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Solving some problems in analytic geometry by means of the barycentric calculus

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The problems that will be solved in what follows with the help of the barycentric calculus that Herrn Prof. **Möbius** discovered are merely elimination problems for that method. Due to their importance, it would seem interesting to solve them in that way.

1. – Suppose that the barycentric expressions for three points in a plane are given. One then seeks the area of the triangle that has those three points for its vertices.

Let the expressions for the three points be:

$$q\,P = a\,A + b\,B + c\,C\,, \qquad q'\,P' = a'\,A + b'\,B + c'\,C\,, \qquad q''\,P'' = a''\,A + b''\,B + c''\,C\,.$$

If one eliminates the fundamental point A then one will get:

$$a'\,q\,P - a\,q'\,P' = \beta'\,B + \gamma'\,C \quad \text{and} \quad a''\,q'\,P' - a'\,q''\,P'' = \beta''\,B + \gamma''\,C\,,$$

in which:

$$\beta' = b\,a' - b'\,a\,, \qquad \gamma' = c\,a' - c'\,a\,, \qquad \beta'' = b'\,a'' - b''\,a'\,, \qquad \gamma'' = c'\,a'' - c''\,a'\,.$$

If one further eliminates the fundamental point B then that will give:

$$\beta''\,a'\,q\,P - (\beta''\,a + \beta'\,a'')\,q'\,P' + \beta'\,a'\,q''\,P'' = \beta''\,\gamma' - \gamma''\beta' \cdot C\,.$$

It follows from those equations that:

$$\begin{aligned} ABC &: BCP = q : a\,, \\ BCP &: CPP' = -a\,q' : \beta'\,, \\ CPP' &: PP'P'' = \beta'\,a'\,q'' : \beta''\,\gamma' - \beta'\,\gamma'' \end{aligned}$$

Hence:

$$ABC : PP'P'' = a' \cdot q\,q'\,q'' : \beta'\,\gamma'' - \beta''\,\gamma'\,.$$

The quantity $(\beta'\gamma'' - \beta''\gamma')/a'$ is equal to:

$$a(b'c'' - b''c') + a'(b''c - bc'') + a''(bc' - b'c) \ .$$

If one denotes $a(b'c'' - b''c')$ by *A*, $a'(b''c - bc'')$ by A', and $a''(bc' - b'c)$ by A'' then one will see that *A* goes to A' when one increases the number of primes in the expression for *A* by one unit, and always writes *c* in place of c'''; A'' also arises from A' in a similar way.

One will then have:

I. $$\frac{P\,P'\,P''}{ABC} = \frac{A + A' + A''}{q\,q'\,q''} \ .$$

2. – Let the expressions for three lines in a plane be given. One shall now find the area of the triangle that they include.

Let their expressions be:

$$A + x\,B + (a + b\,x)\,C\,, \qquad A + x\,B + (a' + b'\,x)\,C\,, \qquad A + x\,B + (a'' + b''\,x)\,C\ .$$

The intersection points of the first and second, second and third, and third and first lines will be called Π, Π′, Π″, respectively. One finds that:

$$q\,\Pi = (b - b')\,A - (a - a')\,B + (a'b - ab')\,C\ .$$

In the same way:

$$q'\,\Pi' = (b' - b'')\,A - (a' - a'')\,B + (a''b' - a'b'')\,C\ ,$$
$$q''\,\Pi'' = (b'' - b')\,A - (a'' - a')\,B + (ab'' - a''b)\,C\ .$$

If one now denotes the quantity $a(b' - b'') + a'(a' - a'') + a''(b - b')$ by *M* then one will get the first factor *A* of the numerator in I:

$$A = -a''(b - b')\,M\ .$$

Similarly:

$$A' = -a(b' - b'')\,M\,, \quad A'' = -a'(b'' - b)\,M\ .$$

As one easily sees, it follows from this that:

II. $$\frac{\Pi\,\Pi'\,\Pi''}{ABC} = \frac{M^2}{-q\,q'\,q''} \ .$$

3. – Let four points in space be given. Find the volume of the tetrahedron that has them for its vertices.

Call the points P, P', P'', P''', and let:

$$q\,P = a\,A + b\,B + c\,C + d\,D\,.$$

P', P'', P''' have similar coefficients, but provided with the primes ′, ″, ‴, resp. Just as before, one finds by eliminating A, B, C that:

III. $$\frac{P\,P'\,P''\,P'''}{A\,B\,C\,D} = \frac{A + A' + A''}{q\,q'\,q''\,q'''\cdot a'\,a''}\,,$$

whereby one will have:

$$A = \delta'(\beta''\gamma''' - \beta'''\gamma''), \qquad A' = \delta''(\beta'''\gamma' - \beta'\gamma'''), \qquad A'' = \delta'''(\beta'\gamma'' - \beta''\gamma'),$$

and

$$\begin{aligned}
\beta' &= a'b - a\,b', & \gamma' &= a'c - a\,c', & \delta' &= a'd - a\,d', \\
\beta'' &= a''b' - a'b'', & \gamma'' &= a''c' - a'c'', & \delta'' &= a''d' - a'd'', \\
\beta''' &= a'''b'' - a''b''', & \gamma''' &= a'''c'' - a''c''', & \delta''' &= a'''d'' - a''d'''.
\end{aligned}$$

One can easily develop the expression $A + A' + A''$, and it will then be found to be divisible by $a'\,a''$ then. In any event, one is free to set $a = a' = a'' = a''' = 1$.

4. – Four planes are given. Find the volume of the tetrahedron that they include.

We denote an arbitrary point of the first plane by P and points of the other three by P', P'', P''', resp.

Let:

$$P \equiv A + x\,B + y\,C + (a + b\,x + c\,y)\,C\,.$$

In order to express the other three planes, we put one, two, and three primes, resp., on a, b, c.

The intersection points of the planes 123, 234, 341, 412 are called Π, Π′, Π″, Π‴, respectively.

One finds that:

$$q\,\Pi = (b\,c')A + (c\,a')\,B + (a\,b')\,C + (a\,b\,c)\,D\,,$$

where

$$\begin{aligned}
(b\,c') &= b\,(c' - c'') + b'(c'' - c') + b''(c - c')\,, \\
(c\,a') &= a\,(c'' - c') + a'(c - c'') + a''(c' - c)\,, \\
(a\,b') &= a\,(b' - b'') + a'(b'' - b) + a''(b - b')\,,
\end{aligned}$$

$$(abc) = a\,(b\,c') + b\,(a\,c') + c\,(a\,b') = a\,(b'c'' - b''c') + a'(b''c - b\,c'') + a''(b\,c' - b'c)\,,$$

$$q = (b\,c') + (c\,a') + (a\,b') + (a\,b\,c)\,.$$

In order to find expressions for Π′, Π″, Π″′, one must only increase the number of primes in the coefficients of the expression for Π by one unit, and always write *a* instead of a^{IV}, *b* instead of b^{IV}, and *c* instead of c^{IV}.

One finds from III that:

$$\frac{\Pi\,\Pi'\,\Pi''\,\Pi'''}{A\,B\,C\,D} = \frac{A + A' + A''}{q\,q'\,q''\,q'''(b'\,c'')(b''\,c''')} .$$

In that expression, one has $A = \delta'(\beta''\gamma''' - \beta'''\gamma'')$, as before, and similarly for A' and A''.

For the quantities β', γ', δ', etc., one has:

$$\begin{aligned}
\beta' &= (b'\,c'')(c\,a') - (b\,c')(c'\,a'') , \\
\gamma' &= (b'\,c'')(a\,b') - (a'\,b'')(b\,c') , \\
\delta' &= (b'\,c'')(a\,b\,c) - (b\,c')(a'\,b'\,c') .
\end{aligned}$$

One must only increase the number of primes in order to get β'', β''', γ'', γ''', δ'', δ''', resp. One sets:

$$a'''(b\,c') - a(b'\,c'') + a'(b''\,c''') - a''(b'''\,c) = M .$$

After the necessary reductions, one will get:

$$\begin{aligned}
\beta' &= -(c'' - c')\,M , & \gamma' &= -(b' - b'')\,M , & \delta' &= -(b'\,c'' - b''\,c)\,M , \\
\beta'' &= +(c''' - c'')\,M , & \gamma'' &= +(b'' - b''')\,M , & \delta'' &= +(b''\,c''' - b'''\,c'')\,M , \\
\beta''' &= -(c - c''')\,M , & \gamma''' &= -(b''' - b)\,M , & \delta''' &= -(b'''\,c - b\,c''')\,M .
\end{aligned}$$

One will further get from this that:

$$\begin{aligned}
A &= -M^3(b''\,c' - b'\,c'')\cdot(b''\,c''') , \\
A' &= -M^3(b'''\,c'' - b''\,c''')\cdot[(b''\,c''') - (b'''\,c)] , \\
A'' &= -M^3(b\,c''' - b'''\,c)\cdot(b'\,c'') .
\end{aligned}$$

A further reduction yields:

$$(b\,c''' - b'''\,c)(b'\,c'') - (b'''\,c)(b'''\,c'' - b''\,c''') = (c'''\,b' - c'\,b''')(b''\,c''') .$$

It then follows from this that $A + A' + A'' = M^3(b'\,c'')\cdot(b''\,c''')$, and as a result:

IV.
$$\frac{\Pi\,\Pi'\,\Pi''\,\Pi'''}{A\,B\,C\,D} = \frac{M^3}{q\,q'\,q''\,q'''} .$$

5. – The expressions II and IV require easy modifications in the cases where the basic barycentric expression for a straight line or plane is not immediately applicable. In order to leave no gaps, we shall now discuss those cases.

One sees that if the line whose expression is:

$$A + x\,B + (a + b\,x)\,C$$

is supposed to coincide with the fundamental line AB then one will need only to set $a = b = 0$.

By contrast, should that line coincide with AC then one would next consider the general expression $A + e\,x\,B + (a + b\,x)\,C$, where e is a constant coefficient. The line that is represented by that expression will coincide with AC when $e = 0$. If one now sets $e\,x = y$ then the expression will go to $A + y\,B + \left(a + \frac{b}{e}\,y\right)C$, from which, one sees that when the given line is supposed to be AC itself, only b will be regarded as an infinite quantity in the expression II. One will then get:

$$\frac{(a'' - a')^2}{(1 + a')(1 + a'')(b' - b'' - a' + a'' + a''\,b' - a'\,b'')}$$

for the area of the triangle that is included by the fundamental line AC and the other two $A + x\,B + (a' + b'\,x)\,C$, $A + x\,B + (a'' + b''\,x)\,C$. Should the line coincide with BC, then one would only need to regard a as infinite.

Should that same line go through the fundamental point A, then one would need to have $a = 0$. Should it go through B, then $b = 0$. However, should it go through C then its expression would actually have to be:

$$A + g\,B + y\,C\,.$$

However, from the basic form, one will find the correct result when one only sets $a = -\,g\,b = \infty$. From II, that will then imply that:

$$\frac{\Pi\,\Pi'\,\Pi''}{ABC} = \frac{(a' - a'' + g\,(b' - b''))^2}{1 + g + a' + g\,b' \cdot 1 + g + a'' + g\,b'' \cdot q'}\ .$$

One can also calculate the area of the triangle for this case directly from the expressions for the three lines, and one would then find exactly the same result.

6. – Should the plane $A + x\,B + y\,C + (a + b\,x + c\,y)\,D$ coincide with the fundamental plane then one would need to have $a = b = c = 0$. Should it coincide with ABC then one would need only to consider the c in the expression IV to be infinite. Should it coincide with ACD then $b = \infty$, and should it coincide with BCD, then one would need to set $a = \infty$.

Should that plane go through one of the fundamental points A, B, C, then one would only need to set a, b, $c = 0$, resp. Should it go through D, then its expression would need to have the form:

$$A + x\, B + (e + f\, x)\, C + y\, D\ .$$

In that case, one would need to set $b = - f\, c$, $a = - e\, c$ in the expression IV and regard c as infinite.

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