, \ldots, y_n$  of x that are compatible with the conditions of the problem, but will depend upon the value that the first of those functions assumes for  $x = x_1$ , and the fundamental formula (33) will not be applicable to Problem VII at all then.