"Ueber die Kriterien des Maximums und Minimums der einfache Integrale," J. reine. angew. Math. **69** (1868), 238-263,

On the criteria for maxima and minima of simple integrals

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The only essential results of the following investigations are the ones that I already published two years ago in my Habilitationsschrift (*). The logic of the presentation was not changed essentially either.

However, the cited paper was undermined by isolated lapses in precision and many detours were made that one could avoid. Those defects made it desirable to me to treat the same subject here once more in a somewhat-altered form while leaving out those considerations that are not absolutely necessary for addressing the main question.

For the sake of completeness and logical continuity, a summary of the transformations by which the second variation first takes on a form that is suitable for the investigation of its sign can probably not be circumvented. In order to derive those formulas of **Clebsch** (**), in what follows I will appeal to the ingenious method that **Lipschitz** communicated in the treatise "Beiträge zur Theorie der Variation der einfachen Integrale" (***), and which has not been extended to the case of relative maxima and minima up to now.

§ 1.

As is known, the most general problem in the calculus of variations for one independent variable (since one can reduce all of the others to it) is the following one:

Determine the variables $y_1, y_2, \ldots, y_n$, which are subject to the m first-order differential equations:

(1) $$\varphi_1 = 0 , \qquad \varphi_2 = 0 , \qquad \ldots, \qquad \varphi_m = 0 ,$$

as functions of x such that the integral:

$$V = \int_{x_0}^{x_1} f(x, y_1, y_1', \ldots, y_n, y_n')\, dx$$

(*) "Beiträge zur Theorie der Maxima und Minima der einfachen Integrale," Leipzig, Teubner, 1866.

(**) Bd. 55, pps. 254 and 335 of this journal.

(***) Bf. 65, pp. 26 of this journal.

will be a maximum or a minimum.

Obviously, one must have $m < n$ in that. Moreover, in order to make the problem well-determined, certain boundary conditions must be given. I shall assume that the limits x_0 and x_1 are all given, as well as the limiting values of the variables y. One can reduce all remaining cases to that one by dividing the problem into separate parts.

Using the **Lagrange** process, one considers the following problem instead of the present one: Determine the functions y such that the integral:

$$J = \int_{x_0}^{x_1} \Omega \, dx$$

will be a maximum or a minimum, in which:

$$\Omega = f + \lambda_1 \varphi_1 + \lambda_1 \varphi_1 + \ldots + \lambda_m \varphi_m , \tag{2}$$

and the λ are undetermined functions of x, about which one has to demand that the given conditions (1) will be satisfied. That problem will be equivalent to the first one when one adds that only those functions y shall be taken into consideration that fulfill those conditions.

If one generally sets:

$$y_h + \varepsilon \mathfrak{z}_h , \qquad \text{instead of} \qquad y_h ,$$

where ε means a sufficiently-small number and the $\mathfrak{z}$ are arbitrary functions of x that must, however, be finite and continuous within the integration limits, along with their differential quotients $\mathfrak{z}'$, and one then develops the integral J in powers of ε then when one neglects terms of order ε^3, it will go to:

$$J + \varepsilon \, \delta J + \frac{\varepsilon^2}{2} \delta^2 J ,$$

whereas since one always considers only those functions y that satisfy equations (1), the variations $\mathfrak{z}$ imply the m equations of constraint:

$$\delta\varphi_k = \sum_{h=1}^{n} \left\{ \frac{\partial \varphi_k}{\partial y_h} \mathfrak{z}_h + \frac{\partial \varphi_k}{\partial y'_h} \mathfrak{z}'_h \right\} = 0 . \tag{3}$$

Since the limiting values of the variables y should remain unvaried, moreover, the function functions $\mathfrak{z}$ must further assume the value zero at the limits x_0 and x_1 .

Now if an actual relative maximum or minimum for the integral V or J is to exist then the first variation δJ would have to vanish for all arbitrary variations $\mathfrak{z}$ that satisfy the given conditions, and the second variation $\delta^2 J$ would have to possess a constant sign.

The first condition leads to the $n + m$ differential equations:

$$(4) \qquad \frac{\partial \Omega}{\partial y_h} = \frac{d}{dx}\frac{\partial \Omega}{\partial y'_h}, \qquad \varphi_h = 0 .$$

Integrating them yields the $n + m$ unknowns y and λ as functions of x and a certain number of arbitrary constants that are determined in such a way that the solutions y will be equal to the given limiting values at the limits.

In order for the determination of those constants to be possible, the solutions y must contain $2n$ mutually-independent arbitrary constants. That assumption is then based upon the following grounds: They can be fulfilled, in any event, only when the system of differential equations (4) has order $2n$, and as one easily sees, for that to be true, it is necessary and sufficient that the determinant:

$$(5) \qquad R = \begin{vmatrix} \frac{\partial^2 \Omega}{\partial y'_1 \partial y'_1} & \cdots & \frac{\partial^2 \Omega}{\partial y'_n \partial y'_1} & \frac{\partial \varphi_1}{\partial y'_1} & \cdots & \frac{\partial \varphi_m}{\partial y'_1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\partial^2 \Omega}{\partial y'_1 \partial y'_n} & \cdots & \frac{\partial^2 \Omega}{\partial y'_n \partial y'_n} & \frac{\partial \varphi_1}{\partial y'_n} & \cdots & \frac{\partial \varphi_m}{\partial y'_n} \\ \frac{\partial \varphi_1}{\partial y'_1} & \cdots & \frac{\partial \varphi_1}{\partial y'_n} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\partial \varphi_m}{\partial y'_1} & \cdots & \frac{\partial \varphi_m}{\partial y'_n} & 0 & \cdots & 0 \end{vmatrix}$$

is not identically zero.

Since the identical vanishing of that determinant will then be excluded on the basis of the assumption that was made, one can determine the $n + m$ quantities y' and λ as functions of the y and v from the equations:

$$\frac{\partial \Omega}{\partial y'_h} = 0 , \qquad \varphi_k = 0 ,$$

and as a result, replace the system (4) with the $2n$ first-order differential equations:

$$(6) \qquad \frac{dy_h}{dx} = \frac{\partial H}{\partial v_h}, \qquad \frac{dv_h}{dx} = -\frac{\partial H}{\partial y_h},$$

in which H denotes the function of the y and v that emerges from the expression:

$$\sum_{h=1}^{n} y'_h v_h - f$$

upon introducing the value of y'.

From them, one will get:

(8) $$y_h = [y_h]\,, \qquad v_h = \left[\frac{\partial\Omega}{\partial y'_h}\right] = [v_h]$$

as the complete solutions of the system (6). The $2n$ integration constants of the solutions $[y]$ and $[\lambda]$ might be denoted by:

$$a_1\,, \quad a_2\,, \quad \ldots, \quad a_{2n}\,.$$

They are determined in such a way that the n functions $[y_h]$ take on the given limiting values y_{h0} and y_{h1} for $x = x_0$ and $x = x_1$, resp., and therefore they will be considered to be well-defined quantities that will be given by those limiting values in what follows. For certain special assumptions on the values of y_{h0} and y_{h1} (e.g., if one would like to set all of them equal to zero), certain special exceptional cases can occur, here as well as later. I will always overlook them and therefore assume, for example, that the determinant $[R]$, which arises from R by the substitutions (7), will also be non-zero after introducing those well-defined values of the integration constants. That assumption is permissible, because since the determinant R should be non-zero, in its own right, and the second differential quotients of the y, as well as the first differential quotients of the λ, do not enter into it at all, the complete integration of equations (4) can never have the equation $R = 0$ as a general consequence, either.

§ 2.

In order to decide whether the solutions (7) of the given integral represent an actual relative maximum or minimum, the sign of the second variation $\delta^2 J$ must be examined.

If one lets $2\,\delta^2\,\Omega$ denote the homogeneous function of degree two in $\mathfrak{z}$ and $\mathfrak{z}'$ that arises from Ω when one generally lets $[y_h] + \varepsilon\,\mathfrak{z}_h$ enter in place of y_h and takes the coefficients of $\frac{1}{2}\varepsilon^2$ in the development in powers of ε then one will have:

(9) $$\delta^2 J = \int_{x_0}^{x_1} 2\,\delta^2\Omega\,dx\,.$$

In order to be able to investigate the sign of that expression, it is generally necessary to put the function $2\,\delta^2\Omega$, and with it, the conditions (3), as well, into a simpler form.

It will be shown that this function can be represented as an aggregate of three functions, namely, a homogeneous function of degree two in only n arguments η_1 , η_2 , …, η_n , which are functions of the $\mathfrak{z}$ and their differential quotients, the differential quotients of a homogeneous function of degree two in the $\mathfrak{z}$, and finally a function that is linear with respect to the $\delta\varphi_k$, while the $\delta\varphi_k$ themselves will be converted into a linear homogeneous function in the η .

Instead of the function $2\,\delta^2\Omega$, we will next consider another function, namely, the function:

$$(10)\qquad \begin{cases} 2\Omega_2 = & 2\sum_{k=1}^{m}\mu_k\sum_{h=1}^{n}\left\{\left[\dfrac{\partial\varphi_k}{\partial y_h}\right]\mathfrak{z}_h+\left[\dfrac{\partial\varphi_k}{\partial y_h'}\right]\mathfrak{z}_h'\right\} \\ & +\sum_{h=1}^{n}\sum_{i=1}^{n}\left\{\left[\dfrac{\partial^2\Omega}{\partial y_h\,\partial y_i}\right]\mathfrak{z}_h\,\mathfrak{z}_i+2\left[\dfrac{\partial^2\Omega}{\partial y_h\,\partial y_i'}\right]\mathfrak{z}_h\,\mathfrak{z}_i'+\left[\dfrac{\partial^2\Omega}{\partial y_h'\,\partial y_i'}\right]\mathfrak{z}_h'\,\mathfrak{z}_i'\right\}, \end{cases}$$

which is the coefficient of $\frac{1}{2}\varepsilon^2$ in the development of the function that one will get from Ω when one replaces the variables y and λ with the quantities $[y] + \varepsilon\,\mathfrak{z}$ and $[\lambda] + \varepsilon\,\mu$.

The functions $2\,\delta^2\Omega$ and $2\,\Omega_2$ are coupled by the relationship:

$$(11)\qquad 2\,\delta^2\Omega = 2\,\Omega_2 - 2\sum_{k=1}^{m}\mu_k\,\delta\varphi_k\,.$$

One can also convert the function $2\,\Omega_2$ then, instead of the function $2\,\delta^2\Omega$, and thus preserve the freedom to be able to add the quantities μ at will.

The solutions of a certain system of differential equations that is closely connected with the system (4) will be used to carry out that conversion.

If one differentiates equations (4), which are identical because of the substitutions (7), with respect to a_i, and when one generally understands $\Omega_2\,(\xi,\eta)$ to mean the value that the function Ω_2 will assume for the values ξ_h, η_k of $\mathfrak{z}_h$, μ_k, resp., then those equations can be represented as:

$$\frac{\partial\Omega_2\left(\dfrac{\partial[y]}{\partial a_i},\dfrac{\partial[\lambda]}{\partial a_i}\right)}{\partial\,\dfrac{\partial[y_h]}{\partial a_i}} = \frac{d}{dx}\,\frac{\partial\Omega_2\left(\dfrac{\partial[y]}{\partial a_i},\dfrac{\partial[\lambda]}{\partial a_i}\right)}{\partial\,\dfrac{\partial[y_h]'}{\partial a_i}}\,, \qquad \frac{\partial\Omega_2\left(\dfrac{\partial[y]}{\partial a_i},\dfrac{\partial[\lambda]}{\partial a_i}\right)}{\partial\,\dfrac{\partial[\lambda_k]}{\partial a_i}} = 0\,.$$

One sees from this that the expressions:

$$(12)\qquad u_h = \sum_{i=1}^{2n}\gamma_i\,\frac{\partial[y_h]}{\partial a_i}\,, \qquad r_k = \sum_{i=1}^{2n}\gamma_i\,\frac{\partial[\lambda_k]}{\partial a_i}\,,$$

with the $2n$ arbitrary constants γ, are the general solutions of the linear differential equations:

$$(13)\qquad \frac{\partial\Omega_2(u,r)}{\partial u_h} = \frac{d}{dx}\,\frac{\partial\Omega_2(u,r)}{\partial\,\dfrac{du_h}{dx}}\,, \qquad \frac{\partial\Omega_2(u,r)}{\partial r_k} = 0\,,$$

to which one will also arrive when one considers the second variation $\delta^2 J$ itself to be an integral whose maximum or minimum is to be sought.

The solutions of those differential equations possess two properties that are extremely important for us.

First of all, since $2\Omega_2$ is a homogeneous function of degree two in the $\mathfrak{z}$, $\mathfrak{z}'$, and μ:

$$2\Omega_2 = \sum_{h=1}^{n}\left\{\frac{\partial\Omega_2}{\partial\mathfrak{z}_h}\mathfrak{z}_h + \frac{\partial\Omega_2}{\partial\mathfrak{z}'_h}\mathfrak{z}'_h\right\} + \sum_{k=1}^{m}\frac{\partial\Omega_2}{\partial\mu_k}\mu_k$$

$$= \sum_{h=1}^{n}\left\{\frac{\partial\Omega_2}{\partial\mathfrak{z}_h} - \frac{d}{dx}\frac{\partial\Omega_2}{\partial\mathfrak{z}'_h}\right\}\mathfrak{z}_h + \frac{d}{dx}\sum_{h=1}^{n}\frac{\partial\Omega_2}{\partial\mathfrak{z}'_h}\mathfrak{z}_h + \sum_{k=1}^{m}\frac{\partial\Omega_2}{\partial\mu_k}\mu_k .$$

It then follows from this that any system of solutions of equations (13) will fulfill the equation:

$$2\Omega_2(u, r) = \frac{d}{dx}\sum_{h=1}^{n} u_h \frac{\partial\Omega_2(u,r)}{\partial\dfrac{du_h}{dx}} . \tag{14}$$

For any two systems of solutions of those differential equations u, r and ω, τ, one will further find that the identity exists that:

$$\sum_{h=1}^{n}\left\{\omega_h\frac{\partial\Omega_2(u,r)}{\partial u_h} + \frac{d\omega_h}{dx}\frac{\partial\Omega_2(u,r)}{\partial\dfrac{du_h}{dx}}\right\} + \sum_{k=1}^{m}\tau_k\frac{\partial\Omega_2(u,r)}{\partial r_k} = \frac{d}{dx}\sum_{h=1}^{n}\omega_h\frac{\partial\Omega_2(u,r)}{\partial\dfrac{du_h}{dx}} ,$$

as well as the one that arises from it when one switches the u, r with the ω, τ. However, from a known property of homogeneous functions of degree two, the first part of the equation above will remain unchanged by the exchange. One must also have:

$$\frac{d}{dx}\sum_{h=1}^{n}\omega_h\frac{\partial\Omega_2(u,r)}{\partial\dfrac{du_h}{dx}} = \frac{d}{dx}\sum_{h=1}^{n}u_h\frac{\partial\Omega_2(\omega,\tau)}{\partial\dfrac{d\omega_h}{dx}}$$

then, from which it will follow by integration that:

$$\sum_{h=1}^{n}\left\{\omega_h\frac{\partial\Omega_2(u,r)}{\partial u_h} - u_h\frac{\partial\Omega_2(\omega,\tau)}{\partial\dfrac{d\omega_h}{dx}}\right\} = \text{const.} \tag{15}$$

In order to convert the function Ω_2, we employ n systems of complete solutions of equations (13):

$$(16) \qquad u_h^\sigma = \sum_{i=1}^{2n} \gamma_i^\sigma \frac{\partial [y_h]}{\partial a_i}, \qquad r_k^\sigma = \sum_{i=1}^{2n} \gamma_i^\sigma \frac{\partial [\lambda_k]}{\partial a_i},$$

and introduce n new variables g in place of the $n + m$ quantities $\mathfrak{z}$ and μ, by means of the $n + m$ equations:

$$(17) \qquad \mathfrak{z}_h = \sum_{\sigma=1}^{n} g_\sigma u_h^\sigma, \qquad \mu_k = \sum_{\sigma=1}^{n} g_\sigma r_k^\sigma ;$$

in other words, we set the μ equal to certain linear functions of the $\mathfrak{z}$.

Those substitutions will imply that:

$$(18) \qquad \begin{cases} \dfrac{d\mathfrak{z}_h}{dx} = \zeta_h + \eta_h, \\ \zeta_h = \displaystyle\sum_{\sigma=1}^{n} g_\sigma \frac{du_h^\sigma}{dx}, \\ \eta_h = \displaystyle\sum_{\sigma=1}^{n} \frac{dg_\sigma}{dx} u_h^\sigma. \end{cases}$$

One now arrives at mainly the problem of defining the function Ω_2^0 that arises when one gives the $\mathfrak{z}$ and μ the values (17) in Ω_2, but one does not set $d\mathfrak{z} / dx$ equal to $\zeta + \eta$, but only to ζ.

To that end, one remarks that when the new variables g are constants, the expressions (17) will become solutions of the differential equations (13), and therefore from (14), one will have:

$$2\Omega_2^0 = \frac{d}{dx} \sum_{h=1}^{n} \left\{ \sum_{\sigma=1}^{n} g_\sigma u_h^\sigma \sum_{\rho=1}^{n} g_\rho \frac{\partial \Omega_2(u^\rho, r^\rho)}{\partial \dfrac{du_h^\rho}{dx}} \right\}.$$

However, since the function $2\Omega_2^0$ does not contain the differential quotients of the g at all, and as a result, they must have the same form, regardless of whether the quantities g are constants or functions of x, the foregoing equation will be true for all arbitrary values of those quantities, as long as one only performs the suggested differentiations on the right-hand side with respect to x, as if the g were independent of x. However, one can also arrive at that result in such a way that one considers the g to be functions of x and subtracts the terms on the right-hand side that originate in the differentiation of g with respect to x. One will then find the following equation to be true for all arbitrary values of the g :

(19)
$$\left\{\begin{aligned}
2\Omega_2^0 = \;& \frac{d}{dx}\sum_{h=1}^{n}\left\{\sum_{\sigma=1}^{n} g_\sigma\, u_h^\sigma \sum_{\rho=1}^{n} g_\rho\, \frac{\partial\,\Omega_2(u^\rho, r^\rho)}{\partial\,\dfrac{du_h^\rho}{dx}}\right\} \\
& -\sum_{h=1}^{n}\left\{\sum_{\sigma=1}^{n} \frac{dg_\sigma}{dx}\, u_h^\sigma \sum_{\rho=1}^{n} g_\rho\, \frac{\partial\,\Omega_2(u^\rho, r^\rho)}{\partial\,\dfrac{du_h^\rho}{dx}}\right\} \\
& -\sum_{h=1}^{n}\left\{\sum_{\sigma=1}^{n} g_\sigma\, u_h^\sigma \sum_{\rho=1}^{n} \frac{dg_\rho}{dx}\, \frac{\partial\,\Omega_2(u^\rho, r^\rho)}{\partial\,\dfrac{du_h^\rho}{dx}}\right\}.
\end{aligned}\right.$$

Now the value that the function 2 Ω_2 will assume under the substitutions (17) will arise from $2\Omega_2^0$ in such a way that one generally replaces:

$$\zeta_h + \eta_h \qquad \text{with} \qquad \zeta_h$$

in it. Since only the first and second powers of ζ_h enter into $2\Omega_2^0$, one will get:

(20)
$$2\,\Omega_2 = 2\Omega_2^0 + 2\sum_{h=1}^{n}\frac{\partial\Omega_2^0}{\partial\zeta_h}\eta_h + \sum_{h=1}^{n}\sum_{i=1}^{n}\frac{\partial^2\Omega_2^0}{\partial\zeta_h\,\partial\zeta_i}\eta_h\,\eta_i\,.$$

However, the original definition of the function Ω_2^0 implies immediately that:

$$\frac{\partial\Omega_2^0}{\partial\zeta_h} = \sum_{\rho=1}^{n} g_\rho\, \frac{\partial\,\Omega_2(u^\rho, r^\rho)}{\partial\,\dfrac{du_h^\rho}{dx}}\,.$$

From (17), (18), and (19), one will then have:

$$\begin{aligned}
2\Omega_2^0 + 2\sum_{h=1}^{n}\frac{\partial\Omega_2^0}{\partial\zeta_h}\eta_h = \;& \frac{d}{dx}\sum_{h=1}^{n}\mathfrak{z}_h\sum_{\rho=1}^{n} g_\rho\, \frac{\partial\,\Omega_2(u^\rho, r^\rho)}{\partial\,\dfrac{du_h^\rho}{dx}} \\
& + \sum_{h=1}^{n}\left\{\sum_{\sigma=1}^{n} \frac{dg_\sigma}{dx}\, u_h^\sigma \sum_{\rho=1}^{n} g_\rho\, \frac{\partial\,\Omega_2(u^\rho, r^\rho)}{\partial\,\dfrac{du_h^\rho}{dx}}\right\}
\end{aligned}$$

$$- \sum_{h=1}^{n} \left\{ \sum_{\sigma=1}^{n} g_\sigma \, u_h^\sigma \sum_{\rho=1}^{n} \frac{dg_\rho}{dx} \frac{\partial \Omega_2(u^\rho, r^\rho)}{\partial \dfrac{du_h^\rho}{dx}} \right\}.$$

The last two terms can then be combined and written as:

$$\sum_{\sigma=1}^{n} \sum_{\rho=1}^{n} g_\rho \, \frac{dg_\sigma}{dx} \sum_{h=1}^{n} \left\{ u_h^\sigma \frac{\partial \Omega_2(u^\rho, r^\rho)}{\partial \dfrac{du_h^\rho}{dx}} - u_h^\rho \frac{\partial \Omega_2(u^\sigma, r^\sigma)}{\partial \dfrac{du_h^\sigma}{dx}} \right\}.$$

If one then subjects the n systems of solutions to the differential equations (13) that were introduced to the $n\,(n - 1) / 2$ equations of constraint:

$$\sum_{h=1}^{n} \left\{ u_h^\sigma \frac{\partial \Omega_2(u^\rho, r^\rho)}{\partial \dfrac{du_h^\rho}{dx}} - u_h^\rho \frac{\partial \Omega_2(u^\sigma, r^\sigma)}{\partial \dfrac{du_h^\sigma}{dx}} \right\} = 0 , \tag{21}$$

which are independent of x, from (15), and as a result, they will produce only equations of constraint between the $2n^2$ constants γ_h^0, then those last two terms will drop out, and since one obviously has:

$$\frac{\partial^2 \Omega_2^0}{\partial \zeta_h \, \partial \zeta_i} = \frac{\partial^2 \Omega_2^0}{\partial \mathfrak{z}'_h \, \partial \mathfrak{z}'_i} = \left[\frac{\partial^2 \Omega}{\partial y'_h \, \partial y'_i} \right],$$

one will get:

$$2\,\Omega_2 = \frac{d}{dx} \sum_{h=1}^{n} \mathfrak{z}_h \sum_{\rho=1}^{n} g_\rho \frac{\partial \Omega_2(u^\rho, r^\rho)}{\partial \dfrac{du_h^\rho}{dx}} + \sum_{h=1}^{n} \sum_{i=1}^{n} \left[\frac{\partial^2 \Omega}{\partial y'_h \, \partial y'_i} \right] \eta_h \, \eta_i \, .$$

We thus arrive at the following transformation of the function $2\,\delta^2 \Omega$:

$$2\,\delta^2 \Omega = \sum_{h=1}^{n} \sum_{i=1}^{n} \left[\frac{\partial^2 \Omega}{\partial y'_h \, \partial y'_i} \right] \eta_h \, \eta_i + \frac{d}{dx} \sum_{h=1}^{n} \mathfrak{z}_h \sum_{\rho=1}^{n} g_\rho \frac{\partial \Omega_2(u^\rho, r^\rho)}{\partial \dfrac{du_h^\rho}{dx}} - 2 \sum_{k=1}^{m} \mu_k \delta \varphi_k ,$$

in which the quantities g, μ, and η are defined by the equations:

$$\mathfrak{z}_h = \sum_{\sigma=1}^{n} u_h^\sigma g_\sigma , \qquad \mu_k = \sum_{\sigma=1}^{n} r_h^\sigma g_\sigma , \qquad \eta_k = \frac{d\mathfrak{z}_h}{dx} - \sum_{\sigma=1}^{n} g_\sigma \frac{du_h^\sigma}{dx} .$$

However, at the same time, the expressions $\delta\varphi_k$ will also be transformed. namely, if one introduces the foregoing values for $\mathfrak{z}$ and $d\mathfrak{z} / dx$ in them, then they will go to:

$$\delta\varphi_k = \sum_{\sigma=1}^{n} g_\sigma \sum_{\sigma=1}^{n} \left\{ \left[\frac{\partial \varphi_k}{\partial y_h} \right] u_h^\sigma + \left[\frac{\partial \varphi_k}{\partial y_h'} \right] \frac{du_h^\sigma}{dx} \right\} + \sum_{h=1}^{n} \left[\frac{\partial \varphi_k}{\partial y_h'} \right] \eta_h .$$

However, the coefficient of g_σ in that is equal to:

$$\frac{\partial \Omega_2(u^\sigma, r^\sigma)}{\partial r_k^\sigma} ,$$

and therefore, from (13), it will be equal to zero, such that the formula will reduce to:

$$\delta\varphi_k = \sum_{h=1}^{n} \left[\frac{\partial \varphi_k}{\partial y_h'} \right] \eta_h .$$

All that remains now is to express everything in terms of the $\mathfrak{z}$ and $d\mathfrak{z} / dx$. From (17), one first has:

$$\mathfrak{z}_i = u_i^1 g_1 + u_i^2 g_2 + \cdots + u_i^n g_n .$$

If one sets $i = 1, 2, \ldots, n$ in that and couples those n equations with the equation arises from (18):

$$-\eta_h + \frac{d\mathfrak{z}_h}{dx} = \frac{du_i^1}{dx} g_1 + \frac{du_i^2}{dx} g_2 + \cdots + \frac{du_i^n}{dx} g_n$$

then upon eliminating the g, one will get:

$$(22) \qquad U \cdot \eta_h = \begin{vmatrix} \frac{d\mathfrak{z}_h}{dx} & \frac{du_h^1}{dx} & \cdots & \frac{du_h^n}{dx} \\ \mathfrak{z}_1 & u_1^1 & \cdots & u_1^n \\ \vdots & \vdots & \cdots & \vdots \\ \mathfrak{z}_n & u_n^1 & \cdots & u_n^n \end{vmatrix} = U_h ,$$

in which:

$$(23) \qquad U = \sum \pm u_1^1 u_2^2 \cdots u_n^n .$$

$$
(24)\quad \left\{
\begin{aligned}
& U \sum_{\rho=1}^{n} g_\rho \sum_{h=1}^{n} \mathfrak{z}_h \frac{\partial \Omega_2(u^\rho, r^\rho)}{\partial \dfrac{du_h^\rho}{dx}} \\
& = - \begin{vmatrix}
0 & \displaystyle\sum_{h=1}^{n} \mathfrak{z}_h \frac{\partial \Omega_2(u^1, r^1)}{\partial \dfrac{du_h^1}{dx}} & \cdots & \displaystyle\sum_{h=1}^{n} \mathfrak{z}_h \frac{\partial \Omega_2(u^n, r^n)}{\partial \dfrac{du_h^n}{dx}} \\
\mathfrak{z}_1 & u_1^1 & \cdots & u_1^n \\
\vdots & \vdots & \cdots & \vdots \\
\mathfrak{z}_n & u_n^1 & \cdots & u_n^n
\end{vmatrix} = -2B,
\end{aligned}
\right.
$$

and if one would like to learn the values of the μ, in addition, then one will finally find from the second equation in (17), in the same way, that:

$$
(25)\qquad U \cdot \mu_k = - \begin{vmatrix}
0 & r_k^1 & \cdots & r_k^n \\
\mathfrak{z}_1 & u_1^1 & \cdots & u_1^n \\
\vdots & \vdots & \cdots & \vdots \\
\mathfrak{z}_n & u_n^1 & \cdots & u_n^n
\end{vmatrix} = - R_k ,
$$

such that one will ultimately obtain **Clebsch**'s formula:

$$
(26)\qquad 2\,\delta^2\Omega = \sum_{h=1}^{n}\sum_{i=1}^{n} \left[\frac{\partial^2 \Omega}{\partial y_h' \, \partial y_i'}\right] \frac{U_h U_i}{U^2} + \frac{d}{dx}\frac{2B}{U} + 2\sum_{k=1}^{m} \frac{R_k}{U}\,\delta\varphi_k .
$$

Since one has, at the same time:

$$
(27)\qquad \delta\varphi_k = \sum_{h=1}^{m} \left[\frac{\partial \varphi_k}{\partial y_h'}\right] \frac{U_h}{U} ,
$$

one will see that the conversion that was obtained will achieve everything that was promised at the beginning of this section.

The definitions of the quantities that appear in it are contained in formulas (22), (23), (24), (25), and (16).

§ 3.

Formulas (26), (27), are valid identically for all values of the $2\,n^2$ constants γ_i^σ that satisfy the $n\,(n-1)/2$ equations of constraint:

(21) $$\sum_{h=1}^{n}\left\{u_h^\sigma \frac{\partial \Omega_2(u^\rho, r^\rho)}{\partial \frac{du_h^\rho}{dx}} - u_h^\rho \frac{\partial \Omega_2(u^\sigma, r^\sigma)}{\partial \frac{du_h^\sigma}{dx}}\right\} = 0 ,$$

which are independent of *x*, and the determinant

$$U = \sum \pm u_1^1 u_2^2 \cdots u_n^n$$

is not identically zero for them.

We are now dealing with the problem of actually determining those constants so that they will be consistent with the stated requirements. Under the assumption that the integration constants *a* are the solutions [*y*] and [λ] are the so-called *canonical constants*, **Clebsch** gave the most general values for the γ_i^σ that would satisfy those conditions. However, for our purpose, it is not at all necessary to know those most general values. Rather, we will employ only a well-defined special system of values of the γ_i^σ.

In order to discover it, we recall the fact that:

(8) $$y_h = [y_h] , \qquad v_h = [v_h] = \left[\frac{\partial \Omega}{\partial y_h'}\right]$$

are the complete solutions of the system of differential equations:

(6) $$\frac{dy_h}{dx} = \frac{\partial H}{\partial v_h} , \quad \frac{dv_h}{dx} = -\frac{\partial H}{\partial y_h} .$$

The 2*n* equations (8) must necessarily be soluble for the 2*n* integration constants:

$$a_1 , \quad a_2 , \quad \ldots, \quad a_{2n}$$

then. Let:

$$(a_1) , \quad (a_2) , \quad \ldots, \quad (a_{2n})$$

be the values that are obtained that way. If one back-substitutes them in equations (8) then they will become identical, and one then differentiates them with respect to *y* and *v*. If one does that and substitutes the values (8) for the *y* and *v* after differentiating then one will get the following system of identities:

$$(28)\quad \begin{cases} \dfrac{\partial y_h}{\partial y_\sigma} = \sum_{i=1}^{2n} \dfrac{\partial [y_h]}{\partial a_i}\left[\dfrac{\partial (a_i)}{\partial y_\sigma}\right], \\ 0 = \sum_{i=1}^{2n} \dfrac{\partial [y_h]}{\partial a_i}\left[\dfrac{\partial (a_i)}{\partial v_\sigma}\right], \\ 0 = \sum_{i=1}^{2n} \dfrac{\partial [v_h]}{\partial a_i}\left[\dfrac{\partial (a_i)}{\partial y_\sigma}\right], \\ \dfrac{\partial v_h}{\partial v_\sigma} = \sum_{i=1}^{2n} \dfrac{\partial [v_h]}{\partial a_i}\left[\dfrac{\partial (a_i)}{\partial v_\sigma}\right], \end{cases}$$

in which $\dfrac{\partial y_h}{\partial y_\sigma} = \dfrac{\partial v_h}{\partial v_\sigma} = 0$ or 1 according to whether the indices h and σ are different or equal, resp.

Now, one has:

$$u_h^\sigma = \sum_{i=1}^{2n} \gamma_i^\sigma \left[\frac{\partial [y_h]}{\partial a_i}\right],$$

and as one can easily explain:

$$\frac{\partial \Omega_2(u^\sigma, r^\sigma)}{\partial \dfrac{du_h^\sigma}{dx}} = \sum_{i=1}^{2n} \gamma_i^\sigma \frac{\partial}{\partial a_i}\left[\frac{\partial \Omega}{\partial y_h'}\right] = \sum_{i=1}^{2n} \gamma_i^\sigma \frac{\partial [v_h]}{\partial a_i}.$$

If one then sets:

$$(29)\qquad \gamma_i^\sigma = \left[\frac{\partial (a_i)}{\partial v_\sigma}\right]_\omega,$$

in general (i.e., equal to the value that the function $\left[\dfrac{\partial (a_i)}{\partial v_\sigma}\right]$ assumes for any well-defined value x_ω of x), then one will have:

$$u_h^\sigma = 0, \qquad \frac{\partial \Omega_2(u^\sigma, r^\sigma)}{\partial \dfrac{du_h^\sigma}{dx}} = 0 \text{ or } 1$$

for $x = x_\omega$ according to whether $\sigma \neq \eta$ or $\sigma = \eta$, resp.

The $n (n - 1) / 2$ equations of constraint (21) will then be fulfilled identically for $x = x_\omega$ with the values (29) of the constants γ_i^σ, and as a result, they will be fulfilled for any x at all, since they are independent of x.

However, the system of values (29) for the γ_i^σ possesses yet a second important property, namely, that it allows one to express the determinant U in terms of another determinant that plays a very prominent role in the criteria for a maximum or minimum, as one will see.

Namely, if one multiplies the determinant:

$$(30) \qquad \Delta\,(x, x_\omega) = \begin{vmatrix} \frac{\partial[y_1]}{\partial a_1} & \frac{\partial[y_1]}{\partial a_2} & \cdots & \frac{\partial[y_1]}{\partial a_{2n}} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial[y_n]}{\partial a_1} & \frac{\partial[y_n]}{\partial a_2} & \cdots & \frac{\partial[y_n]}{\partial a_{2n}} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial[y_n]_\omega}{\partial a_1} & \frac{\partial[y_n]_\omega}{\partial a_2} & \cdots & \frac{\partial[y_n]_\omega}{\partial a_{2n}} \end{vmatrix},$$

in which $\frac{\partial[y_i]_\omega}{\partial a_h}$ stands for $\left[\frac{\partial[y_i]}{\partial a_h}\right]_{x=x_\omega}$, by A_ω , where A_ω means the value of the determinant:

$$A = \sum \pm \left[\frac{\partial(a_1)}{\partial v_1}\right] \cdots \left[\frac{\partial(a_n)}{\partial v_n}\right]\left[\frac{\partial(a_{n+1})}{\partial y_1}\right] \cdots \left[\frac{\partial(a_{2n})}{\partial v_n}\right]$$

for $x = x_\omega$, and applies the law of multiplication of determinants to the product then, due to the identities (28), one will get the formula:

$$U\,(x, x_\omega) = A_\omega\,\Delta\,(x, x_\omega)\,,$$

in which $U\,(x, x_\omega)$ denotes the determinant that emerges from U by the substitutions (29).

When one introduces the reciprocal determinant of A :

$$\nabla = \sum \pm \frac{\partial[v_1]}{\partial a_1} \cdots \frac{\partial[v_n]}{\partial a_n}\frac{\partial[y_1]}{\partial a_{n+1}} \cdots \frac{\partial[y_m]}{\partial a_{2n}}\,,$$

that relation can also be written:

$$\nabla_\omega\,U\,(x, x_\omega) = \Delta\,(x, x_\omega)\,,$$

which will show that the determinants U and $\Delta\,(x, x_\omega)$ will be equal, up to a factor that is independent of x, as soon as one gives the γ_i^σ the values (29).

It likewise follows from this that those values also satisfy the second condition, so the determinant U cannot be identically zero.

Therefore, the determinant $\Delta\,(x, x_\omega)$ cannot be identically zero as long as one leaves the $2n$ integration constants a undetermined, since otherwise the basic condition that the functions y should assume given values at both limits could not be satisfied. Thus, from what was established at the conclusion of § **1**, they cannot vanish identically when one replaces those constants with the values that follow from the stated condition either.

The determinant $U(x, x_\omega)$ can then reduce to zero only when $A_\omega = 0$ or $\nabla_\omega = \infty$. However, that is impossible, because the determinants A and ∇ are independent of x.

One can deduce that indirectly from the **Poisson-Jacobi** or **Lagrange** laws for the theory of perturbations. However, it is easier to show it directly.

Namely, if one differentiates the determinant ∇ with respect to x then that will produce:

$$\frac{d\nabla}{dx} = \sum_{h=1}^{n}\sum_{i=1}^{2n}\left\{\frac{\partial\nabla}{\partial\frac{\partial(v_h)}{\partial a_i}}\frac{d}{dx}\frac{\partial|v_h|}{\partial a_i}+\frac{\partial\nabla}{\partial\frac{\partial(y_h)}{\partial a_i}}\frac{d}{dx}\frac{\partial|y_h|}{\partial a_i}\right\}.$$

However, when one differentiates the identities (6) that are obtained by substituting the solutions (8) with respect to a_i , one will get:

$$\frac{d}{dx}\frac{\partial[y_h]}{\partial a_i} = \sum_{k=1}^{n}\left\{\left[\frac{\partial^2 H}{\partial v_h\,\partial y_k}\right]\frac{\partial|y_k|}{\partial a_i}+\left[\frac{\partial^2 H}{\partial v_h\,\partial v_k}\right]\frac{\partial|v_k|}{\partial a_i}\right\},$$

$$\frac{d}{dx}\frac{\partial[v_h]}{\partial a_i} = -\sum_{k=1}^{n}\left\{\left[\frac{\partial^2 H}{\partial y_h\,\partial y_k}\right]\frac{\partial|y_k|}{\partial a_i}+\left[\frac{\partial^2 H}{\partial y_h\,\partial v_k}\right]\frac{\partial|v_k|}{\partial a_i}\right\}.$$

If one substitutes those values in the foregoing equation and combines the terms that are multiplied by the same second partial differential quotient of the function H then one will find that:

$$\frac{d\nabla}{dx} = \nabla\sum_{h=1}^{n}\left\{\left[\frac{\partial^2 H}{\partial y_h\,\partial y_h}\right]-\left[\frac{\partial^2 H}{\partial y_h\,\partial v_h}\right]\right\} = 0$$

when one applies the first law of determinants. The determinant ∇ will then be, in fact, independent of x, and it cannot be zero or infinite because the $2n$ integration constants a must be expressible in terms of v and y by the $2n$ equations:

$$v_h = [v_h]\,, \qquad y_h = [y_h]$$

if one is to define a complete system of integrals of the differential equations (6).

We can then combine our results into the following theorem, whose great importance will first appear clearly in what follows:

When one gives the values:

$$(31) \qquad \gamma_i^\sigma = \left[\frac{\partial(a_i)}{\partial v_\sigma}\right]_{x=x_\omega}$$

to the $2n^2$ *constants* γ_i^σ *, in which* x_ω *denotes an arbitrarily-chosen value of x, then the constraint equations* (21) *will be fulfilled identically, and one will have:*

$$U = C \cdot \Delta\,(x, x_\omega)$$

identically, moreover, where C is a non-zero constant that is independent of x_ω .

§ 4.

We are now fully equipped to be able to go on to the actual goal of this treatise, namely, presenting the criteria for a maximum or minimum.

If one applies the identity conversion (26) that was obtained to the second variation:

$$(9) \qquad \delta^2 J = \int_{x_0}^{x_1} 2\,\delta^2\Omega\, dx$$

then when one recalls the conditions that the variations $\mathfrak{z}$ are subject to, one will get the formula:

$$(32) \qquad \delta^2 J = \int_{x_0}^{x_1} dx \sum_{h=1}^{n}\sum_{i=1}^{n}\left[\frac{\partial^2\Omega}{\partial y_h\,\partial y_i'}\right]\frac{U_h\,U_i}{U^2} ,$$

whereas, from (27), the equations of constraint $\delta\varphi_k = 0$ will go to:

$$(33) \qquad \sum_{h=1}^{n}\left[\frac{\partial\varphi_k}{\partial y_h'}\right]U_h = 0 .$$

However, in the derivation of formula (32) from the transformation (26) for the definite integral:

$$\int_{x_0}^{x_1} dx\,\frac{d}{dx}\frac{2B}{U} ,$$

since the difference between the limiting values is set equal to the undetermined value $2B / U$, which will vanish because the variations $\mathfrak{z}$ must vanish at the limits, that formula can be applied only as long as the function $2B / U$ that is defined by (24) remains finite between the limits x_0 and x_1 .

If we now assume that the limits are so close that none of the coefficients in the original function $\delta^2 J$ nor any of the quantities $\frac{\partial[y]}{\partial a}$ and $\frac{\partial[\lambda]}{\partial a}$ become infinite between them then that function will suffer a break in finitude only when its denominator U becomes zero.

However, the vanishing of U depends upon the values of the $2n^2$ arbitrary constants γ_i^σ , and when we consider only the values of those constants that satisfy the equations of constraint (21) (which is self-explanatory under the assumption that was made), we can then state the following theorem:

I. – *As long as one can determine the arbitrary constants* γ_i^σ *in such a way that the determinant* U *is nowhere-vanishing between the limits* x_0 *and* x_1 *, the formula* (32), *in which those constants are assigned just those values that satisfy the aforementioned requirement, will be true identically for all variations* $\mathfrak{z}$ *that come under consideration.*

However, as soon as one extends the upper limit further than the condition that U must not vanish allows (and we will see that in general that condition will imply an extreme limit that cannot be exceeded, nor even attained) formula (32) will then become false, and one must once more revert to the original formula (9), in which the limits were subject to no sort of restriction.

§ 5.

If the second variation is to yield a definite criterion for a maximum or a minimum then it cannot change sign or even vanish unless all of the variations $\mathfrak{z}$ are identically zero. That is because when the second variation vanishes, the change in the integral will be of order three and can therefore be made positive or negative, in general.

Assuming that fact, the following conclusion can be inferred from formula (32):

II. – *If one can determine the arbitrary constants* γ_i^σ *in such a way that the determinant* U *does not vanish between the limits* x_0 *and* x_1 *, and if, in addition, the homogeneous function of degree two:*

$$2W = \sum_{h=1}^{n}\sum_{i=1}^{n}\left[\frac{\partial^2\Omega}{\partial y'_h\,\partial y'_i}\right]U_h\,U_i\,,$$

between whose n arbitrary arguments U_h *the m linear equations of constraint* (33) *exist, does not change sign between those limits then the second variation* $\delta^2 J$ *can neither change sign nor* *vanish, and as a result one will certainly find that a maximum or minimum for the functions* [y] *exists then.*

Since the assumptions that were made make the expression under the integral sign remain finite and incapable of experiencing a change of sign, they immediately imply that the integral (32) must possess an unvarying sign and can vanish only when the function $2W$ is identically zero.

In order to prove the last part of the theorem, one must investigate when that function vanishes identically.

It is known from algebra that any homogeneous function of degree two in n arguments between which m linear homogeneous equations of constraint exist can be transformed into a sum of $n - m$ squares.

If one sets:

$$U_h = p_h^1 V_1 + p_h^2 V_2 + \cdots + p_h^{n-m} V_{n-m} \tag{34}$$

then one can determine the n ($n - m$) coefficients p_h^i in such a way that the values of the U_h satisfy the m equations of constraint (33) and make the following two equations become identities:

$$\begin{aligned} 2W &= \rho_1 V_1^2 + \rho_2 V_2^2 + \cdots + \rho_{n-m} V_{n-m}^2 \,, \\ \sum_{h=1}^{n} U_h^2 &= \quad V_1^2 + \quad V_2^2 + \cdots + \qquad V_{n-m}^2 \,, \end{aligned}$$

moreover. The quantities ρ are then the roots of the equation:

$$\begin{vmatrix} \left[\dfrac{\partial^2\Omega}{\partial y_1' \, \partial y_1'}\right] - \rho & \cdots & \left[\dfrac{\partial^2\Omega}{\partial y_n' \, \partial y_1'}\right] & \left[\dfrac{\partial \varphi_1}{\partial y_1'}\right] & \cdots & \left[\dfrac{\partial \varphi_m}{\partial y_1'}\right] \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ \left[\dfrac{\partial^2\Omega}{\partial y_1' \, \partial y_n'}\right] & \cdots & \left[\dfrac{\partial^2\Omega}{\partial y_n' \, \partial y_n'}\right] - \rho & \left[\dfrac{\partial \varphi_1}{\partial y_n'}\right] & & \left[\dfrac{\partial \varphi_m}{\partial y_n'}\right] \\ \left[\dfrac{\partial \varphi_1}{\partial y_1'}\right] & \cdots & \left[\dfrac{\partial \varphi_1}{\partial y_n'}\right] & 0 & \cdots & 0 \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ \left[\dfrac{\partial \varphi_m}{\partial y_1'}\right] & \cdots & \left[\dfrac{\partial \varphi_m}{\partial y_n'}\right] & 0 & \cdots & 0 \end{vmatrix} = 0 \,.$$

The term that is free of ρ in that equation is the determinant [R] exactly, which is not identically zero, by assumption.

It follows from this that no root ρ of the equation above can be identically zero. The function $2W$ cannot be transformed into a sum of less than $n - m$ squares for undetermined x then. Should it not change its sign between the limits x_0 and x_1, as the theorem demands, then the $n - m$ coefficients ρ must always be positive or always be negative between those limits. Therefore, the function $2W$ can vanish for every value of x between the limits x_0 and x_1 only in such a way that all of the $n - m$ quantities V_i will be identically zero, and as a result of the substitutions (34), that would imply the vanishing of the n quantities U_h .

When the conditions for our theorem are fulfilled, the integral (320 can vanish only for those functions $\mathfrak{z}$ that are solutions of the n first-order linear differential equations:

$$(35) \qquad U_1 = 0 , \qquad U_2 = 0 , \qquad \ldots, \qquad U_n = 0 .$$

However, since one can permit only those functions $\mathfrak{z}$ that vanish at the two limits, one must further add that those solutions must vanish at the two limits or (since one also gives the variations $\mathfrak{z}$ the values that follow from (35) only in a subset of the interval from x_0 to x_1, and one can assume that they assume the constant value zero in the remaining subset of it, assuming that a continuous transition between those two types of values takes place) for any two values of x between the limits at all.

However, if one recalls the values (22) of U_h then one will see that the general solutions for the system of differential equations (35) are the following ones:

$$\mathfrak{z}_h = c_1 u_h^1 + c_2 u_h^2 + \cdots + c_n u_h^n ,$$

in which the quantities c denote arbitrary constants. Therefore, if those n solutions should vanish simultaneously for any value of x without vanishing identically then the determinant:

$$U = \sum \pm u_1^1 u_2^2 \cdots u_n^n$$

would have to likewise vanish for that value of x. However, the arbitrary constants u_h^σ in the formula were supposed to be determined in such a way that this determinant would be nowhere-vanishing between the limits x_0 and x_1 . Hence, the n expressions U_h can never vanish simultaneously between those limits for any one value of x, let alone two of them. From I, the last part of our theorem is proved with that. One sees that the second variation can neither change sign nor vanish as along as the assumptions of Theorem II are fulfilled.

By contrast, if the function $2W$ can change sign between the limits that Theorem I demands then there must be at least one coefficient ρ that possess a different sign in any interval between x_0 and x_1 than it has in the remaining one, which easily explains the fact that the integral $\delta^2 J$ can then be made positive or negative at will then.

The condition that the function $2W$ must have an unvarying sign within a limited interval for which formula (32) is valid is therefore not only sufficient, but also unconditionally necessary for the existence of an actual relative maximum or minimum of the integral J or V when it is extended over that interval.

On those grounds, and in order to be able to express myself more concisely, in what follows, I will always assume that this function is not capable of changing sign between x_0 and x_1 without abandoning the equations of constraint (33).

Under that assumption, one can remark that the second variation can no longer vanish when the conditions of Theorem II are fulfilled (which is a remark that originally goes back to **Richelot** and which deserves to be regarded as a special theorem, due to its importance), which can be expressed more briefly in the form:

III. – *As long as one can determine the arbitrary constants* γ_i^σ *in such a way that the determinant U is nowhere-zero between the limits* x_0 *and* x_1 *, the second variation itself cannot vanish either.*

§ 6.

It follows easily from that theorem that in general there is always an extreme limit x' that the upper limit x_1 cannot exceed or even attain if it is to be possible to determine the constants γ_i^σ in accordance with the stated requirement.

In fact, formulas (13) and (14) show that for every system of solutions to the differential equations:

$$\frac{\partial \Omega_2}{\partial \mathfrak{z}_h} = \frac{d}{dx}\frac{\partial \Omega_2}{\partial \mathfrak{z}_h'}, \qquad \frac{\partial \Omega_2}{\partial \mu_k} = 0\,,$$

one will have:

$$2\,\delta^2\Omega = 2\,\Omega_2 = \frac{d}{dx}\sum_{h=1}^{n} \mathfrak{z}_h \frac{\partial \Omega_2}{\partial \mathfrak{z}_h'}$$

identically. However, it follows from this that the second variation:

$$\delta^2 J = \int_{x_0}^{x_1} 2\,\delta^2\Omega\,dx$$

can always be made to vanish for values of the $\mathfrak{z}$ that satisfy the equations:

$$\delta\varphi_k = \frac{\partial \Omega_2}{\partial \mu_k} = 0$$

and are zero at both limits when $x_1 \geq x'$, where x' denotes the nearest root to x_0 of the equation:

(36)
$$\Delta(x, x_0) = \begin{vmatrix} \frac{\partial [y_1]}{\partial a_1} & \frac{\partial [y_1]}{\partial a_2} & \cdots & \frac{\partial [y_1]}{\partial a_{2n}} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial [y_n]}{\partial a_1} & \frac{\partial [y_n]}{\partial a_2} & \cdots & \frac{\partial [y_n]}{\partial a_{2n}} \\ \frac{\partial [y_1]_0}{\partial a_1} & \frac{\partial [y_1]_0}{\partial a_2} & \cdots & \frac{\partial [y_1]_0}{\partial a_{2n}} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial [y_n]_0}{\partial a_1} & \frac{\partial [y_n]_0}{\partial a_2} & \cdots & \frac{\partial [y_n]_0}{\partial a_{2n}} \end{vmatrix} = 0 .$$

Obviously, one always assumes that $x_0 < x_1$. In fact, from (12), the general solutions of the differential equations above are:

(37)
$$\mathfrak{z}_h = \gamma_1 \frac{\partial [y_h]}{\partial a_1} + \gamma_2 \frac{\partial [y_h]}{\partial a_2} + \cdots + \gamma_{2n} \frac{\partial [y_h]}{\partial a_{2n}} ,$$

(38)
$$\mu_h = \gamma_1 \frac{\partial [\lambda_h]}{\partial a_1} + \gamma_2 \frac{\partial [\lambda_h]}{\partial a_2} + \cdots + \gamma_{2n} \frac{\partial [\lambda_h]}{\partial a_{2n}} ,$$

and since the determinant $\Delta(x, x_0)$ is supposed to vanish for $x = x'$, one can determine the $2n$ arbitrary constants γ such that the n functions (37) all vanish for $x = x_0$ and $x = x'$.

When one then assigns the values (37) and (38) to the $\mathfrak{z}$ and μ for the interval from x_0 to x' , and assumes that the $\mathfrak{z}$ are constantly equal to zero outside of it, one will have:

$$\delta^2 J = \left[\sum_{h=1}^{n} \mathfrak{z}_h \frac{\partial \Omega_2}{\partial \mathfrak{z}'_h} \right]_{x_0}^{x'} = 0 ,$$

under the assumption that $x_1 \geq x'$. That would no longer be correct if the function:

$$\sum_{h=1}^{n} \mathfrak{z}_h \frac{\partial \Omega_2}{\partial \mathfrak{z}'_h}$$

became infinite between the limits x_0 and x' for the stated values of $\mathfrak{z}$ and μ . However, that case was excluded from the outset by the assumptions that were introduced in § **4**.

Now since the second variation can never vanish for those limits x_0 and x' , between which the assumption of Theorem III is fulfilled, the extreme limit x_1 for which that assumption is still valid must necessarily lie between x_0 and x'. However, as soon as it is $\geq x'$, one can certainly no longer satisfy that condition.

From the fact that the second variation can always vanish as long as one extends the upper limit x_1 beyond x', or also only extended it up to x', one likewise sees that a maximum or minimum can generally exist only as long as the upper limit remains between x_0 and x'.

§ 7.

However, one must now ask, conversely, if the upper limit lies between x_0 and x' then can the arbitrary constants γ_i^σ can always be actually determined in such a way that the determinant U will be nowhere-zero between the limits x_0 and x_1 ?

The reasoning that was employed in the previous section shows that the second variation can always be made to vanish as soon as the determinant $\Delta(x_0', x_1')$ vanishes for any two values x_0' and x_1' that lie in the interval from x_0 to x_1.

Therefore, when the condition of Theorem III is fulfilled, the determinant $\Delta(x_0', x_1')$ can never be zero as long as x_0' and x_1' remain between the limits x_0 and x_1 and do not coincide. In particular, we then have the theorem:

IV. – *As long as it is possible to determine the constants γ_i^σ in such a way that the determinant U is nowhere equal to zero between the limits x_0 and x_1, the determinant:*

$$\Delta(x, x_1)$$

cannot vanish for any value of x that is less than x_1 and greater than or equal to x_0 either.

However, one must now recall the theorem (31), according to which the $2n^2$ constants γ_i^σ can be determined in such a way that the equations of constraint (21) are satisfied, and at the same time one will have:

$$U = C \cdot \Delta(x, x_\omega) .$$

Let:

$$x_0 < x^0 < x_1 < x' .$$

The determinant $\Delta(x, x_0)$ can never vanish between the limits x' and x_1 then because x^0 and x_1 must lie between two successive roots of the equation $\Delta(x, x_0) = 0$. From the theorem that was just cited, the constants γ_i^σ can be determined such that one has:

$$U = C \cdot \Delta(x, x_0)$$

identically, so the determinant U will be nowhere-vanishing between the limits x^0 and x_1 . It will then follow from IV that the determinant $\Delta(x, x_1)$ can also be nowhere-vanishing as long as $x^0 \leq x < x_1$. However, that determinant will also be non-zero for $x = x_0$, because one has:

$$\Delta(x_0, x_1) = (-1)^n \Delta(x_1, x_0) .$$

Furthermore, nothing prevents us from letting x^0 get as close as we would like to x_0 . Therefore, $\Delta(x, x_1)$ cannot vanish between x_0 and x^0 either.

If we let $x_1 + \varepsilon$ enter in place of x_1 then we can express the result that we found as:

The determinant $\Delta(x, x_1 + \varepsilon)$ can never be zero between the limits x_0 and x_1 as long as one has $x_0 < x_1 < x_1 + \varepsilon < x'$.

Now since we can assume that $x_1 + \varepsilon$ is arbitrarily close to x' , and ε is arbitrarily small, since we further know that for the values:

$$\gamma_i^\sigma = \left[\frac{\partial(a_i)}{\partial v_\sigma}\right]_{x=x_1+\varepsilon}$$

of the $2n^2$ constants γ_i^σ that satisfy the conditions (21), we will have:

$$U = C \cdot \Delta(x, x_1 + \varepsilon)$$

identically, we will see that not only can those constants always be determined in such a way that the determinant U is nowhere-vanishing between the limits x_0 and x_1 as long as x_1 remains between x_0 and x' , but we will even find a solution to that problem.

If we recall the assumptions by which we arrived at that result and couple them with Theorem II then we will ultimately obtain the following general criterion for the maximum and minimum:

V. – *As long as the upper limit x_1 remains between x_0 and the root x' of the limit equation:*

$$\Delta(x, x_0) = 0$$

that lies closest to x_0 , the given integral will always be a maximum or minimum for those functions y that make the first variation vanish, when one assumes that the homogeneous function:

$$2W = \sum_{h=1}^{n}\sum_{i=1}^{n}\left[\frac{\partial^2\Omega}{\partial y'_h\,\partial y'_i}\right]U_h\,U_i ,$$

whose n arbitrary arguments U_h are subject to the m equations of constraint:

$$\sum_{h=1}^{n} \left[\frac{\partial \varphi_k}{\partial y'_h} \right] U_h = 0 ,$$

cannot vary in sign between those limits. By contrast, neither a maximum nor a minimum will generally exist as soon as $x_1 \geq x'$.

However, one should not overlook the fact that the coefficients of the original function $\delta^2 J$ cannot become infinite between the limits x_0 and x_1, moreover, and that the finiteness of the functions $\frac{\partial [y]}{\partial a}$ and $\frac{\partial [\lambda]}{\partial a}$ is assumed in that interval.

The first condition is necessary because otherwise the entire development of the function Ω in a **Taylor** series would no longer be allowable. By contrast, the second assumption does not seem to be necessary. even though it cannot generally be abandoned in the reasoning that is employed here. At least, **Jacobi** (*) has shown, in the simplest case of the calculus of variations, by completely different considerations that are independent of the finitude or infinitude of those functions, that the limit equation will produce the widest limit interval inside of which the denominator *U* in the reduction can be prevented from vanishing by a suitable determination of its arbitrary constants, and the same thing can also be proved by a similar argument for **Jacobi**'s example of the integral of least action for the elliptic motion of a planet in space, where one of those functions will become infinite within the interval in question.

§ 8.

In the general case where the solutions [*y*] are not linear with respect to their 2*n* integration constants *a*, the limit equation $\Delta\,(x, x_0) = 0$ will admit a very simple interpretation, and it is important to derive it in order to show the agreement between the criteria that were obtained with the ones that **Jacobi** gave without proof in Bd. **17** of this journal on pp. 73.

The 2*n* constants *a* are assigned those values that they get from the 2*n* equations:

$$y_{h0} = [y_h]_0 , \qquad\qquad y_{h1} = [y_h]_1 ,$$

whose left-hand sides are the given limiting values and whose right-hand sides are the values of the solutions for $x = x_0$ ($x = x_1$, respectively).

Now when those equations are not linear with respect to the unknowns *a*, one will generally obtain several different systems of values for those constants from them that correspond to the same limiting values of the variable *y*, and it might be possible that those two systems will coincide for a certain value of x_1 , or as one can also say, come infinitely close to each other.

Let $a_1, a_2, \ldots, a_{2n}$ be a system of values for the constants *a* that satisfies the equations above. Should those equations be also fulfilled by values that differ from them by infinitely little:

(*) Cf., **Hesse**, Bd. 54, pp. 256-260 of this journal.

$$a_1 + \delta a_1\,,\; a_2 + \delta a_2\,,\; \ldots,\; a_{2n} + \delta a_{2n}\,,$$

then the $2n$ equations:

$$0 = \frac{\partial [y_h]_0}{\partial a_1}\,\delta a_1 + \frac{\partial [y_h]_0}{\partial a_2}\,\delta a_2 + \cdots + \frac{\partial [y_h]_0}{\partial a_{2n}}\,\delta a_{2n}\;,$$

$$0 = \frac{\partial [y_h]_1}{\partial a_1}\,\delta a_1 + \frac{\partial [y_h]_1}{\partial a_2}\,\delta a_2 + \cdots + \frac{\partial [y_h]_1}{\partial a_{2n}}\,\delta a_{2n}$$

would have to be true. However, that cannot be true unless:

$$\Delta\,(x, x_0) = 0\;.$$

As a result, the equation $\Delta\,(x, x_0) = 0$ is the condition for the two different systems of roots for the $2n$ equations:

$$y_{h0} = [y_h]_0\,, \qquad\qquad y_{h1} = [y_h]_1\,,$$

from which the integration constants of the solutions are determined from the limiting values of the variables, to be equal to each other.

However, if one uses that interpretation of the limit equation as a basis then Theorem V will give one the case that **Jacobi** treated when he sought the maximum or minimum of the integral:

$$\int_{x_0}^{x_1} f\,(x, y, y', \ldots, y^{(n)})\,dx\,,$$

in which the homogeneous function $2W$ reduces to:

$$\left[\frac{\partial^2 f}{\partial y^{(n)}\,\partial y^{(n)}}\right] U_n^2$$

here (*), in which one sets:

$$\begin{array}{llll} y_1 = y\,, & y_2 = y'\,, & \ldots, & y_n = y^{(n-1)}, \\ \mathfrak{z}_1 = \mathfrak{z}\,, & \mathfrak{z}_2 = \mathfrak{z}'\,, & \ldots, & \mathfrak{z}_n = \mathfrak{z}^{(n-1)}, \end{array}$$

which is the same criterion that one finds to be expressed in the given location.

(*) Cf., **Clebsch**, Bd. 55, pp. 267 of this journal.

§ 9.

In the foregoing, one always considered only the case in which the limits x_0 and x_1 were both given, as well as the limiting values of the y. However, as is known, one can reduce all other cases to that one when one divides the problem into two separate parts, the first of which belongs to the actual calculus of variations, while the second belongs to differential calculus. One first assumes that the limiting values of the variables are all given and looks for the maximum or minimum of the integral in question under that assumption. The criteria for it to be an actual maximum or minimum under that assumption are known from the foregoing, and from theorem V, they depend upon the limits and the values of the integration constants of the problem, which in turn depend upon the limiting values of the variables. Now if one has found the maximum or minimum value of the integral then if the limiting values are not all given, one secondly comes to the problem of determining them when one considers the given limit conditions such that the value of the integral, which is now a given function of the limiting values, will be a maximum or minimum. The substitution of the limiting values, thus determined, in the criteria above will yield the criteria for a maximum or minimum in the present case.

That way of deriving the criteria for a maximum or minimum for the limiting values that are not given always seems to require that one must perform a quadrature. However, since that does not amount to finding the greatest or least value of the integral, but only variation that it experiences when one varies the limits and the integration constants, one will easily see that one can always avoid that quadrature (or more precisely, it can be replaced with another one that can be performed immediately) as long as the limiting values do not enter under the integral sign.

Leipzig, in March 1868.