"Sulla transformation delle equazioni differentiali di Hamilton, Nota I," Rend. Real. Accad. Lincei (5) **12** (1903), 113-122.

On the transformation of Hamilton's differential equations. Note I.

By **G. MORERA**

Translated by D. H. Delphenich

The transformation of a system of Hamiltonian differential equations into another similar system was amply and profoundly treated by Sophus Lie in the paper entitled: "Die Störungstheorie und die Berührungstransformationen," that was included in the second volume of the Archiv for Mathematatik og Naturvidenskab (Christiania, 1877).

The treatment of the argument will increase considerably in *generality*, simplicity, and elegance when one bases it upon the consideration of the *bilinear covariant* of a certain differential expression, as I propose to show in this article ([1]).

1. – Consider the differential expression:

$$E_d = q_1\, dp_1 + q_2\, dp_2 + \ldots + q_n\, dp_n + U\, dt\,, \tag{I}$$

in which q_1, p_1 ; q_2, p_2 ; … ; q_n , p_n, and t denote $2n + 1$ independent variables, and U is a given function of them.

Form the bilinear covariant of that expression:

$$\delta E_d - d\, E_\delta = \sum_{i=1}^{n} (\delta q_i\, dp_i - dq_i\, \delta p_i) + \delta U\, dt - dU\, \delta t$$

and equate the coefficients of δp_i , δq_i , and δt to zero. One will then have the system of differential equations:

$$\frac{dp_i}{dt} = -\frac{\partial U}{\partial q_i}\,, \qquad \frac{dq_i}{dt} = \frac{\partial U}{\partial p_i} \qquad (i = 1, 2, \ldots, n)\,, \tag{II}$$

$$0 = \sum_i \left(\frac{\partial U}{\partial p_i}\frac{dp_i}{dt} + \frac{\partial U}{\partial q_i}\frac{dq_i}{dt} \right),$$

([1]) A complete exposition of the final results that are obtained in the theory of perturbations is found in the beautiful monograph of E. O. Lovett that was published in volume XXX of the Quarterly Journal of Mathematics (pp. 47-149): "The theory of perturbations and Lie's Theory of contact transformations."

the last of which is a consequence of the preceding $2n$.

Therefore: *The Hamiltonian system* (II) *is the first Pfaff system of the differential expression* (I), *and as is known, it is invariantly linked with that differential expression* ([1]).

Note that *if two differential expressions in the same independent variables have identical bilinear covariants then they can differ by only an exact differential*. Hence, if one is given the Hamiltonian system (II) then it will not determine a unique differential expression that is invariantly linked with it, but an infinitude of them that differ from each other by exact differentials.

2. – If the differential expression (I) reduces to the canonical Pfaff form:

$$E_d = \sum_i Q_i \, dP_i + d\Phi$$

then the system (II) will become:

$$dP_i = dQ_i = 0$$

since the P_i, Q_i are the integrals of the Hamiltonian system.

In addition, reducing the differential expression E_d to the canonical Pfaffian form according to the Pfaff method of integration is equivalent to finding a complete integral of the Hamilton-Jacobi partial differential equation:

$$\text{(III)} \qquad \frac{\partial f}{\partial t} = U\left(p_1, p_2, \dots, p_n ; \frac{\partial f}{\partial p_1}, \frac{\partial f}{\partial p_2}, \dots, \frac{\partial f}{\partial p_n} ; t\right).$$

In summary, we can state the following *fundamental theorem:*

The Hamiltonian system:

$$\frac{dp_i}{dt} = -\frac{\partial U}{\partial q_i}, \qquad \frac{dq_i}{dt} = \frac{\partial U}{\partial p_i}$$

is the first Pfaff system of the differential expression:

$$\sum_i q_i \, dp_i + U \, dt \ .$$

The integration of the Hamiltonian system is equivalent to reducing the preceding differential expression to the canonical Pfaff form, or to completely integrating the partial differential equation:

([1]) Cf., Darboux, *Sur la problème de Pfaff*, Paris, Gauthier-Villars, 1882, pp. 7.

$$\frac{\partial f}{\partial t} = U\left(p_1, p_2, \ldots, p_n; \frac{\partial f}{\partial p_1}, \frac{\partial f}{\partial p_2}, \ldots, \frac{\partial f}{\partial p_n}; t\right).$$

3. – Let:

(IV) $$X_0\,dx_0 + \sum_{j=1}^{2n} X_j\,dx_j$$

be a differential expression of class $2n + 1$, in which the X denote functions of x. Set:

$$(i, k) = \frac{\partial X_i}{\partial x_k} - \frac{\partial X_k}{\partial x_i} \qquad (i, k = 0, 1, 2, \ldots, n).$$

Not all of the sub-determinants of order 1 (degree $2n$) of the determinant of the bilinear covariant can be zero, and consequently, due to a known theorem of Frobenius ([1]), not all of the principal sub-determinants of degree $2n$ can vanish. Suppose that the principal sub-determinant:

$$\begin{vmatrix} (1,1) & (1,2) & \cdots & (1,2n) \\ (2,1) & (2,2) & \cdots & (2,2n) \\ \vdots & \vdots & \ddots & \vdots \\ (2n,1) & (2n,2) & \cdots & (2n,2n) \end{vmatrix}$$

is non-zero.

The differential expression $\sum_{j=1}^{2n} X_j\,dx_j$, in which only the $x_1, x_2, \ldots, x_{2n}$ are considered to be variables, then has class $2n$, and is therefore reducible to the canonical Pfaff form:

$$q_1\,dp_1 + q_2\,dp_2 + \ldots + q_n\,dp_n .$$

However, if one considers only x_0 to be variable, which one replaces with the symbol t, then one will have:

$$X_0\,dt + \sum_{j=1}^{2n} X_j\,dx_j = \sum_{i=1}^{n} q_i\,dp_i + U\,dt$$

identically, in which:

$$U \equiv X_0\,dt + \sum_{j=1}^{2n} X_j \frac{\partial x_j}{\partial t} .$$

([1]) Crelle's Journal, t. 82, pp. 244. Cf., my note "Sulle proprietà invariantive del sistema di una forma lineare e di una bilineare alternata," Atti della R. Accad. delle Scienze di Torino **18** (1883).

The first Pfaff system of the last differential expression is a Hamiltonian system: That Hamiltonian system is the transform of the first Pfaff system of (IV).

Conversely, if one substitutes the new, mutually-independent variables $x_0, \ldots, x_{2n}$ for p_i, q_i, t in (I) then it will assume the form (IV), and consequently the Hamiltonian system (III) will transform into the first Pfaff system of (IV), which is:

$$(i, 0)\, dx_0 + (i, 1)\, dx_1 + (i, 2)\, dx_2 + \ldots + (i, 2n)\, dx_{2n} = 0 \qquad (i = 0, 1, 2, \ldots, 2n),$$

which is comprised of $2n$ mutually-independent differential equations.

4. – The most-general change of variables that converts a Hamiltonian system (II) into another similar system is therefore the one that converts the differential expression:

$$\sum_i q_i\, dp_i + U\, dt$$

into another one of the type:

$$d\,\Omega + \sum_{i=1}^{n} q_i^* dp_i^* + U^* dt^*,$$

in which Ω denotes an arbitrary function of the new independent variables p_i^*, q_i^*, t^*, and U^* is a function of the same variables.

Above all, one proposes to find the most-general change of just the variables p_i and q_i (but not t) that will satisfy the desired condition.

From the preceding discussion, such a change *is a transformation between two systems of independent variables:*

$$(p_i, q_i)\ ;\ (p_i^*, q_i^*)$$

that also depends upon the parameter t, and is such that when one regards the latter as constant, one will have:

$$\sum_i q_i\, dp_i - \sum_i q_i^* p_i^* = d\,\Omega$$

identically.

If one adds one new independent variable to each system, z and z^*, resp., and one poses the relation:

$$z = \Omega + z^*$$

then *one will have a contact transformation between the two systems of variables that depends upon the parameter t.*

Such a transformation is found immediately in its most general form by the following procedure: Assume that Ω is an arbitrary function of p_i, p_i^*, t, and agree, as one pleases, that no equation, or one of them, or even more equations exist between just p_i, p_i^*, t, but in such a way that it is not possible to eliminate the p_i or the p_i^* from them. One will then once more find the relations that are needed for defining the transformation by the procedure that one uses in analytical mechanics for deducing the equilibrium conditions for a constrained system from the equation of virtual work when the constraints are expressed by finite equations between the position parameters and time, and therefore by the classical *method of Lagrange multipliers*.

Therefore, establish the equations:

$$\text{(V)} \qquad \Omega_\nu (p_1, \ldots, p_\nu ;\; p_1^*, \ldots, p_n^*; t) = 0 \qquad (\nu = 1, 2, \ldots, q)$$

in which $q \leq n$, and the mutually-independent functions Ω_ν relate to either the p_i or the p_i^*. One then combines the preceding equations with the $2n - \nu$ other ones that result by eliminating the multipliers λ from the following equations:

$$\text{(VI)} \qquad \begin{cases} q_i = \dfrac{\partial \Omega}{\partial p_i} + \sum\limits_\nu \lambda_\nu \dfrac{\partial \Omega}{\partial p_i}, \\[2ex] q_i^* = \dfrac{\partial \Omega}{\partial p_i^*} + \sum\limits_\nu \lambda_\nu \dfrac{\partial \Omega}{\partial p_i^*} \end{cases} \qquad (i = 1, 2, \ldots, n).$$

From the theory of functional determinants, it is easy to point out what the limitation would be that Ω would have to be subject to in order for equations (V) and (VI) to be soluble with respect to either λ_ν, p_i, q_i or to λ_ν, p_i^*, q_i^*, or for them *to define the desired transformation*.

Set:

$$W \equiv \Omega + \lambda_1 \, \Omega_1 + \ldots + \lambda_q \, \Omega_q \, .$$

The aforementioned limitation is that because of (V), the following determinant, which is an entire function of degree $n - q$ in the λ:

$$\begin{vmatrix} \dfrac{\partial \Omega_1}{\partial p_1} & \dfrac{\partial \Omega_1}{\partial p_2} & \cdots & \dfrac{\partial \Omega_1}{\partial p_n} & 0 & 0 & \cdots & 0 \\ \dfrac{\partial \Omega_2}{\partial p_1} & \dfrac{\partial \Omega_2}{\partial p_2} & \cdots & \dfrac{\partial \Omega_2}{\partial p_n} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ \dfrac{\partial \Omega_q}{\partial p_1} & \dfrac{\partial \Omega_q}{\partial p_2} & \cdots & \dfrac{\partial \Omega_q}{\partial p_n} & 0 & 0 & \cdots & 0 \\ \dfrac{\partial^2 \Omega}{\partial p_1^* \, \partial p_1} & \dfrac{\partial^2 \Omega}{\partial p_1^* \, \partial p_2} & \cdots & \dfrac{\partial^2 \Omega}{\partial p_1^* \, \partial p_n} & \dfrac{\partial \Omega_1}{\partial p_1^*} & \dfrac{\partial \Omega_2}{\partial p_1^*} & \cdots & \dfrac{\partial \Omega_q}{\partial p_1^*} \\ \dfrac{\partial^2 \Omega}{\partial p_2^* \, \partial p_1} & \dfrac{\partial^2 \Omega}{\partial p_2^* \, \partial p_2} & \cdots & \dfrac{\partial^2 \Omega}{\partial p_2^* \, \partial p_n} & \dfrac{\partial \Omega_1}{\partial p_2^*} & \dfrac{\partial \Omega_2}{\partial p_2^*} & \cdots & \dfrac{\partial \Omega_q}{\partial p_2^*} \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ \dfrac{\partial^2 \Omega}{\partial p_n^* \, \partial p_1} & \dfrac{\partial^2 \Omega}{\partial p_n^* \, \partial p_2} & \cdots & \dfrac{\partial^2 \Omega}{\partial p_n^* \, \partial p_n} & \dfrac{\partial \Omega_1}{\partial p_n^*} & \dfrac{\partial \Omega_2}{\partial p_n^*} & \cdots & \dfrac{\partial \Omega_q}{\partial p_n^*} \end{vmatrix}$$

must not be identically zero. [Cf., Lie-Engel, *Transformationsgruppen*, II Abschn. Kap. 6, Abt. I].

Set:

$$V = \frac{\partial \Omega}{\partial t} + \sum_{\nu} \lambda_\nu \frac{\partial \Omega_\nu}{\partial t}$$

and imagine how one can eliminate the λ in that equation by means of (VI). One has:

$$\sum_i q_i \, dp_i - \sum_i q_i^* \, dp_i^* + V \, dt = d\,\Omega$$

identically.

One notes that once the expressions for p_i and q_i in terms of p_i^*, q_i^*, and t have been obtained, the aforementioned Ω^* and V^* will be the transforms of Ω and V in terms of those variables, and one will have:

$$V^* \equiv \frac{\partial \Omega^*}{\partial t} - \sum_i q_i \frac{\partial p_i}{\partial t} .$$

Therefore, with the transformation thus-found, one will have the identity:

$$\text{(VII)} \qquad \sum_i q_i \, dp_i + U \, dt = d\,\Omega + \sum_i q_i^* \, dp_i^* + (U^* - V^*) \, dt ,$$

in which U^* denotes the transform of U in terms of p_i^*, q_i^*, and t.

In particular, if $\nu = 0$ then one will get the solubility conditions for (VI) in the form of requiring that the functional determinant:

$$\frac{\partial\left(\dfrac{\partial\Omega}{\partial p_1^*},\dfrac{\partial\Omega}{\partial p_2^*},\dots,\dfrac{\partial\Omega}{\partial p_n^*}\right)}{\partial(p_1,p_2,\dots,p_n)}$$

must not be identically zero. (VI) will then become:

$$q_i = \frac{\partial\Omega}{\partial p_i}; \qquad -\,q_i^* = \frac{\partial\Omega}{\partial p_i^*},$$

and the expression for V will become:

$$V = \frac{\partial\Omega}{\partial t}.$$

One can assume that V *is an arbitrarily-given function of the original variables* p_i, q_i, t*: One then assumes that* Ω *is a complete integral of the partial differential equation:*

$$\frac{\partial\Omega}{\partial t} = V\left(p_1,\dots,p_n;\frac{\partial V}{\partial p_1},\dots,\frac{\partial V}{\partial p_n};t\right)$$

with non-additive arbitrary constants:

$$p_1^*,\ p_2^*,\ \dots,\ p_n^*.$$

In any case, from (VII), *one sets:*

$$H^* = U^* - V^*,$$

and the transform of the Hamiltonian system (II) *will be:*

$$\frac{\partial p_i^*}{\partial t} = -\frac{\partial H^*}{\partial q_i^*}; \qquad \frac{\partial q_i^*}{\partial t} = \frac{\partial H^*}{\partial p_i}.$$

The conventional theory of perturbations is based upon the particular transformation that was just pointed out.

5. – In order to find the most-general transformation that converts the Hamiltonian system (II) into another Hamiltonian system:

$$\frac{\partial p_i^*}{\partial t^*} = -\frac{\partial U^*}{\partial q_i}; \qquad \frac{\partial q_i^*}{\partial t^*} = \frac{\partial U^*}{\partial p_i^*},$$

one considers the most general transformation between two systems of *independent variables:*

$$(p_1, q_1; p_2, q_2; \dots; p_n, q_n; t, u),$$

$$(p_1^*, q_1^*; p_2^*, q_2^*; \ldots; p_n^*, q_n^*; t^*, u^*),$$

and that will give rise to an identity of the form:

$$\text{(VIII)} \qquad \sum_i q_i\, dp_i + u\, dt = d\,\Omega + \sum_i q_i^*\, dp_i^* + u^*\, dt.$$

For the transformation in question, that relation will be converted into a relation:

$$F(p_1^*, q_1^*; p_2^*, q_2^*; \ldots; p_n^*, q_n^*; t^*, u^*) = 0,$$

which cannot be an identity. Suppose that F contains u^*: That hypothesis is basically *non-restrictive*, since the pair of conjugate variables t^*, u^* is an arbitrary pair of the new variables, and furthermore it is always legitimate to set: $\Omega_1 = \Omega + t^*\, u^*$ in place of Ω and set t^* and u^* equal to $t_1^* = -\, u^*$ and $u_1^* = t^*$. The preceding equation can then be solved for u^*, and it can then be put into the form:

$$u^* - U^*(p_1^*, \ldots\ p_n^*; q_1^*, \ldots, q_n^*; t^*) = 0.$$

Since one can eliminate the auxiliary variable from the new variables from the expressions for the original variables p_i, q_i, and t, and therefore one can eliminate the u from the formula that gives the inverse transformation. Because neither du nor du^* appears in the identity (VIII), the form of that identity will not change with the substitution of the function U for u and U^* for u^*. Therefore, *one will have:*

$$\sum_i q_i\, dp_i + U\, dt = d\,\Omega + \sum_i q_i^*\, dp_i^* + U^*\, dt$$

identically. The desired transformation is thus found.

For example, choose Ω in any way:

$$\Omega \equiv \Omega(p_1, p_2, \ldots, p_n; p_1^*, \ldots\ p_n^*; t, t^*),$$

and assume that one has $n + 1$ relations of the form:

$$\Omega(p_1, p_2, \ldots, p_n; p_1^*, \ldots\ p_n^*; t, t^*) = 0 \qquad (\nu = 0, 1, 2, \ldots, n)$$

that satisfy the condition of being soluble for either $p_1, \ldots, p_n$, t or for $p_1^*, \ldots, p_n^*$, t^*. The desired transformation is defined by the preceding $n + 1$ equations and the other $n + 1$ equations that one obtains by eliminating the multipliers λ from the following 2 (n + 1) equations:

$$\begin{cases} q_i + \sum_{\nu=0}^{n} \lambda_\nu \dfrac{\partial \Omega_\nu}{\partial p_i} = \dfrac{\partial \Omega}{\partial p_i}, \quad (i = 1, 2, \ldots, n), \\ u + \sum_{\nu=0}^{n} \lambda_\nu \dfrac{\partial \Omega_\nu}{\partial t} = \dfrac{\partial \Omega}{\partial t}, \\ -q_i^* + \sum_{\nu=0}^{n} \lambda_\nu \dfrac{\partial \Omega_\nu}{\partial p_i^*} = \dfrac{\partial \Omega}{\partial p_i^*}, \quad (i = 1, 2, \ldots, n), \\ -u^* + \sum_{\nu=0}^{n} \lambda_\nu \dfrac{\partial \Omega_\nu}{\partial t^*} = \dfrac{\partial \Omega}{\partial t^*}. \end{cases}$$

Under such a transformation, (IX) will be converted into an equation that cannot contain just $p_1^*, \ldots, p_n^*, t$. With no loss of generality, one can then keep the demand that the transformation of (IX) must contain u^*.

6. – Consider an arbitrary system of $2n$ first-order differential equations in the $2n + 1$ variables:

$$\frac{dx_0}{X_0} = \frac{dx_1}{X_1} = \frac{dx_2}{X_2} = \ldots = \frac{dx_{2n}}{X_{2n}}. \tag{X}$$

We propose to examine whether such a system is reducible to the canonical Hamiltonian form under a transformation of the variables x into other ones y, which are mutually independent.

From the previous discussion, we need to see whether there exists a differential expression of class $2n + 1$:

$$Y_0\,dx_0 + Y_1\,dx_1 + \ldots + Y_{2n}\,dx_{2n}$$

whose first Pfaff system coincides with the given system: In other words, whether it is possible to determine the Y as functions of the x in such a way that when one sets:

$$(i, k) = \frac{\partial Y_i}{\partial x_k} - \frac{\partial Y_k}{\partial x_i} \qquad (i, k = 0, 1, 2, \ldots, 2n),$$

one will have:

$$X_0\,(i, 0) + X_1\,(i, 1) + X_2\,(i, 2) + \ldots + X_{2n}\,(i, 2n) = 0 \tag{XI}$$

$$(i = 0, 1, 2, \ldots, 2n).$$

If one assumes that this is possible then the aforementioned differential expression will reduce to the form (**§ 3**):

$$\sum_i q_i\,dp_i + U\,dt,$$

and the differential equations (X) will assume the Hamiltonian form (II) under the introduction of new independent variables.

The (XI) are not mutually independent, since multiplying them by $X_0, \ldots, X_{2n}$ and summing will give rise to the identity:

$$\sum_i \sum_k (i,k)\, X_i\, X_k = 0 .$$

Therefore, if one supposes, for example, that $X_0 \neq 0$ then the first of them will be a consequence of the remaining $2n$.

The system of $2n$ homogeneous, linear, first-order partial differential equations (XI) is known to admit an infinitude of solutions. Indeed, if one chooses, e.g., Y_0 arbitrarily then Cauchy (*Œuvres*, pp. I; t. VII, page 33) proved that the $Y_1, \ldots, Y_{2n}$ can be determined in such a way that they satisfy the $2n$ differential equations. For a particular value of the independent variable x_0, they will then become equal to arbitrarily-given functions of the $x_1, x_2, \ldots, x_{2n}$: The single condition that must be satisfied is that the functions considered must be regular analytic functions.

Moreover, the same conclusion can obviously be drawn from our invariant theory.

If one integrates (X) and introduces new variables in place of the $x_1, \ldots, x_{2n}$ by way of a system of $2n$ independent integrals of those equations:

$$y_i = y_i\,(x_0, x_1, \ldots, x_{2n}) \qquad (i = 1, 2, \ldots, 2n)$$

then they will assume the form:

$$dx_0 = \frac{dy_1}{0} = \frac{dy_2}{0} = \ldots = \frac{dy_{2n}}{0} ,$$

and consequently (XI) will become:

$$\frac{\partial Y_i}{\partial x_0} - \frac{\partial Y_0}{\partial y_i} = 0 .$$

The most-general solution of those equations is:

$$Y_0 = \frac{\partial \Phi}{\partial x_0} , \qquad Y_i = \frac{\partial \Phi}{\partial y_i} + \eta_i ,$$

in which Φ is an arbitrary function of all the variables, and the η_i are arbitrary functions of just $y_1, y_2, \ldots, y_{2n}$. Consequently, (X) will constitute the first Pfaff system of the differential expression:

$$d\,\Phi + \sum_{i=1}^{2n} \eta_i\, dy_i ,$$

in which one must set the η_i equal to functions of the y_i such that the skew determinant that is formed by $\frac{\partial \eta_i}{\partial y_k} - \frac{\partial \eta_k}{\partial y_i}$ will not be zero in order for it to have class $2n + 1$. For example, it is enough to choose:

$$\eta_j = y_{n+j}\,, \qquad \eta_j = 0 \qquad\qquad (j = 1, 2, \ldots, n).$$

Therefore, one has the following *theorem* that seems more curious than useful:

Any system of 2*n differential equations is always reducible to the canonical Hamiltonian form.*

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On the transformation of Hamilton's differential equations. Note II.

By **G. MORERA**

Translated by D. H. Delphenich

7. – According to the last section (6) in my preceding note ([1]), if one is given $2n$ first-order differential equations then there will exist an infinitude of differential expressions that are linear in their differentials, *and not only for exact differentials*, and admit those equations as their first Pfaff system.

Therefore, if one observes that the differential expression $\sum_{i=1}^{2n} \eta_i \, dy_i$, in which the η_i are functions of only y_i, is always reducible to the form $\sum_{j=1}^{n} Y_j^* \, dy_j^*$ then one can conclude that *any system of* $2n$ *first-order differential equations is the first Pfaff system of a differential expression of the type:*

[I] $$E_d \equiv \sum_{j=1}^{n} y_{n+j} \, dy_j + d\Phi ,$$

in which the y are an arbitrary system of independent integrals of those differential equations.

Consider a Hamiltonian system:

[II] $$\frac{dp_i}{dt} = -\frac{\partial U}{\partial q_i}, \qquad \frac{dq_i}{dt} = \frac{\partial U}{\partial p_i} \qquad (i = 1, 2, \dots, n),$$

and take $y_1, \dots, y_n$; $y_{n+1}, \dots, y_{2n}$ to be those integrals that become equal to the initial values of $p_1, p_2, \dots, p_n$; $q_1, \dots, q_n$, respectively, for a given initial value t_0 of the independent variable t. As is known ([2]), one will then have:

$$\sum_{j=1}^{n} y_{n+j} \, dy_j \equiv U \, dt + \sum_{j=1}^{n} q_j \, dp_j + d\varphi ,$$

([1]) Cf., page 113.

([2]) Cf., my note "Il metodo di Pfaff per l'integrazione delle equazioni a derivate parziali del 1º ordino," which was inserted in the Rend. dell'Istituto Lombardo **16** (1883), 637-644.

and one therefore finds that E_d has precisely the form that was considered in **§ 1** of the preceding note.

However, if one takes the y to be those integrals that become equal to arbitrary, but mutually-independent, functions of the initial values of the p and q for $t = t_0$ then the form of E_d will not be known *a priori*, *in general*, and its determination cannot be carried out with no integrations.

Let us examine, in particular, the case in which U does not depend upon t. We choose a canonical group to which U belongs, which will be written U_1, for the sake of symmetry. Let one such group be:

$$U_1, U_2, \ldots, U_n ; V_1, V_2, \ldots, V_n .$$

The canonical integrals of [II] are then ([1]):

$$U_1, U_2, \ldots, U_n ; V_1 - t, V_2, \ldots, V_n ,$$

and in [I], one can assume that:

$$y_1 = V_1 - t ; y_1 = f_2 (U_1, U_2, \ldots, U_n ; V_1, V_2, \ldots, V_n) ; \ldots ; y_n = f_n (U_1, U_2, \ldots, U_n ; V_1, V_2, \ldots, V_n) ;$$

$$y_{n+j} = f_{n+j} (U_1, U_2, \ldots, U_n ; V_1, V_2, \ldots, V_n) \qquad (j = 1, 2, \ldots, n),$$

in which the f are arbitrary functions of their respective arguments, but subject to the single limitation that they must be independent with respect to $U_1, \ldots, U_n ; V_1, \ldots, V_n$. Having done that, [I] will become:

$$E_d = \sum_{j=1}^{n} f_{n+j} \cdot df_j - f_{n+1} \cdot dt + d\Phi ,$$

in which one writes f_1, in place of V_1, for the sake of symmetry. Therefore, the transformation:

$$\begin{array}{llll} p_1^* = V_1 , & p_2^* = f_2 , & \ldots, & p_n^* = f_n ; \\ q_1^* = f_{n+1} , & q_2^* = f_{n+2} , & \ldots, & q_n^* = f_{2n} \end{array}$$

will convert the system [II] into another Hamiltonian system. That transformation will still be independent of t if the f_{n+j} do not contain $V_1 - t$. (Cf., Lie, paper cited, page 155, Theorem III) As was known to Lie, such a transformation does not, in general, reduce to a contact transformation, i.e., a transformation of only the p and q of the type that was considered in **§ 4** of my preceding note.

([1]) Keep in mind that an integral is a solution of the partial differential equation:

$$\frac{\partial y}{\partial t} + \sum_{j=1}^{n} \left(\frac{\partial U_1}{\partial p_j} \frac{\partial y}{\partial q_j} - \frac{\partial U_1}{\partial p_j} \frac{\partial y}{\partial q_j} \right) \equiv \frac{\partial y}{\partial t} + (U_1, Y) = 0 \quad \text{and that } (U_1, V_1) = 1.$$

8. – Consider the system:

[III] $$\delta x_i = X_i\, \delta z \qquad (i = 0, 1, 2, \ldots, 2n),$$

in which the X are functions of only the x.

As in **§ 6**, set:

$$(i, k) = \frac{\partial Y_i}{\partial x_k} - \frac{\partial Y_k}{\partial x_i} \qquad (i, k = 0, 1, \ldots, 2n).$$

One can determine the Y as functions of the x by means of the system of equations:

$$(i, 0)\, X_0 + (i, 1)\, X_1 + \ldots + (i, 2n)\, X_{2n} = 0\ .$$

According to Lie's terminology, the differential form:

[IV] $$Y_{dx} \equiv Y_0\, dx_0 + Y_1\, dx_1 + \ldots + Y_{2n}\, dx_{2n}$$

is (at least up to the exact differential $d \sum_i X_i Y_i$) *an invariant* for the infinitesimal transformation of [III], or for the infinitesimal transformation:

[V] $$U f \equiv \sum_{i=0}^{2n} X_i \frac{\partial f}{\partial x_i}\ .$$

Indeed, one will find that (1):

$$U\, Y_{dx} \equiv \sum_i \sum_k (ik)\, X_k\, dx_i + d \sum_i X_i Y_i \equiv d \sum_i X_i Y_i\ .$$

The function $\sum_i X_i Y_i$ is a simultaneous invariant of the differential form [IV] and the infinitesimal transformation [V] for arbitrary changes of values.

According to the terminology that Poincaré used in the book *Les méthodes nouvelles de la mécanique celeste* (t. III, pp. 9), *the integral of* [IV] *is a relative integral invariant for the closed line*. As a consequence, the differential expression [IV] will differ from an absolute linear invariant by an exact differential (*ibid.*, pp. 14). As one sees immediately from the fact that:

$$U\,(Y_{dx} + df) \equiv \sum_i \sum_k (ik)\, X_k\, dx_i + d\, [U\, f + \sum_i X_i Y_i],$$

that exact differential is the differential of a solution to the partial differential equation:

(1) Cf., Lie, "Einige Bemerkungen über Pfaff'sche Ausdrücke und Gleichungen," Leipziger Berichte (1896).

$$U\,f + \sum_i X_i\,Y_i = 0\,.$$

Such a solution is obtained by a quadrature when one knows $2n$ independent solutions of the partial differential equation:

$$U\,f = 0$$

or the integrals of the system [III] that are independent of z. Express the $x_1, \ldots, x_{2n}$ as functions of x_0 and the given integrals, which one regards as constant. One then calculates the function:

$$f = -\int \sum_i X_i\,Y_i\,\frac{dx_0}{X_0}\,.$$

With a subsequent quadrature, one replaces the integrals with their expressions in terms of the x. f, thus-obtained, will be the desired solution.

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