"Il metodo di Pfaff per l'integrazione delle equazioni a derivatie parziali del $1^{\circ}$ ordine," Rend. Reale Instituto Lombardo di scienze (2) 16 (1883), 637-644, 691-699.

# The Pfaff method for integrating first-order partial differential equations. 

By G. Morera,<br>presented by M. E. Prof. E. Beltrami<br>Translated by D. H. Delphenich

In Darboux's paper: "Sur le problème de Pfaff" (Bull. des Sc. math. et astr. an. 1882, fasc. de Janv. et de Fév.), he has, I believe, indicated the true path that one must follow at this point in time in order to arrive at a regular analytical exposition of all the results that have been obtained up to now in the theory of first-order partial differential equations. That path is nothing but the old one that Pfaff once blazed, and nowadays one can easily pursue that path due to the great progress that has been made in the Pfaff problem in recent times ( ${ }^{*}$ ).

The Pfaff method, when suitably applied to the problem of integrating just one first-order partial differential equation, will easily lead to Cauchy's method, as Darboux had observed. This article is intended to show that Pfaff's method can be applied with equal success to a Jacobian system of first-order partial differential equations. That application will imply a method of integration that is a natural extension of the so-called Jacobi-Hamilton method ( ${ }^{* *}$ ), and if one utilizes Mayer's beautiful theorem on the integration of linear equations in total differentials then that method will naturally imply the classical theorem of Lie that "the complete integration of a Jacobian system of $m$ first-order partial differential equations in $n$ independent variables can be performed by completely integrating just one first-order partial differential equation with only $n-$ $m+1$ independent variables."

In that way, one can develop a natural connection between the aforementioned theorems of Lie and Mayer, which would be reasonable to predict insofar as the two theorems are based upon essentially the same transformations of variables, but that has not been pointed out explicitly up to now.

In this article, I shall rapidly recall some theorems on the Pfaff problem, and for greater clarity, I shall start by applying the Pfaff method to the case of just one first-order partial differential equation in which the unknown function does not occur. That will lead me directly to the JacobiHamilton method of integration and to the successive improvements that Mayer made to it ( ${ }^{* * *}$ ). After that, I will apply the same method to Jacobian systems of partial differential equations.

[^0]
## Some theorems that relate to the Pfaff problem.

Let:

$$
u=u_{d x}=u_{1} d x_{1}+u_{2} d x_{2}+\ldots+u_{n} d x_{n}
$$

be the differential expression to be reduced to canonical form.
One forms its bilinear covariant:

$$
\Theta=\delta u_{d x}-d u \delta_{x}=\sum_{i, k} \Theta_{i k}\left(d x_{i} \delta x_{k}-d x_{k} \delta x_{i}\right),
$$

in which:

$$
\Theta_{i k}=\frac{\partial u_{i}}{\partial x_{k}}-\frac{\partial u_{k}}{\partial x_{i}} .
$$

In order to recognize the canonical form to which the differential expression $u$ is reducible, one needs only to study the invariant properties of $u$ and $\Theta$, what are regarded as comprising a system of two algebraic forms, one of which is linear with respect to the differentials $d x$ and $\delta x$, while the other is alternating bilinear in them.

As is known, those properties are given by the number that one calls the class of the system (*), and in our present case, that number is called the class of the proposed differential expression. One has the following criteria for finding that class.

One forms the two skew determinants:

$$
\Delta=\left|\begin{array}{cccc}
\Theta_{11} & \Theta_{12} & \cdots & \Theta_{1 n} \\
\Theta_{21} & \Theta_{22} & \cdots & \Theta_{2 n} \\
\vdots & \vdots & \cdots & \vdots \\
\Theta_{n 1} & \Theta_{n 1} & \cdots & \Theta_{n n}
\end{array}\right|, \quad \Delta_{u}=\left|\begin{array}{cccc}
\Theta_{11} & \cdots & \Theta_{1 n} & u_{1} \\
\Theta_{21} & \cdots & \Theta_{2 n} & u_{2} \\
\vdots & \cdots & \vdots & \vdots \\
\Theta_{n 1} & \cdots & \Theta_{n n} & u_{n} \\
-u_{1} & \cdots & -u_{m} & 0
\end{array}\right|,
$$

the first of which is called the determinant of the bilinear covariant, and the second of which is called the bordered determinant. The system of forms $u$ and $\Theta$ is said to have class $2 r$ if $2 r$ is the maximum degree of the subdeterminants of $\Delta$ and $\Delta_{u}$ that do not all vanish, while it is said to have class $2 r+1$ if $2 r$ is the maximum degree of those subdeterminants in $\Delta$ and $2 r+2$ is the maximum degree for the ones in $\Delta_{u}$.

Now, if one calculates the class for a given canonical form in $u$ then one will immediately see that it is equal to precisely the number of (independent) functions that occur in that canonical form. Therefore, if one admits the possibility that a differential expression can be reduced to canonical form then the latter will necessarily contain just as many functions as the number that its class represents.

[^1]As for the number of integrations that are required for the reduction to canonical form of $u_{d x}$, one has the following general theorem:

If the class of a differential expression is $p$ then its reduction to canonical form will require $p$ $-1, p-3, p-5, \ldots$ operations ( ${ }^{*}$ ).

If the expression $u_{d x}$ has class $2 r+1$ then its first Pfaff system:

$$
\Theta_{i 1} d x_{1}+\Theta_{i 2} d x_{2}+\ldots+\Theta_{i n} d x_{n}=0 \quad(i=1,2, \ldots, n)
$$

will contain only $2 r$ mutually-distinct equations, so that system will be soluble for $2 r$ certain differentials, which are taken with respect to:

$$
x_{n-2 r+1}, x_{n-2 r+2}, \ldots, x_{n}
$$

If one integrates the preceding total differential equations completely (it is known how to do that) and one takes the $2 r$ arbitrary integration constants to be the initial values $x_{n-2 r+1}^{0}, x_{n-2 r+2}^{0}, \ldots$, $x_{n}^{0}$, which are chosen arbitrarily from the variables $x_{n-2 r+1}, x_{n-2 r+2}, \ldots, x_{n}$, and correspond to certain initial values $x_{1}^{0}, \ldots, x_{n-2 r}^{0}$ of the remaining variables, then it is clear that the integral equations will be soluble for $x_{n-2 r+1}^{0}, \ldots, x_{n}^{0}$, in such a way that one will have $\left(^{* *}\right)$ :

$$
x_{n-2 r+1}^{0}=\left[x_{n-2 r+1}\right], \ldots, x_{n}^{0}=\left[x_{n}\right],
$$

in which $\left[x_{n-2 r+1}\right], \ldots,\left[x_{n}\right]$ are certain functions of the $x$ that reduce to $x_{n-2 r+1}, \ldots, x_{n}$ identically when one sets $x_{1}=x_{1}^{0}, \ldots, x_{n-2 r}=x_{n-2 r}^{0}$. The given differential expression will then be reducible to the form:

$$
d H+\left[x_{n-2 r+1}\right] d\left[x_{n-2 r+1}\right]+\ldots+\left[u_{n}\right] d\left[x_{n}\right],
$$

in which one intends $\left[u_{k}\right]$ to mean what $u_{k}$ will become when one sets $x_{1}, \ldots, x_{n-2 r}, x_{n-2 r+1}, \ldots, x_{n}$ equal to $x_{1}^{0}, \ldots, x_{n-2 r}^{0},\left[x_{n-2 r+1}\right], \ldots,\left[x_{n}\right]$, respectively, and $H$ denotes a function that reduces to a constant for $x_{1}=x_{1}^{0}, \ldots, x_{n-2 r}=x_{n-2 r}^{0}$, (cf., Darboux, op. cit., pp. 34).

[^2]
## Applying the Pfaff method to the integration of just one first-order partial differential equation that does not contain the unknown function.

One poses the first-order partial differential equation:

$$
\begin{equation*}
p_{1}=f\left(q_{1}, \ldots, q_{n}, p_{2}, \ldots, p_{n}\right), \tag{1}
\end{equation*}
$$

in which the unknown function $z$ is not included explicitly, and as usual, the $p$ denote the partial derivatives of $z$ with respect to the corresponding $q$.

According to the Pfaff method, in order to integrate that equation completely, one must seek the canonical form of the differential expression:

$$
u=d z-f d q_{1}-p_{2} d q_{2}-\ldots-p_{n} d q_{n}
$$

The bilinear covariant of that expression (up to sign):

$$
\begin{aligned}
\Theta & =\delta f d q_{1}-d f \delta q_{1}+\sum_{r=2}^{n}\left(\delta p_{r} d q_{r}-\delta q_{r} d p_{r}\right) \\
& =\sum_{r=2}^{n} \frac{\partial f}{\partial q_{r}}\left(\delta q_{r} d q_{1}-d q_{r} \delta p_{1}\right)+\sum_{r=2}^{n} \frac{\partial f}{\partial p_{r}}\left(\delta p_{r} d q_{1}-d p_{r} \delta p_{1}\right)+\sum_{r=2}^{n}\left(\delta p_{r} d q_{r}-d p_{r} \delta q_{r}\right) .
\end{aligned}
$$

In order to find the class of $u$, it is enough to examine that of $f d q_{1}+p_{2} d q_{2}+\ldots+p_{n} d q_{n}$, whose bilinear covariant is precisely $\Theta$. The bordered determinant in that case is:

$$
\left|\begin{array}{cccccccccc}
0 & \frac{\partial f}{\partial q_{2}} & \frac{\partial f}{\partial q_{2}} & \cdots & \frac{\partial f}{\partial q_{n}} & \frac{\partial f}{\partial q_{2}} & \frac{\partial f}{\partial q_{2}} & \cdots & \frac{\partial f}{\partial q_{n}} & f \\
-\frac{\partial f}{\partial q_{2}} & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & p_{2} \\
-\frac{\partial f}{\partial q_{2}} & 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 & p_{2} \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots & & \cdots & \vdots & \vdots \\
-\frac{\partial f}{\partial q_{n}} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & p_{n} \\
-\frac{\partial f}{\partial q_{2}} & -1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\
-\frac{\partial f}{\partial q_{2}} & 0 & -1 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \cdots & -1 & 0 & 0 & \cdots & 0 & 0 \\
-\frac{\partial f}{\partial q_{n}} & 0 & 0 & \cdots & -1 & 0 & 0 & \cdots & 0 & 0 \\
-f & -p_{2} & -p_{2} & \cdots & -p_{n} & 0 & 0 & \cdots & 0 & 0
\end{array}\right|=\Delta_{u},
$$

and the determinant of $\Theta$ is obtained from that by suppressing the last row and the last column.
One sees immediately that the maximum degree of the subdeterminants of the determinant $\Theta$ that do not all vanish is $2(n-1)$ and that $\Delta_{u}$ is generally non-zero, so the class of the expression will be $2 n-1$.

However, if one has:

$$
f=p_{2} \frac{\partial f}{\partial p_{2}}+p_{2} \frac{\partial f}{\partial p_{2}}+\cdots+p_{n} \frac{\partial f}{\partial p_{n}},
$$

i.e., if $f$ is a homogeneous function of degree 1 with respect to $p_{2}, p_{2}, \ldots, p_{n}$, then $\Delta_{u}$ will vanish, and the expression $f d q_{1}+p_{2} d q_{2}+\ldots+p_{n} d q_{n}$ will then have class $2(n-1)$.

That is the only case in which that can happen.
That exceptional case is very important insofar as it always presents itself when equation (1) has been deduced from another one in which the unknown function occurs by the usual transformation.

That observation explains the following fact:
If one has a partial differential equation in which the dependent variable occurs, along with $n$ - 1 independent variables, then it is known that the complete integration of that equation will require $2 n-3,2 n-5, \ldots, 1$ operations. However, if one applies the Jacobi transformation to the equation then one will obtain another one with $n$ independent variables whose complete integration will require the higher number of operations $2 n-2,2 n-4, \ldots, 2,0$, according to the general rule. Indeed, it is sufficient to note that in this case, as one saw above, the class of the differential expression $f d q_{1}+p_{2} d q_{2}+\ldots+p_{n} d q_{n}$ is $2(n-1)$, so its reduction to canonical form, and therefore that of $u$ (although its class is $2 n-1$ ), will demand only $2 n-3,2 n-5, \ldots, 3,1$ operations: Therefore, the integration by the Pfaff method will again require the minimum number of operations that were indicated above.

If one equates the partial derivatives of the bilinear covariant $\Theta$ with respect to $\delta p, \delta q$ then one will have the differential equations:

$$
\sum_{r}\left(\frac{\partial f}{\partial q_{r}} d q_{r}+\frac{\partial f}{\partial p_{r}} d p_{r}\right)=0, \quad \frac{\partial f}{\partial q_{r}} d q_{1}=d p_{1}, \quad \frac{\partial f}{\partial p_{r}} d q_{1}=-d q_{r} \quad(r=2,3, \ldots, n)
$$

in which the first one is a consequence of the remaining one, so one has to consider the Hamiltonian system:

$$
\frac{d p_{r}}{d q_{1}}=\frac{\partial f}{\partial q_{r}}, \quad \frac{d q_{r}}{d q_{1}}=-\frac{\partial f}{\partial p_{r}} .
$$

Suppose that system is integrated completely and assume that the integration constants are the initial values of the $p_{r}, q_{r}$ that correspond to the value $q_{1}^{0}$ of $q_{1}$, and let $\left[p_{r}\right],\left[q_{r}\right]$ be the functions of the $p$ and $q$ that will give the complete system of integrals for the preceding equations when they are equated to those constants. From the cited theorem, one will then have:

$$
\begin{equation*}
d z-f d q_{1}-p_{2} d q_{2}-\ldots-p_{n} d q_{n}=d(z-\varphi)-\left[p_{2}\right] d\left[q_{2}\right]-\ldots-\left[p_{n}\right] d\left[q_{n}\right], \tag{2}
\end{equation*}
$$

in which $\varphi$ is a function of only the $p$ and $q$ that is annulled for $q_{1}=q_{1}^{0}$.
Now set $\left[p_{r}\right]=p_{r}^{0},\left[q_{r}\right]=q_{r}^{0}$ in the preceding identity. One will obviously have:

$$
\varphi^{0}=\int_{q_{i}^{0}}^{q_{i}}\left[f-\sum_{r} p_{r} \frac{\partial f}{\partial p_{r}}\right] d q_{1}
$$

then, in which the integration is performed after having expressed everything under the $\int$ sign as functions of $q_{1}$ by means of the preceding equations.

After performing the quadrature, one now replaces the constants $p_{1}^{0}, q_{1}^{0}$ with their expressions $\left[p_{r}\right],\left[q_{r}\right]$ and obtains precisely the function $\varphi$ that makes equations (2) an identity, which is easy to verify by a known calculation process that I shall omit, and all the more so because in what follows, I shall have occasion to develop another one that will include it as a special case.

One will then have:

$$
d z-f d q_{1}-p_{2} d q_{2}-\ldots-p_{n} d q_{n}=d\left[z-\int_{q_{1}^{0}}^{q_{1}}\left(f-\sum_{r} p_{r} \frac{\partial f}{\partial p_{r}}\right) d q_{1}\right]-\sum_{r}\left[p_{r}\right] d\left[q_{r}\right],
$$

and that identity easily leads to the so-called Jacobi-Hamilton integration method.
However, one will notice immediately that, as Mayer pointed out (Math. Ann. Bd. III), such a method presents an exception, and it can then happen that the equations $\left[q_{r}\right]=q_{r}^{0}$ cannot be solved for the $p_{r}$, which is why the method in question will no longer produce the complete integral of the proposed equation. That difficulty will necessarily present itself when $f$ is a homogeneous function of degree one in the $p_{r}$ (so $\varphi$ will be identically zero then).

That difficulty can be quickly removed by observing that the preceding identity gives rise to the other one:

$$
d z-f d q_{1}-\sum_{r} p_{r} d q_{r}=d\left[z-\sum_{r}\left[p_{r}\right]\left[q_{r}\right]-\int_{q_{1}^{0}}^{q_{1}}\left(f-\sum_{r} p_{r} \frac{\partial f}{\partial p_{r}}\right) d q_{1}\right]-\sum_{r}\left[p_{r}\right] d\left[q_{r}\right] .
$$

One can obviously deduce the Jacobi-Hamilton integration method, as improved upon by Mayer, from that, which is valid in any case as long as the equations $\left[p_{r}\right]=p_{r}^{0}$ are always soluble for the $p_{r}$, since they are clearly valid for $q_{1}=q_{1}^{0}$.

# The Pfaff method for integrating first-order partial differential equations. 

By G. Morera,<br>presented by M. E. Prof. E. Beltrami

(continuation and conclusion)
Translated by D. H. Delphenich

## Applying the Pfaff method to the integration of a Jacobian system.

Suppose that one has a Jacobian system of $m$ first-order partial differential equations with $n$ independent variables:

$$
\begin{equation*}
p_{1}=f_{1}, \quad p_{2}=f_{2}, \quad \ldots, \quad p_{m}=f_{m}, \tag{1}
\end{equation*}
$$

in which the $f$ denote functions of $q_{1}, \ldots, q_{n}, p_{m+1}, \ldots, p_{n}$, and the $p$ denote partial derivatives of the unknown function $z$ with respect to the corresponding $q$, as usual.

In addition, the functions $f$ must verify the following $m(m-1) / 2$ equations:

$$
\begin{equation*}
\left(p_{i}-f_{i}, p_{k}-f_{k}\right)=\frac{\partial f_{i}}{\partial q_{k}}-\frac{\partial f_{k}}{\partial q_{i}}+\sum_{r=m+1}^{n}\left(\frac{\partial f_{i}}{\partial p_{r}} \frac{\partial f_{k}}{\partial q_{r}}-\frac{\partial f_{i}}{\partial q_{r}} \frac{\partial f_{k}}{\partial p_{r}}\right)=0 . \tag{2}
\end{equation*}
$$

According to the Pfaff method, the problem of integrating the system (1) completely is equivalent to that of reducing the expression:

$$
u=d z-f_{1} d q_{1}-f_{2} d q_{2}-\ldots-f_{m} d q_{m}-p_{m+1} d q_{m+1}-\ldots-p_{n} d q_{n}
$$

to canonical form. Above all, one must find that class of that expression, and for that, it suffices to find the class of the expression:

$$
f_{1} d q_{1}+f_{2} d q_{2}+\ldots+f_{m} d q_{m}+p_{m+1} d q_{m+1}+\ldots+p_{n} d q_{n}
$$

Now compute the bilinear covariant of the last expression. One has:

$$
\begin{aligned}
\Theta & =\sum_{s=1}^{m}\left(\delta f_{s} d q_{s}-d f_{s} \delta q_{s}\right)+\sum_{r=m+1}^{n}\left(\delta p_{r} d q_{r}-d p_{r} \delta q_{r}\right) \\
& =\sum_{s, s^{\prime}}\left(\frac{\partial f_{s}}{\partial q_{s^{\prime}}}-\frac{\partial f_{s^{\prime}}}{\partial q_{s}}\right)\left(\delta q_{s^{\prime}} d q_{s}-d q_{s^{\prime}} \delta q_{s}\right)+\sum_{r} \sum_{s} \frac{\partial f_{s}}{\partial q_{r}}\left(\delta q_{r} d q_{s}-d p_{r} \delta q_{s}\right)
\end{aligned}
$$

$$
+\sum_{r} \sum_{s} \frac{\partial f_{s}}{\partial q_{r}}\left(\delta p_{r} d q_{s}-d p_{r} \delta q_{s}\right)+\sum_{r}\left(\delta p_{r} d q_{r}-d p_{r} \delta q_{r}\right),
$$

in which the indices $s, s^{\prime}$ must take the values $1,2,3, \ldots, m$, and the index $r$ must take all values $m+1, m+2, \ldots, n$.

Consider the following two systems of elements (i.e., matrices):

$$
\left.\begin{array}{|cccccccccc|}
\frac{\partial f_{1}}{\partial p_{m+1}} & \frac{\partial f_{1}}{\partial p_{m+2}} & \cdots & \frac{\partial f_{1}}{\partial p_{n}} & \frac{\partial f_{1}}{\partial q_{m+1}} & \frac{\partial f_{1}}{\partial q_{m+2}} & \cdots & \frac{\partial f_{1}}{\partial q_{n}} & 0 & f_{1} \\
\vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\
\frac{\partial f_{m}}{\partial p_{m+1}} & \frac{\partial f_{m}}{\partial p_{m+2}} & \cdots & \frac{\partial f_{m}}{\partial p_{n}} & \frac{\partial f_{m}}{\partial q_{m+1}} & \frac{\partial f_{m}}{\partial q_{m+2}} & \cdots & \frac{\partial f_{m}}{\partial q_{n}} & 0 & f_{m} \\
1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & p_{m+1} \\
0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & p_{m+2} \\
\vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 & 0 & p_{n} \\
0 & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & -1 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & -1 & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 & 0
\end{array} \right\rvert\,
$$

If one combines those two systems of elements using the rule for multiplying the determinants (by rows), while taking the relations (2) into account, then one will get the bordered determinant that relates to the expression $f_{1} d q_{1}+f_{2} d q_{2}+\ldots+f_{m} d q_{m}+p_{m+1} d q_{m+1}+\ldots+p_{n} d q_{n}$, and therefore, when one observes that those two systems of elements have $2(n-m)+2$ columns, one can conclude from a well-known theorem regarding determinants that the maximum degree of the subdeterminants of that determinant that do not all vanish will generally be $2(n-m)+2$.

Similarly, if one omits the last line and the last two columns in the preceding two systems of elements and then multiplies them row-wise then one will get the determinant of the bilinear covariant $\Theta$, so the maximum degree of the subdeterminants in that determinant that do not all vanish will be $2(n-m)$.

Thus, the class of:

$$
f_{1} d q_{1}+f_{2} d q_{2}+\ldots+f_{m} d q_{m}+p_{m+1} d q_{m+1}+\ldots+p_{n} d q_{n}
$$

will be $2(n-m)+1$, and therefore that of $u$, as well.
With the cited theorem, one then deduces from the latter fact that the complete integration of a Jacobian system of $m$ equations will require $2(n-m), 2(n-m-1), \ldots, 2,0$ operations, in general.

One observes that in the two systems of elements in question, if the $f$ are all homogeneous functions of degree one in the $p$ then all of the determinants will have degree $2(n-m)+2$, since in that case, the expression $f_{1} d q_{1}+f_{2} d q_{2}+\ldots+f_{m} d q_{m}+p_{m+1} d q_{m+1}+\ldots+p_{n} d q_{n}$ will have class $2(n-m)$, and therefore the complete integration of the Jacobian system (1) will require only 2 ( $n$ $-m)-1,2(n-m)-3, \ldots, 1$ operations.

The first Pfaff system relative to the expression $u$ consists $\left({ }^{*}\right)$ of $2(n-m)$ distinct differential equations, and those equations are obtained by equating the coefficients of $\delta p_{r}, \delta q_{r}$ in the bilinear covariant to zero. One will then have a completely-integrable system of $2(n-m)$ total differential equations:

$$
\left.\begin{array}{l}
\sum_{s} \frac{\partial f_{s}}{\partial q_{r}} d q_{s}=d p_{r}  \tag{3}\\
\sum_{s} \frac{\partial f_{s}}{\partial p_{r}} d q_{s}=-d q_{r}
\end{array}\right\} \quad\binom{s=1,2, \ldots, m}{r=m+1, \ldots, n}
$$

Imagine that one has integrated that system completely and that, as usual, one has chosen the arbitrary constants to be the arbitrary initial values $p_{r}^{0}, q_{r}^{0}$ of the variables $p_{r}, q_{r}$ that correspond to the initial values $q_{s}^{0}$ of the independent variables $q_{s}$, in such a way that:

$$
\begin{equation*}
p_{r}^{0}=\left[p_{r}\right], \quad q_{r}^{0}=\left[q_{r}\right] \tag{4}
\end{equation*}
$$

are those of its integrals that reduce to $p_{r}, q_{r}$, identically for $q_{1}=q_{1}^{0}, \ldots, q_{m}=q_{m}^{0}$. One will then have, with no further analysis:

[^3]\[

$$
\begin{align*}
d z-\sum_{s} f_{s} d q_{s}-\sum_{r} p_{r} d q_{r} & =d(z-\varphi)-\sum_{r}\left[p_{r}\right] d\left[q_{r}\right] \\
& \left.=d(z-\varphi)-\sum_{r}\left[p_{r}\right]\left[q_{r}\right]\right)+\sum_{r}\left[p_{r}\right] d\left[q_{r}\right] \tag{5}
\end{align*}
$$
\]

Meanwhile, here is how one calculates the function $(\varphi)$ that makes that equation an identity: One expresses the $p_{r}, q_{r}$ as functions of $q_{1}, \ldots, q_{m}$ by means of the integral equations (4) and calculates the function:

$$
\varphi=\sum_{s} \int_{q_{s}^{0}}^{q_{s}}\left(f_{s}-\sum_{r} p_{r} \frac{\partial f_{s}}{\partial p_{r}}\right) d q_{s}
$$

by a quadrature (which is certainly practicable), and after having performed the integration, one substitutes the expressions (4) for the $p_{r}^{0}, q_{r}^{0}$. The function $(\varphi)$ thus-obtained will make equation (5) an identity, as will now be proved.

Take the variation $\delta$ of the function $\varphi$, while supposing that only the $p_{r}^{0}, q_{r}^{0}$ are variable. One will then have:

$$
\delta \varphi=\sum_{s} \int_{q_{s}^{0}}^{q_{s}} \sum_{r}\left(\frac{\partial f_{s}}{\partial q_{r}} \delta q_{r}-p_{r} \delta \frac{\partial f_{s}}{\partial p_{r}}\right) d q_{s}
$$

However, from equations (3), one has identically:

$$
\sum_{s} \delta \frac{\partial f_{s}}{\partial p_{r}} d q_{s}=-d \delta q_{r}
$$

and if one takes the first of (3) into account then one will have:

$$
\delta \varphi=\sum_{s} \int_{q_{s}^{0}}^{q_{s}}\left(d p_{r} \delta q_{r}+p_{r} d \delta q_{r}\right)=\sum_{r} \int_{q_{s}^{0}}^{q_{s}} d\left(p_{r} \delta q_{r}\right)=\sum_{r} \int_{q_{s}^{0}}^{q_{s}}\left(p_{r} \delta q_{r}-p_{r}^{0} \delta q_{r}^{0}\right) .
$$

Take the total differential of $\varphi$, while substituting the expressions (4) for the $p_{r}^{0}, q_{r}^{0}$ and noting that $\delta q_{r}^{0}=d\left[q_{r}\right]$ (while $\delta q_{r}=d q_{r}+\sum_{r} \frac{\partial f_{s}}{\partial p_{r}} d q_{s}$ ). One will certainly have:

$$
\begin{aligned}
d \varphi & =\left(\sum_{s} f_{s} d q_{s}+\sum_{r} p_{r} d q_{r}-\sum_{r} p_{r} \delta q_{r}\right)+\sum_{r}\left(p_{r} \delta q_{r}-\left[p_{r}\right] d\left[q_{r}\right]\right) \\
& =\sum_{s} f_{s} d q_{s}+\sum_{r} p_{r} d q_{r}-\sum_{r}\left[p_{r}\right] d\left[q_{r}\right]
\end{aligned}
$$

Therefore, when the $\varphi$ are calculated in the manner that was described above, that will effectively make equation (5) an identity.

One immediately deduces the generalization of the Jacobi-Hamilton method that was alluded to in the introduction from equation (5), and it can be formulated as follows:

One poses the problem of integrating the Jacobian system:

$$
p_{1}=f_{1}, \quad p_{2}=f_{2}, \quad \ldots, \quad p_{m}=f_{m}
$$

in which the f verify the equations:

$$
\left(p_{s}-f_{s}, p_{s^{\prime}}-f_{s^{\prime}}\right)=0 \quad\left(s, s^{\prime}=1,2, \ldots, m\right)
$$

One establishes the system of total differential equations:

$$
d p_{r}=\sum_{s} \frac{\partial f_{s}}{\partial q_{r}} d q_{s}, \quad d q_{r}=-\sum_{s} \frac{\partial f_{s}}{\partial p_{r}} d q_{s},
$$

which is always an integrable system.
One integrates that system completely and expresses the $2(n-m)$ integration constants by way of the values $p_{r}^{0}, q_{r}^{0}$ of the variables $p_{r}$ and $q_{r}$ that correspond to the initial values $q_{1}^{0}, \ldots, q_{m}^{0}$ that are chosen from the other variables, which will put the integral equations into the form:

$$
p_{r}^{0}=\left[p_{r}\right], q_{r}^{0}=\left[q_{r}\right] .
$$

If one expresses the functions:

$$
f_{s}-\sum_{r} p_{r} \frac{\partial f_{s}}{\partial p_{r}}
$$

in terms of only $q_{1}, \ldots, q_{m}$ by means of those equations then the expression $\sum_{s}\left(f_{s}-\sum_{r} p_{r} \frac{\partial f_{s}}{\partial p_{r}}\right) d q_{s}$ will become an exact differential.

Calculate the function:

$$
\psi=\sum_{r} p_{r}^{0} q_{r}^{0}+\sum_{s} \int_{q_{s}^{0}}^{q_{s}}\left(f_{s}-\sum_{r} p_{r} \frac{\partial f_{s}}{\partial p_{r}}\right) d q_{s}
$$

by quadrature. One gets the expressions for the $p_{r}$ in terms of $q$ and $p_{r}^{0}$ from the equations $p_{r}^{0}=$ $\left[p_{r}\right]$ and substitutes them in the equations $q_{r}^{0}=\left[q_{r}\right]$, in such a way that the $q_{r}^{0}$ will prove to be expressible in terms of only the $q$ and $p_{r}^{0}$. One lets $(\psi)$ denote the function that arises from $\psi$ when one replaces the $q_{r}^{0}$ with their expressions in terms of the $q$ and $p_{r}^{0}$. The formula:

$$
z=(\psi)+\text { constant }
$$

will then give a complete solution (with the arbitrary constants $p_{m+1}^{0}, \ldots, p_{n}^{0}$, along with the additive ones) to the Jacobian system that was posed.

One can express all of the integral equations of the system (3) with the function ( $\psi$ ), as is the case for just one first-order partial differential equation.

Indeed, it will result from (5) that:

$$
\delta(\psi)=\sum_{r} p_{r} \delta q_{r}+\sum_{r}\left[q_{r}\right] \delta q_{r}^{0}+\sum_{s} f_{s} \delta q_{s}
$$

but on the other hand, one will have:

$$
\delta(\psi)=\sum_{r} \frac{\partial(\psi)}{\partial q_{r}} \delta q_{r}+\sum_{r} \frac{\partial(\psi)}{\partial p_{r}^{0}} \delta q_{r}^{0}+\sum_{s} \frac{\partial(\psi)}{\partial q_{s}} \delta q_{s} .
$$

If one then observes that the variations $\delta q$ and $\delta p_{r}^{0}$ are essentially independent of each other then one then can conclude that:

$$
\frac{\partial(\psi)}{\partial q_{r}}=p_{r}, \quad \frac{\partial(\psi)}{\partial p_{r}}=\left[q_{r}\right], \quad \frac{\partial(\psi)}{\partial q_{s}}=f_{s} .
$$

The integrals of equations (3) are then:

$$
\frac{\partial(\psi)}{\partial q_{r}}=p_{r}, \quad \frac{\partial(\psi)}{\partial p_{r}^{0}}=q_{r}^{0} .
$$

Finally, observe that the integration of the system (3) is equivalent to the integration of the Jacobian system:

$$
\left(\theta, p_{s}-f_{s}\right)=0
$$

## Lie's theorem.

If one applies Mayer's method (Math. Ann. Bd. V) to the integration of the system (3) then Lie's theorem will follow easily from the preceding theorem.

Indeed, set:

$$
\begin{equation*}
q_{s}=q_{s}^{0}+\left(q_{1}-q_{1}^{0}\right) x_{s}, \tag{6}
\end{equation*}
$$

so it results that:

$$
d q_{s}=x_{s} d q_{s}^{0}+\left(q_{1}-q_{1}^{0}\right) d x_{s},
$$

and with that, equations (3) will transform into the following ones:

$$
\left.\begin{array}{rl}
d p_{r} & =\left(\frac{\partial f_{1}}{\partial q_{r}}+x_{2} \frac{\partial f_{2}}{\partial q_{r}}+\cdots+x_{m} \frac{\partial f_{m}}{\partial q_{r}}\right) d q_{1}+\left(q_{1}-q_{1}^{0}\right)\left(\frac{\partial f_{2}}{\partial q_{r}} d x_{2}+\cdots+\frac{\partial f_{m}}{\partial q_{r}} d x_{m}\right), \\
-d q_{r} & =\left(\frac{\partial f_{1}}{\partial p_{r}}+x_{2} \frac{\partial f_{2}}{\partial p_{r}}+\cdots+x_{m} \frac{\partial f_{m}}{\partial p_{r}}\right) d q_{1}+\left(q_{1}-q_{1}^{0}\right)\left(\frac{\partial f_{2}}{\partial p_{r}} d x_{2}+\cdots+\frac{\partial f_{m}}{\partial p_{r}} d x_{m}\right) .
\end{array}\right\}
$$

In order to integrate that system completely, according to Mayer's theorem, it suffices to integrate the following system of ordinary differential equations:

$$
\left.\begin{array}{l}
d p_{r}=\frac{\partial F}{\partial q_{r}} d q_{1} \\
d q_{r}=-\frac{\partial F}{\partial p_{r}} d q_{1} \tag{7}
\end{array}\right\}
$$

in which $F=f_{1}+x_{2} f_{2}+\ldots+x_{m} f_{m}$, and the $x$ enter only as constants. One then, in turn, introduces the initial values of the dependent variables $p_{r}, q_{r}$ that correspond to the initial value $q_{1}^{0}$ of the independent variable $q_{1}$ as arbitrary constants.

Now, integrating (7) is equivalent to finding an arbitrary complete integral of the partial differential equation:

$$
p_{1}=F,
$$

which contain only the $n-m+1$ independent variables:

$$
q_{1}, q_{m+1}, q_{m+2}, \ldots, q_{n},
$$

and that constitutes Lie's theorem precisely.
Let us see how we can complete that result. Those integrals of (7):

$$
p_{r}^{0}=\left[p_{r}\right], q_{r}^{0}=\left[q_{r}\right]
$$

that reduce to $p_{r}$ and $q_{r}$ identically for $q_{1}=q_{1}^{0}$ will give integrals of the system (3), and it will therefore be possible to find a function $\varphi$ such that one has:

$$
u=d z-F d q_{1}-\left(q_{1}-q_{1}^{0}\right)\left(f_{1} d x_{1}+\ldots+f_{m} d x_{m}\right)=d(z-\varphi)-\sum_{r}\left[p_{1}\right] d\left[q_{r}\right]
$$

identically.
Now, from what we saw before, we will get the function $\varphi$ by calculating the integral:

$$
\varphi=\int_{q_{s}^{0}}^{q_{s}}\left[\left(F-\sum_{r} p_{r} \frac{\partial F}{\partial p_{r}}\right) d q_{1}+\sum_{\lambda=2}^{m}\left(q_{1}-q_{1}^{0}\right)\left(f_{\lambda}-\sum_{r} p_{r} \frac{\partial f_{\lambda}}{\partial p_{r}}\right) d x_{\lambda}\right],
$$

in which the initial values of the variables $x_{\lambda}$ are indeterminate, and from a theorem in integral calculus ( ${ }^{*}$ ) that one can deduce immediately from Mayer's theorem, it suffices to perform the quadrature over $q_{1}$ between the limits $q_{1}$ and $q_{1}^{0}$, i.e., to take:
(*) I shall take this opportunity to state that theorem, which seems quite interesting to me:
One proposes to integrate the total differential:

$$
\varphi=\int_{q_{s}^{o}}^{q_{s}}\left(F-\sum_{r} p_{r} \frac{\partial F}{\partial p_{r}}\right) d q_{1} .
$$

When that function is transformed into the old variables using (6), it will make equation (5) an identity, and it will enjoy the desired property that it will reduce to zero when $q_{1}=q_{1}^{0}, q_{2}=q_{2}^{0}$, $\ldots, q_{m}=q_{m}^{0}$.

The desired solution to the Jacobian system (1) will therefore be given by the complete integral of the equation $p_{1}=F$ that is obtained by applying the Jacobi-Hamilton method, as it was perfected by Mayer.

That is the elegant form that Mayer gave to Lie's theorem. (See the supplement to Mayer's paper: "Die Lie'sche Int. Meth.," Math. Ann., Bd. VI.)

$$
d V=Q_{1} d q_{1}+Q_{2} d q_{2}+\ldots+Q_{m} d q_{m} \quad\left(\frac{\partial Q_{i}}{\partial q_{k}}-\frac{\partial Q_{k}}{\partial q_{i}}\right)
$$

between the limits $\left(q_{1}^{0}, q_{2}^{0}, \ldots, q_{m}^{0}\right),\left(q_{1}, q_{2}, \ldots, q_{m}\right)$. One sets:

$$
q_{2}=q_{2}^{0}+\left(q_{1}-q_{2}^{0}\right) x_{2}, \quad \ldots, \quad q_{m}=q_{m}^{0}+\left(q_{1}-q_{2}^{0}\right) x_{m}
$$

in $Q$, and calculates the function:

$$
V=\int_{q_{1}^{0}}^{q_{1}}\left(Q_{1}+x_{2} Q_{2}+\cdots+x_{m} Q_{m}\right) d q_{1}
$$

by quadrature, while regarding the $x$ as constants. When the function $V$, thus-found, is transformed into the original variables, it will give the desired integral.


[^0]:    (*) Cf., in addition to the cited paper by Darboux, the one by Frobenius in vol. 82 of Crelle's Journal.
    (**) According to Lie, the Jacobi-Hamilton method should also be called Cauchy's method.
    $\left({ }^{* * *}\right) \quad$ Cf., Mayer, "Ueber die Jacobi-Hamilton'sche Integrationsmethode der partiellen Differentialgleichungen 1.O," Math. Ann., Bd. III.

[^1]:    (*) Cf., the cited paper by Frobenius, or also the last § of my article: "Sulle proprietà invariantive, etc." Atti della R. Acc. delle Sc. di Torino, vol. XVIII, in which those properties are regularly studied from viewpoint of the theory of algebraic forms by means of symbolic notations.

[^2]:    (*) See my recent article "Sul problema di Pfaff," Atti della R. Acc. delle Sc. di Torino, vol. XVII.
    (**) In his article: "Neue Int. Meth. eines $2 n$-glied. Pfaff'schen problem," Abh. d. Ges. d. Wiss. zu Christiania (1873), Lie considered those functions to be solutions of the system of linear partial differential equations that is associated with the differential equations, and he called them principal solution with respect to $x_{1}=x_{1}^{0}, \ldots, x_{n-2 r}=$ $x_{n-2 r}^{0}$.

[^3]:    (*) The following also applies to the exceptional case that was mentioned above, since the expression $u$ will always have class $2(n-m)+1$, at any rate.

