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Lagrange's method of multipliers and the most general problem in the calculus of variations for one independent variable

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In the 1885 issue of these Berichte, and then in somewhat more detail in volume 26 of the Mathematischen Annalen, I gave a rigorous basis for LAGRANGE's method of multipliers for the problem that can be regarded as the most general problem of the maximum and minimum of a simple integral ([1]):

Among all continuous functions $y_1, \ldots, y_n$ *of* x *that satisfy* r *given first-order differential equations and possess given values at the two given limits* x_0 *and* x_1, *find the ones for which a given integral*:

$$\int_{x_0}^{x_1} f(x, y_1, \ldots, y_n, y_1', \ldots, y_n')\, dx$$

attains a greatest or least value.

However, if one imagines that the differential equation:

$$y_0' - f = 0 ,$$

in conjunction with the limit condition that y_0 should vanish for $x = x_0$, will give the value of the function y_0 a the location $x = x_1$ as just:

$$y_{01} = \int_{x_0}^{x_1} f\, dx$$

then one will see that this problem is itself once more only a special case of the following one:

([1]) One can always restrict oneself to the first differential quotients of the unknown functions, since when higher differential quotients originally appear, one only needs to set the lower derivatives of each unknown function equal to new variables and then add those defining equations as new condition equations in order to reduce the problem to one with only first differential quotients.

Among all continuous functions y_0, y_1, …, y_n of the independent variable x that fulfill r + 1 given first-order differential equations:

$$(1) \qquad \varphi_k\,(x, y_0, y_1, \ldots, y_n, y_0', y_1', \ldots, y_n') = 0\,, \qquad k = 0, 1, \ldots, r < n$$

identically, and the last n of which possess given values x_0 and x_1 of x, but the first one y_0 possesses a given value for only $x = x_0$, moreover, find the ones that assign a greatest or least value to the function y_0 at the location $x = x_1$.

I have already addressed that most general problem in the calculus of variation for one independent variable previously (these Berichte 1878), but at the time, I simply assumed that the LAGRANGE rule was obvious.

The differential equations that it implies are formed in a completely symmetric way with respect to the variables y, and in that way they immediately make the reciprocity relations that exist between the present problem and the one in which y_0 switches roles with one of the other n variables y_1, …, y_n quite intuitive.

Therefore, although the first special problem, insofar as it is disproportionally more important than all of the interesting problems up to now in the calculus of variations, is still subordinate to it, the general problem, and above all the question of whether its solutions must necessarily satisfy the LAGRANGE differential equations, still attracts some interest.

The goal of the following analysis is to solve that problem. It will be shown that *Lagrange's method of multipliers* can, in fact, also remain valid for the general problem, and indeed, they are even simpler to establish here in some respects than they were for the previously-treated special problem. Namely, whereas certain exceptional cases can occur in the latter problem ([1]), the LAGRANGE rule is true here with no exceptions, and that also immediately explains the fact that the LAGRANGE equations are *reduced* linear differential equations relative to the multipliers in the general problem, but *unreduced* ones in the special one.

Naturally, the derivation of the differential equations of the general problem must also be combined with a discussion of those equations, i.e., one must also answer the question of when a solution to the problem is possible and determinate, and that again induced me to actually carry out the reduction of the differential equations of the problem to a first-order partial differential equation with only n + 1 independent variables that had been only suggested in 1878.

§ 1. – Deriving the differential equations of the problem and the number of its integration constants.

If finite equations occur among the given differential equations (1), or if the differential quotients y_0', y_1', …, y_n' can be eliminated completely from them, then can always imagine that

([1]) A new and highly original way of establishing the LAGRANGE method by TURKSMA will appear shortly in the Math. Ann. and also ignores those exceptions, moreover.

one or more of the unknowns y is determined as a function of x and the remaining ones. Thus, it is no loss of generality when one assumes that the determinant:

$$\sum \pm \varphi_0' \, y_0' \, \varphi_1' \, y_1' \cdots \varphi_r' \, y_r' \tag{2}$$

is neither zero by itself nor does it vanish identically as a consequence of the condition equations (1).

If the problem is soluble at all in general (i.e., with no restriction on the prescribed boundary values) then as long as the latter do not take on entirely special exceptional values, it must, in any event, also admit solutions for which the determinant (2) is not identically zero.

I imagine that those solutions have been found already and denote them by just $y_0, y_1, \ldots, y_n$ (such that in what follows $y_0, y_1, \ldots, y_n$ will mean well-defined continuous functions of x, but also that their first derivatives with respect to x_0 and x_1 will be assumed to be continuous).

Therefore, for all continuous variations $\delta y_1, \ldots, \delta y_n$ that vanish for $x = x_0$ and $x = x_1$ and satisfy the $r + 1$ linear differential equations:

$$\sum_{i=0}^{n} (\varphi_k' \, y_i \, \delta y_i + \varphi_k' \, y_i' \, \delta y_i') = 0 , \tag{3}$$

the variation δy_0 must assume the value zero at the location $x = x_0$, as well as the location $x = x_1$.

One then addresses the problem of deriving differential equations from the demand that our solutions $y_0, y_1, \ldots, y_n$ must necessarily satisfy and which, at the same time, also allows one to determine those solutions, and one will again arrive at the problem of calculating the values of those $r + 1$ variations $\delta y_0, \delta y_1, \ldots, \delta y_r$ from equations (3), which are soluble for the differential quotients of those variations themselves.

To that end, I multiply the $r + 1$ condition equations (3) by the temporarily-undetermined factors λ_k, and then add them and write the sum in the form:

$$\frac{d}{dx}\left(\sum_{i=0}^{n} \delta y_i \sum_{k=0}^{r} \lambda_k \, \varphi_k' \, y_i'\right) + \sum_{i=0}^{n} \delta y_i \sum_{k=0}^{r} \left(\lambda_k \, \varphi_k' \, y_i - \frac{d \lambda_k \, \varphi_k' \, y_i'}{dx}\right) = 0 . \tag{4}$$

Now, from the assumption that was made about the determinant (2), the $r + 1$ equations:

$$\sum_{k=0}^{r} \left(\lambda_k \, \varphi_k' \, y_\rho - \frac{d \lambda_k \, \varphi_k' \, y_\varphi'}{dx}\right) = 0 , \qquad \rho = 0, 1, \ldots, r \tag{5}$$

are soluble for the differential quotients of the multipliers $\lambda_0, \lambda_1, \ldots, \lambda_r$, and they therefore define an abbreviated system of $r + 1$ first-order linear differential equations in the $\lambda_0, \lambda_1, \ldots, \lambda_r$, and x that admit the linearly-independent system of solutions:

$$\lambda_0 = \lambda_0^\sigma,\ \lambda_1 = \lambda_1^\sigma,\ \ldots,\ \lambda_r = \lambda_r^\sigma, \qquad \sigma = 0, 1, \ldots, r.$$

If one successively assigns those $r + 1$ different systems of values to the multipliers then one will get the $r + 1$ equations:

$$\frac{d}{dx}\left(\sum_{i=0}^{n} \delta y_i \sum_{k=0}^{r} \lambda_k \varphi_k' y_i'\right) + \sum_{\tau=r+1}^{n} \delta y_\tau \sum_{k=0}^{r}\left(\lambda_k^\sigma \varphi_k' y_\tau - \frac{d\lambda_k^\sigma \varphi_k' y_\tau'}{dx}\right) = 0$$

from (4), and since all variations δy should vanish for $x = x_0$, the equations:

$$(6) \qquad \sum_{i=0}^{n} \delta y_i \sum_{k=0}^{r} \lambda_k \varphi_k' y_i' = -\sum_{i=0}^{n} \delta y_i \sum_{k=0}^{r} \lambda_k \varphi_k' y_i' - \int_{x_0}^{x_1} dx \sum_{\tau=r+1}^{n} \delta y_\tau \sum_{k=0}^{r}\left(\lambda_k^\sigma \varphi_k' y_\tau - \frac{d\lambda_k^\sigma \varphi_k' y_\tau'}{dx}\right)$$

will arise from them upon integrating between the limits x_0 and x_1. The determinant of those $r + 1$ equations:

$$\sum \pm \lambda_0^0 \lambda_1^1 \cdots \lambda_r^r \sum \pm \varphi_0' y_0' \varphi_1' y_1' \cdots \varphi_r' y_r'$$

is not zero, so it immediately yields the desired solutions $\delta y_0, \delta y_1, \ldots, \delta y_r$ to equations (3).

Furthermore, if we choose x_0 and x_1 such that this determinant does not vanish for $x = x_0$ and $x = x_1$ either then equations (6) will also still be soluble for $\delta y_0, \delta y_1, \ldots, \delta y_r$ at those two locations. When applied to the values x_0 and x_1 of x, on the one hand, they will show that $\delta y_0, \delta y_1, \ldots, \delta y_r$ already assume the desired value of zero at the location $x = x_0$ by themselves, and on the other hand, they will imply the $r + 1$ linearly-independent equations:

$$(7) \qquad \sum_{\rho=0}^{r} \delta y_{\rho i}\left[\sum_{k=0}^{r} \lambda_k^0 \varphi_k' y_\rho'\right]_{x=x_1} = -\int_{x_0}^{x_1} dx \sum_{\tau=r+1}^{n} \delta y_\tau \sum_{k=0}^{r}\left(\lambda_k^\sigma \varphi_k' y_\tau - \frac{d\lambda_k^\sigma \varphi_k' y_\tau'}{dx}\right)$$

for the values $\delta y_{01}, \delta y_{11}, \ldots, \delta y_{r1}$ of those variations at the location $x = x_1$. One will obtain values for the $\delta y_{\rho 1}$ themselves that have the form:

$$\delta y_{\rho 1} = -\sum_{\sigma=0}^{r} c_\rho^\sigma \int_{x_0}^{x_1} dx \sum_{\tau=r+1}^{n} \delta y_\tau \sum_{k=0}^{r}\left(\lambda_k^\sigma \varphi_k' y_\tau - \frac{d\lambda_k^\sigma \varphi_k' y_\tau'}{dx}\right)$$

from them. However, one can bring the constant factors c_ρ^σ under the integral sign and then combine all integrals into one. In that way, when one sets:

$$(8) \qquad \sum_{\sigma=0}^{r} c_\rho^\sigma \lambda_k^\sigma \equiv \mu_k^\sigma,$$

one will find that:

$$\delta y_{\rho 1} = -\int_{x_0}^{x_1} dx \sum_{\tau=r+1}^{n} \delta y_\tau \sum_{k=0}^{r} \left(\mu_k^\sigma \varphi_k' \, y_\tau - \frac{d\mu_k^\sigma \varphi_k' \, y_\tau'}{dx} \right), \tag{9}$$

and from (8), the:

$$\lambda_0 = \mu_0^\rho, \qquad \lambda_1 = \mu_1^\rho, \qquad \ldots, \qquad \lambda_r = \mu_r^\rho$$

in that are, in turn, solutions to the differential equations (5).

The basic demand of our problem that δy_0 must vanish for $x = x_1$ at the same time as $\delta y_1, \ldots, \delta y_n$ will then reduce to the demand that the equation:

$$W_{\delta y}^0 \equiv \int_{x_0}^{x_1} dx \sum_{\tau=r+1}^{n} \delta y_\tau \sum_{k=0}^{r} \left(\mu_k^0 \varphi_k' \, y_\tau - \frac{d\mu_k^0 \varphi_k' \, y_\tau'}{dx} \right) = 0 \tag{10}$$

must be fulfilled by all continuous functions $\delta y_{r+1}, \ldots, \delta y_n$ that vanish at the two limits x_0 and x_1 and fulfill the r conditions ([1]):

$$W_{\delta y}^\rho \equiv \int_{x_0}^{x_1} dx \sum_{\tau=r+1}^{n} \delta y_\tau \sum_{k=0}^{r} \left(\mu_k^\rho \varphi_k' \, y_\tau - \frac{d\mu_k^\rho \varphi_k' \, y_\tau'}{dx} \right) = 0, \qquad \rho = 1, 2, \ldots, r. \tag{11}$$

However, if one sets:

$$\delta y_\tau = z_\tau + \sum_{\sigma=1}^{r} \alpha_\sigma u_\tau^\sigma, \tag{12}$$

in which the α_σ are constants and z_τ and u_τ^σ mean continuous functions of x that vanish for $x = x_0$ and $x = x_1$, and of which the z_μ should remain as arbitrary as possible, while the u_τ^σ are chosen suitably, then that demand will go to the one that the equation:

$$W_s^0 + \sum_{\sigma=1}^{r} \alpha_\sigma W_{u^\sigma}^0 = 0 \tag{10$'$}$$

must be a mere consequence of the r equations:

([1]) Naturally, one can also conclude immediately from equations (7) that one of the $n + 1$ equations:

$$\int_{x_0}^{x_1} dx \sum_{\tau=r+1}^{n} \delta y_\tau \sum_{k=0}^{r} \left(\lambda_k^0 \varphi_k' \, y_\tau - \frac{d\lambda_k^0 \varphi_k' \, y_\tau'}{dx} \right) = 0$$

must be a consequence of the remaining ones.

(11′) $$W_s^{\rho} + \sum_{\sigma=1}^{r} \alpha_\sigma W_{u^\sigma}^{\rho} = 0 .$$

If one can now choose the functions u_τ^σ in such a way that the *determinant:*

(13) $$\Delta_r \equiv \sum \pm W_{u^1}^1 W_{u^2}^2 \cdots W_{u^r}^r$$

is non-zero then equations (11′) will not restrict the arbitrariness of the z in any way, but only associate every given system of functions $z_{r+1}, \ldots, z_n$ with a system of constants $\alpha_1, \ldots, \alpha_r$ that satisfy those r equations.

Therefore, a relation of the form:

$$W_s^0 \equiv \beta^1 W_s^1 + \cdots + \beta^r W_s^r$$

must exist in that case for all continuous functions z that vanish at both limits, in which the β are constants that are independent of the choice of functions z.

However, if one sets:

(14) $$-\mu_k^0 + \sum_{\rho=1}^{r} \beta^\rho \mu_k^\rho \equiv \nu_k$$

then, from (10) and (11), one can write that relation as:

$$\int_{x_0}^{x_1} dx \sum_{\tau=r+1}^{n} z_\tau \sum_{k=0}^{r} \left(\nu_k \varphi_k' y_\tau - \frac{d\nu_k \varphi_k' y_\tau'}{dx} \right) \equiv 0 .$$

Their existence then demands that the coefficient of each independent z_τ in it must vanish.

However, from (14), along with the μ_k^0, the functions:

$$\lambda_0 = \nu_0 , \qquad \lambda_1 = \nu_1 , \qquad \ldots, \qquad \lambda_r = \nu_r$$

are again solutions of the differential equations:

(5) $$\sum_{k=0}^{r} \left(\lambda_k \varphi_k' y_\rho - \frac{d\lambda_k \varphi_k' y_\rho'}{dx} \right) = 0 , \qquad \rho = 0, 1, \ldots, r .$$

Therefore, as long as the determinant (13) is not always zero for all continuous functions u_τ^σ that vanish at both limits, it must necessarily give solutions $\lambda_0, \lambda_1, \ldots, \lambda_r$ to the $r + 1$ equations (5) that also simultaneously fulfill the $n - r$ equations:

$$(15) \qquad \sum_{k=0}^{r} \left(\lambda_k \, \varphi'_k \, y_\tau - \frac{d\lambda_k \, \varphi'_k \, y'_\tau}{dx} \right) = 0 \,, \qquad \tau = r + 1, \ldots, n \,.$$

If we now assume, conversely, *that the determinant* Δ_r *is not zero, no matter what continuous functions that vanish at both limits one might also set the* u_τ^σ *equal to* then that might originate in the fact that each individual element:

$$W_{u^\sigma}^\rho \equiv \int_{x_0}^{x_1} dx \sum_{\tau=r+1}^{n} u_\tau^\sigma \sum_{k=0}^{r} \left(\mu_k^\rho \, \varphi'_k \, y_\tau - \frac{d\mu_k^\rho \, \varphi'_k \, y'_\tau}{dx} \right)$$

is always zero in its own right. However, that requires that each of the sums must satisfy:

$$\sum_{k=0}^{r} \left(\mu_k^\rho \, \varphi'_k \, y_\tau - \frac{d\mu_k^\rho \, \varphi'_k \, y'_\tau}{dx} \right) \equiv 0 \,,$$

which then leads directly to the previous result once more.

However, it can also be the case that only all of the sub-determinants of degree p of the determinant Δ_r vanish identically for any $1 < p \leq r$, while any sub-determinant of degree $p - 1$, say:

$$\Delta_{p-1} \equiv \sum \pm W_{u^1}^1 \, W_{u^2}^2 \cdots W_{u^{p-1}}^{p=1} \,,$$

is not zero for all u_τ^σ.

If one then sets:

$$u_\tau^1, \qquad u_\tau^2, \qquad \ldots, \qquad u_\tau^{p-1}; \quad \tau = r + 1, \ldots, n$$

equal to any continuous functions of x that vanish at both limits and do not make Δ_{p-1} vanish and then develops the determinant:

$$\Delta_p \equiv \sum \pm W_{u^1}^1 \, W_{u^2}^2 \cdots W_{u^p}^p$$

in the elements $W_{u^p}^\rho$ then that will give:

$$(16) \qquad \Delta_p \equiv \sum_{\rho=1}^{p} c_\rho \, W_{u^p}^\rho \,,$$

in which the coefficients:

$$c_\rho \equiv \frac{\partial \Delta_p}{\partial W_{u^p}^\rho}$$

of the p elements:

$$W_{u^p}^{\rho} \equiv \int_{x_0}^{x_1} dx \sum_{\tau=r+1}^{n} u_\tau^\sigma \sum_{k=0}^{r} \left(\mu_k^\rho \varphi_k' y_\tau - \frac{d\mu_k^\rho \varphi_k' y_\tau'}{dx} \right), \qquad \rho = 1, 2, \ldots, p$$

and therefore the functions:

$$u_{r+1}^p, \ldots, u_n^p$$

as well, are themselves independent constants, and $c \equiv \Delta_{p-1}$ is non-zero in each case.

However, if one sets:

(17) $$\sum_{\rho=1}^{p} c_\rho \mu_k^\rho \equiv \pi_k$$

then one can write formula (16) as:

$$\Delta_p \equiv \int_{x_0}^{x_1} dx \sum_{\tau=r+1}^{n} u_\tau^p \sum_{k=0}^{r} \left(\pi_k \varphi_k' y_\tau - \frac{d\pi_k \varphi_k' y_\tau'}{dx} \right).$$

Now, by assumption, Δ_p is zero for all arbitrary functions $u_{r+1}^p, \ldots, u_n^p$ that vanish at both limits, so the quantities:

$$\lambda_0 = \pi_0, \qquad \lambda_1 = \pi_1, \qquad \ldots, \qquad \lambda_r = \pi_r,$$

which already satisfy the differential equations (5), due to (17), will, at the same time, also be solutions to the differential equations (15), and one then sees that:

In every case, the solutions $y_0, y_1, \ldots, y_n$ to our problem must possess the property that the n + 1 differential equations:

(18) $$\sum_{k=0}^{r} \left(\lambda_k \varphi_k' y_\tau - \frac{d\lambda_k \varphi_k' y_\tau'}{dx} \right), \qquad i = 0, 1, \ldots, n$$

produce common solutions $\lambda_0, \lambda_1, \ldots, \lambda_r$.

r + 1 new unknowns will be added to the original n + 1 unknowns $y_0, y_1, \ldots, y_n$ in the multipliers $\lambda_0, \lambda_1, \ldots, \lambda_r$. However, along with the n + 1 differential equations (18) for them that were just obtained, one will also have the r + 1 given differential equations (1) themselves and therefore precisely as many equations as unknowns.

However, in order for be able to assign given values to the solution y_0 at $x = x_0$ and the solutions $y_1, \ldots, y_n$ at the two given places x_0 and x_1, those solutions must include $2n + 1$ arbitrary constants.

Now, equations (18) will remain unchanged when one replaces $\lambda_0, \lambda_1, \ldots, \lambda_r$ with $c\,\lambda_0, c\,\lambda_1, \ldots, c\,\lambda_r$, in which c is an arbitrary constant. Of the arbitrary constants that come about when one completely integrates the system of differential equations (1) and (18), only one of them will appear as a common factor in the solutions λ then. As a result, solving our problem can be possible and determinate only when the system of differential equations has order $2n + 2$.

However, if one sets:

(19) $$\Omega \equiv \lambda_0\, \varphi_0 + \lambda_1\, \varphi_1 + \ldots + \lambda_r\, \varphi_r$$

then one can write the differential equations (18) more concisely as:

(20) $$\frac{d}{dx}\frac{\partial \Omega}{\partial y_i'} = \frac{\partial \Omega}{\partial y_i}, \qquad i = 0, 1, \ldots, n,$$

and one easily sees that the system of differential equations (1), (20) will have order $2n + 2$ if and only if the $n + r + 2$ equations:

(21) $$\frac{\partial \Omega}{\partial y_i'} = v_i, \qquad \varphi_k = 0$$

determine the $n + r + 2$ unknowns:

(22) $$y_0', \; y_1', \; \ldots, \; y_n', \; \lambda_0, \lambda_1, \ldots, \lambda_r .$$

In fact, if equations (21) can be solved for those unknowns then let:

(23) $$y_i' = (y_i'), \qquad \lambda_k = (\lambda_k)$$

be the solutions, and their substitution will be indicated by enclosing them in ( ).

One can then replace equations (1), (20) with the $2n + 2$ first-order differential equations in y_0, $y_1, \ldots, y_n$, $v_0, v_1, \ldots, v_n$, and x :

(24) $$\frac{dy_i}{dx} = (y_i'), \qquad \frac{dv_i}{dx} = \left(\frac{\partial \Omega}{\partial y_i}\right),$$

and the $r + 1$ finite equations:

(25) $$\lambda_k = (\lambda_k) .$$

The complete integration of the system (24) implies the solutions y of equations (1), (20) immediately, and upon substituting its solutions in (25), one will also get the solutions λ to those equations. Therefore, along with the system (24), the original system (1), (20) will certainly have order $2n + 2$.

On the other hand, from our assumption on the determinant (2), the last $r + 1$ equations (21) will not allow us to eliminate the y' in any case. Hence, should an equation that is free of the quantities (22) be derivable from all $n + r + 2$ equations (21), then it could not be free of all v in any event. However, if equations (21) imply the equation:

$$\Phi\,(x, y_0, y_1, \ldots, y_n, v_0, v_1, \ldots, v_n) = 0$$

then once one has:

replaced v_i with the corresponding $\frac{\partial\Omega}{\partial y_i'}$,

along with the latter equation, the equation:

$$\frac{\partial\Phi}{\partial x}+\sum_{i=0}^{n}\frac{\partial\Phi}{\partial y_i}y_i'+\sum_{i=0}^{n}\frac{\partial\Phi}{\partial v_i}\frac{d}{dx}\frac{\partial\Omega}{\partial y_i'} = 0$$

will also become an identity because of equations (1), and it will then follow from equations (20) that:

$$\frac{\partial\Phi}{\partial x}+\sum_{i=0}^{n}\frac{\partial\Phi}{\partial y_i}y_i'+\sum_{i=0}^{n}\frac{\partial\Phi}{\partial v_i}\frac{\partial\Omega}{\partial y_i} = 0\,, \tag{26}$$

so one will also have:

$$\sum_{i=0}^{n}\frac{\partial\Phi}{\partial v_i}\left(\frac{\partial\Omega}{\partial y_i}-\frac{d}{dx}\frac{\partial\Omega}{\partial y_i'}\right) = 0\,. \tag{27}$$

Now, equation (26) is, in turn, either a mere consequence of equations (1) or it is not.

In the first case, from (27), one of the n + 1 equations (20) is also a consequence of the remaining ones and equations (1). Equations (1), (20) then reduce to less than $n + r + 2$ equations, so it will no longer suffice to define all $n + r + 2$ unknowns y and λ as functions of x, and the problem will be indeterminate.

That case will always occur, among other things, when one has introduced x as only an auxiliary variable that is extraneous to the original problem. Namely, equations (1) will then be free of x, and at the same time homogeneous of order zero in the differential quotients y_0', y_1', …, y_n'. From (1) (by itself, resp.) one will get:

$$\Omega \equiv 0\,,$$

as well as:

$$\sum_{i=0}^{n} y_i'\frac{\partial\Omega}{\partial y_i'} \equiv 0\,.$$

However, if one differentiates both relations completely with respect to x and subtracts the derivatives from each other then one will see that equations (1) also imply the equation:

$$\sum_{i=0}^{n} y_i'\left(\frac{\partial\Omega}{\partial y_i}-\frac{d}{dx}\frac{\partial\Omega}{\partial y_i'}\right) = 0\,.$$

By contrast, when equation (26) is not merely a consequence of equations (1), from (27), one can replace equation (26) itself with one of the n equations (20). However, the system (1), (20) will then contain only n second-order differential equations, and it is then clear that its integration (in the event that the system determines all unknown functions at all) can only introduce fewer arbitrary constants in each case than it does in the general case. Solving the problem will then be either impossible or, in turn, indeterminate.

§ 2. – Reducing the differential equations of the problem.

From the foregoing, the solution of our problem can be possible and determinate only when the $n + r + 2$ equations:

$$\frac{\partial \Omega}{\partial y_i'} = v_i\,, \qquad \varphi_k = 0 \tag{21}$$

are soluble for the $n + r + 2$ unknowns y_i' and λ_k, and the substitution ( ) of the solutions will take its differential equation to the $2n + 2$ first-order differential equations:

$$\frac{dy_i}{dx} = (y_i')\,, \qquad \frac{dv_i}{dx} = \left(\frac{\partial \Omega}{\partial y_i}\right) \tag{24}$$

Now, it is known that these differential equations possess the canonical form and are therefore equivalent to a first-order partial differential equation with $n + 2$ independent variables, but without the unknown function itself.

However, the peculiarity in the system (24) is that it can always be reduced to a system of only $2n + 1$ first-order differential equations, and then likewise to a first-order partial differential equation with only $n + 1$ independent variables that does not, however, include the unknown function itself.

If one defines H to be a function of the variables $x, y_0, y_1, \ldots, y_n, v_0, v_1, \ldots, v_n$ by the equation:

$$H \equiv \sum_{i=0}^{n} v_i\,(y_i') - (\Omega) \equiv \sum_{i=0}^{n} v_i\,(y_i')\,, \tag{28}$$

and one varies the variables y and v then when one observes the identities:

$$\left(\frac{\partial \Omega}{\partial y_i'}\right) \equiv v_i\,, \qquad (\varphi_k) \equiv 0\,,$$

that will immediately give:

$$\delta H \equiv \sum_{i=0}^{n} \left\{ (y_i')\,\delta v_i - \left(\frac{\partial \Omega}{\partial y_i'}\right) \delta y_i \right\}\,,$$

i.e.:

$$\frac{\partial H}{\partial v_i} \equiv (y_i')\,, \qquad \frac{\partial H}{\partial v_i} \equiv -\left(\frac{\partial \Omega}{\partial y_i}\right). \tag{29}$$

Therefore, equations (24) will possess the canonical form:

$$\frac{dy_i}{dx} = \frac{\partial H}{\partial v_i}, \qquad \frac{dv_i}{dx} = -\frac{\partial H}{\partial y_i}. \tag{30}$$

However, at the same time, from (28) and (29):

$$H \equiv \sum_{i=0}^{n} v_i \frac{\partial H}{\partial v_i}$$

is then homogenous of degree one in the variables $v_0, v_1, \ldots, v_n$.

If one sets:

$$\frac{v_h}{v_0} \equiv -p_h \tag{31}$$

then H will take the form:

$$H \equiv v_0\, F\,(x, y_0, y_1, \ldots, y_n, p_0, p_1, \ldots, p_n)\,, \tag{32}$$

and when one calculates the values of the partial differential quotients of H from that and employs the relation:

$$\frac{dp_h}{dx} \equiv -\frac{1}{v_0}\left(\frac{dv_h}{dx} + p_h \frac{dv_0}{dx}\right)$$

then one will see that the system (30) will, in its own right, again reduce to the $2n + 1$ first-order differential equations between $x, y_0, y_1, \ldots, y_n, p_0, p_1, \ldots, p_n$:

$$\frac{dy_0}{dx} = F - \sum_{h=1}^{n} p_h \frac{\partial F}{\partial y_h}, \qquad \frac{dy_h}{dx} = -\frac{\partial F}{\partial p_h}, \qquad \frac{dp_h}{dx} = \frac{\partial F}{\partial y_h} + p_h \frac{\partial F}{\partial y_0}, \tag{33}$$

whose integration will determine v_0 by the quadrature:

$$\log v_0 = -\int \frac{\partial F}{\partial y_0}\, dx + \text{const.}$$

With that, it is once more shown in a new way that when our problem is indeterminate, the solutions y to its differential equations can no longer include more than $2n + 1$ arbitrary constants. However, at the same time, those differential equations themselves are reduced to a first-order partial differential equation with only $n + 1$ independent variables that include the unknown function itself, at least in general.

Since the system (33) is equivalent to the partial differential equation:

$$\frac{\partial y_0}{\partial x} = F\left(x, y_0, y_1, \ldots, y_n, \frac{\partial y_0}{\partial y_1}, \frac{\partial y_0}{\partial y_2}, \ldots, \frac{\partial y_0}{\partial y_n}\right).$$

If:

$$y_0 = Y_0\,(x, y_0, y_1, \ldots, y_n, \alpha, \alpha_1, \ldots, \alpha_n)$$

is any complete solution of it then the $2n + 1$ equations, with the $2n + 1$ arbitrary constants α, α_1, …, α_n, β, β_1, …, β_n:

$$y_0 = Y_0\,, \qquad \frac{\partial Y_0}{\partial \alpha_h} = \beta_h \frac{\partial Y_0}{\partial \alpha}\,, \qquad p_h = \frac{\partial Y_0}{\partial y_h}$$

will define the complete integral equations of the system (33) ([1]).

However, one will get the function F directly from (21), (28), (31), and (32) when one uses the $n + r + 1$ equations:

$$\frac{\dfrac{\partial \Omega}{\partial y_h'}}{\dfrac{\partial \Omega}{\partial y_0'}} = -p_h\,, \qquad \varphi_k = 0$$

to determine the $n + r + 1$ unknowns:

$$y_0',\ y_1',\ \ldots,\ y_n'\ ;\ \lambda_0 : \lambda_1 : \ldots : \lambda_r$$

and substitute the values of the y' in:

$$F = y_0' - \sum_{h=1}^{n} p_h\, y_h'\ .$$

One then gets the following rule for reducing the differential equations of the problem to a partial differential equation ([2]):

One solves the $n + r + 1$ *equations:*

$$\frac{\partial \Omega}{\partial y_h'} + \frac{\partial \Omega}{\partial y_0'} \frac{\partial y_0}{\partial y_h} = 0\,, \qquad \varphi_k = 0\,, \qquad h = 1, 2, \ldots, n\,; \qquad k = 0, 1, \ldots, r$$

for the $n + r + 1$ *unknowns:*

$$y_0',\ y_1',\ \ldots,\ y_n'\ ;\ \lambda_0 : \lambda_1 : \ldots : \lambda_r\,,$$

and upon substituting the solutions y', *one will convert the equation:*

([1]) JACOBI, *Werke*, Bd. V, pp. 291.
([2]) These Berichte, 1878, pp. 20.

$$\frac{\partial y_0}{\partial x} = y_0' - \sum_{h=1}^{n} \frac{\partial y_0}{\partial y_h} y_h'$$

into a first-order partial differential equation between the unknown function y_0 *and the* $n + 1$ *independent variables* $x, y_1, \ldots, y_n$.

If one has found any complete solution to that partial differential equation:

$$y_0 = Y_0 (x, y_0, y_1, \ldots, y_n, \alpha, \alpha_1, \ldots, \alpha_n)$$

then one will get the complete solutions y to the differential equations (1), (20) *immediately upon solving the* $n + 1$ *equations with the* $2n + 1$ *arbitrary constants* $\alpha, \alpha_1, \ldots, \alpha_n, \beta, \beta_1, \ldots, \beta_n$:

$$y_0 = Y_0 , \qquad \frac{\partial Y_0}{\partial \alpha_h} = \beta_h \frac{\partial Y_0}{\partial \alpha} ,$$

and at the same time, the ratios of their multipliers λ *will be determined, while one can ultimately find the latter from one of equations* (20) *by a mere quadrature*.

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