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# Remark on the unwinding of curved lines on surfaces

(By Herrn Dr. *Minding* in Berlin)

Translated by D. H. Delphenich

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The quantity (cos *i*) / *R*, which refers to the inverse value of the radius of curvature $\rho$ of a curve that is wound in the plane (in the sense of the notation that was used in Article 22 of the previous volume) can be expressed in general in a way that is similar to the measure that **Gauss** called the *mensura curvaturae* and presented in his *Disquisitiones generales circa superficies curvas*. Namely, if one sets:

$$dx = a\,dp + a'\,dq \cdot dy = b\,dp + b'\,dq \cdot dz = c\,dp + c'\,dq\,,$$

as was done in that treatise, in which one regards the coordinates *x*, *y*, *z* as given functions of two variables *p* and *q*, and further sets:

$$a^2 + b^2 + c^2 = E\,, \qquad a\,a' + b\,b' + c\,c' = F\,, \qquad a'^2 + b'^2 + c'^2 = G$$

then $\rho$ will depend upon the quantities *E*, *F*, *G*, and their differentials, as well as on the differential quotients $\frac{dp}{dq}$, $\frac{d^2 p}{dq^2}$, which are determined by the equation of the curve.

If the differential equation of the surface is:

$$X\,dx + Y\,dy + Z\,dz = 0$$

then one will get:

$$X : Y : Z = c\,b' - b\,c' : a\,c' - c\,a' : b\,a' - a\,b'\,,$$

so the tangent plane will be:

$$(c\,b' - b\,c')\,x + (a\,c' - c\,a')\,y + (b\,a' - a\,b')\,z + \alpha = 0\,.$$

For the subsequent (*anschließende*) plane, one has:

$$A\,x + B\,y + C\,z + \beta = 0\,,$$

in which:

$$A = dz\,d^2y - dy\,d^2z\,,\quad B = dx\,d^2z - dz\,d^2x\,,\quad C = dy\,d^2x - dx\,d^2y\,.$$

If one now expresses the coordinates in terms of $p$ and $q$ then one will get:

$$\begin{aligned} A = {} & (c\,db - b\,dc)\,dp^2 + (c'\,db' - b'\,dc')\,dq^2 \\ & + (c\,db' + c'\,db - b\,dc' - b'\,dc)\,dp\,dq + (c\,b' - b\,c')\,(dp\,dq^2 - dq\,d^2p)\,, \end{aligned}$$

$$\begin{aligned} B = {} & (a\,dc - c\,da)\,dp^2 + (a'\,dc' - c'\,da')\,dq^2 \\ & + (a\,dc' + a'\,dc - c\,da' - c'\,da)\,dp\,dq + (a\,c' - c\,a')\,(dp\,dq^2 - dq\,d^2p)\,, \end{aligned}$$

$$\begin{aligned} C = {} & (b\,da - a\,db)\,dp^2 + (b'\,da' - a'\,db')\,dq^2 \\ & + (b\,da' + b'\,da - a\,db' - a'\,db)\,dp\,dq + (b\,a' - a\,b')\,(dp\,dq^2 - dq\,d^2p)\,. \end{aligned}$$

One obtains the numerator $Z$ of cos $i$ with the help of those expressions:

$$\begin{aligned} Z = {} & (c\,b' - b\,c')\,A + (a\,c' - c\,a')\,B + (b\,a' - a\,b')\,C \\ = {} & \{(E\,a' - F\,a)\,da + (E\,b' - F\,b)\,db + (E\,c' - E\,c)\,dc\}\,dp^2 \\ & + \{(F\,a' - G\,a)\,da' + (F\,b' - G\,b)\,db' + (F\,c' - G\,c)\,dc'\}\,dq^2 \\ & + \left\{\begin{array}{ll} & (E\,a' - F\,a)\,da' + (F\,a' - G\,a)\,da \\ + & (E\,b' - F\,b)\,db' + (F\,b' - G\,b)\,db \\ + & (E\,c' - F\,c)\,dc' + (F\,c' - G\,c)\,dc \end{array}\right\} dp\,dq \\ & + (E\,G - F\,F)(dp\,d^2q - dq\,d^2p)\;. \end{aligned}$$

Now, due to the meanings of the quantities $a, b, c, a', b', c'$:

$$\frac{da}{dq} = \frac{da'}{dp}\,,\qquad \frac{db}{dq} = \frac{db'}{dp}\,,\qquad \frac{dc}{dq} = \frac{dc'}{dp}\,,$$

which will then imply that:

$$a\,da' + b\,db' + c\,dc' = \tfrac{1}{2}\frac{dE}{dq}\,dp + \left(\frac{dF}{dq} - \tfrac{1}{2}\frac{dG}{dp}\right)dq\,,$$

$$a'\,da + b'\,db + c'\,dc = \tfrac{1}{2}\frac{dG}{dp}\,dp + \left(\frac{dF}{dp} - \tfrac{1}{2}\frac{dE}{dq}\right)dp\,.$$

Finally, one will therefore get the following value for $Z$:

$$\left\{E\left(\tfrac{1}{2}\frac{dG}{dp}dq+\frac{dF}{dp}dp-\tfrac{1}{2}\frac{dE}{dq}dp\right)-\tfrac{1}{2}F\,dE\right\}dp^2$$
$$+\left\{\tfrac{1}{2}F\,dG-G\left(\tfrac{1}{2}\frac{dE}{dq}dp+\frac{dF}{dq}dq-\tfrac{1}{2}\frac{dG}{dp}dq\right)\right\}dq^2$$
$$+\left\{\tfrac{1}{2}E\,dG-\tfrac{1}{2}G\,dE+F\left(\frac{dF}{dp}dp-\frac{dF}{dq}dq+\frac{dG}{dp}dq-\frac{dE}{dq}dq\right)\right\}dp\,dq$$
$$+\ (EG-FF)(dp\,d^2q-dq\,d^2p)\ .$$

One now has:

$$\cos i=\frac{Z}{\sqrt{(EG-FF)}\cdot\sqrt{(A^2+B^2+C^2)}}\ .$$

Furthermore, one has:

$$R=\frac{dP^2}{\sqrt{(A^2+B^2+C^2)}}$$

for the radius of curvature, in which:

$$dP^2\ =\ E\,dp^2+2F\,dp\,dq+G\,dq^2\ =\ dx^2+dy^2+dz^2\,.$$

Hence:

$$\frac{\cos i}{R}\ =\ \frac{1}{\rho}\ =\ \frac{Z}{\sqrt{(EG-FF)}\,dP^2}\ .$$

If one lets $p$ and $q$ mean polar coordinates on the surface, i.e., one sets $p = s$, $q = \psi$, with the previous notations, then one will get:

$$E=1\,,\qquad F=0\,,\qquad G=\varphi^2\,,\qquad dP^2\ =\ ds^2+\varphi^2\,d\psi^2\,.$$

That will imply that:

$$\frac{dP^3}{\rho}+\varphi\,d\psi\,d^2s-2\frac{d\varphi}{ds}d\psi\,ds^2-\frac{d\varphi}{d\psi}d\psi^2\,ds-\varphi^2\frac{d\varphi}{d\psi}d\psi^3\ =0\ ,$$

when one sets $d^2\psi = 0$. One further has:

$$\frac{d\,\dfrac{dy}{dP}+q\,d\,\dfrac{dz}{dP}}{dx\sqrt{(1+p^2+q^2)}}\ =\ \frac{1}{\rho}\ .$$

It emerges from a comparison between those two forms for $1 / \rho$ that one will also have the following relation for a curve whose equations in included in the expression $s$ = const.:

$$(A) \qquad d\,\frac{dy}{dP} + q\,d\,\frac{dz}{dP} = \frac{d\varphi}{ds} \cdot \frac{1}{\varphi} .$$

In general, in addition to the variable quantity $\psi$, the expression on the right-hand side also includes the coordinates of the midpoint of the curve, which must be eliminated by means of the equations $s$ = const. and $ds = 0$ if ($A$) is to be the general differential equation for a curve of constant radius on the surface. However, the quantities $\alpha$, $\beta$, $\gamma$ will drop out automatically for those surfaces on which the measure of curvature is constant.

If one denotes it by $k$ then § 19 of the aforementioned treatise by **Gauss** will show that in general:

$$\frac{d^2\varphi}{ds^2} + k\,\varphi = 0 .$$

The function $\varphi$ (which was denoted by $m$ there) must then be arranged so that $\varphi = 0$ and $d\varphi / ds = 1$ for $s = 0$. It will then follow from this that when $k$ is constant, $\varphi = \frac{1}{\sqrt{k}} \sin s\sqrt{k}$ . In that case, equation ($A$) will then go to:

$$\frac{d\,\dfrac{dy}{dP} + q\,d\,\dfrac{dz}{dP}}{dx\sqrt{(1 + p^2 + q^2)}} = \sqrt{k}\,\cot s\sqrt{k} .$$

Therefore, the theorem will be true on those surfaces that I was tempted to exhibit in general by requiring it, and one will find the basis for that briefly suggested on page 303 of the previous volume.

Berlin, in May 1830.

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