# The exterior differential forms $\Omega_{n}$ in the calculus of variations 

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Introduction. - Consider the action integral:

$$
I_{1}=\int_{t_{0}}^{t_{1}} L\left(t, q^{\alpha}(t), \dot{q}^{\alpha}(t)\right) d t \quad(\alpha=1,2, \ldots, m)
$$

of a dynamical problem.
One knows the fundamental role that is played by the 1 -fold differential form (viz., the quantity of energy-motion form):

$$
H_{1} \equiv \frac{\partial L}{\partial \dot{q}^{\alpha}} d q^{\alpha}-\left(-L+\frac{\partial L}{\partial \dot{q}^{\alpha}} \dot{q}^{\alpha}\right) d t
$$

in the study of the integral invariants of the extremals of $I_{1}$, as well as in the method that Hilbert $\left({ }^{1}\right)$ introduced into the study of the conditions for a minimum of the integral $I_{1}$.

In a prior work $\left({ }^{2}\right)$, the author considered the case of the $n$-fold integral:

$$
I_{n}=\int_{D_{n}} F\left[x^{i}, y^{\alpha}\left(x^{i}\right), y_{t_{1} t_{2} \cdots t_{k}}^{\alpha}\left(x^{i}\right)\right] d x^{1} \cdots d x^{n} \quad \begin{aligned}
&(i=1,2, \ldots, n) \\
&(\alpha=1,2, \ldots, m) \\
&(k=1,2, \ldots, c)
\end{aligned}
$$

for a problem in the calculus of variations with $n$ independent variables $x^{i}, m$ unknown functions $y^{\alpha}\left(x^{i}\right)$, and their partial derivatives $y_{t_{1} t_{2} \cdots t_{k}}^{\alpha}\left(x^{i}\right)$ up to order $c(n, m, c$ are arbitrary). In that article, the integral $I_{n}$ was associated with an $n$-tuple differential form with exterior multiplication ["exterior differential form" in the elliptic terminology of E. Cartan $\left(^{3}\right.$ ), or an "integral form" according to that of Th. De Donder $\left(^{4}\right)$ ] that the author denoted by $H_{n}$.

The form $H_{n}$ enjoys the same properties with respect to the integral $I_{n}$ that the form $H_{1}$ does with respect to the integral $I_{1}$. That is what permits one to put the first variation of $I_{n}$ with a variable

[^0]boundary into the remarkable form that Cartan $\left({ }^{1}\right)$ gave it in the case of the integral $I_{1}$. In that form, the expression for the first variation of $I_{n}$ will immediately lead to the canonical Volterra-De Donder equations $\left({ }^{2}\right)$ and De Donder's generalized integral invariants $\left({ }^{3}\right)$. As the author has likewise shown $\left({ }^{4}\right)$, the form $H_{n}$ permits one to extend the results the H. Weyl $\left({ }^{5}\right)$ obtained in the theory of geodesic fields in the calculus of variations for the case of $c=1$ (first derivatives) to the integral $I_{n}$.

Inspired by the fundamental paper by Th.-H. Lepage $\left({ }^{6}\right)$, the author will show in the present article that the properties of the form $H_{n}$ extend to any arbitrary form $\Omega_{n}$ on the space of $x^{i}, y^{\alpha}$, $y_{i}^{\alpha}$ (the author limits himself to the case of $c=1$ ) that is subject to the congruence conditions:

$$
\begin{aligned}
\Omega_{n} & \equiv \omega_{n} & & \bmod \theta^{\alpha} \\
d \Omega_{n} & \equiv 0 & & \bmod \theta^{\alpha}
\end{aligned}
$$

with

$$
\omega_{n} \equiv \mathcal{F}\left(x^{i}, y^{\alpha}, y_{i}^{\alpha}\right) d x^{1} \cdots d x^{n}, \quad \theta^{\alpha} \equiv d y^{\alpha}-y_{i}^{\alpha} d x^{i}
$$

in which $d \Omega_{n}$ denotes the symbolic differential of $\Omega_{n}$ in the sense of the theory of exterior differential forms.

In the first section, the author will show that the forms $\Omega_{n}$ that satisfy the first of those congruences are introduced naturally when the multiple integral $I_{n}$ is thought of as an integral on a manifold in the space of $x^{i}, y^{\alpha}, y_{i}^{\alpha}$. Those forms depend upon arbitrary coefficients $X_{\alpha_{1} \cdots \alpha_{p}}^{i_{1} \cdots i_{j}}$ (1.9).

In the second section, the author will apply the formula ( $\delta . d . E$ ) [see (2.6)] to the theorem of integral invariants in the calculation of the first variation of $I_{n}$ with variable boundary. He will then obtain a new expression for that first variation in which the arbitrary coefficients $X_{\alpha_{1}}^{i_{1}}$ appear. Upon choosing them to have the values $\partial \mathcal{F} / \partial y_{i_{1}}^{\alpha_{1}}$, he will find the expression for the first variation in its conventional form. That particular choice of coefficients $X_{\alpha_{1}}^{i_{1}}$ corresponds to the forms $\Omega_{n}$ that satisfy the second of the congruences that were in question above.

In the third section, the author will determine those of the forms $\Omega_{n}$ that have minimum rank. In order to do that, we will first establish a necessary and sufficient condition for an arbitrary exterior differential form to have minimum rank: It is necessary and sufficient that its coefficients must verify the relations (3.2). Those relations are the ones that are called the well-known

[^1]D'Ovidio relations in the theory of linear spaces $\left({ }^{1}\right)$. They belong to the theory of exterior differential forms in affine geometry. As one can show, the theory of exterior differential forms is not distinct from that of fields of multivectors in a locally-affine space: The coefficients of an exterior differential form are the components of a field of covariant $n$-vectors. The D'Ovidio conditions express the idea that an $n$-vector of that field is a simple $n$-vector, i.e., the exterior product, in the Grassmann sense, of covariant $n$-vectors in the tangent affine space. The author has reserved for a later work the development of the relations between the theory of exterior differential forms, the vector calculus, and the generalization that Gassmann gave to it in his Ausdehnungslehre $\left(^{2}\right)$ and showing the importance of those methods in the calculus of variations in its parametric, or "intrinsic," form. That will then lead naturally to the generalization of the geometries of Finsler $\left({ }^{3}\right)$, Cartan $\left({ }^{4}\right)$, Kawaguchi $\left({ }^{5}\right)$.

Finally, in the last section, the author will recall the definition of a geodesic field in the calculus of variations and determine the independent integral and the Weierstrass formula that correspond to it. By assuming the viewpoint that Th. Lepage adopted in the previously-cited paper, the author will specify the nature of the condition $X_{\alpha_{1}}^{i_{1}}=\partial \mathcal{F} / \partial y_{i_{1}}^{\alpha_{1}}$ that was encountered above. To conclude, the author will address the geodesic fields of Caratheory and De Donder-Weyl, and thus recover the form $H_{n}$ that he began with.

1. The integrals of the calculus of variations. - Consider the multiple integral:

$$
\begin{equation*}
I_{n} \equiv \int_{D_{n}} F\left[x^{i}, y^{\alpha}\left(x^{i}\right), y_{t_{1} t_{2} \cdots t_{k}}^{\alpha}\left(x^{i}\right)\right] d x^{1} \cdots d x^{n} \tag{1.1}
\end{equation*}
$$

that relates to a problem in the calculus of variations for $n$ independent variables $x^{i}, m$ unknown functions $y^{\alpha}\left(x^{i}\right)$, and the first derivatives $y_{i}^{\alpha}(x)$ of those functions. The value of the integral $I_{n}$ depends upon the choice of domain $D$ in the space of variables:

$$
\begin{equation*}
x^{i} \quad(i=1,2, \ldots, n) \tag{1.2}
\end{equation*}
$$

and the choice of functions:

$$
\begin{equation*}
y^{\alpha}\left(x^{i}\right) \equiv y^{\alpha}\left(x^{1}, \ldots, x^{n}\right) \quad(\alpha=1,2, \ldots, m) \tag{1.3}
\end{equation*}
$$

with

$$
\begin{equation*}
y_{i}^{\alpha}(x) \equiv \frac{\partial y^{\alpha}(x)}{\partial x^{i}} . \tag{1.4}
\end{equation*}
$$

[^2]In the space $\mathcal{E}_{n+m+n m}$ of the arguments $x^{i}, y^{\alpha}, y_{i}^{\alpha}$, consider the $n$-tuple exterior differential form:

$$
\begin{equation*}
\omega_{n} \equiv \mathcal{F}\left(x^{i}, y^{\alpha}, y_{i}^{\alpha}\right) d x^{1} \cdots d x^{n} \tag{1.5}
\end{equation*}
$$

and the portion of the $n$-fold manifold $\Sigma_{n}$ that is defined by the equations:

$$
\begin{align*}
& x^{i}=x^{i}, \\
& y^{\alpha}=y^{\alpha}(x) \quad(x \in D),  \tag{1.6}\\
& y_{i}^{\alpha}=y_{i}^{\alpha}(x) \quad\left(\equiv \frac{\partial y^{\alpha}(x)}{\partial x^{i}}\right) .
\end{align*}
$$

The multiple integral $I_{n}$ is equal to the integral on the manifold in the space $\mathcal{E}_{n+m+n m}$ that is defined by $\omega_{n}$ and $\Sigma_{n}$ :

$$
\begin{equation*}
I_{n}=\int_{\Sigma_{n}} \omega_{n} . \tag{1.7}
\end{equation*}
$$

The latter integral is characterized by the particular nature of the form $\omega_{n}$, whose coefficients are zero, except for one, and by the particular nature of the manifold $\Sigma_{n}$, which is an integral of the Pfaff system ( ${ }^{1}$ ):

$$
\begin{equation*}
\theta^{\alpha} \equiv d y^{\alpha}-y_{i}^{\alpha} d x^{i}=0, \quad \alpha=1,2, \ldots, m . \tag{1.8}
\end{equation*}
$$

It results from the latter characteristic that the integral in the manifold (1.7) will not change value if one replaces the form $\omega_{n}$ with another form on the space $\mathcal{E}_{n+m+n m}$ that is congruent to $\omega_{n}$ $\bmod \theta^{1}, \ldots, \theta^{m}$, i.e., one that reduces to the form $\omega_{n}$ by virtue of equations (1.8).

The most general form in the space $\mathcal{E}_{n+m+n m}$ that enjoys that property is the form:

$$
\begin{equation*}
\Omega_{n} \equiv \mathcal{F} d(x)+\sum_{p=1}^{l} \frac{1}{(p!)^{2}} X_{\alpha_{1}, \ldots, \alpha_{p}}^{i_{1} \cdots i_{p}} \theta^{\alpha_{1}} \cdots \theta^{\alpha_{p}}(-1)^{i_{1} \cdots+i_{p}-1} d\left(i_{1} \cdots i_{p}\right) \tag{1.9}
\end{equation*}
$$

in which $l$ is the smaller of the two numbers $n$ and $m$, and the $X_{\alpha_{1}, \ldots, \alpha_{p}}^{i_{1} \cdots i_{p}}$ are arbitrary analytic functions of the arguments $x^{i}, y^{\alpha}, y_{i}^{\alpha}$ that are completely-antisymmetric in their indices $i$ and $\alpha$. On the other hand, one has set:

$$
\begin{gather*}
d(x) \equiv d x^{1} \cdots d x^{n} \\
d\left(i_{1}, \ldots, i_{p}\right) \equiv d x^{i_{1}-1} \cdots d x^{i_{1}+1} \cdots d x^{i_{p}-1} d x^{i_{p}+1} \cdots d x^{n} \tag{1.10}
\end{gather*}
$$

[^3]in (1.9), in which the indices $i_{1}, i_{2}, \ldots, i_{p}(p=1, \ldots, l)$ and $\alpha_{1}, \alpha_{2}, \ldots, \alpha_{p}(p=1, \ldots, l)$ independently take one or the other of the values $1,2, \ldots, n$ and $1,2, \ldots, m$, respectively.

The multiple integral $I_{n}$ is then equal to the integral on the manifold:

$$
\begin{equation*}
I_{\Sigma_{n}} \equiv \int_{\Sigma_{n}} \Omega_{n} \tag{1.11}
\end{equation*}
$$

in which $\Omega_{n}$ and $\Sigma_{n}$ are given by (1.9) and (1.6), respectively.
2. First variation of $I_{n}$. - The calculation of the first variation of $I_{n}$ (1.1) introduces the second derivatives:

$$
\begin{equation*}
y_{i j}^{\alpha}\left(x^{1}, \ldots, x^{n}\right) \equiv \frac{\partial^{2} y^{\alpha}}{\partial x^{i} \partial x^{j}} \tag{2.1}
\end{equation*}
$$

of the functions $y^{\alpha}(x)$. That will lead us to modify the viewpoint of the preceding section.
Instead of considering the space $\mathcal{E}_{n+m+n m}$ of the $x^{i}, y^{\alpha}, y_{i}^{\alpha}$, consider the space $\mathcal{E}_{n+m+n m+m}$ of the $\binom{n+2-1}{2}$ variables $x^{i}, y^{\alpha}, y_{i}^{\alpha}, y_{i j}^{\alpha}$. The form $\Omega_{n}$ will be thought as being defined on that space, as we once more let $\Sigma_{n}$ denote the $n$-fold manifold in that space that is obtained by adding the equations:

$$
\begin{equation*}
y_{i j}^{\alpha} \equiv y_{i j}^{\alpha}\left(x^{1}, \ldots, x^{n}\right) \quad\left(\equiv \frac{\partial^{2} y^{\alpha}(x)}{\partial x^{i} \partial x^{j}}\right) \tag{2.1}
\end{equation*}
$$

to equations (1.6).
$\Sigma_{n}$ is an integral of the Pfaff system:

$$
\begin{equation*}
\theta^{\alpha} \equiv d y^{\alpha}-y_{i}^{\alpha} d x^{i}=0, \quad \theta_{i}^{\alpha} \equiv d y_{i}^{\alpha}-y_{i j}^{\alpha} d x^{j}=0 \tag{2.2}
\end{equation*}
$$

Consider an arbitrary family of one parameter $\tau$ of portions of the manifolds $\Sigma_{n}(\tau)$ in the $\binom{n+1}{2}$-fold space $\mathcal{E}_{n+m+n m+m}$; let:

$$
\begin{align*}
x^{i} & =x^{i}\left(\tau, \lambda^{1}, \ldots, \lambda^{n}\right) \\
y^{\alpha} & =y^{\alpha}\left[\tau, x\left(\tau, \lambda^{1}, \ldots, \lambda^{n}\right)\right] \\
y_{i}^{\alpha} & =\frac{\partial y^{\alpha}}{\partial x^{i}} \equiv y_{i}^{\alpha}\left[\tau, x\left(\tau, \lambda^{1}, \ldots, \lambda^{n}\right)\right] \quad\left(\lambda^{1}, \ldots, \lambda^{n} \in \Lambda\right),  \tag{2.3}\\
y_{i j}^{\alpha} & =\frac{\partial y^{\alpha}}{\partial x^{i} \partial x^{j}} \equiv y_{i j}^{\alpha}\left[\tau, x\left(\tau, \lambda^{1}, \ldots, \lambda^{n}\right)\right]
\end{align*}
$$

be the parametric equations of one such family, in which $\Lambda$ denotes the domain of variation of the parameters $\lambda^{1}, \ldots, \lambda^{n}$.

On the other hand, let:

$$
\begin{equation*}
\frac{\delta x^{i}}{X^{i}(\tau, x, y)}=\frac{\delta y^{\alpha}}{Y^{\alpha}(\tau, x, y)}=\frac{\delta y_{i}{ }^{\alpha}}{Y_{i}^{\alpha}(\tau, x, y)}=\frac{\delta y_{i j}^{\alpha}}{Y_{i j}{ }^{\alpha}(\tau, x, y)}=\delta \tau \tag{2.4}
\end{equation*}
$$

be the "variational derivative" of an arbitrary point $\left(x^{i}, y^{\alpha}, y_{i}^{\alpha}, y_{i j}^{\alpha}\right)$ of $\Sigma_{n}(\tau)$ that corresponds to the elementary variation ( $\tau \rightarrow \tau+\delta \tau$ ) of the parameter $\tau$.

Upon referring to the theory of exterior differential forms, the differential of:

$$
\begin{equation*}
I_{\Sigma_{n}} \equiv \int_{\Sigma_{n}} \Omega_{n} \tag{2.5}
\end{equation*}
$$

that conforms to the system (2.4) is given the relation [formula ( $\delta . d . E)]\left({ }^{1}\right)$ :

$$
\begin{equation*}
\delta I_{\Sigma_{n}} \equiv \int_{\Sigma_{n}} E d \Omega_{n}+\int_{\Sigma_{n-1}} E \Omega_{n} . \tag{2.6}
\end{equation*}
$$

In (2.6), $d$ is the symbol for the symbolic differentiation in the theory of exterior differential forms, and $E$ is the symbol for the integral substitution in that same theory. $\Sigma_{n-1}$ denotes the closed manifold that is the boundary of the portion of the manifold $\Sigma_{n}$.

Upon applying the operator $d$ to $\Omega_{n}$, one will get:

$$
\begin{align*}
d \Omega_{n}= & {\left[\frac{\partial \mathcal{F}}{\partial y^{\alpha}} \theta^{\alpha}+\frac{\partial \mathcal{F}}{\partial y_{i}^{\alpha}} \theta_{i}^{\alpha}\right] d(x)+d X_{\alpha_{1}}^{i_{1}} \theta^{\alpha_{1}}(-1)^{i_{1}-1} d\left(i_{1}\right)-X_{\alpha_{1}}^{i_{1}} \theta_{i_{1}}^{\alpha_{1}} d(x) } \\
& -\frac{(-1)^{i_{1}-1}}{(2!)^{2}} X_{\alpha_{1} \alpha_{2}}^{i_{i} i_{2}} d y_{i_{2}}^{\alpha_{1}} \theta^{\alpha_{2}} d\left(i_{1}\right)-\frac{(-1)_{1}^{i_{1}-1}}{(2!)^{2}} X_{\alpha_{1} \alpha_{2}}^{i_{i_{2}}} d y_{i_{1}}^{\alpha_{2}} \theta^{\alpha_{1}} d\left(i_{2}\right)+\ldots \tag{2.7}
\end{align*}
$$

after reduction, in which the unwritten terms each contain at least two of the forms $\theta^{\alpha}$ as a factor.
If one sets:

$$
\begin{equation*}
\omega^{\alpha}=\delta y^{\alpha}-y_{i}^{\alpha} \delta x^{i}, \quad \omega_{i}^{\alpha}=\delta y_{i}^{\alpha}-y_{i j}^{\alpha} \delta x^{j} \tag{2.8}
\end{equation*}
$$

then one will have:

$$
E d \Omega_{n}=\frac{\partial \mathcal{F}}{\partial y^{\alpha}} \omega^{\alpha} d(x)+\left(\frac{\partial \mathcal{F}}{\partial y_{i}^{\alpha}}-X_{i}^{\alpha}\right) \omega_{\alpha}^{i} d(x)
$$

[^4]\[

$$
\begin{equation*}
+(-1)^{i-2} d X_{\alpha_{1}}^{i_{1}} \omega^{\alpha} d\left(i_{1}\right)+\frac{(-1)^{i_{1}-2}}{(2!)^{2}} X_{\alpha_{1} \alpha_{2}}^{i_{1} i_{2}} d y_{i_{2}}^{\alpha_{1}} \theta^{\alpha_{2}} d\left(i_{1}\right)-\frac{(-1)^{i_{1}-2}}{(2!)^{2}} X_{\alpha_{1} \alpha_{2}}^{i_{i}} d y_{i_{1}}^{\alpha_{2}} \theta^{\alpha_{1}} d\left(i_{2}\right)+\ldots \tag{2.9}
\end{equation*}
$$

\]

in which the unwritten terms once more each contain one of the forms $\theta^{\alpha}$ as a factor at least once.
Upon denoting the form on the space of $x^{1}, \ldots, x^{n}$ that is obtained by referring $E d \Omega_{n}$ to the manifold $\Sigma_{n}$ by $\left[E d \Omega_{n}\right]_{\Sigma_{n}}$ then one will have, by virtue of (2.9) and 2.2):
$\left[E d \Omega_{n}\right]_{\Sigma_{n}}$

$$
\begin{equation*}
=\left[\frac{\partial \mathcal{F}}{\partial y^{\alpha}}-\frac{d X_{\alpha}^{i}}{d x^{i}}\right] \omega^{\alpha} d(x)+\left[\frac{\partial \mathcal{F}}{\partial y_{i}^{\alpha}}-X_{\alpha}^{i}\right] \omega_{i}^{\alpha} d(x)+\frac{1}{2!} X_{\alpha \beta}^{i_{1} i_{2}}\left(\frac{d y_{i_{2}}^{\beta}}{d x^{i_{1}}}-\frac{d y_{i_{1}}^{\beta}}{d x^{i_{2}}}\right) \omega^{\alpha} d(x), \tag{2.10}
\end{equation*}
$$

in which $d / d x^{i}$ is the complete partial derivation with respect to $x^{i}$ :

$$
\begin{equation*}
\frac{d}{d x^{i}}=\frac{\partial}{\partial x^{i}}+\frac{\partial}{\partial y^{\alpha}} \frac{\partial y^{\alpha}(x)}{\partial x^{i}}+\frac{\partial}{\partial y_{j}^{\alpha}} \frac{\partial^{2} y^{\alpha}(x)}{\partial x^{i} \partial x^{j}} . \tag{2.11}
\end{equation*}
$$

We note that the third term on the right-hand side of (2.10) is zero, by virtue of the permutability of the second derivatives of the functions $y^{\alpha}(x)$.

On the other hand:

$$
\begin{equation*}
E \Omega_{n}=(-1)^{i-1} \mathcal{F} \delta x^{i} d(i)+X_{\alpha}^{i}(-1)^{i-1} d(i)+\ldots \tag{2.12}
\end{equation*}
$$

in which the unwritten terms each contain one of the $\theta^{\alpha}$ as a factor at least once, so:

$$
\begin{equation*}
\left[E \Omega_{n}\right]_{\Sigma_{n}}=(-1)^{i-1}\left(\mathcal{F} \delta x^{i}+X_{\alpha}^{i}\right) d(i) \tag{2.13}
\end{equation*}
$$

on $\Sigma_{n}$. If one substitutes (2.13) and (2.10) into (2.6) then one will get:

$$
\begin{align*}
& \delta I_{n} \\
& \quad=\int_{D_{n}}\left[\frac{\partial \mathcal{F}}{\partial y^{\alpha}}-\frac{d X_{\alpha}^{i}}{d x^{i}}\right] \omega^{\alpha} d(x)+\left[\frac{\partial \mathcal{F}}{\partial y_{i}^{\alpha}}-X_{\alpha}^{i}\right] \omega_{i}^{\alpha} d(x)+\oint_{D_{n-1}}(-1)^{i-1}\left(\mathcal{F} \delta x^{i}+X_{\alpha}^{i}\right) d(i), \tag{2.14}
\end{align*}
$$

in which $D_{n-1}$ denotes the boundary of the domain $D_{n}$ of the $x^{1}, \ldots, x^{n}$. Formulas (2.10), (2.13), and (2.14) are valid for any values of the functions $X_{\alpha}^{i}$. If one sets:

$$
\begin{equation*}
X_{\alpha}^{i} \equiv \frac{\partial \mathcal{F}}{\partial y_{i}^{\alpha}}, \tag{2.15}
\end{equation*}
$$

in particular, then formulas (2.10) and (2.13) will become:

$$
\begin{equation*}
\left[E d \Omega_{n}\right]_{\Sigma_{n}}=\left(\frac{\partial \mathcal{F}}{\partial y^{\alpha}}-\frac{d}{d x^{i}} \frac{\partial \mathcal{F}}{\partial y_{i}^{\alpha}}\right) \omega^{\alpha} d(x) \tag{2.16}
\end{equation*}
$$

and

$$
\begin{equation*}
\left[E \Omega_{n}\right]_{\Sigma_{n}}=(-1)^{i-1}\left(\mathcal{F} \delta x^{i}+\frac{\partial \mathcal{F}}{\partial y_{i}^{\alpha}} \omega^{\alpha}\right) d(i) \tag{2.17}
\end{equation*}
$$

and (2.14) can be written:

$$
\begin{equation*}
\delta I_{\Sigma_{n}}=\int_{D_{n}}\left(\frac{\partial \mathcal{F}}{\partial y^{\alpha}}-\frac{d}{d x^{i}} \frac{\partial \mathcal{F}}{\partial y_{i}{ }^{\alpha}}\right) \omega^{\alpha} d(x)+\oint_{D_{n-1}}(-1)^{i-1}\left(\mathcal{F} \delta x^{i}+\frac{\partial \mathcal{F}}{\partial y_{i}^{\alpha}} \omega^{\alpha}\right) d(i) \tag{2.18}
\end{equation*}
$$

We thus recover the usual form for the first variation of $I_{n}$.
By contrast, upon setting:

$$
\begin{equation*}
X_{\alpha}^{i} \equiv 0 \tag{2.19}
\end{equation*}
$$

the formula (2.14) will lead to the relation:

$$
\begin{equation*}
\delta I_{n}=\int_{D_{n}}\left(\frac{\partial \mathcal{F}}{\partial y^{\alpha}} \omega^{\alpha}+\frac{\partial \mathcal{F}}{\partial y_{i}^{\alpha}} \omega_{i}^{\alpha}\right) d(x)+\oint_{D_{n-1}}(-1)^{i-1} \mathcal{F} \delta x^{i} d(i) \tag{2.20}
\end{equation*}
$$

or rather, upon applying the generalized Stokes theorem:

$$
\begin{equation*}
\delta I_{n}=\int_{D_{n}}\left(\frac{\partial \mathcal{F}}{\partial y^{\alpha}} \omega^{\alpha}+\frac{\partial \mathcal{F}}{\partial y_{i}^{\alpha}} \omega_{i}^{\alpha}+\frac{d\left(\mathcal{F} \delta x^{i}\right)}{d x^{i}}\right) d(x) \tag{2.21}
\end{equation*}
$$

That form of the first variation of $I_{n}$ is useful in the calculus of the second variation of $I_{n}$.

## Remarks:

a) If one refers to (2.7) then one will see that the condition:

$$
\begin{equation*}
X_{\alpha}^{i} \equiv \frac{\partial \mathcal{F}}{\partial y_{i}^{\alpha}} \tag{2.22}
\end{equation*}
$$

is necessary and sufficient for the congruence:

$$
\begin{equation*}
d \Omega_{n} \equiv 0 \quad \bmod \quad\left(\theta^{1}, \ldots, \theta^{m}\right) \tag{2.23}
\end{equation*}
$$

to be valid.
b) We point out that instead of considering the $X$ in the form $\Omega_{n}$ to be arbitrary functions of the arguments $x^{i}, y^{\alpha}, y_{i}^{\alpha}$, we can consider them to be arbitrary parameters. In particular, we can consider them to be independent variables on a par with the $x^{i}, y^{\alpha}, y_{i}^{\alpha}$. From the latter standpoint, $\Omega_{n}$ is an $n$-tuple form on the space of $x^{i}, y^{\alpha}, y_{i}^{\alpha}, X_{\alpha_{1}, \ldots, \alpha_{p}}^{i_{1} \cdots i_{p}}$, which is a space in which the manifolds $\Sigma_{n}$ will be defined by adding the trivial relations:

$$
\begin{equation*}
X_{\alpha_{1}, \ldots, \alpha_{p}}^{i_{1} \cdots i_{p}}=X_{\alpha_{1}, \ldots, \alpha_{p}}^{i_{1} \ldots i_{p}} \tag{2.24}
\end{equation*}
$$

to equations (1.6).
3. The forms $\Omega_{n}$ of minimum rank. - In the present section, we propose to determine those of the forms $C_{n}$ of minimum rank from among the forms $\Omega_{n}$.

## Theorem:

In order for a n-tuple differential form (that is not identically zero) $\left(^{1}\right.$ ):

$$
\begin{equation*}
A_{n} \equiv \frac{1}{n!} A_{\mu_{1} \cdots \mu_{n}}\left(z^{1}, \ldots, z^{N}\right) d z^{\mu_{1}} \cdots d z^{\mu_{n}} \quad\left(\mu_{1}, \ldots, \mu_{n}=1,2, \ldots, N\right) \tag{3.1}
\end{equation*}
$$

in the space of $z^{1}, \ldots, z^{N}$ to have minimum rank $n$, it is necessary and sufficient that those coefficients verify the D'Ovidio relations:
( ${ }^{1}$ ) Recall that, by definition, the $A_{\mu_{1} \cdots \mu_{n}}$ are completely antisymmetric in their indices $\mu_{1}, \ldots, \mu_{n}$ in such a way that one will likewise have:

$$
A_{n} \equiv \sum_{\mu_{1}<\mu_{2}<\cdots<\mu_{n}} A_{\mu_{1} \cdots \mu_{n}} d z^{\mu_{1}} \cdots d z^{\mu_{n}}
$$

as the expression for $A_{n}$.

$$
\begin{equation*}
A_{\mu_{1} \cdots \mu_{n}} A_{v_{1} \cdots v_{n}}=\sum_{i=1}^{n} A_{v_{i} \mu_{2} \cdots \mu_{n}} A_{v_{1} \cdots v_{i-1} \mu_{1} v_{i+1} \cdots v_{n}} \quad\left(\mu_{1}, \ldots, \mu_{n}, v_{1}, \ldots, v_{n}=1,2, \ldots, N\right) \tag{3.2}
\end{equation*}
$$

Proof. - By definition, the form $A_{n}$ has minimum rank (i.e., rank $n$ ) when there exist $n$ 1-tuple forms (i.e., Pfaff forms):

$$
\begin{equation*}
\omega^{i} \equiv A_{\mu}^{i} d z^{\mu}, \quad \mu=1,2, \ldots, N, \quad i=1,2, \ldots, n \tag{3.3}
\end{equation*}
$$

whose exterior product reduces to $A_{n}$ :

$$
\begin{equation*}
A_{n}=\prod_{i=1}^{n} \omega^{1} \cdot \omega^{2} \cdots \omega^{n} \tag{3.4}
\end{equation*}
$$

a) The condition (3.2) is necessary. - By hypothesis, one has the relation (3.4), from which, one must deduce the relations (3.2).

Consider the associated system to $A_{n}$, namely, the system of Pfaff forms:

$$
\omega^{\mu_{2} \cdots \mu_{n}} \equiv \sum_{\mu_{1}=1}^{N} A_{\mu_{1} \mu_{2} \cdots \mu_{n}} d z^{\mu_{1}} .
$$

By virtue of (3.4), each term in $\omega^{\mu_{2} \cdots \mu_{n}}$ contains one of the forms $\omega^{1}$ as a factor. It will then follow that the exterior product of $A_{n}$ with $\omega^{\mu_{2} \cdots \mu_{n}}$ will be identically zero. One will then have:

$$
\begin{gather*}
\left(\sum_{\mu_{1}=1}^{N} A_{\mu_{1} \mu_{2} \cdots \mu_{n}} d z^{\mu_{1}}\right)\left(\frac{1}{n!} A_{v_{1} \cdots v_{n}} d z^{\nu_{1}} \cdots d z^{v_{n}}\right)=\frac{1}{n!} \sum_{\mu_{1}=1}^{N} A_{\mu_{1} \mu_{2} \cdots \mu_{n}} A_{v_{1} \cdots v_{n}} d z^{\mu_{1}} d z^{v_{1}} \cdots d z^{v_{n}} \\
=\frac{1}{(n+1)!}\left(A_{v_{1} \mu_{2} \cdots \mu_{n}} A_{v_{1} \cdots v_{n}}-\sum_{i=1}^{N} A_{v_{i} \mu_{2} \cdots \mu_{n}} A_{v_{1} \cdots v_{i-1} v_{1} v_{n+1} \cdots v_{n}}\right) d z^{v_{1}} d z^{v_{2}} \cdots d z^{v_{n}}, \tag{3.5}
\end{gather*}
$$

which gives the relations (3.2).
Remark. - We point out that one has:

$$
\prod_{i=1}^{n}\left(A_{\mu}^{i} d z^{\mu}\right)=\frac{1}{n!}\left|\begin{array}{lll}
A_{\mu_{1}}^{1} & \cdots & A_{\mu_{n}}^{1}  \tag{3.6}\\
A_{\mu_{1}}^{n} & \cdots & A_{\mu_{n}}^{n}
\end{array}\right| d z^{\mu_{1}} \cdots d z^{\mu_{n}}
$$

and therefore, by virtue of (3.4)

$$
A_{\mu_{1} \cdots \mu_{n}} \equiv\left|\begin{array}{ccc}
A_{\mu_{1}}^{1} & \cdots & A_{\mu_{n}}^{1} \\
A_{\mu_{1}}^{n} & \cdots & A_{\mu_{n}}^{n}
\end{array}\right| .
$$

The D'Ovidio relations are the relations that necessarily exist between the determinants (3.6').
$b$ ). The condition is sufficient. - By hypothesis, the coefficients of $A_{n}$ verify the D'Ovidio relations (3.2). One must show that one can construct $n$ Pfaff forms whose exterior product reduces to $A_{n}$.

Since, by hypothesis, $A_{n}$ is not identically zero, at least one of its coefficients must be nonzero, namely:

$$
\begin{equation*}
A_{12 \ldots n} \neq 0 . \tag{3.7}
\end{equation*}
$$

Choose $n^{2}$ functions $A_{j}^{i}(i, j=1,2, \ldots, n)$ arbitrarily, but such that:

$$
\left|A_{i}^{j}\right| \equiv\left|\begin{array}{lll}
A_{1}^{1} & \cdots & A_{n}^{1}  \tag{3.8}\\
A_{1}^{n} & \cdots & A_{n}^{n}
\end{array}\right|=A_{12 \ldots n} .
$$

Then consider the system of $n(N-n)$ equations in the $n(N-n)$ unknowns $A_{n+\alpha}^{j}$ :

$$
\begin{gather*}
\left|\begin{array}{ccccccc}
A_{1}^{n} & \cdots & A_{i-1}^{n} & A_{n+\alpha}^{n} & A_{i+1}^{n} & \cdots & A_{n}^{n} \\
A_{1}^{1} & \cdots & A_{i-1}^{1} & A_{n+\alpha}^{1} & A_{n+\alpha}^{1} & \cdots & A_{n}^{1}
\end{array}\right| \equiv \sum_{j=1}^{n} \frac{\partial\left|A_{i}^{j}\right|}{\partial A_{i}^{j}} A_{n+\alpha}^{j}=A_{1 \ldots(i-1)(n+\alpha)(i+1) \ldots n} \\
\alpha=1,2, \ldots, N-n, \quad i=1,2, \ldots, n . \tag{3.9}
\end{gather*}
$$

The system (3.9) is composed of $m \equiv N-n$ subsystems that each have the adjoint determinant to (3.8) for their characteristic determinant. By virtue of (3.7), we are then in the presence of a Cramer system, and we can determine the $A_{n+\alpha}^{j}$ as functions of the $A_{1 \ldots(i-1)(n+\alpha)(i+1) \ldots n}$ and the $A_{i}^{j}$. We are then in possession of $n N$ quantities $A_{\mu}^{i}\binom{i=1,2, \ldots, n}{\mu=1,2, \ldots, N}$ that verify the conditions (3.8) and (3.9).

Now, by hypothesis, the $A_{\mu_{1} \cdots \mu_{n}}$ verify the D'Ovidio relations. Thanks to them, one can effortlessly see all of the $A_{\mu_{1} \cdots \mu_{n}}$ expressed in a well-defined manner as functions of the $A_{12 \ldots n}$ and $A_{1 \ldots(i-1)(n+\alpha)(i+1) \ldots n}$, and therefore as functions of the $A_{\mu}^{i}[(3.8),(3.9)]$.

Upon performing the calculations, one will finally have:

$$
A_{\mu_{1} \cdots \mu_{n}}=\left|\begin{array}{ccc}
A_{\mu_{1}}^{1} & \cdots & A_{\mu_{n}}^{1}  \tag{3.10}\\
A_{\mu_{1}}^{n} & \cdots & A_{\mu_{n}}^{n}
\end{array}\right|,
$$

i.e., by virtue of (3.6):

$$
\begin{equation*}
A_{n}=\prod_{i=1}^{n} A_{\mu}^{i} d z_{\mu} \tag{3.11}
\end{equation*}
$$

Remark. - The reasoning that we just presented shows that the D'Ovidio relations are sufficient for (3.10), when considered to be a system of equations in the unknowns $A_{\mu}^{i}$, to admit a solution. We showed in (3.6') that it is likewise necessary.

## Theorem:

In order for one of the forms $\Omega_{n}(1.9)$ to have minimum rank, it is necessary and sufficient that one has:

$$
X_{\alpha_{1} \cdots \alpha_{p}}^{i_{1} \cdots i_{p}} \equiv \frac{1}{\mathcal{F}^{p-1}}\left|\begin{array}{ccc}
X_{\alpha_{1}}^{i_{1}} & \cdots & X_{\alpha_{p}}^{i_{1}}  \tag{3.12}\\
X_{\alpha_{1}}^{i_{p}} & \cdots & X_{\alpha_{p}}^{i_{p}}
\end{array}\right| .
$$

Proof. - In order for $\Omega_{n}$ to have minimum rank, it is necessary and sufficient that its coefficients verify the D'Ovidio relations. Upon setting:

$$
\begin{align*}
\Omega_{n} \equiv & A_{12 \ldots n} d(x)+A_{\left(n+\alpha_{1}\right) 12 \ldots\left(i_{1}-1\right)\left(i_{1}+1\right) \ldots n} \theta^{\alpha_{1}} d\left(i_{1}\right) \\
& +A_{\left(n+\alpha_{1}\right)\left(n+\alpha_{2}\right) 12 \ldots\left(i_{1}-1\right)\left(i_{1}+1\right) \ldots\left(i_{2}-1\right)\left(i_{2}+1\right) \ldots n} \theta^{\alpha_{1}} \theta^{\alpha_{2}} d\left(i_{1}, i_{2}\right)+\ldots, \tag{3.13}
\end{align*}
$$

one will then have:

$$
\begin{align*}
A_{i_{1} 12 \ldots\left(i_{1}-1\right)\left(i_{1}+1\right) \ldots n} & A_{\left(n+\alpha_{1}\right)\left(n+\alpha_{2}\right) 12 \ldots\left(i_{1}-1\right)\left(i_{1}+1\right) \ldots\left(i_{2}-1\right)\left(i_{2}+1\right) \ldots n}  \tag{3.14}\\
& =A_{i_{1}\left(n+\alpha_{1}\right) 12 \ldots\left(i_{1}-1\right)\left(i_{1}+1\right) \ldots n} A_{i_{1}\left(n+\alpha_{2}\right) 12 \ldots\left(i_{1}-1\right)\left(i_{1}+1\right) \ldots\left(i_{2}-1\right)\left(i_{2}+1\right) \ldots n} \\
& +A_{i_{1}\left(n+\alpha_{2}\right) 12 \ldots\left(i_{1}-1\right)\left(i_{1}+1\right) \ldots n} A_{i_{1}\left(n+\alpha_{1}\right) 12 \ldots\left(i_{1}-1\right)\left(i_{1}+1\right) \ldots\left(i_{2}-1\right)\left(i_{2}+1\right) \ldots n},
\end{align*}
$$

so, upon specifying the coefficients of $\Omega_{n}$ :

$$
\begin{equation*}
\left((-1)^{i_{1}-1} \mathcal{F}\right)\left(X_{\alpha_{1} \alpha_{2}}^{i_{1} i_{2}}\right)=\left((-1)^{i_{1}-1} X_{\alpha_{1}}^{i_{1}}\right)\left((-1)^{i_{2}+i_{1}-1} X_{\alpha_{2}}^{i_{2}}\right) \tag{3.15}
\end{equation*}
$$

and finally:

$$
X_{\alpha_{1} \alpha_{2}}^{i_{i} i_{2}}=\frac{1}{\mathcal{F}}\left|\begin{array}{ll}
X_{\alpha_{1}}^{i_{1}} & X_{\alpha_{2}}^{i_{1}}  \tag{3.16}\\
X_{\alpha_{1}}^{i_{2}} & X_{\alpha_{2}}^{i_{2}}
\end{array}\right|
$$

Similarly, one infers from the D'Ovidio relations:

$$
\begin{align*}
& A_{i_{1} 12 \ldots\left(i_{1}-1\right)\left(i_{1}+1\right) \ldots n} A_{\left(n+\alpha_{1}\right)\left(n+\alpha_{2}\right)\left(n+\alpha_{3}\right) 12 \ldots\left(i_{1}-1\right)\left(i_{1}+1\right) \ldots\left(i_{2}-1\right)\left(i_{2}+1\right) \ldots\left(i_{3}-1\right)\left(i_{3}+1\right) \ldots n}  \tag{3.17}\\
& \quad=A_{\left(n+\alpha_{1}\right) 12 \ldots\left(i_{1}-1\right)\left(i_{1}+1\right) \ldots n} A_{i_{1}\left(n+\alpha_{2}\right)\left(n+\alpha_{3}\right) 12 \ldots\left(i_{1}-1\right)\left(i_{1}+1\right) \ldots\left(i_{2}-1\right)\left(i_{2}+1\right) \ldots\left(i_{3}-1\right)\left(i_{3}+1\right) \ldots n} \\
& \quad+A_{\left(n+\alpha_{2}\right) 12 \ldots\left(i_{1}-1\right)\left(i_{1}+1\right) \ldots . . n} A_{\left(n+\alpha_{1}\right) i_{1}\left(n+\alpha_{3}\right) 12 \ldots\left(i_{1}-1\right)\left(i_{1}+1\right) \ldots\left(i_{2}-1\right)\left(i_{2}+1\right) \ldots\left(i_{3}-1\right)\left(i_{3}+1\right) \ldots n}
\end{align*}
$$

$$
+A_{\left(n+\alpha_{3}\right) 12 \ldots\left(i_{1}-1\right)\left(i_{1}+1\right) \ldots n} A_{\left(n+\alpha_{1}\right)\left(n+\alpha_{2}\right) i_{1} 12 \ldots\left(i_{1}-1\right)\left(i_{1}+1\right) \ldots\left(i_{2}-1\right)\left(i_{2}+1\right) \ldots\left(i_{3}-1\right)\left(i_{3}+1\right) \ldots n}
$$

upon taking (3.16) into account, that:

$$
X_{\alpha_{1} \alpha_{2} \alpha_{3}}^{i_{i} i_{3}}=\frac{1}{\mathcal{F}^{2}}\left|\begin{array}{ccc}
X_{\alpha_{1}}^{i_{1}} & X_{\alpha_{2}}^{i_{1}} & X_{\alpha_{3}}^{i_{1}}  \tag{3.18}\\
X_{\alpha_{1}}^{i_{2}} & X_{\alpha_{2}}^{i_{2}} & X_{\alpha_{3}}^{i_{2}} \\
X_{\alpha_{1}}^{i_{3}} & X_{\alpha_{2}}^{i_{3}} & X_{\alpha_{3}}^{i_{3}}
\end{array}\right|,
$$

Q.E.D.

## Theorem:

The forms $\Omega_{n}$ of minimum rank are given by the relation:

$$
\begin{equation*}
C_{n}=\frac{1}{\mathcal{F}^{n-1}} \prod_{i=1}^{n}\left(\mathcal{F} d x^{i}+X_{\alpha}^{i} \theta^{\alpha}\right) \tag{3.19}
\end{equation*}
$$

## Proof:

Upon referring to (3.8), set $\left(^{1}\right)$ :

$$
\begin{equation*}
A_{i}^{j} \equiv \delta_{i}^{j} \mathcal{F}^{1 / n} \tag{3.20}
\end{equation*}
$$

in such a way that one will have:

$$
\left|A_{i}^{j}\right|=\mathcal{F} .
$$

The analogue of the system (3.9) is:

$$
\sum_{i} A_{n+\alpha}^{j} \bar{A}_{i}^{j}=X_{\alpha}^{i}, \quad \text { with } \quad \bar{A}_{i}^{j} \equiv \frac{\partial\left|A_{i}^{j}\right|}{\partial A_{i}^{j}}
$$

here, so:

$$
A_{n+\alpha}^{j}=X_{\alpha}^{i} \overline{\bar{A}}_{i}^{j}, \quad \text { with } \quad \overline{\bar{A}}_{i}^{j} \equiv \frac{\partial\left|\bar{A}_{i}^{j}\right|}{\partial \bar{A}_{i}^{j}} .
$$

Upon specifying the values of $\overline{\bar{A}}_{i}{ }^{\alpha}$, one will get:

$$
A_{n+\alpha}^{i}=X_{\alpha}^{j} \delta_{j}^{i} \mathcal{F}^{(1-n)}=X_{\alpha}^{i} \mathcal{F}^{(1-n)}
$$

so, by virtue of (3.20):

$$
\begin{equation*}
C_{n}=\prod_{i=1}^{n}\left(\delta_{j}^{i} \mathcal{F}^{1 / n} d x^{j}+X_{\alpha}^{i} \mathcal{F}^{(1-n)} \theta^{\alpha}\right)=\frac{1}{\mathcal{F}^{n-1}} \prod_{i=1}^{n}\left(\mathcal{F} d x^{j}+X_{\alpha}^{i} \theta^{\alpha}\right) \tag{Q.E.D.}
\end{equation*}
$$

$$
\text { ( } \left.{ }^{1}\right) \quad \delta_{j}^{i}=\left\{\begin{array}{ll}
0 & i \neq j \\
1 & i=j
\end{array} .\right.
$$

Remark. - Note that since:

$$
\theta^{\alpha} \equiv d y^{\alpha}-y_{j}^{\alpha} d x^{j},
$$

(3.19) can be put into the form:

$$
\begin{equation*}
C_{n}=\frac{1}{\mathcal{F}^{n-1}} \prod_{i=1}^{n}\left[\left(\mathcal{F} \delta_{i}^{j}-X_{\alpha}^{i} y_{j}^{\alpha}\right) d x^{j}+X_{\alpha}^{i} d y^{\alpha}\right] \tag{3.21}
\end{equation*}
$$

i.e.:

$$
\begin{equation*}
C_{n}=\prod_{i=1}^{n}\left(H_{j}^{i} d x^{j}+p_{\alpha}^{i} d y^{\alpha}\right) \tag{3.22}
\end{equation*}
$$

upon setting:

$$
\begin{equation*}
H_{j}^{i} \equiv \frac{\mathcal{F} \delta_{i}^{j}-X_{\alpha}^{i} y_{j}^{\alpha}}{\mathcal{F}^{n-1}} \quad \text { and } \quad \quad p_{\alpha}^{i} \equiv X_{\alpha}^{i} \tag{3.23}
\end{equation*}
$$

4. Geodesic fields in the calculus of variations. - Once again, consider the integral $I_{n}$ (1.1) and the forms $\Omega_{n}(1.9)$ that we associated with it in § $\mathbf{1}$. Consider the $X_{\alpha_{1}, \ldots, \alpha_{p}}^{i_{i} \cdots i_{p}}$ in those forms to be arbitrary parameters (see § 2, Remark b).

A geodesic field relative to the integral $I_{n}$ will be, by definition, any set of functions:

$$
\begin{equation*}
\bar{y}_{i}^{\alpha}\left(x^{i}, y^{\alpha}\right), \quad \bar{X}_{\alpha_{1}, \ldots, \alpha_{p}}^{i_{1} \cdots i_{p}}\left(x^{i}, y^{\beta}\right) \quad((x, y) \in R) \tag{4.1}
\end{equation*}
$$

that gives rise to the relation:

$$
\begin{equation*}
\oint_{\Sigma_{n}} \bar{\Omega}_{n}=0, \tag{4.2}
\end{equation*}
$$

in which $\bar{\Omega}_{n}$ denotes the differential form on the space $\mathcal{E}_{n+m}$ of $x^{i}, y^{\alpha}$ that is obtained by replacing the arguments $y_{i}^{\alpha}, X_{\alpha_{1}, \ldots, \alpha_{p}}^{i_{1} \cdots i_{p}}$ in $\Omega_{n}$ with the functions $\bar{y}_{i}^{\alpha}(x, y), \bar{A}_{\alpha_{1}, \ldots, \alpha_{p}}^{i_{1} \cdots i_{p}}(x, y)$, and in which $\Sigma_{n}$ denotes an arbitrary closed oriented n-fold manifold in a region $R$ in the space $\mathcal{E}_{n+m}$.

Suppose that the portion of the manifold $\dot{\Sigma}_{n}$ that is defined by the equations:

$$
\begin{align*}
x^{i} & =x^{i} \\
y^{\alpha} & =\dot{y}^{\alpha}\left(x^{i}\right) \tag{4.3}
\end{align*}
$$

is given on $R$.
By definition, a geodesic field $\bar{y}_{i}^{\alpha}(x, y), \bar{A}_{\alpha_{1}, \ldots, \alpha_{p}}^{i_{i} \cdots i_{p}}(x, y)$ incorporates the portion of the manifold $\dot{\Sigma}_{n}$ if one has:

$$
\begin{equation*}
\bar{y}_{i}^{\alpha}(x, y)=\frac{\partial \dot{y}^{\alpha}(x)}{\partial x^{i}} \quad(x \in D), \tag{4.4}
\end{equation*}
$$

in which the domain $D$ is supposed to be interior to the projection of $R$ onto the space of $x^{1}, \ldots$, $x^{n}$ 。

With those definitions, suppose that a portion of the manifold (4.3) is given (but not necessarily an extremal of $\left.I_{n}(!)\right]$ and a geodesic field (4.1) that incorporates $\dot{\Sigma}_{n}$ and is defined in a region $R$ of $\mathcal{E}_{n+m}$ (that does not necessarily contain $\dot{\Sigma}_{n}$ ). Let $\dot{\Sigma}_{n}$ denote a portion of the arbitrary $n$-fold manifold that is interior to $R$ and is defined by the equations:

$$
\begin{align*}
x^{i} & =x^{i} \\
y^{\alpha} & =y^{\alpha}\left(x^{i}\right) \tag{4.5}
\end{align*} \quad(x \in D),
$$

but nonetheless restricted to admitting the same boundary as $\dot{\Sigma}_{n}$, i.e., it is such that:

$$
\begin{equation*}
y^{\alpha}=\dot{y}^{\alpha}\left(x^{i}\right) \quad\left(x \in D_{n-1}\right) \tag{4.6}
\end{equation*}
$$

when one lets $D_{n-1}$ denote the boundary of the domain $D$ in the space of $x^{1}, \ldots, x^{n}$. The manifold in the space $\mathcal{E}_{n+m}$ that is composed of the set of manifolds $\dot{\Sigma}_{n}$ and $\hat{\Sigma}_{n}$ (suitably oriented) is a closed oriented manifold in the region $R$. Now, by virtue of (4.2), and with convenient orientations on $\dot{\Sigma}_{n}$ and $\hat{\Sigma}_{n}$ :

$$
\begin{equation*}
\int_{\dot{\Sigma}_{n}} \bar{\Omega}_{n}=\int_{\hat{\Sigma}_{n}} \bar{\Omega}_{n} . \tag{4.7}
\end{equation*}
$$

On the other hand, by virtue of the incorporation relation (4.4) and the property of $\Omega_{n}$ that is congruent to $\omega_{n} \bmod \theta^{\alpha}$, one can conclude that:

$$
\begin{equation*}
\int_{D} \mathcal{F}\left(x^{i}, y^{\alpha}(x), \frac{\partial \dot{y}^{\alpha}(x)}{\partial x^{i}}\right) d x^{1} \cdots d x^{n}=\int_{\Sigma_{n}^{0}} \bar{\Omega}_{n} \tag{4.8}
\end{equation*}
$$

The properties (4.7) and (4.8) are the ones that define the independent integral that Hilbert (see [footnote $\left({ }^{1}\right)$ on pp. 1] introduced in the case of a simple integral. They suffice for one to construct an excess function and obtain a Weierstrass formula.

Upon setting:

$$
\begin{equation*}
\Delta I \equiv \int_{D} \mathcal{F}\left(x^{i}, y^{\alpha}(x), \frac{\partial y^{\alpha}(x)}{\partial x^{i}}\right) d x^{1} \cdots d x^{n}-\int_{D} \mathcal{F}\left(x^{i}, \dot{y}^{\alpha}(x), \frac{\partial \dot{y}^{\alpha}(x)}{\partial x^{i}}\right) d x^{1} \cdots d x^{n} \tag{4.9}
\end{equation*}
$$

the relations (4.7) and (4.8) will indeed lead to the relation:

$$
\begin{equation*}
\Delta I=\int_{\hat{\Sigma}_{n}} \omega_{n}-\bar{\Omega}_{n}, \tag{4.10}
\end{equation*}
$$

in which $\hat{\Sigma}_{n}$ is defined in the space of the $x^{i}, y^{\alpha}, y_{i}^{\alpha}$ by adding the equation:

$$
\begin{equation*}
y_{i}^{\alpha}=\frac{\partial y^{\alpha}(x)}{\partial x^{i}} \tag{4.11}
\end{equation*}
$$

to equations (4.5). Upon referring to the space of the $x^{1}, \ldots, x^{n}$, (4.10) can be written:

$$
\begin{equation*}
\Delta I=\int_{D} \mathcal{E}\left(x^{i}, y^{\alpha}(x), \frac{\partial y^{\alpha}(x)}{\partial x^{i}}, \bar{y}_{i}^{\alpha}\left(x, y^{\alpha}(x)\right), A_{\alpha_{1}, \ldots, \alpha_{p}}^{i_{1} \cdots i_{p}}\left(x, y^{\alpha}(x)\right)\right) d x^{1} \cdots d x^{n} \tag{4.12}
\end{equation*}
$$

in which the function $\mathcal{E}$ is defined by the relation:

$$
\begin{align*}
\mathcal{E}\left(x^{i}, y^{\alpha}, \bar{y}_{i}^{\alpha}, \bar{X}_{\alpha_{1}, \ldots, \alpha_{p}}^{i_{1} \cdots i_{p}}\right) \equiv \mathcal{F}\left(x^{i}, y^{\alpha}, y_{i}^{\alpha}\right)-\mathcal{F}\left(x^{i}, y^{\alpha}, \bar{y}_{i}^{\alpha}\right) \\
-\bar{X}_{\alpha_{1}}^{i_{1}}\left(y_{i_{1}}^{\alpha_{1}}-\bar{y}_{i_{1}}^{\alpha_{1}}\right)-\sum_{p=2}^{l} \bar{X}_{\alpha_{1}, \ldots, \alpha_{p}}^{i_{1} \cdots i_{p}}\left|\begin{array}{ccc}
\left(y_{i_{2}}^{\alpha_{2}}-\bar{y}_{i_{2}}^{\alpha_{2}}\right) & \cdots & \left(y_{i_{2}}^{\alpha_{p}}-\bar{y}_{i_{p}}^{\alpha_{p}}\right) \\
\left(y_{i_{p}}^{\alpha_{2}}-\bar{y}_{i_{p}}^{\alpha_{2}}\right) & \cdots & \left(y_{i_{p}}^{\alpha_{p}}-\bar{y}_{i_{p}}^{\alpha_{p}}\right)
\end{array}\right|, \tag{4.13}
\end{align*}
$$

in which $l$ denotes the smaller of the two number $n$ and $m$, as always.

Remark. - Upon considering the $X_{\alpha_{1}, \ldots, \alpha_{p}}^{i_{1} \cdots i_{p}}$ to be functions of the $x^{i}, y^{\alpha}, y_{i}^{\alpha}$, the functions $\bar{X}_{\alpha_{1}, \ldots, \alpha_{p}}^{i_{1} \cdots i_{p}}$, which define a geodesic field, are obtained by starting with convenient functions $y_{i}^{\alpha}\left(x, y^{\beta}\right)$ and setting:

$$
\begin{equation*}
\bar{X}_{\alpha_{1}, \ldots, \alpha_{p}}^{i_{1} \cdots i_{p}}\left(x, y^{\beta}\right) \equiv X_{\alpha_{1}, \ldots, \alpha_{p}}^{i_{1} \cdots i_{p}}\left(x, y^{\alpha}, \bar{y}_{i}^{\alpha}\left(x, y^{\beta}\right)\right) . \tag{4.14}
\end{equation*}
$$

That is the point of view that Th. Lepage assumed in his paper that was cited above ${ }^{1}$ ). Upon adopting that viewpoint, any choice of functions $X_{\alpha_{1}, \ldots, \alpha_{p}}^{i_{i} \cdots i_{p}}\left(x, y^{\alpha}, y_{i}^{\alpha}\right)$ will correspond to a particular type of geodesic field and a particular excess function:

$$
\begin{align*}
& \mathcal{E}\left(x^{i}, y^{\alpha}, y_{i}^{\alpha}, \bar{y}_{i}^{\alpha}\right) \equiv \mathcal{F}\left(x^{i}, y^{\alpha}, y_{i}^{\alpha}\right)-\mathcal{F}\left(x^{i}, y^{\alpha}, \bar{y}_{i}^{\alpha}\right) \\
- & \bar{X}_{\alpha_{2}}^{i_{1}}\left(x, y^{\alpha}, \bar{y}_{i}^{\alpha}\right)\left(y_{i_{2}}^{\alpha_{1}}-\bar{y}_{i_{2}}^{\alpha_{1}}\right)-\sum_{p=2}^{l} \bar{X}_{\alpha_{1}, \ldots, \alpha_{p}}^{i_{1} \cdots i_{p}}\left(x, y^{\alpha}, \bar{y}_{i}^{\alpha}\right)\left|\begin{array}{ccc}
\left(y_{i_{2}}^{\alpha_{2}}-\bar{y}_{i_{2}}^{\alpha_{2}}\right) & \cdots & \left(y_{i_{2}}^{\alpha_{p}}-\bar{y}_{i_{2}}^{\alpha_{p}}\right. \\
\left(y_{i_{p}}^{\alpha_{2}}-\bar{y}_{i_{p}}^{\alpha_{2}}\right) & \cdots & \left(y_{i_{p}}^{\alpha_{p}}-\bar{y}_{i_{p}}^{\alpha_{p}}\right)
\end{array}\right| . \tag{4.15}
\end{align*}
$$

[^5]One infers from the relation (4.15) that:

$$
\begin{equation*}
\left(\frac{\partial \mathcal{E}}{\partial y_{i_{1}}^{\alpha_{1}}}\right)_{y_{i_{1}}^{\alpha_{1}}=\bar{y}_{i_{1}}^{\alpha_{1}}}=\frac{\partial \mathcal{F}}{\partial y_{i_{1}}^{\alpha_{1}}}-X_{\alpha_{1}}^{i_{1}} \tag{4.16}
\end{equation*}
$$

and one sees that the condition [see (2.17) and (2.24)]:

$$
\begin{equation*}
X_{\alpha}^{i}=\frac{\partial \mathcal{F}}{\partial y_{i}^{\alpha}} \tag{4.17}
\end{equation*}
$$

expresses the idea that $\mathcal{E}$ is stationary in $y_{i}^{\alpha}$ for $y_{i}^{\alpha}=\bar{y}_{i}^{\alpha}$, which explains the importance of that condition for questions regarding extrema.

To conclude, we remark that if we impose the condition (4.17) on the forms $C_{n}$ of minimum rank then we will obtain the form with respect to which Caratheodory's geodesic fields are defined $\left({ }^{1}\right)$. The De Donder-Weyl fields $\left({ }^{2}\right)$ are defined with respect to the form $H_{n}$ that is obtained by setting:

$$
\begin{equation*}
X_{\alpha_{1}}^{i_{1}}=\frac{\partial \mathcal{F}}{\partial y_{i_{i}}^{\alpha_{i}}}, \quad X_{\alpha_{1}, \ldots, \alpha_{p}}^{i_{1} \cdots i_{p}} \equiv 0 \quad(p=2, \ldots, l) \tag{4.48}
\end{equation*}
$$

in $\Omega_{n}$ and which can be written:

$$
\begin{equation*}
H_{n}=p_{\alpha}^{i} d x^{1} \cdots d x^{i-1} d y^{\alpha} d x^{i+1} \cdots d x^{n}-H d x^{1} \cdots d x^{n} \tag{4.19}
\end{equation*}
$$

upon setting:

$$
\begin{equation*}
p_{\alpha}^{i}=\frac{\partial \mathcal{F}}{\partial y_{i}^{\alpha}} \quad \text { and } \quad H=-\mathcal{F}+\frac{\partial \mathcal{F}}{\partial y_{i}^{\alpha}} y_{i}^{\alpha} . \tag{4.20}
\end{equation*}
$$

The form in (4.19) should be compared to the form in (3.22).

[^6]
[^0]:    (*) Presented by Th. De Donder.
    ${ }^{(1)}$ D. HILBERT, Gesammelte Abhandlungen, v. III, Springer, 1935, pp. 38-55.
    $\left(^{2}\right)$ P. V. PÂQUET, "La forme intégrale $H_{n}$ dans la théorie invariantive du Calcul des variations," Bull. Acad. roy. Belg., Cl. Sc. 12 (1936), 1259-1272.
    $\left({ }^{3}\right)$ E. CARTAN, Leçons sur les invariants intégraux, Paris, Hermann, 1922, ch. VII, pp. 65.
    $\left({ }^{4}\right)$ TH. DE DONDER, Théorie des invariants intégraux, Paris, Gauthier-Villars, 1927.

[^1]:    ( ${ }^{1}$ ) Loc. cit., pp. 17, second-to-last line; formula (88) in the author's note.
    $\left(^{2}\right)$ TH. DE DONDER, Théorie invariantive du Calcul des variations, Paris, Gauthier-Villars (new ed.), 1935, pp. 107, form. (622); form. (39) in the author's note.
    ${ }^{3}$ ) Idem, ibid., pp. 142, form. (612); forms. (47) and (48) in the author's note.
    $\left(^{4}\right)$ Doctoral thesis, Fac. Sc. Univ. of Brussells, June 1937.
    $\left({ }^{5}\right)$ H. WEYL, "The geodesic fields in the Calculus of variations," Ann. Math. 36 (1935), 607-629.
    $\left({ }^{6}\right)$ Th. LEPAGE, "Sur les champs géodésiques du Calcul des variations," Bull. Acad. roy. Belg., Cl. Sc., 10, pp. 716-729; pp. 1036-1046.

    Similarly, see the recent paper by H. BOERNER, "Über die Legendre Bedingung und die Feldtheorien in der Variationsrechnung der mehrfachen Integralen," Math. Zeit. 40, Heft 5 (1940), 720-742.

[^2]:    $\left({ }^{1}\right)$ A. ROSENBLATT, "Sur la variéte de Grassmann qui représente les espaces linéaires à $k$ dimensions contenus dans un espace linéaire à $n$ dimensions," Mém. Soc. roy. des Sc. de Liége 16, fasc. I, (1930), pp. 4, form. (2).
    $\left({ }^{2}\right)$ H. GRASSMANN, Ausdehnungslehre, Berlin, 1862.
    $\left(^{3}\right)$ E. CARTAN, Les espaces de Finsler, Act. Sc. et Ind., no. 79, Paris, Hermann, 1934.
    $\left(^{4}\right)$ E. CARTAN, Les espaces métriques fondés sur la notion d’aire, idem, no. 72, idem, 1933.
    $\left({ }^{5}\right)$ A. KAWAGUCHI and H. HOMEU, "Die Geometrie des Systems der partiellen Differentialgleichungen," J. of Fac. of Sc., Hokkaido, Imp. Univ. (1) 6, no. 1, and the bibliography that those authors gave.

[^3]:    $\left({ }^{1}\right)$ In the course of this paper, we shall make constant use of the well-known convention regarding the summation over dummy indices (i.e., indices that are repeated twice in a monomial).

[^4]:    $\left.{ }^{1}\right)$ Th. De Donder, loc. cit. in footnote $\left({ }^{4}\right)$ on pp. 1 of the present article; pp. 60, form. (138).

[^5]:    $\left({ }^{1}\right)$ Loc. cit., in footnote $\left({ }^{6}\right)$ on pp. 2 of the present article; pp. 720, form. (2.6).

[^6]:    $\left({ }^{1}\right)$ Idem., § 10.
    $\left(^{2}\right)$ Cf., footnote $\left({ }^{5}\right)$ on pp. 2.

