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On systems of first-order partial differential equations in involution.

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Translated by D H. Delphenich

In the note that I read to the R. Istituto Lombardo twenty years ago in which I applied the Pfaff method to the integration of an involutory system of first-order partial differential equations, I arrived at a result that it is a natural extension of Jacobi's first method of integration, as modified by Mayer, or Cauchy's method, as one might say (*).

In that way, any involutory system of such partial differential equations can be associated with a completely-integrable system of total differential equations whose form recalls (or better yet, includes) the Hamiltonian form.

One can deduce a complete system of the Jacobian system from the complete integration of that system, and conversely, as in Jacobi's classical theory (**).

The integration of a completely-integrable system of total differential equations immediately reduces to the integration of a system of ordinary differential equations, thanks to a well-known theorem of Mayer, which implies, in the final analysis, *that one can fix the integration path in the domain of independent variables that starts from the system of initial values that is prescribed for those variables at will*, which is a property that is *characteristic of total differentials*, regardless of whether it is assigned explicitly or given implicitly by means of total differential equations, as I pointed out at the end of 1886 in an article of mine that was included in Mathematische Annalen (***).

If one applies that theorem to the system of total differential equations that is associated with an involutory system of partial differential equations then one will more spontaneously arrive at the theorem of Lie that appears to be, in a sense, *the correlative* of Mayer's theorem, as I observed before in my cited note that I read to the Institute in 1883.

In the present work, permit me to return to that argument in order to communicate to the Institute some noteworthy properties of the total differential equations that are associated with an

(*) See my article in the Rend. dell'Istituto in the year 1883 [(2) **16**, pp. 637 and 691] that was entitled "Il metodo di Pfaff per l'integrazione delle equazioni a derivate parz. del 1e ordine."

(**) Those same results were reproduced by Saltykow in 1899 and published in the *Comptes Rendus* of the Paris Academy. Cf., vol. 128, pp. 166, 225, etc.

(***) Band XXVII: "Ueber die Integration der vollständstigen Differentiale.",

Cf., also my paper: "Sull'integrazione delle equaz. ai diff. tot. del 2e ordine," published in vol. LII (series II) of the r. Accademia delle scienze di Torino, especially § 5.

involutory system of first-order partial differential equations, and in particular, to show how one can extend the theory of *transformations of Hamiltonian canonical equations* to such systems with no further analysis, which is what I developed recently in the Rendiconti della R. Accademia dei Lincei (*).

§ 1.

Consider an involutory system of m first-order partial differential equations with n independent variables: $p_1, p_2, \ldots, p_n$.

Let $q_1, q_2, \ldots, q_n$ denote the partial derivatives of the unknown function V, so that system can be put into the solved form:

$$F_h \equiv q_n - f_h\,(p_1, \ldots, p_n\,; q_{m+1}, \ldots, q_n) = 0 \qquad (h = 1, 2, \ldots, m). \qquad \text{[I]}$$

One then sets:

$$(F, G) = \sum_{k=1}^{n} \left(\frac{\partial F}{\partial p_k}\frac{\partial G}{\partial q_k} - \frac{\partial F}{\partial q_k}\frac{\partial G}{\partial p_k} \right),$$

as usual, and one will have:

$$0 = (F_h, F_i) \equiv \frac{\partial f_i}{\partial p_h} - \frac{\partial f_h}{\partial p_i} + \sum_{j=m+1}^{n} \left(\frac{\partial f_h}{\partial p_j}\frac{\partial f_i}{\partial q_j} - \frac{\partial f_h}{\partial q_j}\frac{\partial f_i}{\partial p_j} \right) \qquad (i, h = 1, 2, \ldots, m), \qquad \text{[II]}$$

identically.

If we would like to apply Jacobi's *Nova Methodus* to the integration of [I] then, above all, we must find a solution F to the linear system:

$$(F, F_h) = 0 \qquad (h = 1, 2, \ldots, m) \qquad \text{[III]}$$

that is independent of the solutions $F_1, F_2, \ldots, F_m$. Now, it is clear that if one is given such a solution then one can always eliminate the $q_1, q_2, \ldots, q_m$ from the remaining ones in [I]: That is because *it can always be expressed in terms of only* $p_1, \ldots, p_n\,; q_{m+1}, \ldots, q_n$, *and when it is expressed in that way, it will again be a solution to the Jacobian system* [III] (**).

If one then regards the F as being independent of the $q_1, q_2, \ldots, q_m$, as is permissible, then [III] will become:

$$\frac{\partial F}{\partial p_h} - \sum_{j=m+1}^{n} \left(\frac{\partial F}{\partial p_j}\frac{\partial f_h}{\partial q_j} - \frac{\partial F}{\partial q_j}\frac{\partial f_h}{\partial p_j} \right) = 0\,. \qquad \text{[III}'\text{]}$$

(*) Cf., my note: "Sulla trasformazione delle equazioni differenziali di Hamilton," *loc. cit.* (5) **12**, pt. 1, pp. 113-122, 149-152, 297-300.

(**) Cf., **Goursat**'s excellent: *Leçons sur l'intégration des equations aux dérivées partielles du premier ordre*, on page 158.

The adjoint system of total differential equations is (*):

$$\left.\begin{aligned} dp_j &= -\sum_{h=1}^{m} \frac{\partial f_h}{\partial q_j}\, dp_h\,, \\ dq_j &= \sum_{h=1}^{m} \frac{\partial f_h}{\partial p_j}\, dp_h \end{aligned}\right\} \qquad (j = m+1, \ldots, n). \qquad \text{[IV]}$$

That system of 2 $(n - m)$ total differential equations is completely integrable. The condition for its complete integrability is verified by [II]. It does not contain the variables $q_1, q_2, \ldots, q_m$ since any one of its integrals would generally be a solution of [III] that is independent of the F_h, which nonetheless contains the $q_1, \ldots, q_n$.

That system is the one whose complete integration I reduced to the determination of a complete solution to the involutory system [I] in my 1883 article.

One calls a complete integrable system of total differential equations of type [IV] a *canonical system*. It is the associated system to the system in involution [I].

§ 2.

For ease of notation, consider a canonical system of 2ν total differential equations with μ independent variables $t_1, t_2, \ldots, t_\mu$, which are written as follows:

$$\left.\begin{aligned} dp_i &= -\sum_{s=1}^{\mu} \frac{\partial f_s}{\partial q_i}\, dt_s\,, \\ dq_i &= \sum_{s=1}^{\mu} \frac{\partial f_s}{\partial p_i}\, dt_s\,, \end{aligned}\right\} \qquad (i = 1, 2, \ldots, \nu) \qquad \text{[I]}$$

in which $f_1, f_2, \ldots, f_\mu$ are given functions of the:

$$t_1, t_2, \ldots, t_\mu\,; p_1, q_1\,; p_2, q_2\,; \ldots, p_\nu, q_\nu$$

that satisfy the conditions:

$$\frac{\partial f_r}{\partial p_s} - \frac{\partial f_s}{\partial p_r} = \sum_{i=1}^{\nu} \frac{\partial (f_r, f_s)}{\partial (p_i, q_i)} \qquad \text{[II]}$$

identically.

That canonical system if the first Pfaff system of the differential equation:

$$E_d \equiv q_1\, dp_1 + \ldots + q_\nu\, dp_\nu + f_1\, dt_1 + \ldots + f_\mu\, dt_\mu + d\varphi\,, \qquad \text{[III]}$$

(*) Cf., **Pascal**, *I gruppi continui di trasformazioni*, Milan, Heopli, 1903, pp. 307-308.

in which φ indicates any function of all the variables.

The bilinear covariant of that expression is:

$$\delta E_d - d E_\delta = \sum_{i=1}^{r} (\delta q_i \, dp_i - dq_i \, \delta p_i) + \sum_{s=1}^{\mu} (\delta f_s \, dt_s - df_s \, \delta t_s) \ ,$$

and when one equates the coefficients of the δp_i and δq_i to zero, one will get [I], while if one equates the coefficients of the δt_s to zero then one will get the equations:

$$df_s - \sum_{r=1}^{\mu} \frac{\partial f_r}{\partial t_s} dt_r = 0 \qquad (s = 1, \ldots, \mu),$$

which are a consequence of [I].

Indeed, when the latter equation is developed, it will become:

$$\sum_{i=1}^{\nu} \left(\frac{\partial f_s}{\partial p_i} dp_i + \frac{\partial f_s}{\partial q_i} dq_i \right) + \sum_{r=1}^{\mu} \left(\frac{\partial f_s}{\partial t_r} - \frac{\partial f_r}{\partial t_s} \right) dt_r = 0 \ ,$$

and from [II]:

$$\sum_{i=1}^{\nu} \left(\frac{\partial f_s}{\partial p_i} dp_i + \frac{\partial f_s}{\partial q_i} dq_i \right) + \sum_{r=1}^{\mu} dt_r \sum_{i=1}^{\nu} \frac{\partial (f_s, f_r)}{\partial (p_i, q_i)} = 0 \ ,$$

which is an equation that can obviously be deduced from [I].

Equations [I] include not only Hamilton's canonical equations as a special case ($\mu = 1$), but it also has the characteristic property that the Poisson parentheses:

$$(\alpha, \beta) \equiv \sum_{i=1}^{\nu} \frac{\partial (\alpha, \beta)}{\partial (p_i, q_i)}$$

is composed of two given integrals or other integrals or constants.

Indeed, the adjoint system to [I] is:

$$\frac{\partial F}{\partial t_r} = (F, f_r) \qquad (r = 1, 2, \ldots, \mu),$$

while from the Jacobi identity, one has:

$$(\alpha, (\beta, f_r)) + (\beta, (f_r, \alpha)) + (f_r, (\alpha, \beta)) = 0 \ ,$$

since if α and β are two solutions of the preceding linear equations, one will also have:

$$\frac{\partial (\alpha, \beta)}{\partial t_r} = \left(\alpha, \frac{\partial \beta}{\partial t_r} \right) + \left(\frac{\partial \alpha}{\partial t_r}, \beta \right) = ((\alpha, \beta), f_r) \ .$$

Since the first Pfaff system of a differential expression is invariantly linked with that, it will follow that *any transformation of the variables will convert the system* [I] *into the first Pfaff system of the transform of* [III].

If one regards φ as arbitrary then E_d will always have class $2\nu + 1$, as I proved in the note that I read to the Institute in 1883. It can then be reduced to the canonical form:

$$E_d = d\,\Phi + \sum_{i=1}^{\nu} Q_i\,dP_i\,,$$

whose first Pfaff system is:

$$d\,P_i = d\,Q_i = 0\,,$$

so the P_i and Q_i are the integrals of the canonical system [I]. To carry out that reduction is equivalent to finding a complete solution to the involutory system of partial differential equations that is associated with the system [I] using the Pfaff procedure, i.e., the system:

$$\frac{\partial V}{\partial t_s} = f_s\left(t_1,\ldots,t_\mu\,;p_1,\ldots,p_\nu\,;\frac{\partial V}{\partial p_1},\ldots,\frac{\partial V}{\partial p_r}\right) \quad (s = 1, 2, \ldots, \mu). \qquad \text{[IV]}$$

We will then have a *fundamental theorem* that is the generalization of the analogous one that applies to Hamiltonian systems:

The complete integration of the canonical system [I] *is analytically equivalent to the search for a complete solution of a complete solution to the involutory system* [IV].

However, one will not only find a complete solution of [IV] that implies the canonical Pfaffian form for E_d, and therefore all of the integrals of (I) are determined from that (*), but conversely, in order to know all of the integrals of those total differential equations, it is enough to introduce the (undetermined) initial values of the dependent values as the arbitrary data in the integration and then form the so-called principle integrals of that system in order to get a complete solution to the involutory system by one quadrature, as one will find explained in my note in 1883 (*loc. cit.*, pp. 695-696).

§ 3.

No matter how one proceeds with the integration of an involutory system [IV] (of the previous §), whether by the Jacobi method, properly speaking, the method that I presented in to the Lombardy Institute in 1883 as an extension of Mayer's modification of the Jacobi-Hamilton

(*) We shall not go into the details regarding this point since we shall recover that result by a different method on a later occasion.

method, or the Cauchy method, if you want to call it that, *one will always encounter the associated canonical system* [I].

With the first procedure, one only seeks an integral of that system with the use of which one continues to use the same procedure in order to form another analogous canonical system with two fewer dependent variables, but one more independent variable, and so on.

However, with the second procedure, the system [I] is integrated completely.

It is mainly the integration of the canonical system [I] that we must address.

If one regards the initial values $t_1^0, \ldots, t_\mu^0$ of the independent variables as fixed (as well as numerically, if you prefer) then one establishes the path of integration that starts from them, and in order to do that, it is sufficient to perform a *transformation of polar type:*

$$t_s = t_s^0 + \varphi_s(t; \xi_1, \xi_2, \ldots, \xi_{\mu-1}) \qquad (s = 1, 2, \ldots, \mu),$$

in which the φ_s are mutually-independent functions of the variables $t, \xi_1, \ldots, \xi_{\mu-1}$ that are annulled for $t = 0$, and one regards the ξ as constants.

Set:

$$U = \sum_{s=1}^{\mu} f_s \frac{\partial \varphi_s}{\partial t}, \qquad \Xi_\lambda = \sum_{s=1}^{\mu} f_s \frac{\partial \varphi_s}{\partial \xi_\lambda} \qquad (\lambda = 1, 2, \ldots, m-1), \qquad \text{(V)}$$

so the differential expression E_d will become:

$$E_d = d\varphi + \sum_{i=1}^{\nu} q_i\, dp_i + U\, dt + \Xi_1\, d\xi_1 + \cdots + \Xi_{\mu-1}\, d\xi_{\mu-1} .$$

The Ξ_λ are functions of t and the ξ that are annulled for $t = 0$.

If the path of integration is fixed then it will be sufficient to integrate the Hamiltonian system:

$$\frac{dp_i}{dt} = -\frac{\partial U}{\partial q_i}, \qquad \frac{dq_i}{dt} = \frac{\partial U}{\partial p_i} \qquad (i = 1, 2, \ldots, \nu), \qquad \text{[VI]}$$

instead of the transform of [I], in which the ξ are regarded as constant parameters, and the integration is performed in such a way that the p_i and q_i assume the arbitrary values p_i^0 and q_i^0, respectively, for $t = 0$.

In place of the involutory system [IV], one will then the likewise-involutory system:

$$\left.\begin{aligned}\frac{\partial V}{\partial t} &= U\left(t, \xi_1, \ldots, \xi_{\mu-1}; p_1, \ldots, p_\nu; \frac{\partial V}{\partial p_1}, \ldots, \frac{\partial V}{\partial p_\nu}\right), \\ \frac{\partial V}{\partial \xi_\lambda} &= \Xi_\lambda\left(t, \xi_1, \ldots, \xi_{\mu-1}; p_1, \ldots, p_\nu; \frac{\partial V}{\partial p_1}, \ldots, \frac{\partial V}{\partial p_\nu}\right), \\ &(\lambda = 1, 2, \ldots, \mu-1).\end{aligned}\right\} \qquad \text{[VII]}$$

Now, the search for a complete solution to the first of those partial differential equations is a problem that is analytically equivalent to the complete integration of the Hamiltonian system [VI]. In addition, if one finds one such solution V with arbitrary, non-additive constants p_1^*, p_2^*, …, p_r^* then it is well known that the canonical integrals of [VI] are subordinate to the equations:

$$\frac{\partial V}{\partial p_i} = q_i, \qquad \frac{\partial V}{\partial p_i^*} = -\,q_i^*, \qquad [A]$$

in which the q_i^* denote some other arbitrary constants. One introduces the initial values p_i^0, q_i^0 of p_i and q_i, resp., into those integral equations in place of p_i^*, q_i^*, resp. That is achieved by performing the contact transformation:

$$\frac{\partial V_0}{\partial p_i^0} = q_i^0, \qquad \frac{\partial V_0}{\partial p_i^*} = -\,q_i^*, \qquad [B]$$

in which V_0 denotes the function of p_i^0, p_i^*, and ξ_λ that is obtained from V by setting $t = 0$, $p_i = p_i^0$.

If one introduces new variables p_i^0, q_i^0 into the differential expression E_d in place of the p_i, q_i, resp., when one sets $\varphi = 0$ for the sake of convenience, one will immediately find from (A) and (B) that:

$$E_d = d\,(V - V_0) + \sum_{i=1}^{r} q_i^0\, dp_i^0 + \sum_{\lambda=1}^{\mu-1}\left(\Xi_\lambda - \frac{\partial (V - V_0)}{\partial \xi_\lambda}\right) d\xi_\lambda\,.$$

However, from Mayer's theorem, p_i^0, q_i^0 are integrals of the first Pfaff system of that differential expression, or of the system:

$$\left.\begin{aligned} dp_i^0 &= -\sum_\lambda \frac{\partial H}{\partial q_i^0}\, d\xi_\lambda, \\ dq_i^0 &= \quad \sum_\lambda \frac{\partial H}{\partial p_i^0}\, d\xi_\lambda, \\ H_\lambda &= \Xi_\lambda - \frac{\partial (V - V_0)}{\partial \xi_\lambda} \end{aligned}\right\} \qquad \begin{array}{c} (i = 1, 2, \ldots, \nu) \\ (\lambda = 1, 2, \ldots, \mu - 1), \end{array}$$

and consequently:

$$\frac{\partial H}{\partial p_i^0} = \frac{\partial H}{\partial q_i^0} = 0 .$$

Now, if the coefficient of *dt* is zero then from the integrability condition [II], it will follow that *H* cannot depend upon *t*, while on the other hand, that quantity will be annulled for $t = 0$, so:

$$\frac{\partial (V - V_0)}{\partial \xi_\lambda} = \Xi_\lambda \qquad (\lambda = 1, 2, \ldots, \mu - 1)$$

or

$$E_d = d(V - V_0) + \sum_i q_i^0\, dp_i^0 .$$

Therefore, E_d has the canonical Pfaffian form, and when that is combined with canonical Pfaffian form of the involutory system [VII], one will know *a complete solution, in the broad sense of Lie*, and any other one can be deduced from it without integration.

In particular, if the equations:

$$\frac{\partial V_0}{\partial p_i^*} = \frac{\partial V}{\partial p_i^*}$$

that one obtains by eliminating p^* the from the second two groups of equations (*A*) and (*B*) are soluble for the $p_1^*, \ldots, p_r^*$ are used to eliminate the p_i^* from $V - V_0$ then the resulting function *W will be a complete solution in the Lagrangian sense*. Indeed, one will then have:

$$\frac{\partial (V - V_0)}{\partial p_i^*} = 0 , \qquad \frac{\partial W}{\partial t} = \frac{\partial V}{\partial t} + \sum_k \frac{\partial (V - V_0)}{\partial p_k^*} \frac{\partial p_k^*}{\partial t} = U ,$$

$$\frac{\partial W}{\partial \xi_\lambda} = \frac{\partial (V - V_0)}{\partial \xi_\lambda} = \Xi_\lambda , \qquad \frac{\partial W}{\partial p_i} = \frac{\partial V}{\partial p_i} = q_i .$$

We have then obtained what can be called *Lie's theorem*, in full generality:

If one has found a complete solution for the first of the partial differential equations [VII] *in which the ξ play the role of constants then one will get a complete solution of the involutory system* $[\mathrm{VII}]_n$ *with no further integrations.*

Therefore, when one such solution is transformed into the old variables $t_1, \ldots, t_\mu$, that will give a complete solution to the original involutory system [IV]. Ordinarily, Lie's theorem is proved for a special transformation of polar type, which is a loss of generality, and in particular, by the transformation (*):

$$t_1 = t_1^0 + t\,, \qquad t_2 = t_2^0 + \xi_1 t\,, \quad \ldots, \qquad t_\mu = t_\mu^0 + \xi_{\mu-1} t\,.$$

In addition, it is known that when one has found an integral to [VI], equating it to the value that it assumes for:

$$t = 0\,, \qquad p_i = p_i^0, \qquad q_i = q_i^0$$

will give an integral equation of [I] from which one can deduce one or more integrals of the equations with no further integrations, as was shown by Mayer (Math. Annalen, Band 5: "Ueber unbesch. int. Systeme, etc.," § **5**).

A simple and symmetric way of fixing the integration path that goes from the initial system $(t_1^0, \ldots, t_\mu^0)$ to the arbitrary final system $(t_1, \ldots, t_\mu)$ of values of the independent values is the following one: In place of t_s, one takes:

$$t_s^0 + (t_s - t_s^0)\,t$$

and makes it run through all values of the auxiliary variable t from 0 to 1. One then sets:

$$U = \sum (t_s - t_s^0)\, f_s\,(t_1^0 + (t_1 - t_1^0)\,t\,; \ldots, t_\mu^0 + (t_\mu - t_\mu^0)\,t\,; p_1, \ldots, q_r)\,,$$

so the system [I] will be converted into the Hamiltonian system:

$$\frac{dp_i}{dt} = -\frac{\partial U}{\partial q_i}\,, \qquad \frac{dq_i}{dt} = \frac{\partial U}{\partial p_i}\,,$$

in which $t_1, \ldots, t_\mu$ now play the role of constants. That system is integrated in such a way that the p_i, q_i take the values p_i^0, q_i^0 for $t = 0$: The integral equations will give those of the canonical system [I] when one sets $t = 1$.

A complete solution of the partial differential equation that is associated with the preceding Hamiltonian system, and therefore of the equation:

(*) Cf., **Lie**, Math. Annalen, Bd. IX, pp. 286; **Goursat**, *Leçons sur l'int. des équat. aux dériv. part. du 1er ordre*, page 171; **Von Weber**, *Vorl. über das Pfaff'sche Problem*, pp. 512.

$$\frac{\partial V}{\partial t} = U \qquad \left(q_i = \frac{\partial V}{\partial p_i}\right),$$

will imply a complete solution to the involutory system [VI] with no integrations, which results from a proportion that will be proved in what follows.

The transformation in question will lead to a symmetric procedure for constructing a complete solution of the involutory system when the Hamiltonian system has been integrated completely, which is a procedure whose development I shall defer to another occasion.

§ 4.

Consider a *contact transformation of the* p_i, q_i *that depends upon the parameters* $t_1, \ldots, t_\mu$, and therefore a transformation of the type:

$$\left.\begin{aligned} p_i &= p_i(p_1^*, \ldots, p_\nu^*; q_1^*, \ldots, q_\nu^*; t_1, \ldots, t_\mu), \\ q_i &= q_i(p_1^*, \ldots, p_\nu^*; q_1^*, \ldots, q_\nu^*; t_1, \ldots, t_\mu), \end{aligned}\right\} \qquad (i = 1, 2, \ldots, \nu)$$

such that when the $t_1, \ldots, t_\mu$ are regarded as constants, the relation:

$$\sum_i q_i \, dp_i = d\Omega + \sum_i q_i^* \, dp_i^*$$

will be verified identically, in which Ω denotes an arbitrary function of all the variables.

One can obtain such a transformation in its more-general form by a known procedure that one will find described, e.g., in § **4** of my article "Sulla trasformazione delle equazioni differenziali di Hamilton," that was included vol. XII, pt. 1 (series 5.a) of the Rendiconti della R. Accademia dei Lincei (pp. 117).

One establishes the equations:

$$\Omega_\kappa(p_1, \ldots, p_\nu, p_1^*, \ldots, p_\nu^*; t_1, \ldots, t_\mu) = 0 \qquad (\kappa = 1, 2, \ldots, \rho \leq \nu), \quad \text{[VIII]}$$

in which the Ω_κ are mutually-independent functions, regardless of whether they are referred to the p_i or the p_i^*. One combines the preceding ρ equations to the other $2\nu - \rho$ ones that result from eliminating the ρ multipliers from the following equations:

$$\left.\begin{aligned} q_i &= \frac{\partial \Omega}{\partial p_i} + \sum_\kappa \lambda_\kappa \frac{\partial \Omega_\kappa}{\partial p_i}, \\ -q_i^* &= \frac{\partial \Omega}{\partial p_i^*} + \sum_\kappa \lambda_\kappa \frac{\partial \Omega_\kappa}{\partial p_i^*}, \end{aligned}\right\} \qquad (i = 1, 2, \ldots, \nu). \qquad \text{[IX]}$$

However, one must impose an ultimate limitation on the functions Ω, Ω_1, …, Ω_r in order for the preceding equations to effectively define a transformation, which is a limitation that is expressed by the non-vanishing of a certain determinant that would be pointless to write out here.

In particular, one will then have contact transformations that are defined by equations of the type:

$$q_i = \frac{\partial \Omega}{\partial p_i}, \qquad - q_i^* = \frac{\partial \Omega}{\partial p_i^*}, \tag{X}$$

in which Ω denotes a function of the p_i, p_i^*, t_r, no matter how they are chosen, with the limitation that the functional determinant of the $\frac{\partial \Omega}{\partial p_i^*}$ with respect to the p_i^* is generally non-zero.

Now, if the t_1, …, t_μ are also considered to be variable and one sets:

$$V_s = \frac{\partial \Omega}{\partial t_s} + \sum_\kappa \lambda_\kappa \frac{\partial \Omega_\kappa}{\partial t_s} \qquad (s = 1, 2, \ldots, \mu) \tag{XI}$$

then it will obviously follow from [IX] and [VIII] that:

$$\sum_i (q_i\, dp_i - q_i^*\, dp_i^*) + \sum_s V_s\, dt_s = d\,\Omega\,.$$

Thus, under the transformation in question, it will result identically that:

$$E_d = \sum_i q_i\, dp_i + \sum_s f_s\, dt_s = d\,\Omega + \sum_i q_i^*\, dp_i^* + \sum_s (f_s^* - V_s^*)\, dt_s, \tag{XII}$$

in which one intends f_s^*, V_s^* to mean the transforms of the functions f_s, V_s in terms of the p_i^*, q_i^*.

Set:

$$f_s - V_s = H_s \qquad (s = 1, 2, \ldots, \mu)$$

and construct the first Pfaff system of the differential expression [XII]. One will then have that *the transform of* [I] *is again a canonical system*, which implies the following theorem:

For the most-general contact transformation that is defined by [VIII] *and* [IX], *the canonical system* [I] *will transform into the canonical system:*

$$dp_i^* = -\sum_{s=1}^{\mu} \frac{\partial H_s^*}{\partial q_i^*} dt_s, \qquad dq_i = \sum_{s=1}^{\mu} \frac{\partial H_s^*}{\partial p_i^*} dt_s, \tag{XIII}$$

in which the H_s^* *are the transforms of the* H_s *into the new variables.*

In particular, if one employs the contact transformation [X] then one will have:

$$H_s = f_s - \frac{\partial \Omega}{\partial t_s} .$$

This proposition follows almost immediately:

If Ω *is a complete solution of the involutory system:*

$$\frac{\partial \Omega}{\partial t_s} = f_s\left(t_1, \ldots, t_\mu ; p_1, \ldots, p_\nu ; \frac{\partial \Omega}{\partial p_1}, \ldots, \frac{\partial \Omega}{\partial p_\nu} \right) , \qquad \text{[XIV]}$$

with the arbitrary, non-additive constants $p_1^*, \ldots, p_\nu^*$, *then the integral equations of the canonical system* [I] *will be* [X]:

$$\frac{\partial \Omega}{\partial p_i} = q_i , \qquad \frac{\partial \Omega}{\partial p_i^*} = - q_i^* .$$

Indeed, [XIV] are verified under the transformation [X], so E_d will assume the canonical Pfaff form, and consequently, [XIII[will take *the solved form:*

$$dp_i^* = dq_i = 0 .$$

We shall now prove the following proposition, which is the converse of the one that was just established.

If one knows a system of integrals of a canonical system [I] *that define a contact transformation of the* p_i, q_i *when the independent variables are regarded as constants then the canonical form of the differential expression* E_d *will also be known by one quadrature, and consequently one can get a complete solution of the associated involutory system* [IV] *by one quadrature.*

Let p_i^*, q_i^* be the aforementioned integrals of [I].

As I have observed before (Rend. Lincei, *loc. cit.*, pp. 298-299), if one replaces any pair (p_i, q_i), (p_k^*, q_k^*) with $(q_i, -p_i)$, $(q_k^*, -p_k^*)$, resp., then the contact transformation can always be reduced to one of the type:

$$q_i = \frac{\partial \Omega}{\partial p_i} , \qquad - q_i^* = \frac{\partial \Omega}{\partial p_i^*} .$$

Having done that, one will then have:

$$E_d = d\,\Omega + \sum_k q_k^* \, dp_k^* + \sum_s H_s^* \, dt_s \, ,$$

in which:

$$H_s = f_s - \frac{\partial \Omega}{\partial t_s} \, .$$

However, when one introduces the new variables p_i^*, q_i^*, the system [I] must reduce to the solved form, and therefore:

$$\frac{\partial H_s^*}{\partial p_i^*} = \frac{\partial H_s^*}{\partial q_i^*} = 0,$$

or

$$H_s^* = H_s = \frac{\partial \psi}{\partial t_s} \, ,$$

in which ψ is a function of only $t_1, \ldots, t_\mu$.

A complete solution to the involutory system is therefore:

$$V = \Omega + \psi \, ,$$

and it is calculated by a quadrature when one knows the derivatives.

A system of integrals that defines a contact transformation of the p_i and q_i is that of the principal integrals, or the integrals that reduce to p_i, q_i ($i = 1, 2, \ldots, \nu$), respectively, for $t_1 = t_1^0, \ldots, t_\mu = t_\mu^0$.

§ 5.

The following proposition results from the property that was established in the preceding §, as it does in the case of an ordinary canonical system (see my article in Lincei, §§ **9** and **10**):

Any canonical system [I] *can be converted into any other one with the same number of dependent variables by a contact transformation of the* p_i, q_i *that depends upon the* $t_1, \ldots, t_\mu$.

The canonical system of total differential equations has the characteristic property that it admits systems of integrals that will define contact transformations when the independent variables are regarded as constants.

However, it is known that *there are also non-contact transformations that transform one canonical system into another.* Indeed, one observes that when it is reduced to the solved form:

$$dp_i^* = dq_i^* = 0 \, ,$$

any transformation of the type:

$$p_i^* = p_i\,(P_1, \dots, P_n, Q_1, \dots, Q_\nu), \qquad q_i^* = q_i\,(P_1, \dots, P_n, Q_1, \dots, Q_\nu) \tag{XV}$$

will convert it into another one in solved form, and that, in turn, can be converted into an arbitrary canonical system by a general contact transformation.

One says: If S is a transformation of the infinite group [XV], and T is a contact transformation that reduces a canonical system to the solved form then the transformation $T\,S\,T^{-1}$ will transform that system into itself.

There is then an infinite group of transformations that is similar to the group of S that transform a given canonical system into itself. That group contains a subgroup of contact transformations that is similar to the one that is contained in the group of S.

Therefore, the infinite group of contact transformations that depends upon the parameters t_1, $\dots$, t_μ *contains infinite subgroups that are all similar to each other, and any one of which is characterized by the property that it leaves a well-defined canonical system invariant.*

Observation.

Contact transformations were implicitly adopted by Prof. **Sciacca** in his 1882 paper "Teorema fondamentale nella Teoria delle equazioni canoniche del moto," Mem. della r. Acc. dei Lincei, series III, t. XII), in which he showed the simplicity and elegance that the use of a theorem in analysis that he deemed to be fundamental conferred upon the theory of dynamical equations.

In regard to that theorem, I had the honor of entertaining the Institute with a note that I read to them in December of that same year. As it appeared in the proof that I communicated to the Institute then, that theorem can be formulated as follows:

The most-general n relations between $2n$ *variables:* $x_1, y_1\,; \dots, x_n, y_n$ *that makes the equation:*

$$\sum_{i=1}^{n} (\delta y_i\, dx_i - dy_i\, \delta x_i) = 0$$

into an identity, or that makes $\sum_i y_i\, dx_i$ *into an exact differential* $d\varphi$*, can always be obtained by arbitrarily establishing none, one, or more relations between just the x:*

$$\psi_\kappa\,(x_1, \dots, x_n) = 0 \qquad (\kappa = 1, 2, \dots, k < n), \tag{A}$$

and combining them with the other n – k relations that are obtained from the equations:

$$y_i = \frac{\partial \varphi}{\partial x_i} + \lambda_1 \frac{\partial \psi_1}{\partial x_i} + \cdots + \lambda_k \frac{\partial \psi_k}{\partial x_i} \qquad (i = 1, 2, \ldots, n)$$

by eliminating the multipliers λ.

It is easy to exhibit the limitation that must be imposed arbitrarily in the choice of φ, ψ_1, …, ψ_k in order for the previous equations to be soluble for the λ and n of the x, y with the theorem of functional determinants. That limitation can be written: If $W = \varphi + \sum_\kappa \lambda_\kappa \psi_\kappa$ then by reason of (A), the determinant:

$$\begin{vmatrix}
\frac{\partial \psi_1}{\partial x_1} & \cdots & \frac{\partial \psi_k}{\partial x_1} & \frac{\partial^2 W}{\partial x_1 \partial x_1} & \frac{\partial^2 W}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 W}{\partial x_1 \partial x_n} \\
\vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\
\frac{\partial \psi_1}{\partial x_n} & \cdots & \frac{\partial \psi_k}{\partial x_n} & \frac{\partial^2 W}{\partial x_n \partial x_1} & \frac{\partial^2 W}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 W}{\partial x_n \partial x_n} \\
\vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\
0 & \cdots & 0 & \frac{\partial \psi_1}{\partial x_1} & \frac{\partial \psi_1}{\partial x_2} & \cdots & \frac{\partial \psi_1}{\partial x_n} \\
\vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\
0 & \cdots & 0 & \frac{\partial \psi_k}{\partial x_1} & \frac{\partial \psi_k}{\partial x_2} & \cdots & \frac{\partial \psi_k}{\partial x_n}
\end{vmatrix}$$

will not prove to be zero identically.

That theorem, along with the further one that *the aforementioned n relations are always soluble for n of the variables x*, *y with different indices*, was used repeatedly by **Sophus Lie** in the theory of first-order partial differential equations in order to show how to determine all of the contact transformation in finite form. (Cf., the two papers: "Begründung einer Invarianten-Theorie der Berührungsformationen; Allgemeine Theorie der partiellen Differentialgleichungen I. O," Math. Ann. vols. VIII and IX.)

Turin, May 1903.